1 Problem Statement

You are in a city where only N in a hundred people wear glasses. The people who wear glasses behave identically to those who do not wear glasses. That is, wearing glasses is a random attribute of a person in this city. Therefore the probability of a person wearing glasses having done any particular action is $N/100 = x$. You know that a person helped your family, and you would like to find them. Name the helper S. You know that neighbor O saw the person who helped you. This neighbor O will 100% of the time identify a person with glasses as a person with glasses. However, they will also say that a person who is not wearing glasses, is wearing glasses M times in a hundred (so $c = M/100$ is the probability of a false positive). Given that this neighbor O says that the person who helped you wears glasses, how confident should be that your helper S actually does wear glasses?

2 Answer

This is a problem that explores Bayesian reasoning. I don't believe in unconditional probabilities, so we will use the conditionalized Bayesian relation

$$\Pr\left(H|E,B\right) = \frac{\Pr(E|H,B)\Pr(H|B)}{\Pr(E|B)} \tag{1}$$

where H stands for hypothesis, E for evidence, and B is background situation. Thus the above says that the probability of the hypothesis given the evidence and background situation is the probability of the evidence given the hypothesis and background situation times the probability of they hypothesis given only the background situation divided by the probability of the evidence given only the background situation.

Here the hypothesis H is that the person who helped your family S wears glasses. The evidence E is that the neighbor O says that the helper S wore glasses. The background situation B is that you are in a city where people who wear glasses are randomly distributed among the population at a rate of N in a hundred and that the neighbor O identifies glasses wearers 100% of the time, but misidentifies people who don't wear glasses as glasses wearers M in a hundred times (or with probability $c = M/100$).

Let's put some numbers to this to make it easier to see the idea. First we'll use an approximation that lets us deal with nice whole numbers and then I'll explain the true numbers. After that we'll translate this into probability with the above formula and show the results in general.

Suppose $N = 1$ and $M = 10$. Then we imagine the neighbor actually sees 100 randomly selected people. They would identify about 11 people out of the 100 as wearing glasses (1 in 10 of the not wearers misidentified and 1 true glasses wearer). Out of the eleven, only 1 person would actually be wearing glasses, so there is only a $1/11 \approx .091$ or about 9 in 100 chance that the helper wears glasses. Thus we went from a 1% expectation to an 11% expectation.

Now in reality the 10 people misidentified is too high. In fact we should use that the neighbor O would identify 1 person as wearing glasses and of the remaining 99 there would be a 1 in 10 shot, so 9.9 people on average are misidentified and so we should use $1/10.9 \approx 0.092$. This is a small correction in this problem, but you should always keep this distinction clear in your mind. If the difference is small, then you can use the round numbers to get an intuitive result.

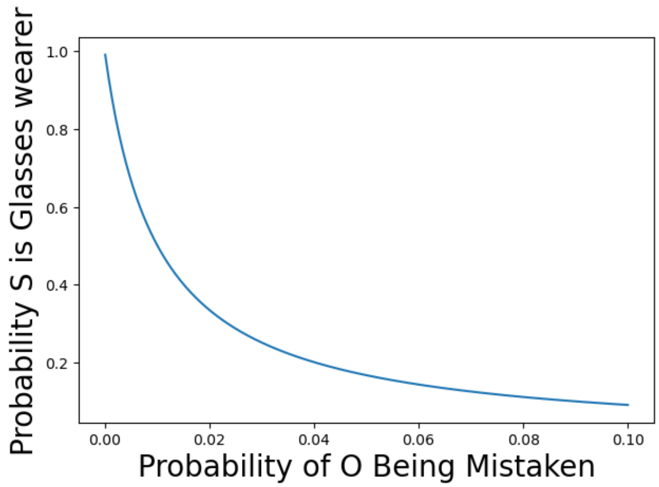

Figure 1: This shows the probability of the helper S wearing glasses as a function of c the probability neighbor O misidentifies a person as wearing glasses (when they do not) when $\Pr(H|B) = x = 0.01$. Clearly, if O is never mistaken, we should believe S is a glasses wearer, but we see even a small false positive rate (being mistaken) should lower our confidence quite a bit.

In general, we write

$$\Pr(H|E,B) = \frac{\Pr(E|H,B)\Pr(H|B)}{\Pr(E|B)} \tag{2}$$

$$\Pr(H|E,B) = \frac{(1)(x)}{x + c(1-x)} = \frac{x}{(1-c)x + c} \tag{3}$$

Let's test this with our numbers before, $x = 0.01$ and $c = 0.1$, so

$$\Pr(H|E,B) = \frac{0.01}{(1-0.1)(0.01) + 0.1} = \frac{0.01}{0.009 + 0.1} \approx 0.092 \tag{4}$$

as it should. We can then plot $\Pr(H|E,B)$ as a function of x and c. Let's fix $x = 0.01$ and plot it as a function of c. We get Figure 1.

It is also of interest to fix c and and plot $\Pr(H|E,B)$ as a function of x as in 2.

Suppose another neighbor Q then came forward and said that they also saw S was a glasses wearer. We can simply apply the conditional Bayesian formalism, but now with the background situation B including that neighbor O previously saw S as a glasses wearer which we can denote B_S. If these two neighbor observations were independent of each other (that is, O and Q didn't know about each other's testimony) then it will be easy to update the probability using

$$\Pr(H|E,B_S) = \frac{\Pr(E|H,B_S)\Pr(H|B_S)}{\Pr(E|B_S)} \tag{5}$$

because the probabilities including E are unaffected by O's observation. Otherwise, one has to think about how likely Q was to say the same thing as O.

The main thing to take away is that you can iteratively apply Bayes' theorem and get the estimate you are after.

Here is some code to generate the above plots:

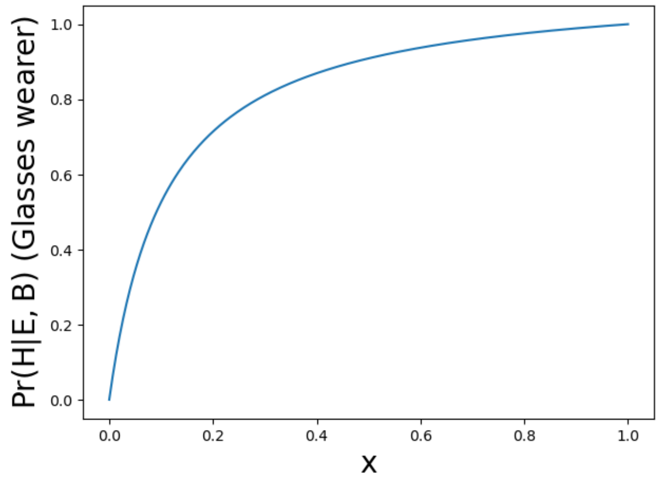

Figure 2: This shows the probability of the helper S wearing glasses as a function of x the probability of a person wearing glasses in the city, for fixed false positive rate $c = 0.1$. If $x = 1$ then there can never be a mistake, but as x approaches zero we should strongly lower our confidence because it is so rare to see a person with glasses.

glasses.py

```
#!/usr/bin/env python2
import numpy as np
import matplotlib.pyplot as plt

# Suppose you heard of a kind soul in a city who
# helped your family.  They did so anonymously,
# but a neighbor happened to see them. Call the
# helper S.
#
# An observant neighbor O saw the helper.  They
# claim that S wore glasses.  But you know the
# neighbor thinks almost everyone wears glasses.
# Suppose they always tell you correctly when
# a person actually wears glasses.
# But O also tells you that a person wears glasses
# when they don't some of the time.  They do so
# such that the chance they tell you a person
# is wearing glasses when they don't is probability
# c.  (If c=0.1 then 1 in 10 times they misidentify
# a person as wearing glasses when they do not.)
#
# But you live in a city that hates glasses.
# Only 1 in 100 people on average wear glasses
# in the city.
#
# How much trust should you put in their
# assignment that S wears glasses?

# S is the person who helped, O is the observer of S
# A is S is a glasses wearer
# B is O saw a glasses wearer
# suppose P(B|A)=1 and P(B|~A)=c
# P(A) = 0.01, P(~A)=0.99, P(A|B)=P(A)*P(B|A)/P(B)
# P(B) = P(B|A)*P(A) + P(B|~A)*P(~A) = P(A) + c*0.99
# So if we want
# P(A|B) = 0.01/(0.01+c*0.99)
pa=0.01
c=np.linspace(1e-4,1e-1,301)
pab=pa/(pa+c*(1-pa))

fig=plt.figure()
```

```
ax=fig.add_subplot(111)
ax.plot(c,pab)
ax.set_xlabel(r'Probability of O Being Mistaken',fontsize=20)
ax.set_ylabel(r'Probability S is Glasses wearer',fontsize=20)
plt.tight_layout()

plt.savefig('probability_glasses.png',bbox_inches='tight')
plt.clf()

pa=np.linspace(1e-4,1e0,301)
c=0.10
pab=pa/(pa+c*(1-pa))

fig=plt.figure()
ax=fig.add_subplot(111)
ax.plot(pa,pab)
ax.set_xlabel(r'x',fontsize=20)
ax.set_ylabel(r'$\rm{Pr}(H|E,B)$ (Glasses wearer)',fontsize=20)
plt.tight_layout()

plt.savefig('probability_glasses2.png',bbox_inches='tight')
```