1 Problem Statement

Given a 75% probability that a soccer player will make any penalty kick,[1] what is the chance that any shootout will go into extra shots? (There are usually 5 rounds, one shot each per team, and if tied after this, extra rounds until there is a winner)

2 Answer

To find the answer analytically requires thinking about the probability of how the five shots for each team go. We can think of each team as a sequence of five shots such as

$$A = (0, 1, 0, 1, 0) B = (1, 0, 1, 0, 0) \tag{1}$$

where 1 means made the penalty kick and 0 means missed.

Let's first do the case of equal probability of making and missing to get a feel for the problem. In that case we want the probability that the sum of sequences A and B are equal after 5 shots. We can summarize this with the the binomial theorem

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \tag{2}$$

where a is the probability of making a shot and b is the probability of missing a shot. We see there is only one way to miss all shots and one way to make all shots. There are 5 ways of making 4 and missing one (and vice versa) and 10 ways of making 3 shots and missing 2 (and vice versa).

For any number A gets we can just use the probability of team B getting the same shot is the corresponding coefficient above. So for A making 3 shots and missing 2, B has a probability $10(0.5)^3(0.5)^2 = 10(0.5)^5 = 10/32 \approx 0.3125$.

This means we can square all of the probabilities and add them up to find our answer.[2] This yields

$$2\frac{1 + 5^2 + 10^2}{2^{10}} \approx 0.246 \tag{3}$$

In the case $a = 0.75$ and $b = 0.25$, then we use the above formula and find

$$\begin{aligned}&(0.75)^{10} + 5^2(0.75)^8(0.25)^2 + 10^2(0.75)^6(0.25)^4 + 10^2(0.75)^4(0.25)^6 \\ &+ 5^2(0.75)^2(0.25)^8 + (0.25)^{10} \approx 0.290\end{aligned} \tag{4}$$

Indeed, for any of possible probability a and b (so $b = 1 - a$) the answer is

$$\sum_{i=0}^{5} \binom{5}{i} a^{2(5-i)} b^{2i} \tag{5}$$

To confirm this, we can use simulation again.

[1] Each kick is thus independent of the other kicks.

[2] When team A gets 4 shots in, it has a probability of $5a^4b$. The chance team B then gets 4 shots in is also $5a^4b$. So the probability of these happening when each kick is independent is simply multiplying the probabilities together.

penalty_kicks.py

```
import numpy as np
import matplotlib.pyplot as plt

# Inspired from fivethirtyeight Riddler
# from July, 13, 2018
np.random.seed(1)

# calculates probability via simulation
# of many penalty kicks
# not worth optimizing to remove
# cases where we do not need to do all
# five kicks. (In any case, these
# should not affect the probability)
# Using array operations with numpy
# will almost certainly be faster
# than any loops
def penalty_kicks(trials):
  trials=int(trials)
  a=np.zeros((5,trials))
  b=np.zeros((5,trials))
  # put ones in for making goals
  mask1=np.random.rand(5,trials)
  mask2=np.random.rand(5,trials)
  mask1=np.where(mask1<.75)
  mask2=np.where(mask2<.75)
  a[mask1]=1
  b[mask2]=1
  # find total goals
  a=np.sum(a,0)
  b=np.sum(b,0)
  #determine who wins
  c=a-b
  # find where c=0
  nowins=1-np.count_nonzero(c)/float(trials)
  print(nowins)
  return nowins


a=penalty_kicks(5e5)
b=penalty_kicks(5e5)
c=penalty_kicks(5e5)
```