Stellarator and Heliotron Devices Notes

Kyle J. Bunkers

June 10, 2016

Contents

Chapter 2

Design Principles of Coil Systems in the Stellarator and Heliotron

2.2 The Magnetic Surface and the Rotational Transform

Note that (W-2.3) is inconsistent with (W-2.4) because of the factor of 2π issue. I will use $\iota\!\!\!-=\iota/(2\pi)$ which Wakatani often refers to as ι.

2.3 The Magnetic Well and Magnetic Shear

Note that $\boldsymbol{\kappa}=\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$ (W-2.24) with $\hat{\mathbf{b}}=\mathbf{B}/|\mathbf{B}|=\mathbf{B}/B$ also has the form

$$\boldsymbol{\kappa}=\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}=-\hat{\mathbf{b}}\times(\boldsymbol{\nabla}\times\hat{\mathbf{b}}) \tag{2.3.1}$$

using $\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A})=\nabla(A^2/2)-\mathbf{A}\cdot\nabla\mathbf{A}$. And so (W-2.25)

$$\begin{aligned}\boldsymbol{\kappa}&=\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}=-\hat{\mathbf{b}}\times(\boldsymbol{\nabla}\times\hat{\mathbf{b}})=-\hat{\mathbf{b}}\times(\boldsymbol{\nabla}\times\mathbf{B}/B)=-\hat{\mathbf{b}}\times\left(\frac{1}{B}\boldsymbol{\nabla}\times\mathbf{B}+\nabla\frac{1}{B}\times\mathbf{B}\right)\\&=\frac{-1}{B^2}(\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})-\hat{\mathbf{b}}\times(\nabla B\times\mathbf{B}))=\underbrace{\frac{\mathbf{B}\cdot\nabla\mathbf{B}-\nabla(B^2/2)}{B^2}}_{-\mathbf{B}\times\mathbf{J}/B^2}+\frac{B\,\nabla B-\mathbf{B}(\hat{\mathbf{b}}\cdot\nabla B)}{B^2}\\&=\frac{1}{B^2}\left[\nabla(B^2/2)-\hat{\mathbf{b}}(\mathbf{B}\cdot\nabla B)+\mathbf{J}\times\mathbf{B}\right]\end{aligned} \tag{2.3.2}$$

and for $\mathbf{J}=\mathbf{0}$ we then have (W-2.25). We have $\hat{\mathbf{b}}\cdot\boldsymbol{\kappa}=0$ from

$$\hat{\mathbf{b}}\cdot\boldsymbol{\kappa}=\hat{\mathbf{b}}\cdot([\boldsymbol{\nabla}\times\hat{\mathbf{b}}]\times\hat{\mathbf{b}})=-(\boldsymbol{\nabla}\times\hat{\mathbf{b}})\cdot\left(\hat{\mathbf{b}}\times\hat{\mathbf{b}}\right)=-(\boldsymbol{\nabla}\times\hat{\mathbf{b}})\cdot\mathbf{0}=0 \tag{2.3.3}$$

We have κ in (ψ,β,ϕ) coordinates with $\nabla\psi\times\nabla\beta=\mathbf{B}=\nabla\phi$ then yielding

$$\boldsymbol{\kappa}=\kappa_\psi\,\nabla\psi+\kappa_\beta\,\nabla\beta \tag{2.3.4}$$

since $\nabla\phi\propto\hat{\mathbf{b}}$ so that $0=\boldsymbol{\kappa}\cdot\nabla\phi=\kappa_\phi|\nabla\phi|^2=B^2\kappa_\phi=0$ and B^2 is not allowed to be zero.

Using that $\mathcal{J} = 1/\nabla\xi^1 \times \nabla\xi^2 \cdot \nabla\xi^3$

$$\mathbf{A} = A_1 \nabla\xi^1 + A_2 \nabla\xi^2 + A_3 \nabla\xi^3 \tag{2.3.5}$$
$$A_1 = \mathcal{J}\mathbf{A} \cdot \nabla\xi^2 \times \nabla\xi^3 \tag{2.3.6}$$
$$A_2 = \mathcal{J}\mathbf{A} \cdot \nabla\xi^3 \times \nabla\xi^1 \tag{2.3.7}$$
$$A_3 = \mathcal{J}\mathbf{A} \cdot \nabla\xi^1 \times \nabla\xi^2 \tag{2.3.8}$$

This is seen simply by recalling $\nabla\xi^j \times \nabla\xi^k$ isolates the correct component and the $\mathcal{J}$ normalizes correctly. Thus using $\mathcal{J} = 1/B^2$,

$$\kappa_\psi = \frac{1}{B^2}\boldsymbol{\kappa} \cdot (\nabla\beta \times \nabla\phi) = \frac{1}{B^2}\boldsymbol{\kappa} \cdot (\nabla\beta \times B\hat{\mathbf{b}}) = -(\hat{\mathbf{b}} \times \nabla\beta \cdot \boldsymbol{\kappa})/B \tag{2.3.9}$$
$$\kappa_\beta = \frac{1}{B^2}\boldsymbol{\kappa} \cdot (\nabla\phi \times \nabla\psi) = \frac{1}{B^2}\boldsymbol{\kappa} \cdot (B\hat{\mathbf{b}} \times \nabla\psi) = (\hat{\mathbf{b}} \times \nabla\psi \cdot \boldsymbol{\kappa})/B \tag{2.3.10}$$

Using (W-2.25) one can also find that

$$\begin{aligned} \boldsymbol{\kappa} &= \frac{1}{B}\frac{\partial B}{\partial\psi}\nabla\psi + \frac{1}{B}\frac{\partial B}{\partial\beta}\nabla\beta + \frac{1}{B}\frac{\partial B}{\partial\phi}\nabla\phi - \frac{\nabla\phi}{B^3}\left[\nabla\phi \cdot \left(\frac{\partial B}{\partial\psi}\nabla\psi + \frac{\partial B}{\partial\beta}\nabla\beta + \frac{\partial B}{\partial\phi}\nabla\phi\right)\right] \\ &= \frac{1}{B}\frac{\partial B}{\partial\psi}\nabla\psi + \frac{1}{B}\frac{\partial B}{\partial\beta}\nabla\beta + \cancel{\frac{1}{B}\frac{\partial B}{\partial\phi}\nabla\phi} - \cancel{\frac{\nabla\phi}{B^3}\frac{\partial B}{\partial\phi}B^2} = \boxed{\frac{1}{B}\frac{\partial B}{\partial\psi}\nabla\psi + \frac{1}{B}\frac{\partial B}{\partial\beta}\nabla\beta} \end{aligned} \tag{2.3.11}$$

so $\kappa_\psi = \frac{1}{B}\frac{\partial B}{\partial\psi}$ and $\kappa_\beta = \frac{1}{B}\frac{\partial B}{\partial\beta}$.

A simpler relationship for (W-2.29) is to just use the form directly

$$\begin{aligned} \oint \kappa_\psi \frac{\mathrm{d}\ell}{B} &= \oint \frac{1}{B}\frac{\partial B}{\partial\psi}\frac{\mathrm{d}\ell}{B} = \oint \frac{1}{B^2}\frac{\partial B}{\partial\psi}\,\mathrm{d}\ell = \oint \frac{\partial}{\partial\psi}\left(-\frac{1}{B}\right)\mathrm{d}\ell \\ &= -\frac{\partial}{\partial\psi}\oint\frac{\mathrm{d}\ell}{B} \end{aligned} \tag{2.3.12}$$
$$\oint \kappa_\psi \frac{\mathrm{d}\ell}{B} > 0 \Rightarrow \frac{\partial}{\partial\psi}\oint\frac{\mathrm{d}\ell}{B} < 0 \tag{2.3.13}$$

Then note that because (α being the angle between $\nabla\phi$ and $\mathrm{d}\mathbf{x}$ with $|\mathrm{d}\mathbf{x}|\sin(\alpha)$ the distance along a magnetic field line since $\nabla\phi = \mathbf{B}$)

$$\mathrm{d}\phi = \nabla\phi \cdot \mathrm{d}\mathbf{x} = \underbrace{|\nabla\phi|}_{B}\underbrace{|\mathrm{d}\mathbf{x}|\sin(\alpha)}_{\mathrm{d}\ell} = B\,\mathrm{d}\ell \tag{2.3.14}$$

Note that because $L = \phi_0/B$ for these integrals, the factor of $-1/2$ in the integral with $\mathrm{d}\phi$ still works out to the same as with the integral with $\mathrm{d}\ell$.

The condition (W-2.34) comes from using $\frac{\partial\phi_0}{\partial\psi} = 0$ with $\phi_0 > 0$ and so

$$\frac{1}{\phi_0}\frac{\partial}{\partial\psi}\oint\frac{\mathrm{d}\ell}{B} < 0 \tag{2.3.15}$$
$$\frac{\partial}{\partial\psi}\frac{\oint\frac{\mathrm{d}\ell}{B}}{\phi_0} < 0 \tag{2.3.16}$$

$$\frac{\partial}{\partial\psi}\underbrace{\left(\frac{\oint\frac{\mathrm{d}\ell}{B}}{\oint\frac{B^2\,\mathrm{d}\ell}{B}}\right)}_{\langle B^2\rangle}=\frac{\partial}{\partial\psi}\frac{\oint\frac{\mathrm{d}\ell}{B}}{\oint\mathrm{d}\ell\;B}<0 \tag{2.3.17}$$

$$\frac{\partial}{\partial\psi}\frac{1}{\langle B^2\rangle}=\frac{-1}{\langle B^2\rangle^2}\frac{\partial}{\partial\psi}\left\langle B^2\right\rangle<0 \tag{2.3.18}$$

$$\frac{\partial}{\partial\psi}\left\langle B^2\right\rangle>0 \tag{2.3.19}$$

with the definition

$$\langle Q\rangle\equiv\oint\frac{Q}{B}\,\mathrm{d}\ell\Big/\oint\frac{\mathrm{d}\ell}{B} \tag{2.3.20}$$

The book tries to be too clever, defining $B^\zeta=\mathbf{B}\cdot\nabla\zeta=1/(2\pi\mathcal{J})$ not $1/(\mathcal{J})$ based on (W-2.22). What Wakatani has done, is used $\mathcal{J}=\nabla\psi\times\nabla\beta\cdot\nabla\phi$ and then defined $\zeta/(2\pi)=\phi,\ \nabla\zeta=2\pi\,\nabla\phi$ so that $2\pi\mathcal{J}=2\pi\,\nabla\psi\times\nabla\beta\cdot\nabla\phi=\nabla\psi\times\nabla\beta\cdot\nabla\zeta=\mathcal{J}_\zeta$.

It still follows that with $\mathcal{J}=\Re\{\sum_{m,n}\mathcal{J}_{m,n}e^{i(m\beta-n\zeta)}\}$

$$\frac{\mathrm{d}V}{\mathrm{d}\psi}=\int_0^{2\pi}\mathrm{d}\zeta\;\int_0^{2\pi}\mathrm{d}\beta\;\mathcal{J} \tag{2.3.21}$$

$$\frac{\mathrm{d}V}{\mathrm{d}\psi}=(2\pi)^2\mathcal{J}_{0,0} \tag{2.3.22}$$

I have no idea what is meant by (W-2.40) with the comment about (W-2.7). We just showed that (W-2.37) is exactly given by the $m=n=0$ component only, so although the written form is correct it seems somewhat odd to write out

$$\int_0^{2\pi}\mathrm{d}\zeta\;\int_0^{2\pi}\mathrm{d}\beta\;\mathcal{J}=\underbrace{\frac{\mathrm{d}V}{\mathrm{d}\psi}}_{\int_0^{2\pi}\int_0^{2\pi}\mathcal{J}_{0,0}\,\mathrm{d}\beta\,\mathrm{d}\zeta}+\Re\left\{\sum_{m,n}{}'\int_0^{2\pi}\mathrm{d}\zeta\;\int_0^{2\pi}\mathrm{d}\beta\;\mathcal{J}_{m,n}e^{i(m\beta-n\zeta)}\right\} \tag{2.3.23}$$

So we then find for (W-2.41) with $\beta=\beta_0+\iota(\psi)\zeta$ that passes through $\beta=\beta_0$ and $\zeta=0$ with N circuits around the torus. Wakatani seems to have normalized by N, the number of toroidal circuits. I am unsure why he chose this definition, as it is unusual for flux tube specific volume. Also, (W-2.41) and (W-2.42) are inconsistent, but I am pretty certain there is not supposed to be the factor of $1/(2\pi)$ in (W-2.41).

$$
\begin{aligned}
U(\psi,\beta_0) &= \frac{1}{N}\int_0^L \frac{\mathrm{d}\ell}{B} = \frac{1}{N}\int_0^{2\pi N}\frac{\mathrm{d}\zeta}{B^\zeta} = \int_0^{2\pi N} 2\pi\mathcal{J}\,\mathrm{d}\zeta = 2\pi\int_0^{2\pi N}\Re\{\sum_{m,n}\mathcal{J}_{m,n}e^{i(m\beta-n\zeta)}\}\,\mathrm{d}\zeta \\
&= \frac{2\pi}{N}\int_0^{2\pi N}\Re\{\sum_{m,n}\mathcal{J}_{m,n}e^{i(m(\beta_0+\iota\zeta)-n\zeta)}\}\,\mathrm{d}\zeta = \frac{2\pi}{N}\int_0^{2\pi N}\Re\{\sum_{m,n}\mathcal{J}_{m,n}e^{im\beta_0}e^{i(m\iota-n)\zeta)}\}\,\mathrm{d}\zeta \\
&= \frac{2\pi}{N}\Re\left\{\sum_{m,n}\mathcal{J}_{m,n}e^{im\beta_0}\int_0^{2\pi N}e^{i(m\iota-n)\zeta)}\,\mathrm{d}\zeta\right\} \\
&= (2\pi)^2\mathcal{J}_{0,0} + \Re\left\{\sum_{m,n}{}'\mathcal{J}_{m,n}e^{im\beta_0}\frac{e^{i(m\iota-n)\zeta}}{i(m\iota-n)}\Big|_0^{2\pi N}\right\} \\
&= (2\pi)^2\mathcal{J}_{0,0} + \Re\left\{\sum_{m,n}{}'\mathcal{J}_{m,n}e^{im\beta_0}\frac{e^{2\pi N i(m\iota-n)}-1}{i(m\iota-n)}\Big|_0^{2\pi N}\right\} \\
&= \frac{\mathrm{d}V}{\mathrm{d}\psi} + \Re\left\{\sum_{m,n}{}'\mathcal{J}_{m,n}e^{im\beta_0}\frac{e^{2\pi N i(m\iota-n)}-1}{i(m\iota-n)}\right\} \\
&= \frac{\mathrm{d}V}{\mathrm{d}\psi}\left[1+\frac{1}{(2\pi N)\mathcal{J}_{0,0}}\Re\left\{\sum_{m,n}{}'\mathcal{J}_{m,n}e^{im\beta_0}\frac{e^{2\pi N i(m\iota-n)}-1}{i(m\iota-n)}\right\}\right]
\end{aligned}
\tag{2.3.24}
$$

In the case that $\iota = \frac{n_0}{m_0}$, $N = n_0$ then we would find

$$
\begin{aligned}
U(\psi,\beta_0) &= \frac{\mathrm{d}V}{\mathrm{d}\psi}\left[1+\frac{2\pi n_0}{(2\pi n_0)\mathcal{J}_{0,0}}\Re\left\{\mathcal{J}_{m_0,n_0}e^{im_0\beta_0}+\mathcal{J}_{-m_0,-n_0}e^{-im_0\beta_0}\right\}\right] \\
&= \frac{\mathrm{d}V}{\mathrm{d}\psi}\left[1+\frac{1}{\mathcal{J}_{0,0}}\Re\left\{\mathcal{J}_{m_0,n_0}e^{im_0\beta_0}+\mathcal{J}_{-m_0,-n_0}e^{-im_0\beta_0}\right\}\right]
\end{aligned}
\tag{2.3.25}
$$

2.4 The Average Magnetic Surface

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = f_k(x_i,t,\alpha) \tag{W-2.45}$$

$$x_k = \xi_k + \epsilon g_{1k}(\xi_i,t,\alpha) + \epsilon^2 g_{2k}(\xi_i,t,\alpha) + \cdots \tag{W-2.47}$$

$$\frac{\mathrm{d}\xi_k}{\mathrm{d}t} = h_{0k}(\xi_i,t) + \epsilon h_{1k}(\xi_i,t) + \epsilon^2 h_{2k}(\xi_i,t) + \cdots \tag{W-2.48}$$

Note for (W-2.45) through (W-2.49) that what we are using is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\xi_k + \epsilon g_{1k}(\xi_i,t,\alpha) + \epsilon^2 g_{2k}(\xi_i,t,\alpha) + \cdots\right] = f_k(\xi_i + \epsilon g_{1i}(\xi_j,t,\alpha) + \epsilon^2 g_{2k}(\xi_j,t,\alpha) + \cdots, t, \alpha) \tag{2.4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\partial\alpha}{\partial t}\frac{\partial}{\partial\alpha} + \frac{\partial x_i}{\partial t}\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + \frac{1}{\epsilon}\frac{\partial}{\partial\alpha} + \frac{\partial\xi_i}{\partial t}\frac{\partial}{\partial t} \tag{2.4.2}$$

where we are using that for each $\frac{\partial}{\partial t}$ that we have $g_{\ell,k}(\xi_i,t,\alpha)$ and that we have

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = \frac{\partial \xi_i}{\partial t} \tag{2.4.3}$$

So that we in fact have suppressing variable dependencies $g_{\gamma,\ell} = g_{\gamma,\ell}(\xi_\delta, t, \alpha)$, $h_{\gamma,k} = h_{\gamma,k}(\xi_\delta, t)$ and $f_k = f_k(\xi_i, t, \alpha)$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\xi_k + \epsilon g_{1k} + \epsilon^2 g_{2k} + \epsilon^3 g_{3k} + \cdots\right] = \Bigg\{ & f_k(\xi_i) + \left(\epsilon g_{1i} + \epsilon^2 g_{2i} + \cdots\right)\frac{\partial f_k}{\partial \xi_i} \\ & + \frac{1}{2}\left(\epsilon g_{1i} + \epsilon^2 g_{2i} + \cdots\right)\left(\epsilon g_{1j} + \epsilon^2 g_{2j} + \cdots\right)\frac{\partial f_k}{\partial x_i \partial x_j} + \cdots \Bigg\} \end{aligned} \tag{2.4.4}$$

$$\begin{aligned} \frac{\mathrm{d}\xi_k}{\mathrm{d}t} + \epsilon\left(\frac{\partial g_{1k}}{\partial t} + \frac{\mathrm{d}\xi_i}{\mathrm{d}t}\frac{\partial g_{1k}}{\partial \xi_i}\right) + \frac{\partial g_{1k}}{\partial \alpha} + \epsilon^2\left(\frac{\partial g_{2k}}{\partial t} + \frac{\mathrm{d}\xi_i}{\mathrm{d}t}\frac{\partial g_{2k}}{\partial t}\right) + \epsilon\frac{\partial g_{2k}}{\partial \alpha} + \epsilon^2\frac{\partial g_{3k}}{\partial \alpha} + \cdots \\ = f_k + \epsilon g_{1i}\frac{\partial f_k}{\partial \xi_i} + \epsilon^2 g_{2i}\frac{\partial f_k}{\partial \xi_i} + \frac{1}{2}\epsilon^2 g_{1i}g_{1j}\frac{\partial f_k}{\partial x_i \partial x_j} + \cdots \end{aligned} \tag{2.4.5}$$

$$\begin{aligned} h_{0k} + \epsilon h_{1k} + \epsilon^2 h_{2k} + \epsilon\left(\frac{\partial g_{1k}}{\partial t} + (h_{0i} + \epsilon h_{1i})\frac{\partial g_{1k}}{\partial \xi_i}\right) + \frac{\partial g_{1k}}{\partial \alpha} + \epsilon^2\left(\frac{\partial g_{2k}}{\partial t} + h_{0i}\frac{\partial g_{2k}}{\partial \xi_i}\right) + \epsilon\frac{\partial g_{2k}}{\partial \alpha} + \epsilon^2\frac{\partial g_{3k}}{\partial \alpha} + \cdots \\ = f_k + \epsilon g_{1i}\frac{\partial f_k}{\partial \xi_i} + \epsilon^2 g_{2i}\frac{\partial f_k}{\partial \xi_i} + \frac{1}{2}\epsilon^2 g_{1i}g_{1j}\frac{\partial f_k}{\partial x_i \partial x_j} + \cdots \end{aligned} \tag{2.4.6}$$

$$\begin{aligned} \left(h_{0k} + \frac{\partial g_{1k}}{\partial \alpha}\right) + \epsilon\left(h_{1k} + \frac{\partial g_{1k}}{\partial t} + h_{0i}\frac{\partial g_{1k}}{\partial \epsilon_i} + \frac{\partial g_{2k}}{\partial \alpha}\right) + \epsilon^2\left(h_{2k} + h_{1i}\frac{\partial g_{1k}}{\partial \xi_i} + \frac{\partial g_{2k}}{\partial t} + h_{0i}\frac{\partial g_{2k}}{\partial \xi_i} + \frac{\partial g_{3k}}{\partial \alpha}\right) \\ = f_k + \epsilon g_{1i}\frac{\partial f_k}{\partial \xi_i} + \epsilon^2\left(g_{2i}\frac{\partial f_k}{\partial \xi_i} + \frac{1}{2}g_{1i}g_{1j}\frac{\partial f_k}{\partial x_i \partial x_j}\right) + \cdots \end{aligned} \tag{2.4.7}$$

Or collecting like powers of ϵ

$$\begin{aligned} & h_{0k} + \frac{\partial g_{1k}}{\partial \alpha} = f_k \\ & h_{1k} + \frac{\partial g_{1k}}{\partial t} + h_{0i}\frac{\partial g_{1k}}{\partial \xi_i} + \frac{\partial g_{2k}}{\partial \alpha} = g_{1i}\frac{\partial f_k}{\partial \epsilon_i} \\ & h_{2k} + h_{1i}\frac{\partial g_{1k}}{\partial \xi_i} + \frac{\partial g_{2k}}{\partial t} + h_{0i}\frac{\partial g_{2k}}{\partial \xi_i} + \frac{\partial g_{3k}}{\partial \alpha} = g_{2i}\frac{\partial f_k}{\partial \epsilon_i} + \frac{1}{2}g_{1i}g_{1j}\frac{\partial f_k}{\partial x_i \partial x_j} \end{aligned} \tag{W-2.49}$$

Note that with $f = \overline{f} + \widetilde{f}$ and applying averaging to both sides again.

$$\overline{f} = \overline{\overline{f} + \widetilde{f}} = \overline{\overline{f}} + \overline{\widetilde{f}} = \overline{f} + \overline{\widetilde{f}} \tag{2.4.8}$$

$$0 = \overline{\widetilde{f}} \tag{2.4.9}$$

However, the note below (W-2.51) that $\overline{\widehat{f}} = \overline{\int_0^\alpha \mathrm{d}\alpha'\ \widetilde{f}(\alpha')} = 0$ is clearly false. First, let's change this into a better looking formula by exchanging the integral order and changing dummy variables

$$\overline{\widehat{f}} = \overline{\int_0^\alpha \mathrm{d}\alpha'\ \widetilde{f}(\alpha')} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\alpha\ \int_0^\alpha \mathrm{d}\alpha'\ \widetilde{f}(\alpha') = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\alpha'\ \int_{\alpha'}^{2\pi}\mathrm{d}\alpha\ \widetilde{f}(\alpha') \tag{2.4.10}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\alpha'\ (2\pi - \alpha')\widetilde{f}(\alpha') = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\alpha\ (2\pi - \alpha)\widetilde{f}(\alpha) = \cancel{2\pi \overline{\widetilde{f}}} - \overline{\alpha\widetilde{f}} \tag{2.4.11}$$

Now as a counterexample, take the periodic function $f = 1 + \sin\alpha$ (or $f = \sin\alpha$ if you prefer). Then $\overline{f} = 1$ (or $\overline{f} = 0$), $\widetilde{f} = \sin\alpha$ and so

$$\overline{\widehat{f}} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\alpha \int_0^{\alpha} \mathrm{d}\alpha' \ \sin\alpha' = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\alpha \ \left(-\cos\alpha'\right)_0^{\alpha} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\alpha \ (1-\cos\alpha) = 1 \quad (2.4.12)$$

$$\begin{aligned}\overline{\alpha\widetilde{f}} &= \frac{-1}{2\pi}\int_0^{2\pi} \mathrm{d}\alpha \ \alpha\sin(\alpha) = \frac{-1}{2\pi}\frac{\partial}{\partial t}\int_0^{2\pi} \mathrm{d}\alpha \ -\cos(t\alpha) = \frac{1}{2\pi}\frac{\partial}{\partial t}\left(\frac{\sin(t\alpha)}{t}\right)_{\alpha=0}^{2\pi} \\ &= \frac{1}{2\pi}\frac{\partial}{\partial t}\left(\frac{\sin(2\pi t)}{t}\right) = \frac{2\pi t\cos(2\pi t) - \sin(2\pi t)}{2\pi t^2} = 1\end{aligned} \quad (2.4.13)$$

If instead we choose $\widehat{f} = \int_{\alpha_0}^{\alpha} \mathrm{d}\alpha' \ \widetilde{f}(\alpha')$ so that we get the indefinite integral (i.e. if $g(\alpha) = \int \mathrm{d}\alpha \ \widetilde{f}(\alpha)$ then $g(\alpha_0) = 0$ and so $\widehat{f} = g(\alpha)$) this then works. This is equivalent to choosing the C to be 0 for each integration in a Fourier series. To prove this, we recognize that the periodic function can be expanded in a Fourier series, with the $m = 0$ component being $\overline{f}$ so that (with the $'$ indicating skipping $m = 0$)

$$\widetilde{f}(\alpha) = \Re\left\{\sum_{m=-\infty}^{\infty}{}'\widetilde{f}_m e^{-im\alpha}\right\} \quad (2.4.14)$$

and so using that we've chosen α_0 such that the constant of integration is zero we then have

$$\widehat{f}(\alpha) = \int^{\alpha} \mathrm{d}\alpha' \ \Re\left\{\sum_m{}'\widetilde{f}_m e^{-im\alpha'}\right\} = \Re\left\{\sum_m{}'\widetilde{f}_m \frac{e^{-im\alpha}}{-im}\right\} = \Re\left\{\sum_m{}'i\widetilde{f}_m \frac{e^{-im\alpha}}{m}\right\} \quad (2.4.15)$$

Note then that this is in fact a new Fourier series with $h_m = i\widetilde{f}_m/m$, and that taking $\alpha \to \alpha + 2\pi M$ for M an integer gives the same answer. So then $\widehat{f}(\alpha)$ is a periodic function with period of 2π as well.

Now if we average this

$$\overline{\widehat{f}(\alpha)} = \frac{1}{2\pi}\int_0^{2\pi} \Re\left\{\sum_m{}'\frac{i\widetilde{f}_m}{m}e^{-im\alpha}\right\} = \Re\left\{\sum_m{}'\frac{i\widetilde{f}_m}{2\pi m}\frac{e^{-im(2\pi)} - e^{-im(0)}}{-im}\right\} = 0 \quad (2.4.16)$$

as $e^{-im(2\pi)} - e^{-im(0)} = 1 - 1 = 0$ and there is no $m = 0$ component by definition. This actually leads to a rather nifty identity.

We use

$$\overline{\widehat{f}} = \overline{\int_{\alpha_0}^{\alpha} \mathrm{d}\alpha' \ \widetilde{f}(\alpha')} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\alpha \int_{\alpha_0}^{\alpha} \mathrm{d}\alpha' \ \widetilde{f}(\alpha') = \frac{1}{2\pi}\left[-\int_0^{\alpha_0} \mathrm{d}\alpha' \int_0^{\alpha'} \mathrm{d}\alpha \ \widetilde{f}(\alpha') + \int_{\alpha_0}^{2\pi} \mathrm{d}\alpha' \int_{\alpha'}^{2\pi} \mathrm{d}\alpha \ \widetilde{f}(\alpha')\right] \quad (2.4.17)$$

$$= \frac{1}{2\pi}\left[-\int_0^{\alpha_0} \alpha'\widetilde{f}(\alpha) + \int_{\alpha_0}^{2\pi} \mathrm{d}\alpha' \ (2\pi - \alpha')\widetilde{f}(\alpha')\right] = \frac{1}{2\pi}\left[-\int_0^{\alpha_0} \alpha'\widetilde{f}(\alpha') + \int_{\alpha_0}^{2\pi} \mathrm{d}\alpha' \ (2\pi - \alpha')\widetilde{f}(\alpha')\right] \quad (2.4.18)$$

$$= \frac{1}{2\pi}\left[2\pi\int_{\alpha_0}^{2\pi} \mathrm{d}\alpha' \ \widetilde{f}(\alpha') - \int_{\alpha_0}^{2\pi} \mathrm{d}\alpha' \ \alpha'\widetilde{f}(\alpha') - \int_0^{\alpha_0} \alpha'\widetilde{f}(\alpha')\right] = \frac{1}{2\pi}\left[2\pi\widehat{f}(2\pi) - \int_0^{2\pi} \mathrm{d}\alpha' \ \alpha'\widetilde{f}(\alpha')\right] \quad (2.4.19)$$

$$= \widehat{f}(2\pi) - \overline{\alpha'\widetilde{f}} \quad (2.4.20)$$

Since this must equal 0, we find (using $M \in \mathbb{Z}$)

$$\widehat{f}(0) = \widehat{f}(2\pi) = \widehat{f}(0 + 2\pi M) = \overline{\alpha \widetilde{f}} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\alpha\ \alpha \widetilde{f}(\alpha) \tag{2.4.21}$$

Let us now prove

$$\overline{\frac{\partial}{\partial \alpha}\left(\widehat{a}\widehat{b}\right)} = 0 \tag{W-2.56}$$

Remember that $\frac{\partial}{\partial\alpha}\widehat{f} = \widetilde{f}$. Now let's just write out

$$\overline{\frac{\partial}{\partial \alpha}\left(\widehat{a}\widehat{b}\right)} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\alpha\ \frac{\partial}{\partial\alpha}(\widehat{a}\widehat{b}) = \frac{\widehat{a}(2\pi)\widehat{b}(2\pi) - \widehat{a}(0)\widehat{b}(0)}{2\pi} \tag{2.4.22}$$

Now both $\widehat{a}$ and $\widehat{b}$ are periodic with period 2π so that $\widehat{a}(2\pi) = \widehat{a}(0)$ and $\widehat{b}(2\pi) = \widehat{b}(0)$ so that (W-2.56) indeed holds.

This then implies as

$$\overline{\widehat{a}b} = -\overline{a\widehat{b}} \tag{W-2.55}$$

from

$$\overline{\frac{\partial}{\partial\alpha}(\widehat{a}\widehat{b})} = \overline{\frac{\partial\widehat{a}}{\partial\alpha}\widehat{b} + \widehat{a}\frac{\partial\widehat{b}}{\partial\alpha}} = \overline{\widetilde{a}\widehat{b}} + \overline{\widehat{a}\widetilde{b}} = 0 \tag{2.4.23}$$

$$\overline{\widehat{a}\widetilde{b}} = -\overline{\widetilde{a}\widehat{b}} \tag{2.4.24}$$

Now because $\widetilde{b} = b - \overline{b}$ and $\widetilde{a} = a - \overline{a}$ we see that

$$\overline{\widehat{a}b} = \overline{\widehat{a}\overline{b} + \widehat{a}\widetilde{b}} = \cancel{\overline{b}\overline{\widehat{a}}} + \overline{\widehat{a}\widetilde{b}} \tag{2.4.25}$$

$$\overline{a\widehat{b}} = \overline{\overline{a}\widehat{b} + \widetilde{a}\widehat{b}} = \cancel{\overline{a}\overline{\widehat{b}}} + \overline{\widetilde{a}\widehat{b}} \tag{2.4.26}$$

and so we can exchange the $\widetilde{a} \to a$ and $\widetilde{b} \to b$ in the above to get (W-2.55).

Let us get $\frac{\mathrm{d}\bar{r}}{\mathrm{d}z}$ (W-2.59) to give a flavor for the replacements to (W-2.62). First we note that

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}z} = \frac{\overline{b_r}}{B_0} - \frac{\overline{b_r b_z}}{B_0^2} + \overline{\frac{\partial}{\partial\bar{r}}\left(\frac{b_r}{B_0}\right)\frac{\widehat{b}_r}{B_0}} + \overline{\frac{\partial}{\partial\bar{\theta}}\left(\frac{b_r}{B_0}\right)\frac{\widehat{b}_\theta}{\bar{r}B_0}} \tag{W-2.59}$$

$$\boldsymbol{\nabla}\cdot\mathbf{b} = \frac{1}{r}\frac{\partial}{\partial r}(rb_r) + \frac{1}{r}\frac{\partial b_\theta}{\partial\theta} + \frac{\partial b_z}{\partial z} = 0 \tag{2.4.27}$$

Note that $\boldsymbol{\nabla}\cdot\overline{\mathbf{b}} = 0$ is tacitly assumed so that we can switch $r \to \bar{r}$ without any higher order terms. So then

$$\frac{\partial b_r}{\partial r} + \frac{b_r}{r} + \frac{1}{r}\frac{\partial b_\theta}{\partial\theta} + \frac{\partial b_z}{\partial z} = 0 \tag{2.4.28}$$

$$\overline{\frac{\widehat{b}_r}{B_0^2}\frac{\partial b_r}{\partial\bar{r}}} = -\overline{\frac{\widehat{b}_r}{B_0^2}\left(\frac{b_r}{\bar{r}} + \frac{1}{\bar{r}}\frac{\partial b_\theta}{\partial\theta} + \frac{\partial b_z}{\partial z}\right)} \tag{2.4.29}$$

The reason we can replace $\frac{\partial}{\partial r}$ with $\frac{\partial}{\partial \bar{r}}$ is because we view some function $f(r(\bar{r}, r_{1k}, \dots))$ so that

$$\frac{\partial f}{\partial \bar{r}} = \frac{\partial r}{\partial \bar{r}}\frac{\partial f}{\partial r} \tag{2.4.30}$$

$$\frac{\partial f}{\partial r_{1k}} = \frac{\partial r}{\partial r_{1k}}\frac{\partial f}{\partial r} \tag{2.4.31}$$

$$\tag{2.4.32}$$

etc., and because

$$r = \bar{r} + \epsilon r_{1k} + \epsilon^2 r_{2k} + \cdots \tag{2.4.33}$$

$$\mathrm{d}r = \mathrm{d}\bar{r} + \epsilon\, \mathrm{d}r_{1k} + \epsilon^2\, \mathrm{d}r_{2k} + \cdots \tag{2.4.34}$$

$$\frac{\partial r}{\partial \bar{r}} = 1\,, \qquad \frac{\partial r}{\partial r_{1k}} = \epsilon\,, \qquad \frac{\partial r}{\partial r_{2k}} = \epsilon^2\,, \qquad \cdots \tag{2.4.35}$$

we see that

$$\frac{\partial f}{\partial \bar{r}} = \frac{\partial f}{\partial r}\,, \qquad \frac{\partial f}{\partial r_{1k}} = \epsilon\frac{\partial f}{\partial r}\,, \qquad \frac{\partial f}{\partial r_{2k}} = \epsilon^2\frac{\partial f}{\partial r}\,, \qquad \cdots \tag{2.4.36}$$

(and similarly for $\theta = \bar{\theta} + \epsilon\theta_{1k} + \epsilon^2\theta_{2k} + \cdots$).

Taking the terms in order remembering that $\frac{\partial}{\partial z} \to \frac{\partial}{\partial \alpha}$ so that

$$-\overline{\frac{\widehat{b}_r b_r}{\bar{r} B_0^2}} = 0 \tag{2.4.37}$$

from $\overline{\widehat{b}_r b_r} = -\overline{b_r \widehat{b}_r}$. Then leave the second term alone and use $\frac{\partial \widehat{f}}{\partial \alpha} = \widetilde{f}$ (remembering that $\overline{b_z} = 0$ by problem setup)

$$-\overline{\frac{\widehat{b}_r}{B_0^2}\frac{\partial b_z}{\partial z}} = \overline{\frac{b_r}{B_0^2}\frac{\partial \widehat{b}_z}{\partial z}} = \overline{\frac{b_r}{B_0^2}\widetilde{b}_z} = \frac{\overline{b_r b_z}}{B_0^2} - \frac{\overline{b_r}\,\cancel{\overline{b_z}}}{B_0^2} = \frac{\overline{b_r b_z}}{B_0^2} \tag{2.4.38}$$

Thus,

$$\begin{aligned}
\frac{\mathrm{d}\bar{r}}{\mathrm{d}z} &= \frac{\overline{b_r}}{B_0} - \cancel{\frac{\overline{b_r b_z}}{B_0^2}} - \overline{\frac{\widehat{b}_r}{\bar{r} B_0^2}\frac{\partial b_\theta}{\partial \theta}} + \cancel{\frac{\overline{b_r b_z}}{B_0^2}} + \overline{\frac{\widehat{b}_\theta}{\bar{r} B_0^2}\frac{\partial b_r}{\partial \bar{\theta}}} \\
&= \frac{\overline{b_r}}{B_0} - \overline{\frac{\widehat{b}_r}{\bar{r} B_0^2}\frac{\partial b_\theta}{\partial \theta}} - \overline{\frac{b_\theta}{\bar{r} B_0^2}\frac{\partial \widehat{b}_r}{\partial \bar{\theta}}} \\
&= \frac{\overline{b_r}}{B_0} - \overline{\frac{1}{\bar{r} B_0^2}\left(\widehat{b}_r\frac{\partial b_\theta}{\partial \bar{\theta}} + b_\theta\frac{\partial \widehat{b}_r}{\partial \bar{\theta}}\right)} \\
&= \frac{\overline{b_r}}{B_0} - \overline{\frac{1}{\bar{r} B_0^2}\frac{\partial(\widehat{b}_\theta b_r)}{\partial \bar{\theta}}} = \frac{\overline{b_r}}{B_0} - \frac{1}{\bar{r} B_0^2}\frac{\partial\overline{(\widehat{b}_\theta b_r)}}{\partial \bar{\theta}}
\end{aligned} \tag{2.4.39}$$

So then using $\Psi = \bar{A}_z - \frac{\overline{\widehat{b}_r b_\theta}}{B_0}$ with $\bar{\mathbf{A}} = \bar{A}_z \hat{\mathbf{z}}$ so that

$$\begin{aligned}
\overline{b_r} &= \frac{1}{r}\frac{\partial \bar{A}_z}{\partial \theta} \\
\overline{b_\theta} &= -\frac{\partial \bar{A}_z}{\partial r}
\end{aligned} \tag{W-2.64}$$

Here we use that

$$\overline{\overline{b_r}} = \overline{b_r} = \overline{\frac{1}{r}\frac{\partial \bar{A}_z}{\partial \theta}} = \overline{\frac{1}{r}\frac{\partial \bar{A}_z}{\partial \bar{\theta}}} = \overline{\frac{1}{r}}\frac{\partial \bar{A}_z}{\partial \bar{\theta}} = \frac{1}{\bar{r}}\frac{\partial \bar{A}_z}{\partial \bar{\theta}} \tag{2.4.40}$$

We see that

$$\begin{aligned}\frac{\mathrm{d}\bar{r}}{\mathrm{d}z} &= \frac{1}{B_0}\frac{1}{\bar{r}}\frac{\partial \bar{A}_z}{\partial \theta} - \frac{1}{\bar{r}B_0^2}\overline{\frac{\partial(\widehat{b}_\theta b_r)}{\partial \bar{\theta}}} = \frac{1}{\bar{r}B_0}\frac{\partial}{\partial \bar{\theta}}\left[\bar{A}_z - \frac{\widehat{b}_\theta b_r}{B_0}\right] \\ &= \frac{1}{\bar{r}B_0}\frac{\partial \Psi}{\partial \bar{\theta}}\end{aligned} \tag{2.4.41}$$

as desired.

If we assume Ψ is a function of $\bar{r}$ alone we can simplify somewhat. We see that this yields from our original equations (looking at $g_{1k} = \widehat{f}_k$) that

$$r = \bar{r} + \frac{\widehat{b}_r}{B_0} \tag{2.4.42}$$

$$\theta = \bar{\theta} + \frac{\widehat{(b_\theta/r)}}{B_0} \tag{2.4.43}$$

But with our simplification that $\Psi = \Psi(\bar{r})$ then $\frac{\mathrm{d}\bar{r}}{\mathrm{d}z} = 0$ and so $\bar{r} = C$ for some constant $C = \bar{r}_c$. Then using that $r = \bar{r}_c + \frac{\widehat{b}_r}{B_0}$ we see that

$$\widehat{(\widehat{b}_\theta/r)}/B_0 = \left(\frac{\widehat{b}_\theta}{B_0\bar{r}_c} + \mathcal{O}([b/B_0]^2)\right) = \frac{\widehat{b}_\theta}{\bar{r}_c B_0} + \mathcal{O}([b/B_0]^2) = \frac{\widehat{b}_\theta}{rB_0} + \mathcal{O}([b/B_0]^2) \tag{2.4.44}$$

Calling $\frac{1}{\bar{r}B_0}\frac{\partial \Psi}{\partial \bar{r}} = \omega$ which is independent of z then $\bar{\theta} = \omega z + C_1$ where we choose $C_1 = 0$ for convenience. Then

$$\begin{aligned} r &= \bar{r}_c + \frac{\widehat{b}_r}{B_0} \\ \theta &= \omega z + \frac{\widehat{b}_\theta}{rB_0}\end{aligned} \tag{W-2.69}$$

In general we may of course write

$$\begin{aligned}\Psi &= \int^{\bar{r}} \bar{r}' B_0 \frac{\mathrm{d}\bar{\theta}(\bar{r},\bar{\theta},z)}{\mathrm{d}z}\,\mathrm{d}\bar{r}' \\ \frac{\mathrm{d}\bar{\theta}}{\mathrm{d}z} &= \omega(\bar{r},\bar{\theta},z)\end{aligned} \tag{W-2.70}$$

with ω the mean rotation angle of the line of force.

2.5 The Helically Symmetric Magnetic Field

Note that with $\nabla\varphi = \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ and

$$\varphi = B_0 z + \frac{1}{\alpha}\sum_{n=1}^{\infty} b_n I_n(n\alpha r)\sin n\zeta \tag{2.5.1}$$

$$\zeta = \theta - \alpha z \tag{2.5.2}$$

with α a constant and I_n a modified Bessel function that using $A_z = 0$ as the gauge condition for $\mathbf{A}$ yields

$$\begin{aligned} \frac{\partial \varphi}{\partial r} &= -\frac{\partial A_\theta}{\partial z} \\ \frac{1}{r}\frac{\partial \varphi}{\partial \theta} &= \frac{\partial A_r}{\partial z} \end{aligned} \tag{2.5.3}$$

Let's prove that φ satisfies $\nabla^2\varphi$ in (r, θ, z) coordinates. First note the identities (for $n \geq 1$)

$$I_{n-1}(z) + I'_{n+1}(z) = 2I'_n(z) \tag{2.5.4}$$

$$I_{n-1}(z) - I_{n+1}(z) = \frac{2n}{z} I_n(z) \tag{2.5.5}$$

$$I'_n(z) = I_{n-1}(z) - \frac{n}{z} I_n(z) \tag{2.5.6}$$

$$I'_n(z) = I_{n+1}(z) + \frac{n}{z} I_n(z) \tag{2.5.7}$$

We then have

$$\varphi = B_0 z + \frac{1}{\alpha}\sum_{n=1}^{\infty} b_n I_n(n\alpha r)\sin(n(\theta - \alpha z)) \tag{2.5.8}$$

$$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} \tag{2.5.9}$$

$$\begin{aligned} &= \frac{1}{\alpha}\sum_{n=1}^{\infty} n^2\alpha^2 I''_n(n\alpha r)\sin(n(\theta-\alpha z)) + \frac{1}{r\alpha}\sum_{n=1}^{\infty} b_n n\alpha I'_n(n\alpha r)\sin(n(\theta-\alpha z)) \\ &+ \frac{1}{r^2\alpha}\sum_{n=1}^{\infty} I_n(n\alpha r)(-n^2)\sin(n(\theta-\alpha z)) + \frac{1}{\alpha}\sum_{n=1}^{\infty} b_n I_n(n\alpha r)(-n^2\alpha^2)\sin(n(\theta-\alpha z)) \end{aligned} \tag{2.5.10}$$

$$= \sum_{n=1}^{\infty} b_n \sin(n\zeta)\left(n^2\alpha I''_n + \frac{n}{r}I'_n - \frac{n^2}{r^2\alpha}I_n - n^2\alpha I_n\right) \tag{2.5.11}$$

$$= \sum_{n=1}^{\infty} b_n \sin(n\zeta)\left(n^2\alpha\frac{\left[I'_{n-1} + I'_{n+1}\right]}{2} + \frac{n(I_{n+1} + I_{n-1})}{2r} - n^2\left(\frac{1}{r^2\alpha} - \alpha\right) I_n\right) \tag{2.5.12}$$

$$= \sum_{n=1}^{\infty} b_n \sin(n\zeta)\left(n^2\alpha\frac{\left[I_n + \frac{n-1}{n\alpha r}I_{n-1} + I_n - \frac{n+1}{n\alpha r}I_{n+1}\right]}{2} + \frac{n(I_{n+1} + I_{n-1})}{2r} - n^2\left(\frac{1}{r^2\alpha} - \alpha\right) I_n\right) \tag{2.5.13}$$

$$= \sum_{n=1}^{\infty} b_n \sin(n\zeta)\left(\cancel{n^2\alpha I_n} + \frac{n^2}{2r}\left(I_{n-1} - I_{n+1}\right) - \cancel{\frac{n}{2r}\left(I_{n-1} + I_{n+1}\right)} + \cancel{\frac{n(I_{n+1} + I_{n-1})}{2r}} - n^2\left(\frac{1}{r^2\alpha} - \cancel{\alpha}\right) I_n\right) \tag{2.5.14}$$

$$= \sum_{n=1}^{\infty} b_n \sin(n\zeta)\left(\frac{n^2}{\cancel{2r}}\frac{\cancel{2n}}{n\alpha r} I_n - \frac{n}{\cancel{r^2}\alpha} I_n\right) \tag{2.5.15}$$

$$= 0 \tag{2.5.16}$$

as desired. Note that for

$$\varphi = B_0 z + \frac{b}{\alpha} I_n(n\alpha r) \sin\left(n[\theta - \alpha z]\right) = B_0 z + \frac{b}{\alpha} I_n(n\alpha r) \sin n\zeta \tag{2.5.17}$$

$$\begin{aligned}\nabla\varphi &= bnI_n'(n\alpha r)\sin(n[\theta-\alpha z])\,\nabla r + \frac{bn}{\alpha r} I_n(n\alpha r)\cos(n[\theta-\alpha z])\frac{\nabla\theta}{|\,\nabla\theta|} + (B_0 - bnI_n(n\alpha r)\cos n\zeta)\,\nabla z \\ &= B_r\,\nabla r + B_\theta \frac{\nabla\theta}{|\,\nabla\theta|} + B_z\,\nabla z\end{aligned} \tag{2.5.18}$$

We see that this is a flux function through the use of

$$\zeta = \theta - \alpha z \tag{2.5.19}$$

$$\nabla\zeta = \nabla\theta - \alpha\,\nabla z \tag{2.5.20}$$

$$|\,\nabla\zeta| = \sqrt{\frac{1}{r^2} + \alpha^2} \tag{2.5.21}$$

$$|\,\nabla\theta| = \frac{1}{r} \tag{2.5.22}$$

and so with $\mathbf{B} = B_r\,\nabla r + B_\zeta \frac{\nabla\zeta}{|\,\nabla\zeta|}$ we find

$$B_\theta = B_\zeta \frac{\nabla\zeta}{|\,\nabla\zeta|}\cdot\frac{\nabla\theta}{|\,\nabla\theta|} = \frac{B_\zeta}{r|\,\nabla\zeta|} \tag{2.5.23}$$

$$B_z = B_\zeta \frac{\nabla\zeta}{|\,\nabla\zeta|}\cdot\nabla z = -\frac{\alpha B_\zeta}{|\,\nabla\zeta|} \tag{2.5.24}$$

so that

$$\alpha r B_z - B_\theta = -\frac{\alpha^2 r B_\zeta}{|\,\nabla\zeta|} - \frac{B_\zeta}{r|\,\nabla\zeta|} = \frac{-rB_\zeta}{|\,\nabla\zeta|}\left(\alpha^2 + \frac{1}{r^2}\right) = \frac{-rB_\zeta}{|\,\nabla\zeta|}|\,\nabla\zeta|^2 = \frac{\partial\Psi}{\partial r} \tag{2.5.25}$$

$$B_\zeta = \frac{-1}{r|\,\nabla\zeta|}\frac{\partial\Psi}{\partial r} = \frac{1}{|\,\nabla\zeta|}\left(\frac{B_\theta}{r} - \alpha B_z\right) \tag{2.5.26}$$

Thus,

$$\begin{aligned}\mathbf{B}\cdot\nabla\Psi &= B_r\frac{\partial\Psi}{\partial r}\,\nabla r\cdot\nabla r + B_\zeta|\,\nabla\zeta|\frac{\partial\Psi}{\partial\zeta}\frac{\nabla\zeta\cdot\nabla\zeta}{|\,\nabla\zeta|^2} = B_r\frac{\partial\Psi}{\partial r} + B_\zeta\frac{\partial\Psi}{\partial\zeta}|\,\nabla\zeta| \\ &= \frac{1}{r}\frac{\partial\Psi}{\partial\zeta}\frac{\partial\Psi}{\partial r} + \frac{-1}{r\cancel{|\nabla\zeta|}}\frac{\partial\Psi}{\partial r}\frac{\partial\Psi}{\partial\zeta}\cancel{|\nabla\zeta|} \\ &= \frac{1}{r}\frac{\partial\Psi}{\partial\zeta}\frac{\partial\Psi}{\partial r}\,(1-1) = 0\end{aligned} \tag{2.5.27}$$

Alternatively, one can use

With $B_r = \frac{1}{r}\frac{\partial\Psi}{\partial\zeta} = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = -\frac{1}{r\alpha}\frac{\partial\Psi}{\partial z}$ and $\alpha r B_z - B_\theta = \alpha\frac{\partial}{\partial r}(rA_\theta) + \frac{\partial A_z}{\partial r} = \frac{\partial\Psi}{\partial r}$ with $\Psi = \alpha r A_\theta + A_z$ we see that

$$\mathbf{B}\cdot\nabla\Psi = B_r\frac{\partial\Psi}{\partial r} + \frac{B_\theta}{r}\frac{\partial\Psi}{\partial\theta} + B_z\frac{\partial\Psi}{\partial z} = \frac{1}{r}\frac{\partial\Psi}{\partial\zeta}\frac{\partial\Psi}{\partial r} + \frac{B_\theta}{r}\frac{\partial\Psi}{\partial\zeta} - \alpha B_z\frac{\partial\Psi}{\partial\zeta} \tag{2.5.28}$$

$$= \frac{1}{r}\frac{\partial\Psi}{\partial\zeta}\frac{\partial\Psi}{\partial r} + \frac{1}{r}(B_\theta - \alpha r B_z)\frac{\partial\Psi}{\partial\zeta} = \frac{1}{r}\left(\frac{\partial\Psi}{\partial\zeta}\frac{\partial\Psi}{\partial r} - \frac{\partial\Psi}{\partial r}\frac{\partial\Psi}{\partial\zeta}\right) = 0 \tag{2.5.29}$$

2.6 The Magnetic Island and Destruction of the Magnetic Surface

Given (note there is an error in the book in this equation)

$$\begin{aligned} B_z\frac{\mathrm{d}r}{\mathrm{d}z} &= bnI_n'(n\alpha r)\sin n\zeta \\ B_z\frac{\mathrm{d}\zeta}{\mathrm{d}z} &= -\alpha B_0 + bn\left(\alpha+\frac{1}{\alpha r^2}\right)I_n(n\alpha r)\cos(n\zeta) \end{aligned} \tag{W-2.118}$$

with the $'$ denoting differentiation with respect to $n\alpha r$ and $\zeta = \theta - \alpha z$ and $B_z(r,z) = \frac{\partial\varphi}{\partial z}(r,z)$. We then can note that these look similar to a Hamiltonian, the integral of the above (W-2.118), with

$$H = \frac{1}{2}\alpha^2 r^2 - \frac{b}{B_0}\alpha r I_n'(n\alpha r)\cos(n\zeta) \tag{2.6.1}$$

For then

$$\frac{\mathrm{d}H}{\mathrm{d}z} = \frac{\partial H}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}z} + \frac{\partial H}{\partial \zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z} = 0 \tag{2.6.2}$$

$$\begin{aligned} \frac{\partial H}{\partial r} &= \alpha^2 r - \frac{b}{nB_0}\frac{\partial(n\alpha r)}{\partial r}\frac{\partial}{\partial(n\alpha r)}(n\alpha r I_n')\cos n\zeta = \alpha^2 r - \frac{b}{B_0}\alpha\cos n\zeta\,(I_n' + n\alpha r I_n'') \\ &= \alpha^2 r - \frac{b}{B_0}\alpha(n\alpha r)\cos n\zeta\left(\frac{I_n'}{n\alpha r} + I_n''\right) = \alpha^2 r - \frac{b}{B_0}\alpha(n\alpha r)\cos n\zeta\left(1+\frac{1}{\alpha^2 r^2}\right)I_n \end{aligned} \tag{2.6.3}$$

$$\frac{\partial H}{\partial \zeta} = \frac{b}{B_0}n\alpha r I_n'(n\alpha r)\sin(n\zeta) \tag{2.6.4}$$

So then

$$\begin{aligned} \frac{\partial H}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}z} &= \left(\alpha^2 r - \frac{b}{B_0}\alpha(n\alpha r)\cos n\zeta\left(1+\frac{1}{\alpha^2r^2}\right)I_n\right)\frac{bnI_n'(n\alpha r)\sin n\zeta}{B_z} \\ &= \frac{bn\alpha r I_n'(n\alpha r)\sin n\zeta}{B_z}\left(\alpha - \frac{b}{B_0}n\left[\alpha+\frac{1}{\alpha r^2}\right]I_n(n\alpha r)\cos n\zeta\right) \end{aligned} \tag{2.6.5}$$

$$\begin{aligned} \frac{\partial H}{\partial \zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z} &= \frac{\left(\frac{b}{B_0}n\alpha r I_n'(n\alpha r)\sin(n\zeta)\right)}{B_z}\left(-\alpha B_0 + bn\left(\alpha B_0 + \frac{1}{\alpha r^2}\right)I_n(n\alpha r)\cos(n\zeta)\right) \\ &= -\frac{bn\alpha r I_n'(n\alpha r)\sin n\zeta}{B_z}\left(\alpha - \frac{b}{B_0}n\left[\alpha+\frac{1}{\alpha r^2}\right]I_n(n\alpha r)\cos n\zeta\right)\frac{\cancel{B_0}}{\cancel{B_0}} \end{aligned} \tag{2.6.6}$$

Hence we see the easy cancellation. How would one ever guess this, though? One uses

$$\frac{\mathrm{d}r}{\mathrm{d}z} = \frac{\mathbf{B}\cdot\nabla r}{\mathbf{B}\cdot\nabla z} = \frac{B^r}{B^z} \tag{2.6.7}$$

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \frac{\mathbf{B}\cdot\nabla \zeta}{\mathbf{B}\cdot\nabla z} = \frac{B^\zeta}{B^z} \tag{2.6.8}$$

with $B^r = B_r$ and $B^z = B_z$. We have

$$B^\zeta = \mathbf{B}\cdot\nabla\zeta = B_\zeta\frac{\nabla\zeta\cdot\nabla\zeta}{|\nabla\zeta|} = B_\zeta|\nabla\zeta| \tag{2.6.9}$$

with the B_ζ we've used previously in (2.5.26). Thus we confirm that

$$\begin{aligned} B^\zeta = \frac{-1}{r}\frac{\partial\Psi}{\partial r} = \frac{B_\theta}{r} - \alpha B_z &= \frac{bn}{\alpha r^2} I_n(n\alpha r)\cos n\zeta - \alpha B_0 + \alpha bnI_n(n\alpha r)\cos n\zeta \\ &= -\alpha B_0 + bnI_n(n\alpha r)\cos n\zeta\left(\alpha + \frac{1}{\alpha r^2}\right) \end{aligned} \tag{2.6.10}$$

which reduces these previous equations to

$$\frac{\mathrm{d}r}{\mathrm{d}z} = \frac{1}{B_z}\frac{1}{r}\frac{\partial\Psi}{\partial\zeta} = \frac{1}{rB_z}\frac{\partial\Psi}{\partial\zeta} \tag{2.6.11}$$

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \frac{1}{B_z}\frac{-1}{r|\nabla\zeta|}\frac{\partial\Psi}{\partial r}|\nabla\zeta| = -\frac{1}{rB_z}\frac{\partial\Psi}{\partial r} \tag{2.6.12}$$

To see how the Hamiltonian could be formed, choose a new $t = z/(rB_z)$ and then

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial\Psi}{\partial\zeta} \tag{2.6.13}$$

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = -\frac{\partial\Psi}{\partial r} \tag{2.6.14}$$

Since

$$\Psi = \frac{B_0}{2\alpha}\left\{(\alpha r)^2 - \frac{2b}{B_0}\alpha r I_n'(n\alpha r)\cos(n\zeta)\right\} \tag{2.6.15}$$

We see that Ψ is the Hamiltonian in this case. Because with $\Psi = H$, $\frac{\mathrm{d}H}{\mathrm{d}t} = 0$ then $\frac{\mathrm{d}H}{\mathrm{d}z} = 0$. We also see that multiplying ψ by α/B_0 will not change that $\frac{\mathrm{d}H}{\mathrm{d}z} = 0$ and so calling the new Hamiltonian $\mathcal{H} = \alpha H/B_0$ we find

$$\mathcal{H} = \frac{\alpha^2 r^2}{2} - \frac{b}{B_0}\alpha r I_n'(n\alpha r)\cos n\zeta \tag{2.6.16}$$

which is, of course, identical to the Hamiltonian given above.

Now the book instead chooses $tB_z = z$ and simultaneously chooses $\rho = \frac{1}{2}\alpha r^2$. Let us change to ρ coordinates first, using $r = \sqrt{2\rho/\alpha}$. Then

$$\frac{\mathrm{d}r}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}\sqrt{\frac{2\rho}{\alpha}} = \sqrt{\frac{2}{\alpha}}\frac{1}{2\sqrt{\rho}}\frac{\mathrm{d}\rho}{\mathrm{d}z} = \sqrt{\frac{1}{2\alpha\rho}}\frac{\mathrm{d}\rho}{\mathrm{d}z} = \sqrt{\frac{1}{2\alpha\frac{1}{2}\alpha r^2}}\frac{\mathrm{d}\rho}{\mathrm{d}z} = \frac{1}{\alpha r}\frac{\mathrm{d}\rho}{\mathrm{d}z} \tag{2.6.17}$$

$$\frac{\partial}{\partial r} = \frac{\partial\rho}{\partial r}\frac{\partial}{\partial\rho} = \alpha r\frac{\partial}{\partial\rho} \tag{2.6.18}$$

Thus, with the Hamiltonian $\mathcal{H} = H/B_0$ from before, we would then find

$$\begin{aligned} B_z\frac{\mathrm{d}r}{\mathrm{d}z} &= \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{\alpha r}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{r}\frac{\partial\Psi}{\partial\zeta} \\ B_z\frac{\mathrm{d}\zeta}{\mathrm{d}z} &= \frac{\mathrm{d}\zeta}{\mathrm{d}t} = -\frac{1}{r}\frac{\partial\Psi}{\partial r} = -\frac{\alpha \not{r}}{\not{r}}\frac{\partial\Psi}{\partial\rho} \end{aligned} \tag{2.6.19}$$

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} &= \alpha\frac{\partial\Psi}{\partial\zeta} \\ \frac{\mathrm{d}\zeta}{\mathrm{d}t} &= -\alpha\frac{\partial\Psi}{\partial\rho} \end{aligned} \tag{2.6.20}$$

We easily see that $H = \alpha\Psi$ would then do, but if we normalize by B_0 again this will also do no harm, and so the new Hamiltonian will be the same as $\mathcal{H}$ above, yielding with ρ

$$H = \frac{\alpha\Psi}{B_0} = \frac{\alpha^2 r^2}{2} - \frac{b}{B_0}\alpha r I_n'(n\alpha r)\cos n\zeta = \alpha\rho - \frac{b}{B_0}\sqrt{2\alpha\rho}I_n'(n\sqrt{2\alpha\rho})\cos n\zeta \tag{2.6.21}$$

This, then, of course will yield

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\partial\mathcal{H}}{\partial\zeta}\\ \frac{\mathrm{d}\zeta}{\mathrm{d}t} &= -\frac{\partial\mathcal{H}}{\partial\rho}\end{aligned} \tag{2.6.22}$$

For the action-angle, use a type 1 generating function $W(q,Q)$ which in our case is $W(q,Q) = W(\zeta,\Theta)$ [I use $(p,q) \to (P,Q)$ which in our case yields $(\rho,\zeta) \to (J,\Theta)$]. Then using

$$\begin{aligned}\delta W &= \frac{\partial W}{\partial q}\delta q + \frac{\partial W}{\partial Q}\delta Q\\ &= p\delta q - P\delta Q = \rho\delta\zeta - J\delta\Theta\end{aligned} \tag{2.6.23}$$

Now we are periodic in ζ with period 2π so that W must be periodic. Note also that in these new action angle coordinates, that the Hamiltonian H is independent of the action angle Θ (that is, we select W in such a way that H is now independent of Θ and see if this is in fact possible). So then $H = H(J)$ and J must be independent of Θ as well. If we integrate through one period we find

$$\oint \mathrm{d}W = 0 = \oint \rho\,\mathrm{d}\zeta - \oint J\,\mathrm{d}\Theta = \oint \rho\,\mathrm{d}\zeta - 2\pi J \tag{2.6.24}$$

$$J = \frac{1}{2\pi}\oint \rho\,\mathrm{d}\zeta \tag{2.6.25}$$

Now the new Hamiltonian equations say

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \frac{\partial H}{\partial\Theta} = 0 \tag{2.6.26}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \frac{\partial H}{\partial J} \tag{2.6.27}$$

So then J is a constant of motion. Since $H = H(J)$, then H is a constant of motion, and so is $\frac{\partial H}{\partial J} = \nu(J) \equiv \alpha\omega(J)$. This implies an integration of Θ with respect to t yields

$$\Theta = \alpha\omega(J)t + \text{constant} \tag{2.6.28}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \alpha\omega(J) \tag{2.6.29}$$

with the constant simply setting the zero for this coordinate.

We can also see that

$$\int \mathrm{d}W = \int \rho\,\mathrm{d}\zeta - \int J\,\mathrm{d}\Theta = \int \rho\,\mathrm{d}\zeta - J\Theta \tag{2.6.30}$$

$$W = \underbrace{\int \rho\,\mathrm{d}\zeta}_{\equiv S} - J\Theta \tag{2.6.31}$$

$$\cancel{\frac{\partial W(\zeta,\Theta)}{\partial J}} = \frac{\partial S}{\partial J} - \Theta \tag{2.6.32}$$

$$\Theta = \frac{\partial S}{\partial J} \tag{2.6.33}$$

2.7 The Magnetic Surface by Line Tracing Calculation

Let's convince ourselves of the

$$\mathbf{R}(\Psi,\theta_B,\phi_B) = \sum_{mn} \mathbf{R}_{mn}(\Psi) e^{i(m\theta_B - n\phi_B)} \tag{W-2.169}$$

conversion to

$$\mathbf{R}(\Psi,\theta_B,\phi_B) = \sum_{mn} R_{mn}(\Psi) \exp\left[i\frac{(mI^e_{\mathrm{pol}} + nI^i_{\mathrm{tor}})\theta_{B0} + (m\iota - n)\frac{2\pi\chi}{\mu_0}}{I^i_{\mathrm{tor}}\iota + I^e_{\mathrm{pol}}}\right] \tag{W-2.173}$$

via

$$\chi = \frac{\mu_0}{2\pi}\left(I^i_{\mathrm{tor}}\theta_B + I^e_{\mathrm{pol}}\phi_B\right) \tag{W-2.171}$$

$$\theta_B = \iota\phi_B + \theta_{B0} \tag{W-2.172}$$

This implies that

$$\frac{2\pi\chi}{\mu_0} - I^i_{\mathrm{tor}}\theta_B = I^e_{\mathrm{pol}}\phi_B \tag{2.7.1}$$

$$I^e_{\mathrm{pol}}\phi_B = \frac{2\pi\chi}{\mu_0} - I^i_{\mathrm{tor}}\iota\phi_B - I^i_{\mathrm{tor}}\theta_{B0} \tag{2.7.2}$$

$$\phi_B = \frac{\frac{2\pi\chi}{\mu_0} - I^i_{\mathrm{tor}}\theta_{B0}}{I^e_{\mathrm{pol}} + \iota I^i_{\mathrm{tor}}} \tag{2.7.3}$$

So then

$$\begin{aligned} m\theta_B - n\phi_B &= m\iota\phi_B + m\theta_{B0} - n\frac{\frac{2\pi\chi}{\mu_0} - I^i_{\mathrm{tor}}\theta_{B0}}{I^e_{\mathrm{pol}} + \iota I^i_{\mathrm{tor}}} \\ &= m\iota\frac{\frac{2\pi\chi}{\mu_0} - I^i_{\mathrm{tor}}\theta_{B0}}{I^e_{\mathrm{pol}} + \iota I^i_{\mathrm{tor}}} + m\theta_{B0} - n\frac{\frac{2\pi\chi}{\mu_0} - I^i_{\mathrm{tor}}\theta_{B0}}{I^e_{\mathrm{pol}} + \iota I^i_{\mathrm{tor}}} \\ &= \frac{(mI^e_{\mathrm{pol}} + \cancel{m\iota I^i_{\mathrm{tor}}} - \cancel{m\iota I^i_{\mathrm{tor}}} + nI^i_{\mathrm{tor}})\theta_{B0} + (m\iota - n)\frac{2\pi\chi}{\mu_0}}{I^e_{\mathrm{pol}} + \iota I^i_{\mathrm{tor}}} \\ &= \frac{(mI^e_{\mathrm{pol}} + nI^i_{\mathrm{tor}})\theta_{B0} + (m\iota - n)\frac{2\pi\chi}{\mu_0}}{I^e_{\mathrm{pol}} + \iota I^i_{\mathrm{tor}}} \end{aligned} \tag{2.7.4}$$

as desired.

Chapter 3

A Description of Magnetically Confined Plasmas

3.3 The Vlasov Equation for Describing Incompressible Phase Fluid and Moment Equations

A quick proof from (W-3.36) to (W-3.37). using

$$n\langle Q\rangle \equiv \int_{-\infty}^{\infty} \mathrm{d}v_x \; Qf \tag{3.3.1}$$

$$Q = \langle Q\rangle + \delta Q \tag{3.3.2}$$

$$\langle Q\rangle = \langle\langle Q\rangle\rangle + \langle \delta Q\rangle = \langle Q\rangle + \langle \delta Q\rangle \tag{3.3.3}$$

$$\langle \delta Q\rangle = 0 \tag{3.3.4}$$

$$a_x = \hat{\mathbf{x}}\cdot(q\mathbf{E} + q\mathbf{v}\times\mathbf{B}) = qE_x + qv_yB_z - qv_zB_y \tag{3.3.5}$$

$$\int_{-\infty}^{\infty} \mathrm{d}v_x \; v_x\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} v_x f = \frac{\partial}{\partial t}\left(n\langle v_x\rangle\right) \tag{3.3.6}$$

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}v_x \; v_x^2\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \mathrm{d}v_x \; v_x^2 f = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \mathrm{d}v_x \; (\langle v_x\rangle + \delta v_x)^2 f \\ &= \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \mathrm{d}v_x \; \left[\langle v_x\rangle^2 f + \cancel{2\langle v_x\rangle \delta v_x f} + (\delta v_x)^2 f\right] = \frac{\partial}{\partial x}\left(\langle v_x\rangle^2 n + n\left\langle(\delta v_x)^2\right\rangle\right) \\ &= \frac{\partial}{\partial x}\left(n\langle v_x\rangle^2\right) + \frac{\partial}{\partial x}\left(n\left\langle(\delta v_x)^2\right\rangle\right)\end{aligned} \tag{3.3.7}$$

$$\int_{-\infty}^{\infty} \mathrm{d}v_x \; a_x v_x\frac{\partial f}{\partial v_x} = \cancel{f a_x v_x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \mathrm{d}v_x \; f\frac{\partial a_x v_x}{\partial v_x} = -\int_{-\infty}^{\infty} \mathrm{d}v_x \; a_x f = -n\langle a_x\rangle \tag{3.3.8}$$

Where the last follows because a_x has no v_x dependence. (So technically $n\langle a_x\rangle = a_x$).

In general, noting that $\frac{\partial E_i}{\partial v_i} = 0$ and that

$$\epsilon_{ilk}B_k\frac{\partial v_l}{\partial v_i} = \epsilon_{ilk}B_k\delta_{li} = \epsilon_{llk}B_k = 0 \tag{3.3.9}$$

we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}^3v\ \mathbf{v}\mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} &= \int_{-\infty}^{\infty} \mathrm{d}^3v\ v_j a_i \frac{\partial f}{\partial v_i} = \int_{-\infty}^{\infty} \mathrm{d}^3v\ \left[\frac{\partial}{\partial v_i}\left(v_j a_i f\right) - \frac{\partial (v_j a_i)}{\partial v_i} f\right] \\
&= \cancel{\int_{-\infty}^{\infty} \mathrm{d}^2v\ n_i\cdot v_j a_i} - \int_{-\infty}^{\infty} \mathrm{d}^3v\ \delta_{ij} a_i f - \int_{-\infty}^{\infty} \mathrm{d}^3v\ v_j \frac{\partial a_i}{\partial v_i} f \\
&= -\int_{-\infty}^{\infty} \mathrm{d}^3v\ a_j f - q\int_{-\infty}^{\infty} \mathrm{d}^3v\ v_j \frac{\partial}{\partial v_i}\left[E_i + \epsilon_{ilk} v_l B_k\right] f \qquad (3.3.10) \\
&= -\int_{-\infty}^{\infty} \mathrm{d}^3v\ a_j f - q\cancel{\int_{-\infty}^{\infty} \mathrm{d}^3v\ \epsilon_{llk} v_j B_k f} \\
&= -\int_{-\infty}^{\infty} \mathrm{d}^3v\ \mathbf{a} f = -n\left\langle \mathbf{a}\right\rangle
\end{aligned}$$

Where the cancellation on the d^2v term is because $n_i v_j a_i = \mathbf{v}\hat{\mathbf{n}}\cdot\mathbf{a} \to 0$ at the surface at infinity. So we see in general we do get the $\langle \mathbf{a}\rangle$ term, although in one dimension this is trivial.

Thus, putting it all together

$$\int_{-\infty}^{\infty} \mathrm{d}v_x\ \left[v_x\frac{\partial f}{\partial t} + v_x^2\frac{\partial f}{\partial x} + a_x v_x \frac{\partial f}{\partial v_x}\right] = 0 \qquad \text{(W-3.36)}$$

$$\frac{\partial}{\partial t}\left(n\left\langle v_x\right\rangle\right) + \frac{\partial}{\partial x}\left(n\left\langle v_x\right\rangle^2\right) + \frac{\partial}{\partial x}\left(n\left\langle(\delta v_x)^2\right\rangle\right) - n\left\langle a_x\right\rangle = 0 \qquad (3.3.11)$$

$$\frac{\partial}{\partial t}\left(mn\left\langle v_x\right\rangle\right) = -\frac{\partial}{\partial x}\left(mn\left\langle v_x\right\rangle^2\right) - \frac{\partial}{\partial x}\left(mn\left\langle(\delta v_x)^2\right\rangle\right) + mn\left\langle a_x\right\rangle \qquad \text{(W-3.37)}$$

as required.

Let's now do the $\ell = 2$ moment equation ourselves piece by piece. (Use $\mathbf{X}^2 = \mathbf{X}\cdot\mathbf{X} = X_j^2 = X_j X_j$ for short hand). Wakatani seems to prefer a mixed vector, index notation, while I will just use a pure index notation (although with $\left\langle X_j\right\rangle^2 = \left\langle X_j\right\rangle\left\langle X_j\right\rangle$)

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}^3v\ \frac{1}{2}mv^2\frac{\partial f}{\partial t} &= \frac{1}{2}m\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \mathrm{d}^3v\ \left(\left\langle v_j\right\rangle\cdot\left\langle v_j\right\rangle + \cancel{2\left\langle v_j\right\rangle\cdot\delta v_j} + \delta v_j\cdot\delta v_j\right) f \\
&= \frac{1}{2}m\frac{\partial}{\partial t}\left(n\left\langle v_j\right\rangle^2 + n\left\langle \delta v_j^2\right\rangle\right)
\end{aligned} \qquad (3.3.12)$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}^3v\ \frac{1}{2}mv^2 v_i\frac{\partial f}{\partial x_i} &= \frac{1}{2}m\frac{\partial}{\partial x_i}\int_{-\infty}^{\infty} \mathrm{d}^3v\ \left(\left\langle v_j\right\rangle^2 + 2\left\langle v_j\right\rangle\cdot\delta\mathbf{v} + \delta v_j^2\right)\left(\left\langle v_i\right\rangle + \delta v_i\right) f \\
&= \frac{1}{2}m\frac{\partial}{\partial x_i}\left(n\left\langle v_j\right\rangle^2\left\langle v_i\right\rangle + n\left\langle \delta v_j^2\right\rangle\left\langle v_i\right\rangle + n\left\langle v_j\right\rangle^2\left\langle \delta v_i\right\rangle + n\left\langle v_j\right\rangle\cdot\left\langle \delta v_j \delta v_i\right\rangle + n\left\langle \delta v_j^2 \delta v_i\right\rangle\right)
\end{aligned} \qquad (3.3.13)$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}^3v\ \frac{1}{2}mv_j^2\frac{q}{m}\left(E_i + \epsilon_{ilk}v_l B_k\right)\frac{\partial f}{\partial v_i} &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}^3v\ f\frac{\partial}{\partial v_i}\left(v_j^2 q\left(E_i + \epsilon_{ilk}v_l B_k\right)\right) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}^3v\ f\left(2v_j\frac{\partial v_j}{\partial v_i}q\left(E_i + \epsilon_{ilk}v_l B_k\right) + \cancel{v_j^2 q\epsilon_{ilk}B_k\frac{\partial v_l}{\partial v_i}}\right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}^3v\ f\left(v_i q E_i + \cancel{v_i\epsilon_{ilk}v_l B_k}\right)^{\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}) = 0} = qE_i\int_{-\infty}^{\infty} \mathrm{d}^3v\ v_i f \\
&= qn\left\langle v_i\right\rangle E_i = qn\left\langle \mathbf{v}\right\rangle\cdot\mathbf{E}
\end{aligned} \qquad (3.3.14)$$

So we yield the same result. Wakatani's discussion leading to (W-3.46) is sufficiently clear to me that it needs no further explanation here. Just note that all of his identities for averages of δv_i's are due to choosing the Maxwellian as the distribution function.

3.4 Magnetohydrodynamic Equations

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P_i n_i^{-\gamma}\right) = 0 \tag{3.4.1}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P_e n_e^{-\gamma}\right) = 0 \tag{3.4.2}$$

Using quasineutrality $n_e = Zn_i = n$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P_i n_i^{-\gamma}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(P_i Z^{\gamma} n^{-\gamma}\right) = 0 \tag{3.4.3}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P_i n^{-\gamma}\right) = 0 \tag{3.4.4}$$

Thus, when we sum for two species we find

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(P_i n_i^{-\gamma} + P_e n_e^{-\gamma}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(P_i Z^{\gamma} n^{-\gamma} + P_e n^{-\gamma}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(P_i (Z^{\gamma} - 1) n^{-\gamma} + P n^{-\gamma}\right) \\ &= \cancel{(Z^{\gamma-1} - 1)\frac{\mathrm{d}}{\mathrm{d}t}(P_i n^{-\gamma})} + \frac{\mathrm{d}}{\mathrm{d}t}\left(P n^{-\gamma}\right) = 0 \end{aligned} \tag{3.4.5}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P n^{-\gamma}\right) = 0 \tag{3.4.6}$$

We get Ohm's law through

$$m_e n_e \frac{\mathrm{d}\mathbf{v}_e}{\mathrm{d}t} = -\nabla P_e + n_e q_e (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) + \mathbf{R} \tag{W-3.53}$$

and with $\frac{\mathrm{d}\mathbf{v}_e}{\mathrm{d}t} \ll 1$ (negligible electron inertia with

$$\begin{aligned} \mathbf{J} &= n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i \\ \rho_q &= n_e q_e + n_i q_i \approx 0 \\ n_e q_e &= -n_i q_i \\ \mathbf{J} - n_i q_i \mathbf{v}_i &= n_e q_e \mathbf{v}_e \end{aligned} \tag{3.4.7}$$

with $\mathbf{v}_i \approx \mathbf{v}$ (remembering $n_e q_e = -ne$ so $n_i q_i = ne$) or Then

$$0 = -\nabla P_e + n_e q_e \mathbf{E} + (\mathbf{J} - n_i q_i \mathbf{v}_i) \times \mathbf{B} + \mathbf{R} \tag{3.4.8}$$
$$0 = -\nabla P_e - ne\mathbf{E} + \mathbf{J} \times \mathbf{B} - ne\mathbf{v} \times \mathbf{B} + \mathbf{R} \tag{3.4.9}$$
$$ne\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) = -\nabla P_e + \mathbf{J} \times \mathbf{B} + \mathbf{R} \tag{3.4.10}$$
$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{ne}\left(\mathbf{J} \times \mathbf{B} - \nabla P_e + \mathbf{R}\right) \tag{3.4.11}$$

Note that a better way of using notation for conserving phase space "volume" for 2D is to simply say $2\pi S v \, \mathrm{d}v = 2\pi S_0 v_0 \, \mathrm{d}v_0$ and so (both S and S_0 are independent of velocity so we can move them around at will when integrating and S_0 is a constant whereas S can be thought of as a function of t, but independent of v)

$$S v \, \mathrm{d}v = S_0 v_0 \, \mathrm{d}v_0 \tag{3.4.12}$$

$$\int S v \, \mathrm{d}v = \int S_0 v_0 \, \mathrm{d}v_0 \tag{3.4.13}$$

$$\frac{S v^2}{2} = \frac{S_0 v_0^2}{2} \tag{3.4.14}$$

$$S v^2 = S_0 v_0^2 = \text{constant} \tag{3.4.15}$$

and for 1D to say $\ell \, \mathrm{d}v = \ell_0 \, \mathrm{d}v_0$ so that $v\ell = \text{constant}$. With $T_\parallel \propto v_\parallel^2$ and $T_\perp \propto v_\perp^2$ and $S \propto L_\perp^2$ and $\ell \propto L_\parallel$ we find

$$T_\perp \propto v_\perp^2 \propto \frac{1}{S} \propto \frac{1}{L_\perp^2} \tag{3.4.16}$$

$$T_\parallel \propto v_\parallel^2 \propto \frac{1}{\ell^2} \propto \frac{1}{L_\parallel^2} \tag{3.4.17}$$

And so then discover

$$T_\perp^2 T_\parallel \propto L_\perp^{-4} L_\parallel^2 \propto n^2 \tag{3.4.18}$$

because $n \propto L_\perp^{-2} L_\parallel^{-1}$. So then

$$\frac{n^3 T_\perp^2 T_\parallel}{n^3 n^2} \propto 1 \tag{3.4.19}$$

$$\frac{P_\perp^2 P_\parallel}{n^5} \propto 1 \tag{3.4.20}$$

$$\frac{P_\perp^2 P_\parallel}{n^5} = \text{constant} \tag{3.4.21}$$

as said in the book, reproducing (W-3.87).

3.5 MHD WaveS

Let's go through linearization here. Our beginning MHD equations (taking the ideal case)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \tag{W-3.73}$$

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla P + \mathbf{J} \times \mathbf{B} \tag{W-3.74}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} + \gamma P \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \tag{W-3.75}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \tag{W-3.76}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E} \tag{W-3.77}$$

$$\mu_0 \mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B} \tag{W-3.78}$$

and so, linearizing (W-3.73) first, (the zeroth order equation $\frac{\mathrm{d}\rho_0}{\mathrm{d}t} + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{v}_0 = 0$ must be satisfied hence the cancellation)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_0 + \rho_1) + (\rho_0 + \rho_1)\boldsymbol{\nabla} \cdot (\mathbf{v}_0 + \mathbf{v}_1) = 0 \tag{3.5.1}$$

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}t} + \cancel{\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}t} + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{v}_0\right)} + \rho_1 \boldsymbol{\nabla} \cdot \mathbf{v}_0 + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{v}_1 = 0 \tag{3.5.2}$$

$$\frac{\partial \rho_1}{\partial t} + (\mathbf{v}_0 + \mathbf{v}_1) \cdot \nabla \rho_1 + \rho_1 \boldsymbol{\nabla} \cdot \mathbf{v}_0 + \boldsymbol{\nabla} \cdot (\rho_0 \mathbf{v}_1) = 0 \tag{3.5.3}$$

$$\frac{\partial \rho_1}{\partial t} + \mathbf{v}_0 \cdot \nabla \rho_1 + \rho_1 \boldsymbol{\nabla} \cdot \mathbf{v}_0 + \boldsymbol{\nabla} \cdot (\rho_0 \mathbf{v}_1) = 0 \tag{3.5.4}$$

$$\frac{\partial \rho_1}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0) = 0 \tag{3.5.5}$$

Wakatani assumes a zero background flow velocity (although does not explicitly state this) so that $\mathbf{v}_0 = \mathbf{0}$ and we retrieve

$$\frac{\partial \rho_1}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_0 \mathbf{v}_1) = 0 \tag{W-3.89}$$

Now, let's do (W-3.74) (the zeroth order equation equation is $-\nabla P_0 + \mathbf{J}_0 \times \mathbf{B}_0 = \mathbf{0}$, noting that $\mu_0 J_0 = \boldsymbol{\nabla} \times \mathbf{B}_0 = \mathbf{0}$)

$$(\rho_0 + \rho_1)\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} = -\nabla(P_0 + P_1) + (\mathbf{J}_0 + \mathbf{J}_1) \times (\mathbf{B}_0 + \mathbf{B}_1) \tag{3.5.6}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \cancel{-\nabla P_0} - \nabla P_1 + \cancel{\mathbf{J}_0 \times \mathbf{B}_0} + \cancel{\mathbf{J}_0} \times \mathbf{B}_1 + \mathbf{J}_1 \times \mathbf{B}_0 \tag{3.5.7}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{J}_1 \times \mathbf{B}_0 - \nabla P_1 \tag{3.5.8}$$

Now for (W-3.75), (zeroth order is $\frac{\mathrm{d}P_0}{\mathrm{d}t} = 0$)

$$\frac{\mathrm{d}P_1}{\mathrm{d}t} + \gamma(P_0 + P_1)\boldsymbol{\nabla} \cdot \mathbf{v}_1 = 0 \tag{3.5.9}$$

$$\frac{\partial P_1}{\partial t} + \gamma P_0 \boldsymbol{\nabla} \cdot \mathbf{v}_1 = 0 \tag{3.5.10}$$

Now for (W-3.76) (zeroth order is $\mathbf{E}_0 = \mathbf{0}$)

$$\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0 = 0 \tag{3.5.11}$$

And the rest are trivial with these previous calculations.

Wakatani doesn't explain why he neglects the complex conjugate parts when he writes out his primed equations. Here he is tacitly assuming that we take the real part of whatever is left at the end of equations, and hence that is why we can set these quantities so simply.

Using $\Omega = \hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \mathbf{v}_1$ and so $\Omega_k = \hat{\mathbf{b}} \cdot i\mathbf{k} \times \mathbf{v}_k$ we then find when taking $\hat{\mathbf{b}} \cdot i\mathbf{k} \times$(W-3.90') that

$$-i\omega\rho_0 \underbrace{i\hat{\mathbf{b}}\mathbf{k} \times \mathbf{v}_k}_{\Omega_k} = i\hat{\mathbf{b}} \cdot \mathbf{k} \times (\mathbf{J}_k \times \mathbf{B}_0) - \cancel{iP_k i\mathbf{k} \times \mathbf{k}} \tag{3.5.12}$$

$$\omega\rho_0\Omega_k = ii\hat{\mathbf{b}} \cdot (\mathbf{J}_k(\mathbf{k} \cdot \mathbf{B}_0) - \mathbf{B}_0(\mathbf{J}_k \cdot \mathbf{k})) \tag{3.5.13}$$

$$\omega\rho_0\Omega_k = -\frac{(\mathbf{J}_k \cdot \mathbf{B}_0)}{B_0}(\mathbf{k} \cdot \mathbf{B}_0) + B_0(\mathbf{J}_k \cdot \mathbf{k}) \tag{3.5.14}$$

Now using $\mu_0 \mathbf{J}_k = i\mathbf{k} \times \mathbf{B}_k$ we find

$$\omega\rho_0\Omega_k = -\frac{(\mathbf{J}_k \cdot \mathbf{B}_0)}{B_0}(\mathbf{k}\cdot\mathbf{B}_0) + \frac{iB_0}{\mu_0}\cancel{(\mathbf{k}\cdot\mathbf{k}\times\mathbf{B}_k)} \tag{3.5.15}$$

$$-\omega\rho_0\Omega_k = \frac{(\mathbf{J}_k \cdot \mathbf{B}_0)}{B_0}(\mathbf{k}\cdot\mathbf{B}_0) \tag{3.5.16}$$

The rest falls out simply from the manipulations that Wakatani describes.

3.6 The Drift-Kinetic Equation

Let's prove that $(\mathbf{v}_E + \mathbf{v}_G)\cdot(\mu\,\nabla B + q\,\nabla\phi) = 0$ (with $\mathbf{E} = -\,\nabla\phi$).

$$\begin{aligned}\mathbf{v}_E\cdot\mu\,\nabla B + \mathbf{v}_E\cdot q\,\nabla\phi &= \frac{\mathbf{E}\times\mathbf{B}}{B^2}\cdot\mu\,\nabla B + \frac{\mathbf{E}\times\mathbf{B}}{B^2}\cdot q\,\nabla\phi \\ &= \frac{\mu}{B^2}\,\nabla B\cdot\mathbf{E}\times\mathbf{B} + \cancel{\frac{q}{B^2}(-\mathbf{E})\cdot\mathbf{E}\times\mathbf{B}}\end{aligned} \tag{3.6.1}$$

$$\begin{aligned}\mathbf{v}_G\cdot\mu\,\nabla B + \mathbf{v}_G\cdot q\,\nabla\phi &= \cancel{\frac{mv_\perp^2}{2qB^3}\mathbf{B}\times\nabla B\cdot\mu\mathbf{B}} = \frac{mv_\perp^2}{2qB^3}\mathbf{B}\times\,\nabla B\cdot q\,\nabla\phi \\ &= -\frac{mv_\perp^2}{2B^3}\mathbf{E}\cdot\mathbf{B}\times\,\nabla B = -\frac{\mu}{B^2}\mathbf{E}\cdot\mathbf{B}\times\,\nabla B\end{aligned} \tag{3.6.2}$$

And because $\nabla B\cdot\mathbf{E}\times\mathbf{B} = \mathbf{E}\cdot\mathbf{B}\times\,\nabla B$ we see that we get perfect cancellation.

Let us derive (W-3.120) and hence (W-3.121). We begin with ($\mathbf{v}_C = \frac{v_\parallel^2}{\Omega}\boldsymbol{\nabla}\times\hat{\mathbf{b}}$)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = v_\parallel\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_G + \mathbf{v}_C \tag{W-3.112}$$

Then using (W-3.117) with $v_\parallel\sqrt{\frac{m}{2}}\sqrt{K-\mu B-q\phi}$

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{-1}{\sqrt{2m}}\left(\mu\frac{\mathrm{d}B}{\mathrm{d}t} + q\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)(K-\mu B-q\phi)^{-1/2} \tag{W-3.117}$$

$$= -\frac{1}{\sqrt{2m}}\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot(\mu\,\nabla B + q\,\nabla\phi)\right)\frac{1}{v_\parallel\sqrt{\frac{m}{2}}} \tag{3.6.3}$$

$$= -\frac{1}{mv_\parallel}\Big((v_\parallel\hat{\mathbf{b}} + \mathbf{v}_C)\cdot(\mu\,\nabla B + q\,\nabla\phi) + \cancel{(\mathbf{v}_E + \mathbf{v}_G)\cdot(\mu\,\nabla B + q\,\nabla\phi)}\Big) \tag{3.6.4}$$

$$= \left(\hat{\mathbf{b}} + \frac{\mathbf{v}_C}{v_\parallel}\right)\cdot\left(-\frac{\mu}{m}\,\nabla B + \frac{q}{m}\,\nabla\phi\right) \tag{3.6.5}$$

We see

$$\hat{\mathbf{b}}\times\boldsymbol{\kappa} = \hat{\mathbf{b}}\times\left(\hat{\mathbf{b}}\cdot\,\nabla\hat{\mathbf{b}}\right) = \hat{\mathbf{b}}\times\left(-\hat{\mathbf{b}}\times(\boldsymbol{\nabla}\times\hat{\mathbf{b}})\right) = -\hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}) - (\boldsymbol{\nabla}\times\hat{\mathbf{b}})(\hat{\mathbf{b}}\cdot-\hat{\mathbf{b}}) \tag{3.6.6}$$

$$= \cancel{-\hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}})} + \boldsymbol{\nabla}\times\hat{\mathbf{b}} = \boldsymbol{\nabla}\times\hat{\mathbf{b}} \tag{3.6.7}$$

Here we use $B\hat{\mathbf{b}} = \mathbf{B}$ with $\boldsymbol{\nabla}\times\mathbf{B} = \mathbf{0}$ so that

$$\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}} = \hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times(\mathbf{B}/B) = \hat{\mathbf{b}}\cdot\left(\cancel{\frac{1}{B}\boldsymbol{\nabla}\times\mathbf{B}} + \frac{\nabla B}{B^2}\times\mathbf{B}\right) = \frac{\hat{\mathbf{b}}}{B}\cdot\,\nabla B\times\hat{\mathbf{b}} = 0 \tag{3.6.8}$$

3.7 The Averaged Reduced MHD Equations

First, let's get the metric coefficients. The given system is (r, θ, ζ) where $R_0 > 0$ is a constant with

$$r = \sqrt{(\sqrt{x^2+y^2} - R_0)^2 + z^2} \tag{3.7.1}$$

$$\tan\theta = \frac{z}{\sqrt{x^2+y^2} - R_0} \tag{3.7.2}$$

$$\zeta = -R_0 \operatorname{atan}(y/x) \Rightarrow \tan(-\zeta/R_0) = y/x \tag{3.7.3}$$

Thus we find

$$\begin{aligned}
\mathrm{d}r &= \frac{2(\sqrt{x^2+y^2} - R_0)(\frac{2x\,\mathrm{d}x + 2y\,\mathrm{d}y}{2\sqrt{x^2+y^2}}) + 2z\,\mathrm{d}z}{2\sqrt{(\sqrt{x^2+y^2} - R_0)^2 + z^2}} = \frac{(\sqrt{(x^2+y^2} - R_0)(\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{x^2+y^2}}) + z\,\mathrm{d}z}{\sqrt{(\sqrt{x^2+y^2} - R_0)^2 + z^2}} \\
&= \frac{(1 - \frac{R_0}{\sqrt{x^2+y^2}})(x\,\mathrm{d}x + y\,\mathrm{d}y) + z\,\mathrm{d}z}{\sqrt{(\sqrt{x^2+y^2} - R_0)^2 + z^2}}
\end{aligned} \tag{3.7.4}$$

$$\left(\frac{\partial r}{\partial x}\right)_{y,z} = \frac{(1 - \frac{R_0}{\sqrt{x^2+y^2}})x}{\sqrt{(\sqrt{x^2+y^2} - R_0)^2 + z^2}} = \left(1 - \frac{R_0}{\sqrt{x^2+y^2}}\right)\frac{x}{r} \tag{3.7.5}$$

$$\left(\frac{\partial r}{\partial y}\right)_{x,z} = \frac{(1 - \frac{R_0}{\sqrt{x^2+y^2}})y}{\sqrt{(\sqrt{x^2+y^2} - R_0)^2 + z^2}} = \left(1 - \frac{R_0}{\sqrt{x^2+y^2}}\right)\frac{y}{r} \tag{3.7.6}$$

$$\left(\frac{\partial r}{\partial z}\right)_{x,y} = \frac{z}{\sqrt{(\sqrt{x^2+y^2} - R_0)^2 + z^2}} = \frac{z}{r} \tag{3.7.7}$$

$$\sec^2\theta\,\mathrm{d}\theta = \frac{(\sqrt{x^2+y^2} - R_0)\,\mathrm{d}z - z\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{x^2+y^2}}}{(\sqrt{x^2+y^2} - R_0)^2} \Rightarrow \mathrm{d}\theta = \cos^2\theta \frac{(\sqrt{x^2+y^2} - R_0)\,\mathrm{d}z - z\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{x^2+y^2}}}{(\sqrt{x^2+y^2} - R_0)^2} \tag{3.7.8}$$

$$\left(\frac{\partial \theta}{\partial x}\right)_{y,z} = -\frac{zx\cos^2(\theta)}{\sqrt{x^2+y^2}(\sqrt{x^2+y^2} - R_0)^2} \tag{3.7.9}$$

$$\left(\frac{\partial \theta}{\partial y}\right)_{x,z} = -\frac{zy\cos^2(\theta)}{\sqrt{x^2+y^2}(\sqrt{x^2+y^2} - R_0)^2} \tag{3.7.10}$$

$$\left(\frac{\partial \theta}{\partial z}\right)_{x,y} = \frac{\cos^2(\theta)}{\sqrt{x^2+y^2} - R_0} \tag{3.7.11}$$

$$\sec^2(-\frac{\zeta}{R_0})\frac{-\mathrm{d}\zeta}{R_0} = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} \Rightarrow \mathrm{d}\zeta = R_0\cos^2(\frac{\zeta}{R_0})\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2} \tag{3.7.12}$$

$$\left(\frac{\partial \zeta}{\partial x}\right)_{y,z} = \frac{yR_0\cos^2(\zeta/R_0)}{x^2} \tag{3.7.13}$$

$$\left(\frac{\partial \zeta}{\partial y}\right)_{x,z} = \frac{-R_0\cos^2(\zeta/R_0)}{x} \tag{3.7.14}$$

$$\left(\frac{\partial \zeta}{\partial z}\right)_{x,y} = 0 \tag{3.7.15}$$

We form the Jacobian matrix and remember the form of the metric tensor

$$\frac{\partial(r,\theta,\zeta)}{\partial(x,y,z)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{x\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)}{r} & \frac{y\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)}{r} & \frac{z}{r} \\ -\frac{xz\cos^2(\theta)}{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}-R_0\right)^2} & -\frac{yz\cos^2(\theta)}{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}-R_0\right)^2} & \frac{\cos^2(\theta)}{\sqrt{x^2+y^2}-R_0} \\ \frac{R_0 y\cos^2\left(\frac{\zeta}{R_0}\right)}{x^2} & -\frac{R_0\cos^2\left(\frac{\zeta}{R_0}\right)}{x} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)}{r} & \frac{y\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)}{r} & \frac{z}{r} \\ -\frac{xz}{r^2\sqrt{x^2+y^2}} & -\frac{yz}{r^2\sqrt{x^2+y^2}} & \frac{\sqrt{x^2+y^2}-R_0}{r^2} \\ \frac{R_0 y}{x^2+y^2} & -\frac{R_0 x}{x^2+y^2} & 0 \end{bmatrix} \tag{3.7.16}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\zeta)} = \begin{bmatrix} \frac{x\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)}{r} & -\frac{xz}{\sqrt{x^2+y^2}} & \frac{y}{R_0} \\ \frac{y\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)}{r} & -\frac{yz}{\sqrt{x^2+y^2}} & -\frac{x}{R_0} \\ \frac{z}{r} & \sqrt{x^2+y^2}-R_0 & 0 \end{bmatrix} \tag{3.7.17}$$

$$g^{ij} = \frac{\partial \xi^i}{\partial x^k}\frac{\partial \xi^j}{\partial x^k} \tag{3.7.18}$$

$$g_{ij} = \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j} \tag{3.7.19}$$

and remember $(\,\mathrm{d}\ell)^2 = g_{ij}\,\mathrm{d}\xi_i\,\mathrm{d}\xi_j$. So we have

$$(\,\mathrm{d}\ell)^2 = g_{rr}(\,\mathrm{d}r)^2 + g_{\theta\theta}(\,\mathrm{d}\theta)^2 + g_{\zeta\zeta}(\,\mathrm{d}\zeta)^2 + (g_{r\theta}+g_{\theta r})\,\mathrm{d}r\,\mathrm{d}\theta + (g_{r\zeta}+g_{\zeta r})\,\mathrm{d}r\,\mathrm{d}\zeta + (g_{\theta\zeta}+g_{\zeta\theta})\,\mathrm{d}\theta\,\mathrm{d}\theta \tag{3.7.20}$$

Remember that we have

$$r^2 = (\sqrt{x^2+y^2}-R_0)^2 + z^2 = x^2+y^2-2R_0\sqrt{x^2+y^2}+R_0^2+z^2 \tag{3.7.21}$$

$$\cos\theta = \frac{\sqrt{x^2+y^2}-R_0}{\sqrt{z^2+(\sqrt{x^2+y^2}-R_0)^2}} = \frac{\sqrt{x^2+y^2}-R_0}{r} \tag{3.7.22}$$

$$\cos(\zeta/R_0) = \frac{x}{\sqrt{x^2+y^2}} \tag{3.7.23}$$

Let's find the metric tensor components first

$$\begin{aligned} g^{rr} &= \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2 = \left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)^2\left(\frac{x^2+y^2}{r^2}\right) + \frac{z^2}{r^2} \\ &= \left(1-\frac{2R_0}{\sqrt{x^2+y^2}}+\frac{R_0^2}{x^2+y^2}\right)\frac{x^2+y^2}{r^2} + \frac{z^2}{r^2} \\ &= \frac{x^2+y^2+z^2}{r^2} - \frac{2R_0\sqrt{x^2+y^2}}{r^2} + \frac{R_0^2}{r^2} = \frac{x^2+y^2-2R_0\sqrt{x^2+y^2}+R_0^{+}z^2}{r^2} = \frac{r^2}{r^2} = 1 \end{aligned} \tag{3.7.24}$$

$$
\begin{aligned}
g^{r\theta} &= \frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \theta}{\partial z} \\
&= \left(1 - \frac{R_0}{\sqrt{x^2+y^2}}\right)\frac{x}{r}\frac{-zx\cos^2\theta}{\sqrt{x^2+y^2}(\sqrt{x^2+y^2}-R_0)^2} \\
&\qquad + \left(1 - \frac{R_0}{\sqrt{x^2+y^2}}\right)\frac{y}{r}\frac{-zy\cos^2\theta}{\sqrt{x^2+y^2}(\sqrt{x^2+y^2}-R_0)^2} + \frac{z}{r}\frac{\cos^2\theta}{\sqrt{x^2+y^2}-R_0} \\
&= \left(1 - \frac{R_0}{\sqrt{x^2+y^2}}\right)\frac{-z\cos^2\theta\,(x^2+y^2)}{r\sqrt{x^2+y^2}(\sqrt{x^2+y^2}-R_0)^2} + \frac{z\cos^2\theta}{r(\sqrt{x^2+y^2}-R_0)} \\
&= \left(\sqrt{x^2+y^2}-R_0\right)\frac{-z\cos^2\theta}{r(\sqrt{x^2+y^2}-R_0)^2} + \frac{z\cos^2\theta}{r(\sqrt{x^2+y^2}-R_0)} \\
&= \frac{-z\cos^2\theta}{r(\sqrt{x^2+y^2}-R_0)} + \frac{z\cos^2\theta}{r(\sqrt{x^2+y^2}-R_0)} = 0
\end{aligned}
\tag{3.7.25}
$$

$$
\begin{aligned}
g^{r\zeta} &= \frac{\partial r}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \zeta}{\partial z} \\
&= \left(1 - \frac{R_0}{\sqrt{x^2+y^2}}\right)\frac{x}{r}\left(\frac{yR_0\cos^2(\zeta/R_0)}{x^2}\right) + \left(1 - \frac{R_0}{\sqrt{x^2+y^2}}\right)\frac{y}{r}\left(\frac{-R_0\cos^2(\zeta/R_0)}{x}\right) + \cancel{\frac{z}{r}(0)} \\
&= \left(1 - \frac{R_0}{\sqrt{x^2+y^2}}\right)\frac{yR_0\cos^2(\zeta/R_0)[1-1]}{rx} = 0
\end{aligned}
\tag{3.7.26}
$$

$$
\begin{aligned}
g^{\theta\theta} &= \left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 \\
&= \frac{z^2x^2\cos^4\theta}{(x^2+y^2)(\sqrt{x^2+y^2}-R_0)^4} + \frac{z^2y^2\cos^4\theta}{(x^2+y^2)(\sqrt{x^2+y^2}-R_0)^4} + \frac{\cos^4\theta}{(\sqrt{x^2+y^2}-R_0)^2} \\
&= \frac{(x^2+y^2)z^2\cos^4\theta}{(x^2+y^2)(\sqrt{x^2+y^2}-R_0)^4} + \frac{\cos^4\theta}{(\sqrt{x^2+y^2}-R_0)^2} = \frac{\cos^4\theta\left[z^2 + (\sqrt{x^2+y^2}-R_0)^2\right]}{(\sqrt{x^2+y^2}-R_0)^4} \\
&= \frac{r^2}{(\sqrt{x^2+y^2}-R_0)^4}\frac{(\sqrt{x^2+y^2}-R_0)^4}{r^4} = \frac{1}{r^2}
\end{aligned}
\tag{3.7.27}
$$

$$
\begin{aligned}
g^{\theta\zeta} &= \frac{\partial \theta}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \theta}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial \theta}{\partial z}\frac{\partial \zeta}{\partial z} \\
&= \frac{-zx\cos^2\theta}{\sqrt{x^2+y^2}(\sqrt{x^2+y^2}-R_0)^2}\frac{yR_0\cos^2(\zeta/R_0)}{x^2} + \frac{-zy\cos^2\theta}{\sqrt{x^2+y^2}(\sqrt{x^2+y^2}-R_0)^2}\frac{-R_0\cos^2(\zeta/R_0)}{x} \\
&\qquad + \frac{\cos^2\theta}{\sqrt{x^2+y^2}-R_0}(0) \\
&= \frac{-\cos^2\theta\cos^2(\zeta/R_0)\,[zy-zy]\,R_0}{\sqrt{x^2+y^2}(\sqrt{x^2+y^2}-R_0)^2} = 0
\end{aligned}
\tag{3.7.28}
$$

$$
\begin{aligned}
g^{\zeta\zeta} &= \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2 + \left(\frac{\partial\zeta}{\partial z}\right)^2 = \frac{y^2 R_0^2\cos^4(\zeta/R_0)}{x^4} + \frac{R_0^2\cos^4(\zeta/R_0)}{x^2} + 0^2 \\
&= \frac{R_0^2(y^2+x^2)}{x^4}\frac{x^4}{(x^2+y^2)^2} = \frac{R_0^2}{x^2+y^2}
\end{aligned} \tag{3.7.29}
$$

Because the tensor matrix is diagonal, we see that $g_{ii} = 1/g^{ii}$ and so

$$g_{rr} = 1 \tag{3.7.30}$$

$$g_{\theta\theta} = r^2 \tag{3.7.31}$$

$$
\begin{aligned}
g_{\zeta\zeta} &= \frac{x^2+y^2}{R_0^2} = \frac{x^2+y^2}{R_0^2} - \frac{2R_0\sqrt{x^2+y^2}}{R_0^2} + \frac{R_0^2}{R_0^2} + 1 + \frac{2}{R_0}\left(\sqrt{x^2+y^2} - R_0\right) \\
&= \frac{(\sqrt{x^2+y^2} - R_0)^2}{R_0^2} + 1 + \frac{2}{R_0}\left(\sqrt{x^2+y^2} - R_0\right) \\
&= 1 + \frac{r^2}{R_0^2}\frac{(\sqrt{x^2+y^2} - R_0)^2}{r^2} + \frac{2r}{R_0}\left(\sqrt{x^2+y^2} - R_0\right) \\
&= 1 + \frac{2r}{R_0}\cos\theta + \frac{r^2}{R_0^2}\cos^2\theta = \left(1 + \frac{r}{R_0}\cos\theta\right)^2
\end{aligned} \tag{3.7.32}
$$

So we recover (W-3.144) with considerable effort, that is

$$(\,\mathrm{d}\ell)^2 = (\,\mathrm{d}r)^2 + r^2(\,\mathrm{d}\theta)^2 + \left(1 + \frac{r}{R_0}\cos\theta\right)^2 (\,\mathrm{d}\zeta)^2 \tag{W-3.144}$$

Note the inconsistency of defining

$$\nabla_\perp \equiv -\hat{\boldsymbol{\zeta}}\frac{\partial}{\partial\zeta} \tag{W-3.149}$$

but assigning $|\frac{\partial}{\partial\zeta}|/|\,\nabla_\perp| \sim \mathcal{O}(\delta)$. Because $|\,\nabla_\perp| = |-\hat{\boldsymbol{\zeta}}\frac{\partial}{\partial\zeta}| = |\frac{\partial}{\partial\zeta}|$ this is a completely impossible assignment. I believe that Wakatani meant $|\frac{\partial}{\partial\zeta}| = |\,\nabla_\perp| \sim \mathcal{O}(\delta)$ and later $= \mathcal{O}(1)$, as this would make sense.

We combine Ohm's Law and the induction equation

$$\frac{\partial\mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E} \tag{3.7.33}$$

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B} + \eta\mathbf{J} \tag{3.7.34}$$

$$
\begin{aligned}
\frac{\partial\mathbf{B}}{\partial t} &= -\boldsymbol{\nabla}\times(-\mathbf{v}\times\mathbf{B} + \eta\mathbf{J}) \\
&= \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) + -\boldsymbol{\nabla}\times\frac{\eta}{\mu_0}\boldsymbol{\nabla}\times\mathbf{B} \\
&= \mathbf{v}\cancel{\boldsymbol{\nabla}\cdot\mathbf{B}} + \mathbf{B}\cdot\nabla\mathbf{v} - \mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{B} + \frac{\eta}{\mu_0}\left[\cancel{-\nabla(\boldsymbol{\nabla}\cdot\mathbf{B})} + \nabla^2\mathbf{B}\right] \\
&= \mathbf{B}\cdot\nabla\mathbf{v} - \mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{B} + \frac{\eta}{\mu_0}\nabla^2\mathbf{B}
\end{aligned} \tag{3.7.35}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} &= \frac{\partial\mathbf{B}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{v} - \mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{v}\cancel{-\mathbf{v}\cdot\nabla\mathbf{B}} + \frac{\eta}{\mu_0}\nabla^2\mathbf{B} + \cancel{\mathbf{v}\cdot\nabla\mathbf{B}} \\
&= \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{v}) + \mathbf{B}\cdot\nabla\mathbf{v} + \frac{\eta}{\mu_0}\nabla^2\mathbf{B}
\end{aligned} \tag{3.7.36}
$$

We now take the $\hat{\boldsymbol{\zeta}}$ component of this equation so

$$\hat{\boldsymbol{\zeta}}\cdot\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = -\hat{\boldsymbol{\zeta}}\cdot\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{v}) + \hat{\boldsymbol{\zeta}}\cdot(\mathbf{B}\cdot\nabla\mathbf{v}) + \frac{\eta}{\mu_0}\hat{\boldsymbol{\zeta}}\cdot\nabla^2\mathbf{B} \tag{3.7.37}$$

$$\frac{\mathrm{d}(\hat{\boldsymbol{\zeta}}\cdot\mathbf{B})}{\mathrm{d}t} = -B_\zeta(\boldsymbol{\nabla}\cdot\mathbf{v}) + \mathbf{B}\cdot\nabla(\hat{\boldsymbol{\zeta}}\cdot\mathbf{v}) - \mathbf{v}\cdot(\mathbf{B}\cdot\nabla\hat{\boldsymbol{\zeta}}) + \frac{\eta}{\mu_0}\nabla^2(\hat{\boldsymbol{\zeta}}\cdot\mathbf{B}) - \frac{\eta}{\mu_0}\mathbf{B}\cdot\nabla^2\hat{\boldsymbol{\zeta}} \tag{3.7.38}$$

$$\frac{\mathrm{d}B_\zeta}{\mathrm{d}t} = -B_\zeta(\nabla\cdot\mathbf{v}) + \mathbf{B}\cdot\nabla v_\zeta - \mathbf{v}\cdot(\mathbf{B}\cdot\nabla\hat{\boldsymbol{\zeta}}) + \frac{\eta}{\mu_0}\nabla^2 B_\zeta - \frac{\eta}{\mu_0}\mathbf{B}\cdot\nabla^2\hat{\boldsymbol{\zeta}} \tag{3.7.39}$$

Now, Wakatani claims that the terms involving $\mathbf{B}\cdot\nabla\hat{\boldsymbol{\zeta}}$ and $\mathbf{B}\cdot\nabla^2\hat{\boldsymbol{\zeta}}$ are zero (at least implicitly). This is not at all clear, and I believe it is an incorrect statement. It may be true that they are zero to the order of accuracy we are looking for, although it seems a bit unlikely. We can calculate that

$$\nabla\zeta = R_0\frac{y}{x^2+y^2}\nabla x - R_0\frac{x}{x^2+y^2}\nabla y \tag{3.7.40}$$

$$|\nabla\zeta|^2 = R_0^2\frac{y^2+x^2}{(x^2+y^2)^2} = \frac{R_0^2}{x^2+y^2} = \frac{R_0^2}{R^2} \tag{3.7.41}$$

$$\begin{aligned}\nabla\zeta/|\nabla\zeta| &= \frac{\sqrt{x^2+y^2}}{R_0}\left(R_0\frac{y}{x^2+y^2}\nabla x - R_0\frac{x}{x^2+y^2}\nabla y\right) = \frac{y}{\sqrt{x^2+y^2}}\nabla x - \frac{x}{\sqrt{x^2+y^2}}\nabla y\\ &= \sin(\zeta/R_0)\nabla x - \cos(\zeta/R_0)\nabla y = \sin(\zeta/R_0)\hat{\mathbf{x}} - \cos(\zeta/R_0)\hat{\mathbf{y}} = \hat{\boldsymbol{\zeta}}\end{aligned} \tag{3.7.42}$$

Thus

$$\begin{aligned}\mathbf{B}\cdot\nabla\hat{\boldsymbol{\zeta}} &= \mathbf{B}\cdot\nabla(\hat{\boldsymbol{\zeta}})_x\hat{\mathbf{x}} + \mathbf{B}\cdot\nabla(\hat{\boldsymbol{\zeta}})_y\hat{\mathbf{y}} = \mathbf{B}\cdot\nabla(\sin(\zeta/R_0))\hat{\mathbf{x}} - \mathbf{B}\cdot\nabla(\cos(\zeta/R_0))\hat{\mathbf{y}}\\ &= \mathbf{B}\frac{\cos(\zeta/R_0)}{R_0}\cdot\nabla\zeta\,\hat{\mathbf{x}} + \mathbf{B}\frac{\sin(\zeta/R_0)}{R_0}\cdot\nabla\zeta\,\hat{\mathbf{y}}\\ &= \mathbf{B}\frac{\cos(\zeta/R_0)}{\sqrt{x^2+y^2}}\cdot\hat{\boldsymbol{\zeta}}\,\hat{\mathbf{x}} + \mathbf{B}\frac{\sin(\zeta/R_0)}{\sqrt{x^2+y^2}}\cdot\hat{\boldsymbol{\zeta}}\,\hat{\mathbf{y}}\\ &= \frac{B_\zeta}{\sqrt{x^2+y^2}}(\cos(\zeta/R_0)\hat{\mathbf{x}} + \sin(\zeta/R_0)\hat{\mathbf{y}})\end{aligned} \tag{3.7.43}$$

$$\begin{aligned}&= \frac{B_\zeta}{\sqrt{x^2+y^2}}\left\{\cos(\zeta/R_0)\left[\frac{x}{r}\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)\nabla r - \frac{xz}{\sqrt{x^2+y^2}}\nabla\theta + \frac{y}{R_0}\nabla\zeta\right]\right.\\ &\qquad\left. + \sin(\zeta/R_0)\left[\frac{y}{r}\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)\nabla r - \frac{yz}{\sqrt{x^2+y^2}}\nabla\theta - \frac{x}{R_0}\nabla\zeta\right]\right\}\\ &= \frac{B_\zeta}{\sqrt{x^2+y^2}}\left\{\frac{x}{\sqrt{x^2+y^2}}\left[\frac{x}{r}\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)\nabla r - \frac{xz}{\sqrt{x^2+y^2}}\nabla\theta + \frac{y}{R_0}\nabla\zeta\right]\right.\\ &\qquad\left. + \frac{y}{\sqrt{x^2+y^2}}\left[\frac{y}{r}\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)\nabla r - \frac{yz}{\sqrt{x^2+y^2}}\nabla\theta - \frac{x}{R_0}\nabla\zeta\right]\right\}\\ &= \frac{B_\zeta}{\sqrt{x^2+y^2}}\left(\nabla r\left[\frac{\sqrt{x^2+y^2}}{r}\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)\right] - z\nabla\theta\right)\\ &= \frac{B_\zeta}{\sqrt{x^2+y^2}}\left(\hat{\mathbf{r}}\frac{\sqrt{x^2+y^2}-R_0}{\sqrt{(\sqrt{x^2+y^2}-R_0)^2+z^2}} - \frac{z}{r}\hat{\boldsymbol{\theta}}\right)\end{aligned} \tag{3.7.44}$$

$$= B_\zeta \left(\frac{\cos\theta}{\sqrt{x^2+y^2}} \hat{\mathbf{r}} - \frac{z}{r\sqrt{x^2+y^2}} \hat{\boldsymbol{\theta}} \right) = B_\zeta \left(\frac{\cos\theta}{R} \hat{\mathbf{r}} - \frac{z}{rR} \hat{\boldsymbol{\theta}} \right) \tag{3.7.45}$$

with $R = \sqrt{x^2+y^2}$ as a definition.

Now if we wish to rid ourselves of the variable z, we could use

$$z = (\sqrt{x^2+y^2} - R_0)\tan\theta = \tan\theta\sqrt{r^2-z^2} \tag{3.7.46}$$
$$z^2 = \tan^2\theta(r^2-z^2) \Rightarrow z^2(1+\tan^2\theta) = \tan^2\theta r^2 \tag{3.7.47}$$
$$z = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} r = \frac{\sin\theta}{\sqrt{\cos^2\theta+\sin^2\theta}} r = \sin\theta r \tag{3.7.48}$$

So we find

$$\mathbf{B} \cdot \nabla\hat{\boldsymbol{\zeta}} = B_\zeta \left(\frac{\cos\theta}{R} \hat{\mathbf{r}} - \frac{\sin\theta}{R} \hat{\boldsymbol{\theta}} \right) \tag{3.7.49}$$

Note, that if we convert to a pure cylindrical system, (R, z, ζ) (note that $\hat{\boldsymbol{\zeta}} = -\hat{\boldsymbol{\varphi}}$ for normal (R, φ, z) coordinates) we find

$$r^2 = (R-R_0)^2 + z^2 \tag{3.7.50}$$
$$\nabla r = \frac{R-R_0}{r}\,\nabla R + \frac{z}{r}\,\nabla z = \cos\theta\,\nabla R + \sin\theta\,\nabla z \tag{3.7.51}$$
$$\tan\theta = \frac{z}{R-R_0} \Leftrightarrow \cos\theta = \frac{R-R_0}{r} \Leftrightarrow \sin\theta = \frac{z}{r} \tag{3.7.52}$$
$$\begin{aligned} \nabla\theta &= \underbrace{\cos^2\theta}_{(R-R_0)^2/r^2} \frac{(R-R_0)\,\nabla z - z\,\nabla R}{(R-R_0)^2} = \frac{R-R_0}{(R-R_0)^2+z^2}\,\nabla z - \frac{z}{(R-R_0)^2+z^2}\,\nabla R \\ &= \frac{\cos\theta}{r}\,\nabla z - \frac{\sin\theta}{r}\,\nabla R \end{aligned} \tag{3.7.53}$$

Thus, ($\nabla R = \hat{\mathbf{R}}$)

$$\frac{B_\zeta}{R}\left(\cos\theta\,\nabla r - z\,\nabla\theta\right) = \frac{B_\zeta}{R}\left(\cos\theta\left[\cos\theta\,\nabla R + \sin\theta\,\nabla z\right] - z\left[\frac{\cos\theta}{r}\,\nabla z - \frac{\sin\theta}{r}\,\nabla R\right]\right) \tag{3.7.54}$$
$$= \frac{B_\zeta}{R}\left(\nabla r\left[\cos^2\theta + \sin^2\theta\right] + \nabla\theta\left[\sin\theta\cos\theta - \sin\theta\cos\theta\right]\right) \tag{3.7.55}$$
$$= \frac{B_\zeta}{R}\,\nabla R = -\frac{B_\phi}{R}\,\nabla R \tag{3.7.56}$$

in agreement with the plasma formulary for $\mathbf{B} \cdot \nabla A$ in cylindrical coordinates.

Similarly,

$$\begin{aligned}
\nabla^2\hat{\boldsymbol{\zeta}} &= \nabla^2(\hat{\boldsymbol{\zeta}})_x\hat{\mathbf{x}} + \nabla^2(\hat{\boldsymbol{\zeta}})_y\hat{\mathbf{y}} \\
&= \nabla^2(\frac{y}{\sqrt{x^2+y^2}})\hat{\mathbf{x}} - \nabla^2(\frac{x}{\sqrt{x^2+y^2}})\hat{\mathbf{y}} \\
&= \frac{-y}{(x^2+y^2)^{3/2}}\hat{\mathbf{x}} + \frac{x}{(x^2+y^2)^{3/2}}\hat{\mathbf{y}} \\
&= \frac{-y}{(x^2+y^2)^{3/2}}\left[\frac{x}{r}\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)\nabla r - \frac{xz}{\sqrt{x^2+y^2}}\nabla\theta + \frac{y}{R_0}\nabla\zeta\right] \\
&\quad + \frac{x}{(x^2+y^2)^{3/2}}\left[\frac{y}{r}\left(1-\frac{R_0}{\sqrt{x^2+y^2}}\right)\nabla r - \frac{yz}{\sqrt{x^2+y^2}}\nabla\theta - \frac{x}{R_0}\nabla\zeta\right] \\
&= \frac{-(x^2+y^2)}{R_0\,(x^2+y^2)^{3/2}}\nabla\zeta = \frac{-1}{R_0(x^2+y^2)^{1/2}}\hat{\boldsymbol{\zeta}}\frac{R_0}{\sqrt{x^2+y^2}} \\
&= -\frac{1}{(x^2+y^2)^2}\hat{\boldsymbol{\zeta}} = -\frac{1}{R^2}\hat{\boldsymbol{\zeta}}
\end{aligned} \tag{3.7.57}$$

$$\nabla^2\hat{\boldsymbol{\zeta}} = \frac{-1}{R^2}\hat{\boldsymbol{\zeta}} \tag{3.7.58}$$

$$\begin{aligned}
\nabla^2(-\hat{\boldsymbol{\zeta}}) &= \frac{-1}{R^2}(-\hat{\boldsymbol{\zeta}}) \\
\nabla^2\hat{\boldsymbol{\varphi}} &= \frac{-1}{R^2}\hat{\boldsymbol{\phi}}
\end{aligned} \tag{3.7.59}$$

and so we agree with the plasma formulary for cylindrical coordinates, as well.

Thus, in fact our equation should be

$$\frac{\mathrm{d}B_\zeta}{\mathrm{d}t} = -B_\zeta(\boldsymbol{\nabla}\cdot\mathbf{v}) + \mathbf{B}\cdot\nabla v_\zeta - \mathbf{v}\cdot\frac{B_\zeta}{R}\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) + \frac{\eta}{\mu_0}\nabla^2 B_\zeta - \frac{\eta}{\mu_0}\mathbf{B}\cdot\nabla^2\hat{\boldsymbol{\zeta}} \tag{3.7.60}$$

$$\frac{\mathrm{d}B_\zeta}{\mathrm{d}t} = -B_\zeta(\boldsymbol{\nabla}\cdot\mathbf{v}) + \mathbf{B}\cdot\nabla v_\zeta - \frac{B_\zeta}{R}\left(v_r\cos\theta - v_\theta\sin\theta\right) + \frac{\eta}{\mu_0}\nabla^2 B_\zeta + \frac{\eta}{\mu_0}\frac{B_\zeta}{R^2} \tag{3.7.61}$$

If we were to look at orderings after inserting (W-3.147), I believe that we would recover $\frac{\mathrm{d}I_1}{\mathrm{d}t} = 0$ to $\mathcal{O}(\delta^3)$.

Chapter 4

The MHD Equilibrium of a Toroidal Plasma in Three-Dimensional Geometry

4.2 The Generalized Grad-Shafranov Equation

We have with $a = a(\mathbf{r})$ being a flux function so that

$$\mathbf{B} \cdot \nabla a = \boldsymbol{\nabla} \cdot (a\mathbf{B}) = 0 \tag{W-4.5}$$

$$\frac{\mathbf{J}}{\mu_0} \cdot \nabla a = \boldsymbol{\nabla} \times \mathbf{B} \cdot \nabla a = \boldsymbol{\nabla} \cdot (\mathbf{B} \times \nabla a) = 0 \tag{W-4.6}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{W-4.7}$$

We can choose $a = \Psi$ the poloidal flux function. We note that as we are on a particular magnetic surface, so that the derivatives are only along the poloidal and toroidal directions, and not the "radial" direction.

So we then find (assuming $p = p(\Psi)$)

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{4.2.1}$$

$$\mathbf{J} \times \mathbf{B} \cdot \nabla\Psi = \nabla p \cdot \nabla\Psi \tag{4.2.2}$$

$$(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} \cdot \nabla\Psi = \frac{\mathrm{d}p}{\mathrm{d}\Psi} |\nabla\Psi|^2 \tag{W-4.10}$$

$$[(\nabla\Psi)\mathbf{B} - \mathbf{B}(\nabla\Psi)] : \nabla\mathbf{B} = \frac{\mathrm{d}p}{\mathrm{d}\Psi} |\nabla\Psi|^2 \tag{4.2.3}$$

with $\mathbf{AB} : \mathbf{CD} = (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$.

If we take

$$\mathbf{B} = \nabla\Psi \times \mathbf{b}_\theta + F\mathbf{b}_\zeta \tag{W-4.11}$$

with F a poloidal current flux function and $\mathbf{b}_\theta$ and $\mathbf{b}_\zeta$ vectors on the magnetic surface (but with no implied direction, as we will see).

So then

$$\mu_0\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla} \times [\nabla\Psi \times \mathbf{b}_\theta + F\mathbf{b}_\zeta] = \boldsymbol{\nabla} \times (\nabla\Psi \times \mathbf{b}_\theta) + F\boldsymbol{\nabla} \times \mathbf{b}_\zeta + \nabla F \times \mathbf{b}_\zeta \tag{W-4.12}$$

Plugging these into our previous (W-4.7), (W-4.8) ($\mathbf{B}\cdot\nabla\Psi = 0$), and (W-4.9) ($\mathbf{J}\cdot\nabla\Psi = 0$) yields

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{B} &= \boldsymbol{\nabla}\cdot(\nabla\Psi\times\mathbf{b}_\theta) + \boldsymbol{\nabla}\cdot(F\mathbf{b}_\zeta)\\ &= \cancel{\mathbf{b}_\theta\cdot\boldsymbol{\nabla}\times\nabla\Psi} - \nabla\Psi\cdot\boldsymbol{\nabla}\times\mathbf{b}_\theta + \cancel{\nabla F\cdot\mathbf{b}_\zeta} + F\boldsymbol{\nabla}\cdot\mathbf{b}_\zeta \qquad (4.2.4)\\ &= 0\\ F\boldsymbol{\nabla}\cdot\mathbf{b}_\zeta &= \nabla\Psi\cdot\boldsymbol{\nabla}\times\mathbf{b}_\theta \qquad (4.2.5)\end{aligned}$$

where the $\frac{\mathrm{d}F}{\mathrm{d}\Psi}\nabla\Psi\cdot\boldsymbol{\nabla}\times\mathbf{b}_\theta$ vanishing because $\nabla\Psi$ has no variation in either the ζ or θ direction which is all that $\mathbf{b}_\theta$ has (with $\nabla F = \frac{\mathrm{d}F}{\mathrm{d}\Psi}\nabla\Psi$ since F is a flux function).

$$\begin{aligned}\mathbf{B}\cdot\nabla\Psi &= \cancel{(\nabla\Psi\times\mathbf{b}_\theta)\cdot\nabla\Psi} + F\mathbf{b}_\zeta\cdot\nabla\Psi\\ &= 0 \qquad (4.2.6)\\ \mathbf{b}_\zeta\cdot\nabla\Psi &= 0 \qquad (\text{W-4.14})\end{aligned}$$

and

$$\begin{aligned}\mu_0\mathbf{J}\cdot\nabla\Psi &= \nabla\Psi\cdot\boldsymbol{\nabla}\times(\nabla\Psi\times\mathbf{b}_\theta) + F\,\nabla\Psi\cdot\boldsymbol{\nabla}\times\mathbf{b}_\zeta + \cancel{\nabla F\times\mathbf{b}_\zeta\cdot\nabla\Psi}\\ &= 0 \qquad (4.2.7)\end{aligned}$$

$$-\nabla\Psi\cdot\boldsymbol{\nabla}\times(\nabla\Psi\times\mathbf{b}_\theta) = F\,\nabla\Psi\cdot\boldsymbol{\nabla}\times\mathbf{b}_\zeta \qquad (4.2.8)$$

$$\nabla\Psi\cdot[\boldsymbol{\nabla}\cdot(\nabla\Psi\mathbf{b}_\theta - \mathbf{b}_\theta\nabla\Psi)] = F\,\nabla\Psi\cdot\boldsymbol{\nabla}\times\mathbf{b}_\zeta \qquad (4.2.9)$$

where we have used

$$\mathbf{C}\cdot\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{C}\cdot\mathbf{A})\boldsymbol{\nabla}\cdot\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\boldsymbol{\nabla}\cdot\mathbf{A} + \mathbf{C}\cdot(\mathbf{B}\cdot\nabla\mathbf{A}) - \mathbf{C}\cdot(\mathbf{A}\cdot\nabla\mathbf{B}) \qquad (4.2.10)$$

$$\mathbf{C}\cdot\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{C}\cdot[\boldsymbol{\nabla}\cdot(\mathbf{BA} - \mathbf{AB})] \qquad (4.2.11)$$

$$(4.2.12)$$

$$\begin{aligned}\nabla\Psi\cdot\boldsymbol{\nabla}\times(\nabla\Psi\times\mathbf{b}_\theta) &= (\nabla\Psi\cdot\nabla\Psi)\boldsymbol{\nabla}\cdot\mathbf{b}_\theta - (\nabla\Psi\cdot\mathbf{b}_\theta)\boldsymbol{\nabla}\cdot\nabla\Psi + \nabla\Psi\cdot(\mathbf{b}_\theta\cdot\nabla\nabla\Psi) - \nabla\Psi\cdot(\nabla\Psi\cdot\nabla\mathbf{b}_\theta)\\ &= |\nabla\Psi|^2\boldsymbol{\nabla}\cdot\mathbf{b}_\theta + \nabla\Psi\cdot(\mathbf{b}_\theta\cdot\nabla\nabla\Psi) - \nabla\Psi\cdot(\nabla\Psi\cdot\nabla\mathbf{b}_\theta)\\ &= \cancel{|\nabla\Psi|^2\boldsymbol{\nabla}\cdot\mathbf{b}_\theta} + \nabla\Psi\cdot[\boldsymbol{\nabla}\cdot(\mathbf{b}_\theta\nabla\Psi)] - \cancel{|\nabla\Psi|^2\boldsymbol{\nabla}\cdot\mathbf{b}_\theta} - \nabla\Psi\cdot(\nabla\Psi\cdot\nabla\mathbf{b}_\theta)\\ &= \nabla\Psi\cdot[\boldsymbol{\nabla}\cdot(\mathbf{b}_\theta\nabla\Psi)] - \nabla\Psi\cdot[\nabla\Psi\mathbf{b}_\theta] + \cancel{\nabla\Psi\cdot\mathbf{b}_\theta\nabla^2\Psi}\\ &= \nabla\Psi\cdot[\boldsymbol{\nabla}\cdot(\mathbf{b}_\theta\nabla\Psi - \nabla\Psi\mathbf{b}_\theta)]\end{aligned} \qquad (4.2.13)$$

Note that in our case we can use

$$\begin{aligned}\nabla\Psi\cdot\boldsymbol{\nabla}\times(\nabla\Psi\times\mathbf{b}_\theta) &= (\partial_i\Psi)\epsilon_{ijk}\partial_j\epsilon_{klm}(\partial_l\Psi)b_{\theta m} = \epsilon_{ijk}\epsilon_{klm}(\partial_i\Psi)\partial_j(\partial_l\Psi)b_{\theta m}\\ &= \epsilon_{ijk}\epsilon_{klm}\left[\partial_j[b_{\theta m}(\partial_l\Psi)(\partial_i\Psi)] - b_{\theta m}(\partial_l\Psi)\partial_j\partial_i\Psi\right]\end{aligned} \qquad (4.2.14)$$

Now note that (first swapping the order of differentiation and then swapping order in the Levi-Civita tensor. In the second, just exchanging dummy indices i and j completely)

$$\epsilon_{ijk}\epsilon_{klm}b_{\theta m}(\partial_l\Psi)\partial_j\partial_i\Psi = \epsilon_{ijk}\epsilon_{klm}b_{\theta m}(\partial_l\Psi)\partial_i\partial_j\Psi = -\epsilon_{jik}\epsilon_{klm}b_{\theta m}(\partial_l\Psi)\partial_i\partial_j\Psi \qquad (4.2.15)$$

$$\epsilon_{ijk}\epsilon_{klm}b_{\theta m}(\partial_l\Psi)\partial_j\partial_i\Psi = \epsilon_{jik}\epsilon_{klm}b_{\theta m}(\partial_l\Psi)\partial_i\partial_j\Psi \qquad (4.2.16)$$

and so

$$
\begin{aligned}
\epsilon_{jik}\epsilon_{klm}b_{\theta m}(\partial_l\Psi)\partial_i\partial_j\Psi &= -\epsilon_{jik}\epsilon_{klm}b_{\theta m}(\partial_l\Psi)\partial_i\partial_j\Psi && (4.2.17)\\
\epsilon_{jik}\epsilon_{klm}b_{\theta m}(\partial_l\Psi)\partial_i\partial_j\Psi &= 0 && (4.2.18)
\end{aligned}
$$

Continuing, then we find

$$
\begin{aligned}
\nabla\Psi\cdot\boldsymbol{\nabla}\times(\nabla\Psi\times\mathbf{b}_\theta) &= \epsilon_{ijk}\epsilon_{klm}\left[\partial_j[b_{\theta m}(\partial_l\Psi)(\partial_i\Psi)] - \cancel{b_{\theta m}(\partial_l\Psi)\partial_j\partial_i\Psi}\right]\\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\partial_j[b_{\theta m}(\partial_l\Psi)(\partial_i\Psi)]\\
&= \partial_m[b_{\theta m}(\partial_i\Psi)(\partial_i\Psi)] - \cancel{\partial_l[b_{\theta i}(\partial_l\Psi)(\partial_i\Psi)]}\\
&= \partial_m[b_{\theta m}(\partial_i\Psi)(\partial_i\Psi)] = \boldsymbol{\nabla}\cdot(|\nabla\Psi|^2\mathbf{b}_\theta)
\end{aligned}
\quad (4.2.19)
$$

where $b_{\theta i}\partial_i\Psi = \nabla\Psi\cdot\mathbf{b}_\theta = 0$ has been used.

Thus, collecting all we have

$$
\begin{aligned}
F\boldsymbol{\nabla}\cdot\mathbf{b}_\zeta &= \nabla\Psi\cdot\boldsymbol{\nabla}\times\mathbf{b}_\theta && (4.2.20)\\
\nabla\Psi\cdot\mathbf{b}_\zeta &= 0 && (4.2.21)\\
\nabla\Psi\cdot\mathbf{b}_\theta &= 0 && (4.2.22)\\
-\boldsymbol{\nabla}\cdot(|\nabla\Psi|^2\mathbf{b}_\theta) &= F\,\nabla\Psi\cdot\boldsymbol{\nabla}\times\mathbf{b}_\zeta && (4.2.23)
\end{aligned}
$$

Thus, I don't believe that you can get (W-4.13), (W-4.15), (W-4.16), (W-4.17), and (W-4.18) simply from these three relations. The choice of setting all the terms to zero individually will certainly work, assuming that it is consistent to choose all these terms to be zero.

We see that this can work by choosing the relationship between $\mathbf{b}_\theta$ and $\mathbf{b}_\zeta$ as (W-4.19).

$$
\begin{aligned}
\mathbf{b}_\theta &= \mathbf{b}_\zeta + \frac{\nabla\psi\times(\nabla h\times\nabla\Psi)}{|\nabla\Psi|^2} && (4.2.24)\\
\mathbf{b}_\zeta &= \nabla\Psi\times\nabla\lambda && (4.2.25)
\end{aligned}
$$

We then see that (use $\mathbf{f} = \nabla h\times\nabla\Psi$ so that $(\nabla\Psi\times\mathbf{f})\times\nabla\Psi = \mathbf{f}|\nabla\Psi|^2 - \cancel{\nabla\Psi\mathbf{f}\cdot\nabla\Psi}$)

$$
\begin{aligned}
\mathbf{b}_\theta\times\nabla\Psi &= \mathbf{b}_\zeta\times\nabla\Psi + \frac{\nabla\psi\times(\nabla h\times\nabla\Psi)}{|\nabla\Psi|^2}\times\nabla\Psi\\
&= \mathbf{b}_\zeta\times\nabla\Psi + \frac{(\nabla h\times\nabla\Psi)|\nabla\Psi|^2}{|\nabla\Psi|^2}\\
&= \mathbf{b}_\zeta\times\nabla\Psi + \nabla h\times\nabla\Psi
\end{aligned}
\quad (4.2.26)
$$

as required.

It then easily follows (from $\boldsymbol{\nabla}\cdot(\nabla a\times\nabla b) = 0$) that (W-4.22) is true and (W-4.23) is simply the rightmost relation in (W-4.22).

4.3 The Averaged MHD Equilibrium Equation

First let's show that $[f\,,\,g] = \nabla f \times \nabla g \cdot \hat{\boldsymbol{\zeta}}$ is a Poisson bracket (note that $[f\,,\,g] = \nabla f \times \hat{\boldsymbol{\zeta}} \cdot \nabla g = -\nabla f \times \nabla g \cdot \hat{\boldsymbol{\zeta}}$ is also a good Poisson bracket). The required properties for a Poisson bracket are

$$[f\,,\,g] = -\,[g\,,\,f] \tag{4.3.1}$$
$$[f+g\,,\,h] = -\,[f\,,\,h] + [g\,,\,h] \tag{4.3.2}$$
$$[fg\,,\,h] = -f\,[g\,,\,h] + [f\,,\,h]\,g \tag{4.3.3}$$
$$[f\,,\,[g\,,\,h]] + [g\,,\,[h\,,\,f]] + [h\,,\,[f\,,\,g]] = 0 \tag{4.3.4}$$

Let us verify them for f, g, h.

$$[f\,,\,g] = \nabla f \times \nabla g \cdot \hat{\boldsymbol{\zeta}} = -\nabla g \times \nabla f \cdot \hat{\boldsymbol{\zeta}} = -\,[g\,,\,f] \tag{4.3.5}$$
$$[f+g\,,\,h] = \nabla(f+g) \times \nabla h \cdot \hat{\boldsymbol{\zeta}} = \nabla f \times \nabla h \cdot \hat{\boldsymbol{\zeta}} + \nabla g \times \nabla h \cdot \hat{\boldsymbol{\zeta}} = [f\,,\,h] + [g\,,\,h] \tag{4.3.6}$$
$$[fg\,,\,h] = \nabla(fg) \times \nabla h \cdot \hat{\boldsymbol{\zeta}} = f\,\nabla g \times \nabla h \cdot \hat{\boldsymbol{\zeta}} + \nabla f \times \nabla h \cdot \hat{\boldsymbol{\zeta}} g = f\,[g\,,\,h] + [f\,,\,h]\,g \tag{4.3.7}$$

and finally

$$[f\,,\,[g\,,\,h]] + [g\,,\,[h\,,\,f]] + [h\,,\,[f\,,\,g]] = 0 \tag{4.3.8}$$
$$\nabla f \times \nabla(\nabla g \times \nabla h \cdot \hat{\boldsymbol{\zeta}}) \cdot \hat{\boldsymbol{\zeta}} + \nabla g \times \nabla(\nabla h \times \nabla f \cdot \hat{\boldsymbol{\zeta}}) \cdot \hat{\boldsymbol{\zeta}} + \nabla h \times \nabla(\nabla f \times \nabla g \cdot \hat{\boldsymbol{\zeta}}) \cdot \hat{\boldsymbol{\zeta}} = 0 \tag{4.3.9}$$

Using Einstein index notation we see

$$\nabla f \times \nabla(\nabla g \times \nabla h \cdot \hat{\boldsymbol{\zeta}}) \cdot \hat{\boldsymbol{\zeta}} = \zeta_i \epsilon_{ijk} (\partial_j f) \partial_k [\zeta_l \epsilon_{lmn} (\partial_m g)(\partial_n h)] = \zeta_i \epsilon_{ijk} \epsilon_{lmn} (\partial_j f) \partial_k [\zeta_l (\partial_m g)(\partial_n h)] \tag{4.3.10}$$

$$= [\delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}\,(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})]\,\zeta_i \partial_j f \partial_k [\zeta_l (\partial_m g)(\partial_n h)] \tag{4.3.11}$$

$$\begin{aligned} &= \zeta_l(\partial_m f \partial_n [\zeta_l \partial_m g \partial_n h] - \partial_n f \partial_m [\zeta_l \partial_m g \partial_n h]) - \zeta_m(\partial_l f \partial_n [\zeta_l \partial_m g \partial_n h] - \partial_n f \partial_l [\zeta_l \partial_m g \partial_n h]) \\ &+ \zeta_n(\partial_l f \partial_m [\zeta_l \partial_m g \partial_n h] - \partial_m f \partial_l [\zeta_l \partial_m g \partial_n h]) \end{aligned} \tag{4.3.12}$$

Looking at the first term in parantheses, we see (things in {} are not operated on by the differential and $\zeta_l \zeta_l = \hat{\boldsymbol{\zeta}} \cdot \hat{\boldsymbol{\zeta}} = 1$)

$$\zeta_l(\partial_m f \partial_n - \partial_n f \partial_m)\,[\{\zeta_l \partial_m g\}\partial_n h + \{\zeta_l \partial_n h\}\partial_m g + \{\partial_m g \partial_n h\}\zeta_l] \tag{4.3.13}$$
$$\begin{aligned} &= \zeta_l \partial_m f \zeta_l \partial_m g \partial_n \partial_n h + \zeta_l \partial_m f \zeta_l \partial_n h \partial_n \partial_m g + \zeta_l \partial_m f \partial_m g \partial_n h \partial_n \zeta_l \\ &\quad - (\zeta_l \partial_n f \zeta_l \partial_m g \partial_m \partial_n h + \zeta_l \partial_n f \zeta_l \partial_n h \partial_m \partial_m g + \zeta_l \partial_n f \partial_m g \partial_n h \partial_m \zeta_l) \end{aligned} \tag{4.3.14}$$
$$\begin{aligned} &= \nabla f \cdot \nabla g\, \nabla^2 h + (\nabla h\, \nabla f : \nabla \nabla g) + \nabla f \cdot \nabla g (\hat{\boldsymbol{\zeta}}\, \nabla h : \nabla \hat{\boldsymbol{\zeta}}) \\ &\quad - \Big[(\nabla f\, \nabla g : \nabla \nabla h) + \nabla f \cdot \nabla h\, \nabla^2 g + \nabla f \cdot \nabla h (\hat{\boldsymbol{\zeta}}\, \nabla g : \nabla \hat{\boldsymbol{\zeta}}) \Big] \end{aligned} \tag{4.3.15}$$

The second term in parantheses is given by

$$\zeta_m(\partial_l f \partial_n - \partial_n f \partial_l)\,[\{\zeta_l \partial_m g\}\partial_n h + \{\zeta_l \partial_n h\}\partial_m g + \{\partial_m g \partial_n h\}\zeta_l] \tag{4.3.16}$$
$$\begin{aligned} &= \zeta_m \partial_l f \zeta_l \partial_m g \partial_n \partial_n h + \zeta_m \partial_l f \zeta_l \partial_n h \partial_n \partial_m g + \zeta_m \partial_l f \partial_m g \partial_n h \partial_n \zeta_l \\ &\quad - (\zeta_m \partial_n f \zeta_l \partial_m g \partial_l \partial_n h + \zeta_m \partial_n f \zeta_l \partial_n h \partial_l \partial_m g + \zeta_m \partial_n f \partial_m g \partial_n h \partial_l \zeta_l) \end{aligned} \tag{4.3.17}$$
$$\begin{aligned} &= (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla f\, \nabla g)\, \nabla^2 h + \hat{\boldsymbol{\zeta}} \cdot \nabla f (\hat{\boldsymbol{\zeta}}\, \nabla h : \nabla \nabla g) + \hat{\boldsymbol{\zeta}} \cdot \nabla g (\nabla f\, \nabla h : \nabla \hat{\boldsymbol{\zeta}}) \\ &\quad - \Big[\hat{\boldsymbol{\zeta}} \cdot \nabla g (\nabla f \hat{\boldsymbol{\zeta}} : \nabla \nabla h) + \nabla f \cdot \nabla h (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla \nabla g) + (\hat{\boldsymbol{\zeta}} \cdot \nabla g)(\nabla f \cdot \nabla h) \boldsymbol{\nabla} \cdot \hat{\boldsymbol{\zeta}} \Big] \end{aligned} \tag{4.3.18}$$

and finally the third term in parantheses is given by

$$\zeta_n(\partial_l f\partial_m - \partial_m f\partial_l)\left[\{\zeta_l\partial_m g\}\partial_n h + \{\zeta_l\partial_n h\}\partial_m g + \{\partial_m g\partial_n h\}\zeta_l\right] \tag{4.3.19}$$

$$\begin{aligned}&= \zeta_n\partial_l f\zeta_l\partial_m g\partial_m\partial_n h + \zeta_n\partial_l f\zeta_l\partial_n h\partial_m\partial_m g + \zeta_n\partial_l f\partial_m g\partial_n h\partial_m\zeta_l \\ &\quad - (\zeta_n\partial_m f\zeta_l\partial_m g\partial_l\partial_n h + \zeta_n\partial_m f\zeta_l\partial_n h\partial_l\partial_m g + \zeta_n\partial_m f\partial_m g\partial_n h\partial_l\zeta_l)\end{aligned} \tag{4.3.20}$$

$$\begin{aligned}&= \hat{\boldsymbol{\zeta}}\cdot\nabla f(\hat{\boldsymbol{\zeta}}\,\nabla g:\nabla\nabla h) + (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla f\,\nabla h)\,\nabla^2 g + \hat{\boldsymbol{\zeta}}\cdot\nabla h(\nabla f\,\nabla g:\nabla\hat{\boldsymbol{\zeta}}) \\ &\quad - \Big[\nabla f\cdot\nabla g(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla\nabla h) + \hat{\boldsymbol{\zeta}}\cdot\nabla h(\nabla f\hat{\boldsymbol{\zeta}}:\nabla\nabla g) + (\hat{\boldsymbol{\zeta}}\cdot\nabla h)(\nabla f\cdot\nabla g)\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\zeta}}\Big]\end{aligned} \tag{4.3.21}$$

So we would find altogether (using $\boldsymbol{\nabla}\cdot\nabla\zeta = 0$) that we get (then using permutations of f, g, h)

$$\begin{aligned}[f\,,\,[g\,,\,h]] = {}& \nabla f\cdot\nabla g\,\nabla^2 h + (\nabla h\,\nabla f:\nabla\nabla g) + \nabla f\cdot\nabla g(\hat{\boldsymbol{\zeta}}\,\nabla h:\nabla\hat{\boldsymbol{\zeta}}) \\ & - \Big[(\nabla f\,\nabla g:\nabla\nabla h) + \nabla f\cdot\nabla h\,\nabla^2 g + \nabla f\cdot\nabla h(\hat{\boldsymbol{\zeta}}\,\nabla g:\nabla\hat{\boldsymbol{\zeta}})\Big] \\ & + (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla f\,\nabla g)\,\nabla^2 h + \hat{\boldsymbol{\zeta}}\cdot\nabla f(\hat{\boldsymbol{\zeta}}\,\nabla h:\nabla\nabla g) + \hat{\boldsymbol{\zeta}}\cdot\nabla g(\nabla f\,\nabla h:\nabla\hat{\boldsymbol{\zeta}}) \\ & + \Big[\hat{\boldsymbol{\zeta}}\cdot\nabla g(\nabla f\hat{\boldsymbol{\zeta}}:\nabla\nabla h) + \nabla f\cdot\nabla h(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla\nabla g)\Big] \\ & + \hat{\boldsymbol{\zeta}}\cdot\nabla f(\hat{\boldsymbol{\zeta}}\,\nabla g:\nabla\nabla h) + (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla f\,\nabla h)\,\nabla^2 g + \hat{\boldsymbol{\zeta}}\cdot\nabla h(\nabla f\,\nabla g:\nabla\hat{\boldsymbol{\zeta}}) \\ & - \Big[\nabla f\cdot\nabla g(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla\nabla h) + \hat{\boldsymbol{\zeta}}\cdot\nabla h(\nabla f\hat{\boldsymbol{\zeta}}:\nabla\nabla g)\Big]\end{aligned}$$

$$\begin{aligned}[g\,,\,[h\,,\,f]] = {}& \nabla g\cdot\nabla h\,\nabla^2 f + (\nabla f\,\nabla g:\nabla\nabla h) + \nabla g\cdot\nabla h(\hat{\boldsymbol{\zeta}}\,\nabla f:\nabla\hat{\boldsymbol{\zeta}}) \\ & - \Big[(\nabla g\,\nabla h:\nabla\nabla f) + \nabla g\cdot\nabla f\,\nabla^2 h + \nabla g\cdot\nabla f(\hat{\boldsymbol{\zeta}}\,\nabla h:\nabla\hat{\boldsymbol{\zeta}})\Big] \\ & - (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla g\,\nabla h)\,\nabla^2 f + \hat{\boldsymbol{\zeta}}\cdot\nabla g(\hat{\boldsymbol{\zeta}}\,\nabla f:\nabla\nabla h) + \hat{\boldsymbol{\zeta}}\cdot\nabla h(\nabla g\,\nabla f:\nabla\hat{\boldsymbol{\zeta}}) \\ & + \Big[\hat{\boldsymbol{\zeta}}\cdot\nabla h(\nabla g\hat{\boldsymbol{\zeta}}:\nabla\nabla f) + \nabla g\cdot\nabla f(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla\nabla h)\Big] \\ & + \hat{\boldsymbol{\zeta}}\cdot\nabla g(\hat{\boldsymbol{\zeta}}\,\nabla h:\nabla\nabla f) + (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla g\,\nabla f)\,\nabla^2 h + \hat{\boldsymbol{\zeta}}\cdot\nabla f(\nabla g\,\nabla h:\nabla\hat{\boldsymbol{\zeta}}) \\ & - \Big[\nabla g\cdot\nabla h(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla\nabla f) + \hat{\boldsymbol{\zeta}}\cdot\nabla f(\nabla g\hat{\boldsymbol{\zeta}}:\nabla\nabla h)\Big]\end{aligned}$$

$$\begin{aligned}[h\,,\,[f\,,\,g]] = {}& \nabla h\cdot\nabla f\,\nabla^2 g + (\nabla g\,\nabla h:\nabla\nabla f) + \nabla h\cdot\nabla f(\hat{\boldsymbol{\zeta}}\,\nabla g:\nabla\hat{\boldsymbol{\zeta}}) \\ & - \Big[(\nabla h\,\nabla f:\nabla\nabla g) + \nabla h\cdot\nabla g\,\nabla^2 f + \nabla h\cdot\nabla g(\hat{\boldsymbol{\zeta}}\,\nabla f:\nabla\hat{\boldsymbol{\zeta}})\Big] \\ & - (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla h\,\nabla f)\,\nabla^2 g + \hat{\boldsymbol{\zeta}}\cdot\nabla h(\hat{\boldsymbol{\zeta}}\,\nabla g:\nabla\nabla f) + \hat{\boldsymbol{\zeta}}\cdot\nabla f(\nabla h\,\nabla g:\nabla\hat{\boldsymbol{\zeta}}) \\ & + \Big[\hat{\boldsymbol{\zeta}}\cdot\nabla f(\nabla h\hat{\boldsymbol{\zeta}}:\nabla\nabla g) + \nabla h\cdot\nabla g(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla\nabla f)\Big] \\ & + \hat{\boldsymbol{\zeta}}\cdot\nabla h(\hat{\boldsymbol{\zeta}}\,\nabla f:\nabla\nabla g) + (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla h\,\nabla g)\,\nabla^2 f + \hat{\boldsymbol{\zeta}}\cdot\nabla g(\nabla h\,\nabla f:\nabla\hat{\boldsymbol{\zeta}}) \\ & - \Big[\nabla h\cdot\nabla f(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}}:\nabla\nabla g) + \hat{\boldsymbol{\zeta}}\cdot\nabla g(\nabla h\hat{\boldsymbol{\zeta}}:\nabla\nabla f)\Big]\end{aligned}$$

We can now see the cancellation term by term. Remember that $\nabla f\,\nabla g:\nabla\nabla h = \nabla g\,\nabla f:\nabla\nabla h$. Let's mark similar terms with color to see it.

$$
\begin{aligned}
&\nabla f \cdot \nabla g \, \nabla^2 h + (\nabla h \, \nabla f : \nabla\nabla g) + \nabla f \cdot \nabla g(\hat{\boldsymbol{\zeta}} \, \nabla h : \nabla\hat{\boldsymbol{\zeta}}) \\
&\quad - \Big[(\nabla f \, \nabla g : \nabla\nabla h) + \nabla f \cdot \nabla h \, \nabla^2 g + \nabla f \cdot \nabla h(\hat{\boldsymbol{\zeta}} \, \nabla g : \nabla\hat{\boldsymbol{\zeta}})\Big] \\
&\quad - \Big[(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla f \, \nabla g) \, \nabla^2 h + \hat{\boldsymbol{\zeta}} \cdot \nabla f(\hat{\boldsymbol{\zeta}} \, \nabla h : \nabla\nabla g) + \hat{\boldsymbol{\zeta}} \cdot \nabla g(\nabla f \, \nabla h : \nabla\hat{\boldsymbol{\zeta}})\Big] \\
&\quad + \Big[\hat{\boldsymbol{\zeta}} \cdot \nabla g(\nabla f \hat{\boldsymbol{\zeta}} : \nabla\nabla h) + \nabla f \cdot \nabla h(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla\nabla g)\Big] \\
&\quad + \hat{\boldsymbol{\zeta}} \cdot \nabla f(\hat{\boldsymbol{\zeta}} \, \nabla g : \nabla\nabla h) + (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla f \, \nabla h) \, \nabla^2 g + \hat{\boldsymbol{\zeta}} \cdot \nabla h(\nabla f \, \nabla g : \nabla\hat{\boldsymbol{\zeta}}) \\
&\quad - \Big[\nabla f \cdot \nabla g(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla\nabla h) + \hat{\boldsymbol{\zeta}} \cdot \nabla h(\nabla f \hat{\boldsymbol{\zeta}} : \nabla\nabla g)\Big] \\
&\; + \nabla g \cdot \nabla h \, \nabla^2 f + (\nabla f \, \nabla g : \nabla\nabla h) + \nabla g \cdot \nabla h(\hat{\boldsymbol{\zeta}} \, \nabla f : \nabla\hat{\boldsymbol{\zeta}}) \\
&\quad - \Big[(\nabla g \, \nabla h : \nabla\nabla f) + \nabla g \cdot \nabla f \, \nabla^2 h + \nabla g \cdot \nabla f(\hat{\boldsymbol{\zeta}} \, \nabla h : \nabla\hat{\boldsymbol{\zeta}})\Big] \\
&\quad - \Big[(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla g \, \nabla h) \, \nabla^2 f + \hat{\boldsymbol{\zeta}} \cdot \nabla g(\hat{\boldsymbol{\zeta}} \, \nabla f : \nabla\nabla h) + \hat{\boldsymbol{\zeta}} \cdot \nabla h(\nabla g \, \nabla f : \nabla\hat{\boldsymbol{\zeta}})\Big] \\
&\quad + \Big[\hat{\boldsymbol{\zeta}} \cdot \nabla h(\nabla g \hat{\boldsymbol{\zeta}} : \nabla\nabla f) + \nabla g \cdot \nabla f(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla\nabla h)\Big] \\
&\quad + \hat{\boldsymbol{\zeta}} \cdot \nabla g(\hat{\boldsymbol{\zeta}} \, \nabla h : \nabla\nabla f) + (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla g \, \nabla f) \, \nabla^2 h + \hat{\boldsymbol{\zeta}} \cdot \nabla f(\nabla g \, \nabla h : \nabla\hat{\boldsymbol{\zeta}}) \\
&\quad - \Big[\nabla g \cdot \nabla h(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla\nabla f) + \hat{\boldsymbol{\zeta}} \cdot \nabla f(\nabla g \hat{\boldsymbol{\zeta}} : \nabla\nabla h)\Big] \\
&\; + \nabla h \cdot \nabla f \, \nabla^2 g + (\nabla g \, \nabla h : \nabla\nabla f) + \nabla h \cdot \nabla f(\hat{\boldsymbol{\zeta}} \, \nabla g : \nabla\hat{\boldsymbol{\zeta}}) \\
&\quad - \Big[(\nabla h \, \nabla f : \nabla\nabla g) + \nabla h \cdot \nabla g \, \nabla^2 f + \nabla h \cdot \nabla g(\hat{\boldsymbol{\zeta}} \, \nabla f : \nabla\hat{\boldsymbol{\zeta}})\Big] \\
&\quad - \Big[(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla h \, \nabla f) \, \nabla^2 g + \hat{\boldsymbol{\zeta}} \cdot \nabla h(\hat{\boldsymbol{\zeta}} \, \nabla g : \nabla\nabla f) + \hat{\boldsymbol{\zeta}} \cdot \nabla f(\nabla h \, \nabla g : \nabla\hat{\boldsymbol{\zeta}})\Big] \\
&\quad + \Big[\hat{\boldsymbol{\zeta}} \cdot \nabla f(\nabla h \hat{\boldsymbol{\zeta}} : \nabla\nabla g) + \nabla h \cdot \nabla g(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla\nabla f)\Big] \\
&\quad + \hat{\boldsymbol{\zeta}} \cdot \nabla h(\hat{\boldsymbol{\zeta}} \, \nabla f : \nabla\nabla g) + (\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla h \, \nabla g) \, \nabla^2 f + \hat{\boldsymbol{\zeta}} \cdot \nabla g(\nabla h \, \nabla f : \nabla\hat{\boldsymbol{\zeta}}) \\
&\quad - \Big[\nabla h \cdot \nabla f(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\zeta}} : \nabla\nabla g) + \hat{\boldsymbol{\zeta}} \cdot \nabla g(\nabla h \hat{\boldsymbol{\zeta}} : \nabla\nabla f)\Big]
\end{aligned}
$$

The sharp-eyed will notice that there are terms of the form (I have left all of these in black)

$$\hat{\boldsymbol{\zeta}} \cdot \nabla h(\nabla g \, \nabla f : \nabla\hat{\boldsymbol{\zeta}}) - \hat{\boldsymbol{\zeta}} \cdot \nabla h(\nabla f \, \nabla g : \nabla\hat{\boldsymbol{\zeta}}) \tag{4.3.22}$$

which do not obviously cancel each other.Using $\hat{\boldsymbol{\zeta}} = \kappa \nabla\zeta$ with κ some function, then $\nabla\hat{\boldsymbol{\zeta}} = \nabla\kappa \, \nabla\zeta + \kappa \, \nabla\nabla\zeta$. We then have

$$\cancel{\hat{\boldsymbol{\zeta}} \cdot \nabla h(\nabla g \, \nabla f : \nabla\nabla\zeta)} - \cancel{\hat{\boldsymbol{\zeta}} \cdot \nabla h(\nabla f \, \nabla g : \nabla\nabla\zeta)} + \hat{\boldsymbol{\zeta}} \cdot \nabla h(\nabla g \, \nabla f : \nabla\kappa \, \nabla\zeta) - \hat{\boldsymbol{\zeta}} \cdot \nabla h(\nabla f \, \nabla g : \nabla\kappa \, \nabla\zeta) \tag{4.3.23}$$

$$= \hat{\boldsymbol{\zeta}} \cdot \nabla h \left([\nabla f \cdot \nabla\kappa][\nabla g \cdot \nabla\zeta] - [\nabla f \cdot \nabla\zeta][\nabla g \cdot \nabla\kappa]\right) \tag{4.3.24}$$

Thus we see in general that the only way for cancelation is that $\hat{\boldsymbol{\zeta}} \cdot \nabla h = 0$ (or that $\kappa =$ constant). This is in fact implied with f, g, h being independent of ζ so that $\nabla(f, g, h) \cdot \hat{\boldsymbol{\zeta}} = 0$. Thus, we see that all these types of terms are in fact zero.

Where (W-4.32) comes from is a bit of a mystery. Given our previous statements on the definition of a Poisson bracket we might expect

$$\bar{\mathbf{B}} \cdot \nabla f = [f \, , \, \Psi] + \frac{\partial f}{\partial \zeta} \tag{4.3.25}$$

rather than Wakatani's

$$\bar{\mathbf{B}} \cdot \nabla f = [\Psi\,,\, f] + \frac{\partial f}{\partial \zeta} \tag{4.3.26}$$

based off of (W-3.198)

$$\bar{\mathbf{B}} \cdot \nabla = B_0 \frac{\partial}{\partial \bar{\zeta}} + \nabla\Psi \times \hat{\boldsymbol{\zeta}} \cdot \nabla \tag{W-3.198}$$

$$\bar{\mathbf{B}} \cdot \nabla f = B_0 \frac{\partial f}{\partial \bar{\zeta}} + \nabla\Psi \times \hat{\boldsymbol{\zeta}} \cdot \nabla f = B_0 \frac{\partial f}{\partial \bar{\zeta}} + \nabla f \times \nabla\Psi \cdot \hat{\boldsymbol{\zeta}} \tag{4.3.27}$$

Wakatani's definition is fine, although a bit non-standard. Even taking Wakatani's defintion as given, though, it is still a good bit mysterious how one can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{B}} \cdot \nabla f\right) - \bar{\mathbf{B}} \cdot \nabla \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial}{\partial t}[\Psi\,,\, f] - \left[\Psi\,,\, \frac{\partial f}{\partial t}\right] + \left[u\,,\, \frac{\partial f}{\partial \zeta}\right] - \frac{\partial}{\partial \zeta}[u\,,\, f] + [u\,,\,[\Psi\,,\, f]] - [\Psi\,,\,[u\,,\, f]] \tag{4.3.28}$$

From now on, we will use $[r, s] = \nabla s \times \nabla r \cdot \hat{\boldsymbol{\zeta}} = \nabla r \times \hat{\boldsymbol{\zeta}} \cdot \nabla s$ as our definition for the Poisson bracket. As one would expect

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{B}} \cdot \nabla f\right) &= \frac{\mathrm{d}}{\mathrm{d}t}[\Psi\,,\, f] + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \zeta} \\ &= \left[\frac{\mathrm{d}\Psi}{\mathrm{d}t}\,,\, f\right] + \left[\Psi\,,\, \frac{\mathrm{d}f}{\mathrm{d}t}\right] + \nabla\Psi \times \frac{\mathrm{d}\hat{\boldsymbol{\zeta}}}{\mathrm{d}t} \cdot \nabla f + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \zeta} \end{aligned} \tag{4.3.29}$$

$$\bar{\mathbf{B}} \cdot \nabla \frac{\mathrm{d}f}{\mathrm{d}t} = \left[\Psi\,,\, \frac{\mathrm{d}f}{\mathrm{d}t}\right] + \frac{\partial}{\partial \zeta}\frac{\mathrm{d}f}{\mathrm{d}t} \tag{4.3.30}$$

So that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{B}} \cdot \nabla f\right) - \bar{\mathbf{B}} \cdot \nabla \frac{\mathrm{d}f}{\mathrm{d}t} &= \left[\frac{\mathrm{d}\Psi}{\mathrm{d}t}\,,\, f\right] + \nabla\Psi \times \frac{\mathrm{d}\hat{\boldsymbol{\zeta}}}{\mathrm{d}t} \cdot \nabla f + \cancel{\left[\Psi\,,\, \frac{\mathrm{d}f}{\mathrm{d}t}\right]} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \zeta} - \left\{\cancel{\left[\Psi\,,\, \frac{\mathrm{d}f}{\mathrm{d}t}\right]} + \frac{\partial}{\partial \zeta}\frac{\mathrm{d}f}{\mathrm{d}t}\right\} \\ &= \left[\frac{\mathrm{d}\Psi}{\mathrm{d}t}\,,\, f\right] + \nabla\Psi \times \frac{\mathrm{d}\hat{\boldsymbol{\zeta}}}{\mathrm{d}t} \cdot \nabla f + D_{-}f \end{aligned} \tag{4.3.31}$$

Unfortunately, if we use the approximation that follows, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \zeta} - \frac{\partial}{\partial \zeta}\frac{\mathrm{d}f}{\mathrm{d}t} \equiv D_{-}f \neq 0 \tag{4.3.32}$$

This approximation is that $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ and further that $\mathbf{u} \cdot \nabla f = \nabla u \times \hat{\boldsymbol{\zeta}} \cdot \nabla_{\perp} f$ (W-3.165) [also noting that $\nabla_{\perp} f = \nabla f$ for f independent of ζ and that the $\frac{\mathrm{d}\hat{\boldsymbol{\zeta}}}{\mathrm{d}t}$ term is zero when expanded this way] we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{B}} \cdot \nabla f\right) - \bar{\mathbf{B}} \cdot \nabla \frac{\mathrm{d}f}{\mathrm{d}t} = \left[\frac{\partial\Psi}{\partial t} + \nabla u \times \hat{\boldsymbol{\zeta}} \cdot \nabla\Psi\,,\, f\right] + D_{-}f = \left[\frac{\partial\Psi}{\partial t} - [\Psi\,,\, u]\,,\, f\right] + D_{-}f \tag{4.3.33}$$

where we have used Wakatani's reversed Poisson bracket definition.

We find

$$\begin{aligned} D_{-}f &= \cancel{\frac{\partial}{\partial t}\frac{\partial f}{\partial \zeta}} + \nabla u \times \hat{\boldsymbol{\zeta}} \cdot \nabla \frac{\partial f}{\partial \zeta} - \frac{\partial}{\partial \zeta}\left[\cancel{\frac{\partial f}{\partial t}} + \frac{\partial}{\partial \zeta}\left(\nabla u \times \hat{\boldsymbol{\zeta}} \cdot \nabla f\right)\right] \\ &= -\nabla \frac{\partial u}{\partial \zeta} \times \hat{\boldsymbol{\zeta}} \cdot \nabla f - \cancel{\nabla u \times \frac{\partial \hat{\boldsymbol{\zeta}}}{\partial \zeta}} \cdot \nabla f = \left[f\,,\,\frac{\partial u}{\partial \zeta}\right] \end{aligned} \tag{4.3.34}$$

(again using Wakatani's reversed Poisson bracket) because $\nabla u \times \nabla f$ should be purely in the $\hat{\boldsymbol{\zeta}}$ direction and $\frac{\partial \hat{\zeta}}{\partial \zeta}$ has no component in the $\hat{\boldsymbol{\zeta}}$ direction.

Further noting that $\Psi = A + \Psi_h$ with $\frac{\partial \Psi_h}{\partial t} = 0$ we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{B}} \cdot \nabla f\right) - \bar{\mathbf{B}} \cdot \nabla \frac{\mathrm{d}f}{\mathrm{d}t} = \left[\frac{\partial A}{\partial t} - \frac{\partial u}{\partial \zeta} + [u\,,\,\Psi]\,,\,f\right] \tag{4.3.35}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{B}} \cdot \nabla f\right) - \bar{\mathbf{B}} \cdot \nabla \frac{\mathrm{d}f}{\mathrm{d}t} = \left[\frac{\partial A}{\partial t} - \frac{\partial u}{\partial \zeta} - [\Psi\,,\,u]\,,\,f\right] \tag{4.3.36}$$

$$\tag{4.3.37}$$

Note that this is rather different than what Wakatani actually obtains,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{B}} \cdot \nabla f\right) - \bar{\mathbf{B}} \cdot \nabla \frac{\mathrm{d}f}{\mathrm{d}t} = \left[\frac{\partial A}{\partial t} - \frac{\partial u}{\partial \zeta} - [u\,,\,\Psi]\,,\,f\right] \tag{W-4.34}$$

A simpler method of getting this is through approximating immediately that

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial t} + \nabla u \times \hat{\boldsymbol{\zeta}} \cdot \overbrace{\nabla_{\perp} g}^{\nabla g} = \frac{\partial g}{\partial t} - [u\,,\,g] \tag{4.3.38}$$

or using his reverse Poisson bracket definition

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial t} + [u\,,\,g] \tag{4.3.39}$$

So that what is found is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{B}} \cdot \nabla f\right) &= \frac{\mathrm{d}}{\mathrm{d}t}[\Psi\,,\,f] + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \zeta} \\ &= \frac{\partial}{\partial t}[\Psi\,,\,f] + [u\,,\,[\Psi\,,\,f]] + \frac{\partial}{\partial t}\frac{\partial f}{\partial \zeta} + \left[u\,,\,\frac{\partial f}{\partial \zeta}\right] \end{aligned} \tag{4.3.40}$$

$$\bar{\mathbf{B}} \cdot \nabla \frac{\mathrm{d}f}{\mathrm{d}t} = \bar{\mathbf{B}} \cdot \nabla\left(\frac{\partial f}{\partial t} + [u\,,\,f]\right) = \left[\Psi\,,\,\frac{\partial f}{\partial t}\right] + \frac{\partial}{\partial \zeta}\frac{\partial f}{\partial t} + [\Psi\,,\,[u\,,\,f]] + \frac{\partial}{\partial \zeta}[u\,,\,f] \tag{4.3.41}$$

This is a bit suspect compared to our earlier exact answer, but we now find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{B}} \cdot \nabla f\right) - \bar{\mathbf{B}} \cdot \nabla \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial}{\partial t}[\Psi\,,\,f] + [u\,,\,[\Psi\,,\,f]] + \cancel{\frac{\partial}{\partial t}\frac{\partial f}{\partial \zeta}} + \left[u\,,\,\frac{\partial f}{\partial \zeta}\right] \\ &\quad - \left(\left[\Psi\,,\,\frac{\partial f}{\partial t}\right] + \cancel{\frac{\partial}{\partial \zeta}\frac{\partial f}{\partial t}} + [\Psi\,,\,[u\,,\,f]] + \frac{\partial}{\partial \zeta}[u\,,\,f]\right) \end{aligned} \tag{4.3.42}$$

$$= \frac{\partial}{\partial t}[\Psi\,,\,f] - \left[\Psi\,,\,\frac{\partial f}{\partial t}\right] + \left[u\,,\,\frac{\partial f}{\partial \zeta}\right] - \frac{\partial}{\partial \zeta}[u\,,\,f] + [u\,,\,[\Psi\,,\,f]] - [\Psi\,,\,[u\,,\,f]] \tag{4.3.43}$$

So the mystery is demystified. In any case, applying the Jacobi identity for the Poisson bracket, we find

$$[\Psi\,,\,[u\,,\,f]] = -\,[u\,,\,[f\,,\,\Psi]] - [f\,,\,[\Psi\,,\,u]] \tag{4.3.44}$$

Thus,

$$[u\,,\,[\Psi\,,\,f]] - [\Psi\,,\,[u\,,\,f]] = \cancel{[u\,,\,[\Psi\,,\,f]] + [u\,,\,[f\,,\,\Psi]]} + [f\,,\,[\Psi\,,\,u]] = -\,[[\Psi\,,\,u]\,,\,f] \tag{4.3.45}$$

Thus,

$$= \overbrace{\frac{\partial}{\partial t}[\Psi\,,\,f] - \left[\Psi\,,\,\frac{\partial f}{\partial t}\right]}^{=\left[\frac{\partial\Psi}{\partial t}\,,\,f\right]} + \overbrace{\left[u\,,\,\frac{\partial f}{\partial\zeta}\right] - \frac{\partial}{\partial\zeta}[u\,,\,f]}^{=\left[-\frac{\partial u}{\partial\zeta}\,,\,f\right]} - [[\Psi\,,\,u]\,,\,f] \tag{4.3.46}$$

$$= \left[\frac{\partial\Psi}{\partial t} - \frac{\partial u}{\partial\zeta} - [\Psi\,,\,u]\,,\,f\right] \tag{4.3.47}$$

$$= \left[\frac{\partial A}{\partial t} - \frac{\partial u}{\partial\zeta} - [\Psi\,,\,u]\,,\,f\right] \tag{4.3.48}$$

again different from Wakatani declares.

Thus, my two methods agree and Wakatani most likely has a sign error on his $[[\Psi\,,\,u]\,,\,f]$ term.

4.4 A Three-Dimensional MHD Equilibrium Calculation based on the Variational Principle

Let's show how one gets (W-4.78-79) from the linearized equations. Using $Q = Q_0 + \widetilde{Q}$, where Q_0 is independent of time (but not necessarily of spatial coordinates) and $\mathbf{v} = \widetilde{\mathbf{v}}$. We have to first order

$$\frac{\partial\widetilde{\rho}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_0\widetilde{\mathbf{v}}) = 0 \tag{4.4.1}$$

$$\frac{\partial\widetilde{\mathbf{B}}}{\partial t} = \boldsymbol{\nabla}\times(\widetilde{\mathbf{v}}\times\mathbf{B}_0) \tag{4.4.2}$$

with $\widetilde{\mathbf{v}} = \frac{\partial\boldsymbol{\xi}}{\partial t}$ and $\boldsymbol{\xi}(\mathbf{x},0) = \mathbf{0} = \widetilde{\mathbf{B}}(\mathbf{x},0)$ and $\widetilde{\rho}(\mathbf{x},0) = 0$. Thus, integrating

$$\widetilde{\rho}(t) - \cancel{\widetilde{\rho}(0)} = -\boldsymbol{\nabla}\cdot\int_0^t \mathrm{d}t'\ \rho_0\frac{\partial\boldsymbol{\xi}}{\partial t'} = -\boldsymbol{\nabla}\cdot(\rho_0\,[\boldsymbol{\xi}(t) - \cancel{\boldsymbol{\xi}(0)}]) = -\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\xi}) \tag{4.4.3}$$

and

$$\widetilde{\mathbf{B}}(\mathbf{x},t) - \cancel{\widetilde{\mathbf{B}}(\mathbf{x},0)} = -\boldsymbol{\nabla}\times\left[\mathbf{B}_0\times\int_0^t \mathrm{d}t'\ \frac{\partial\boldsymbol{\xi}}{\partial t'}\right] = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B}_0) \tag{4.4.4}$$

just as described.

Note that in this case the adiabatic law being followed can be put as

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\partial P}{\partial t} + \mathbf{v}\cdot\nabla P = -\gamma P\boldsymbol{\nabla}\cdot\mathbf{v} \tag{4.4.5}$$

which will yield $P_0 = \rho_0^\gamma$ then we see that we can get with P_0 independent of time that

$$\frac{\partial \widetilde{P}}{\partial t} + \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \nabla P_0 = -\gamma P_0 \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \tag{4.4.6}$$

$$\int_0^t \mathrm{d}t' \, \frac{\partial \widetilde{P}}{\partial t'} + \int_0^t \mathrm{d}t' \, \frac{\partial \boldsymbol{\xi}}{\partial t'} \cdot \nabla P_0 = \int_0^t \mathrm{d}t' \, -\gamma P_0 \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t'} \tag{4.4.7}$$

$$\widetilde{P}(t) - \cancel{\widetilde{P}(0)} + \boldsymbol{\xi}(t) \cdot \nabla P_0 = -\gamma P_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}(t) \tag{4.4.8}$$

$$\widetilde{P}(t) = -\boldsymbol{\xi}(t) \cdot \nabla P_0 - \gamma P_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}(t) \tag{4.4.9}$$

just as described in (W-4.81).

With

$$W = \int \mathrm{d}^3x \, \left(\frac{B^2}{2\mu_0} + \frac{P}{\gamma - 1} \right) = \int \mathrm{d}^3x \, \left(\frac{\mathbf{B} \cdot \mathbf{B}}{2\mu_0} + \frac{P}{\gamma - 1} \right) \tag{4.4.10}$$

$$\delta W = \int \mathrm{d}^3x \, \left(\frac{\mathbf{B} \cdot \delta \mathbf{B}}{\mu_0} + \frac{\delta P}{\gamma - 1} \right) \tag{4.4.11}$$

For our small perturbations $\delta \mathbf{B} = \widetilde{\mathbf{B}}$ and $\delta P = \widetilde{P}$ so that

$$\delta W = \int \mathrm{d}^3x \, \left(\frac{\mathbf{B} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}_0)}{\mu_0} + \frac{-\boldsymbol{\xi} \cdot \nabla P_0 - \gamma P_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}}{\gamma - 1} \right) \tag{4.4.12}$$

We then use that

$$\mathbf{B} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}_0) = \mathbf{B} \cdot \boldsymbol{\xi} \cancel{\boldsymbol{\nabla} \cdot \mathbf{B}_0} - \mathbf{B}\boldsymbol{\xi} : \nabla \mathbf{B}_0 + \mathbf{B}\mathbf{B}_0 : \nabla \boldsymbol{\xi} - \mathbf{B} \cdot \mathbf{B}_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \tag{4.4.13}$$

If we only take to first order then we see that the $\mathbf{B} \to \mathbf{B}_0$ and thus we find

$$\mathbf{B} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}_0) = -\mathbf{B}_0\boldsymbol{\xi} : \nabla \mathbf{B}_0 + \mathbf{B}_0\mathbf{B}_0 : \nabla \boldsymbol{\xi} - \mathbf{B}_0 \cdot \mathbf{B}_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \tag{4.4.14}$$

$$= -\boldsymbol{\xi} \cdot \nabla (B_0^2/2) + \mathbf{B}_0\mathbf{B}_0 : \nabla \boldsymbol{\xi} - B_0^2 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \tag{4.4.15}$$

So combining, we would find

$$\delta W = -\int \mathrm{d}^3x \, \left(\frac{\boldsymbol{\xi} \cdot \nabla (B_0^2/2) - \mathbf{B}_0\mathbf{B}_0 : \nabla \boldsymbol{\xi} + B_0^2 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}}{\mu_0} + \frac{\boldsymbol{\xi} \cdot \nabla P_0 + \gamma P_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}}{\gamma - 1} \right) \tag{4.4.16}$$

$$\delta W = -\int \mathrm{d}^3x \, \left(\frac{\boldsymbol{\xi} \cdot \nabla (B_0^2/2) - \mathbf{B}_0\mathbf{B}_0 : \nabla \boldsymbol{\xi}}{\mu_0} + \frac{\boldsymbol{\xi} \cdot \nabla P_0}{\gamma - 1} + \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \left[\frac{B_0^2}{\mu_0} + \frac{\gamma P_0}{\gamma - 1} \right] \right) \tag{4.4.17}$$

We can use that $a\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = \boldsymbol{\nabla} \cdot (a\boldsymbol{\xi}) - \nabla a \cdot \boldsymbol{\xi}$ and that the $\boldsymbol{\nabla} \cdot (a\boldsymbol{\xi})$ term disappears due to the boundary condition $\mathbf{n} \cdot \boldsymbol{\xi} = 0$. Thus,

$$\int \mathrm{d}^3x \, \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \left[\frac{B_0^2}{\mu_0} + \frac{\gamma P_0}{\gamma - 1} \right] = \int \mathrm{d}^3x \, \boldsymbol{\nabla} \cdot \left[\left(\frac{B_0^2}{\mu_0} + \frac{\gamma P_0}{\gamma - 1} \right) \boldsymbol{\xi} \right] - \int \mathrm{d}^3x \, \nabla \left[\frac{B_0^2}{\mu_0} + \frac{\gamma P_0}{\gamma - 1} \right] \cdot \boldsymbol{\xi} \tag{4.4.18}$$

$$= \cancel{\int \mathrm{d}^2x \, \mathbf{n} \cdot \left[\left(\frac{B_0^2}{\mu_0} + \frac{\gamma P_0}{\gamma - 1} \right) \boldsymbol{\xi} \right]} - \int \mathrm{d}^3x \, \nabla \left[\frac{B_0^2}{\mu_0} + \frac{\gamma P_0}{\gamma - 1} \right] \cdot \boldsymbol{\xi} \tag{4.4.19}$$

We have then found

$$\delta W = -\int \mathrm{d}^3x \left(\frac{\boldsymbol{\xi}\cdot\nabla(B_0^2/2) - \mathbf{B}_0\mathbf{B}_0 : \nabla\boldsymbol{\xi}}{\mu_0} + \frac{\boldsymbol{\xi}\cdot\nabla P_0}{\gamma-1} - \boldsymbol{\xi}\cdot\nabla(B_0^2/\mu_0) - \frac{\gamma\boldsymbol{\xi}\cdot\nabla P_0}{\gamma-1}\right) \tag{4.4.20}$$

$$= \int \mathrm{d}^3x \left(\frac{\boldsymbol{\xi}\cdot\nabla(B_0^2/2) + \mathbf{B}_0\mathbf{B}_0 : \nabla\boldsymbol{\xi}}{\mu_0} + \frac{(\gamma-1)\boldsymbol{\xi}\cdot\nabla P_0}{\gamma-1}\right) \tag{4.4.21}$$

$$= \int \mathrm{d}^3x \left(\frac{\boldsymbol{\xi}\cdot\nabla(B_0^2/2) + \mathbf{B}_0\mathbf{B}_0 : \nabla\boldsymbol{\xi}}{\mu_0} + \nabla P\cdot\boldsymbol{\xi}\right) \tag{4.4.22}$$

They find that

$$\delta W = -\int \mathrm{d}^3x\ \mathbf{F}\cdot\boldsymbol{\xi} \tag{4.4.23}$$

$$\mathbf{F} = \frac{1}{\mu_0}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\times\mathbf{B} - \nabla P = \frac{\mathbf{B}\cdot\nabla\mathbf{B} - \nabla(B^2/2)}{\mu_0} - \nabla P \tag{4.4.24}$$

We also can now use that

$$\mathbf{B}_0\mathbf{B}_0 : \nabla\boldsymbol{\xi} = \mathbf{B}_0\cdot\nabla(\mathbf{B}_0\cdot\boldsymbol{\xi}) - \boldsymbol{\xi}\mathbf{B}_0 : \nabla\mathbf{B} = \boldsymbol{\nabla}\cdot(\mathbf{B}_0\boldsymbol{\xi}\cdot\mathbf{B}_0) - \boldsymbol{\xi}\cdot[\mathbf{B}_0\cdot\nabla\mathbf{B}_0] \tag{4.4.25}$$

Again, we can use the divergence theorem on the first term with $\mathbf{B}_0\cdot\mathbf{n} = 0$ to find

$$\begin{aligned}\int \mathrm{d}^3x\ \mathbf{B}_0\mathbf{B}_0 : \nabla\boldsymbol{\xi} &= \int \mathrm{d}^3x\ \boldsymbol{\nabla}\cdot(\mathbf{B}_0\boldsymbol{\xi}\cdot\mathbf{B}_0) - \int \mathrm{d}^3x\ [\mathbf{B}_0\cdot\nabla\mathbf{B}_0]\cdot\boldsymbol{\xi} \\ &= \cancel{\int \mathrm{d}^2x\ \mathbf{n}\cdot(\mathbf{B}_0\boldsymbol{\xi}\cdot\mathbf{B}_0)} - \int \mathrm{d}^3x\ [\mathbf{B}_0\cdot\nabla\mathbf{B}_0]\cdot\boldsymbol{\xi}\end{aligned} \tag{4.4.26}$$

Thus, we have

$$\begin{aligned}\delta W &= \int \mathrm{d}^3x \left(\frac{\boldsymbol{\xi}\cdot\nabla(B_0^2/2) + \mathbf{B}_0\mathbf{B}_0 : \nabla\boldsymbol{\xi}}{\mu_0} + \nabla P\right) = \int \mathrm{d}^3x \left(\frac{\nabla(B_0^2/2) - \mathbf{B}_0\cdot\nabla\mathbf{B}_0}{\mu_0} + \nabla P\right)\cdot\boldsymbol{\xi} \\ &= -\int \mathrm{d}^3x \left(\frac{\mathbf{B}_0\cdot\nabla\mathbf{B}_0 - \nabla(B_0^2/2)}{\mu_0} - \nabla P\right)\cdot\boldsymbol{\xi} \\ &= -\int \mathrm{d}^3x\ \mathbf{F}\cdot\boldsymbol{\xi}\end{aligned} \tag{4.4.27}$$

$$\mathbf{F} = \frac{\mathbf{B}_0\cdot\mathbf{B}_0 - \nabla(B_0^2/2)}{\mu_0} - \nabla P = \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B}_0)\times\mathbf{B}_0 - \nabla P \tag{4.4.28}$$

which we note (taking the quantities to be their equilibrium values) matches Wakatani's (W-4.82) and (W-4.83).

4.5 The Solov'ev-Shafranov Equation

Lots of math, but seems straightforward.

4.6 The Pfirsch-Schlüter Current, and Equilibrium with Rational Magnetic Surfaces

We use

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{4.6.1}$$
$$\mathbf{J}_\perp \times \mathbf{B} = \nabla p \tag{4.6.2}$$
$$\boldsymbol{\nabla} \cdot \mathbf{J} = 0 \tag{4.6.3}$$

Using $\mathbf{J}_\perp = -\hat{\mathbf{b}} \times \hat{\mathbf{b}} \times \mathbf{J}$ with $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$, so that, if we solve for $\mathbf{J}_\perp$ via

$$\mathbf{B} \times (\mathbf{J}_\perp \times \mathbf{B}) = \mathbf{B} \times \nabla p \tag{4.6.4}$$
$$-B^2 \hat{\mathbf{b}} \times \left(\hat{\mathbf{b}} \times \mathbf{J}_\perp\right) = \mathbf{B} \times \nabla p \tag{4.6.5}$$
$$(\mathbf{J}_\perp)_\perp = \frac{\mathbf{B} \times \nabla p}{B^2} \tag{4.6.6}$$
$$\mathbf{J}_\perp = \frac{\mathbf{B} \times \nabla p}{B^2} = -\frac{\mathbf{B} \times \nabla \mathcal{V} \frac{\mathrm{d}p}{\mathrm{d}\mathcal{V}}}{B^2} \tag{4.6.7}$$

Thus, we find with $\mathbf{J}_\parallel = \sigma \mathbf{B}$ that $\boldsymbol{\nabla} \cdot \mathbf{J} = 0 \to \boldsymbol{\nabla} \cdot \mathbf{J}_\parallel = -\boldsymbol{\nabla} \cdot \mathbf{J}_\perp$ that

$$\boldsymbol{\nabla} \cdot [\sigma \mathbf{B}] = -\boldsymbol{\nabla} \cdot \left[\frac{\mathbf{B} \times \nabla \mathcal{V} \frac{\mathrm{d}p}{\mathrm{d}\mathcal{V}}}{B^2}\right] \tag{4.6.8}$$
$$\mathbf{B} \cdot \nabla \sigma = -\boldsymbol{\nabla} \cdot \left[\frac{\mathrm{d}P}{\mathrm{d}\mathcal{V}} \frac{\mathbf{B} \times \nabla \mathcal{V}}{B^2}\right] = -\frac{\mathrm{d}P}{\mathrm{d}\mathcal{V}} \boldsymbol{\nabla} \cdot \left[\frac{\mathbf{B} \times \nabla \mathcal{V}}{B^2}\right] + \cancel{\frac{\mathrm{d}^2 P}{\mathrm{d}\mathcal{V}^2} \nabla \mathcal{V} \cdot \frac{\mathbf{B} \times \nabla \mathcal{V}}{B^2}} \tag{4.6.9}$$

We use a $\mathbf{B}$ given in a coordinate system $(\mathcal{V}, \theta, \zeta)$ so that

$$B^{\mathcal{V}} = \mathbf{B} \cdot \nabla \mathcal{V} \equiv \mathbf{B} \cdot \mathbf{e}^{\mathcal{V}} = 0 \tag{4.6.10}$$
$$B^{\theta} = \mathbf{B} \cdot \nabla \theta \equiv \mathbf{B} \cdot \mathbf{e}^{\theta} \tag{4.6.11}$$
$$B^{\zeta} = \mathbf{B} \cdot \nabla \zeta \equiv \mathbf{B} \cdot \mathbf{e}^{\zeta} \tag{4.6.12}$$
$$\mathbf{B} = B_{\mathcal{V}} \nabla \mathcal{V} + B_\theta \nabla \theta + B_\zeta \nabla \zeta \tag{4.6.13}$$

We can also write

$$B_{\mathcal{V}} = \mathbf{B} \cdot \mathcal{J} \nabla \theta \times \nabla \zeta \equiv \mathbf{B} \cdot \mathbf{e}_{\mathcal{V}} \tag{4.6.14}$$
$$B_\theta = \mathbf{B} \cdot \mathcal{J} \nabla \zeta \times \nabla \mathcal{V} \equiv \mathbf{B} \cdot \mathbf{e}_\theta \tag{4.6.15}$$
$$B_\zeta = \mathbf{B} \cdot \mathcal{J} \nabla \mathcal{V} \times \nabla \theta \equiv \mathbf{B} \cdot \mathbf{e}_\zeta \tag{4.6.16}$$

with $\mathcal{J} = 1/(\nabla \mathcal{V} \cdot \nabla \theta \times \nabla \zeta)$ the Jacobian for this system.

With

$$\mathbf{B} \cdot \nabla \sigma = -\frac{\mathrm{d}P}{\mathrm{d}\mathcal{V}} \boldsymbol{\nabla} \cdot \left[\frac{\mathbf{B} \times \nabla \mathcal{V}}{B^2}\right] \tag{4.6.17}$$

It is simple to see

$$\boldsymbol{\nabla} \cdot (\nabla a \times \nabla b) = \nabla b \cdot \boldsymbol{\nabla} \times \nabla a - \nabla a \cdot \boldsymbol{\nabla} \times \nabla b = 0 \tag{4.6.18}$$

for any a and b. Thus, $\boldsymbol{\nabla}\cdot(\mathbf{e}_\theta/\mathcal{J}) = \boldsymbol{\nabla}\cdot(\mathbf{e}_\zeta/\mathcal{J}) = 0$. Thus, using the covariant representation of $\mathbf{B}$ we find

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla\mathcal{V}}{B^2}\right) = \boldsymbol{\nabla}\cdot\left(\frac{(\cancel{B_\mathcal{V}\nabla\mathcal{V}} + B_\zeta\nabla\zeta + B_\theta\nabla\theta)\times\nabla\mathcal{V}}{B^2}\right) \tag{4.6.19}$$

$$= \boldsymbol{\nabla}\cdot\left(\frac{B_\zeta\nabla\zeta\times\nabla\mathcal{V} + B_\theta\nabla\theta\times\nabla\mathcal{V}}{B^2}\right) \tag{4.6.20}$$

$$= \boldsymbol{\nabla}\cdot\left(\frac{B_\zeta\mathbf{e}_\theta/\mathcal{J} - B_\theta\mathbf{e}_\zeta/\mathcal{J}}{B^2}\right) = \boldsymbol{\nabla}\cdot\left(\frac{B_\zeta\mathbf{e}_\theta/\mathcal{J}}{B^2}\right) - \boldsymbol{\nabla}\cdot\left(\frac{B_\theta\mathbf{e}_\zeta/\mathcal{J}}{B^2}\right) \tag{4.6.21}$$

$$= \cancel{\frac{B_\zeta}{B^2}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{e}_\theta}{\mathcal{J}}\right)} + \frac{\mathbf{e}_\theta}{\mathcal{J}}\cdot\nabla\left(\frac{B_\zeta}{B^2}\right) - \cancel{\frac{B_\theta}{B^2}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{e}_\zeta}{\mathcal{J}}\right)} - \frac{\mathbf{e}_\zeta}{\mathcal{J}}\cdot\nabla\left(\frac{B_\theta}{B^2}\right) \tag{4.6.22}$$

$$= \frac{1}{\mathcal{J}}\left[\mathbf{e}_\theta\cdot\nabla\left(\frac{B_\zeta}{B^2}\right) - \mathbf{e}_\zeta\cdot\nabla\left(\frac{B_\theta}{B^2}\right)\right] \tag{4.6.23}$$

We can further simply by the following steps.

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla\mathcal{V}}{B^2}\right) = \frac{B^\theta}{\mathcal{J}B^\theta}\mathbf{e}_\theta\cdot\nabla\left(\frac{B_\zeta}{B^2}\right) - \frac{B^\zeta}{\mathcal{J}B^\zeta}\mathbf{e}_\zeta\cdot\nabla\left(\frac{B_\theta}{B^2}\right) \tag{4.6.24}$$

$$\begin{aligned} &= B^\theta\mathbf{e}_\theta\cdot\nabla\left(\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right) - \frac{B^\theta B_\zeta}{B^2}\mathbf{e}_\theta\cdot\nabla\left(\frac{1}{\mathcal{J}B^\theta}\right) \\ &\qquad - B^\zeta\mathbf{e}_\zeta\cdot\nabla\left(\frac{B_\theta}{\mathcal{J}B^\zeta B^2}\right) + \frac{B_\theta B^\zeta}{B^2}\mathbf{e}_\zeta\cdot\nabla\left(\frac{1}{\mathcal{J}B^\zeta}\right)\end{aligned} \tag{4.6.25}$$

$$\begin{aligned} &= B^\theta\mathbf{e}_\theta\cdot\nabla\left(\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right) - B^\zeta\mathbf{e}_\zeta\cdot\nabla\left(\frac{B_\theta}{\mathcal{J}B^\zeta B^2}\right) \\ &\qquad + \frac{B_\theta B^\zeta}{B^2}\mathbf{e}_\zeta\cdot\nabla\left(\frac{1}{\mathcal{J}B_\zeta}\right) - \frac{B^\theta B_\zeta}{B^2}\mathbf{e}_\theta\cdot\nabla\left(\frac{1}{\mathcal{J}B_\theta}\right)\end{aligned} \tag{4.6.26}$$

We can note that $\mathcal{J}B^\theta = \frac{\mathrm{d}\Psi_P(\mathcal{V})}{\mathrm{d}\mathcal{V}}$ and $\mathcal{J}B^\zeta = \frac{\mathrm{d}\Psi_T(\mathcal{V})}{\mathrm{d}\mathcal{V}}$ so that $\nabla[(\mathcal{J}B^\theta)^{-1}] = \frac{-1}{(\mathcal{J}B^\theta)^2}\nabla(\frac{\mathrm{d}\Psi_P}{\mathrm{d}\mathcal{V}}) = [\frac{\mathrm{d}^2\Psi_p}{\mathrm{d}\mathcal{V}}/(\frac{\mathrm{d}\Psi_p}{\mathrm{d}\mathcal{V}^2})^2]\nabla\mathcal{V}$ and similarly for $\mathcal{J}B^\zeta$. Thus, (using $\mathbf{e}_i\cdot\mathbf{e}^j = \delta_{ij}$)

$$\mathbf{e}_\zeta\cdot\nabla\left(\frac{1}{\mathcal{J}B^\zeta}\right) \propto \mathbf{e}_\zeta\cdot\nabla\mathcal{V} = \mathbf{e}_\zeta\cdot\mathbf{e}^\mathcal{V} = 0 \tag{4.6.27}$$

$$\mathbf{e}_\theta\cdot\nabla\left(\frac{1}{\mathcal{J}B^\theta}\right) \propto \mathbf{e}_\theta\cdot\nabla\mathcal{V} = \mathbf{e}_\theta\cdot\mathbf{e}^\mathcal{V} = 0 \tag{4.6.28}$$

So these terms vanish and we are left with

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla\mathcal{V}}{B^2}\right) = B^\theta\mathbf{e}_\theta\cdot\nabla\left(\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right) - B^\zeta\mathbf{e}_\zeta\cdot\nabla\left(\frac{B_\theta}{\mathcal{J}B^\zeta B^2}\right) \tag{4.6.29}$$

Now we can construct this as $\mathbf{B}\cdot$ term in two different ways. First let's do it based off of the $B^\theta\hat{\mathbf{e}}_\theta$ term.

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla\mathcal{V}}{B^2}\right) = \mathbf{B}\cdot\nabla\left(\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right) - B^\zeta\mathbf{e}_\zeta\cdot\nabla\left(\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right) - B^\zeta\mathbf{e}_\zeta\cdot\nabla\left(\frac{B_\theta}{\mathcal{J}B^\zeta B^2}\right) \tag{4.6.30}$$

$$= \mathbf{B}\cdot\nabla\left(\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right) - B^\zeta\mathbf{e}_\zeta\cdot\nabla\left(\frac{B_\zeta B^\zeta + B_\theta B^\theta}{\mathcal{J}B^\theta B^\zeta B^2}\right) \tag{4.6.31}$$

We can use that $B^2 = B_\theta B^\theta + B_\zeta B^\zeta$ (due to $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_{ij}$) with our identities for $\mathcal{J}B^\theta = \Psi'_P(\mathcal{V})$ and $\mathcal{J}B^\zeta = \Psi'_T(\mathcal{V})$ so that we can write

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla\mathcal{V}}{B^2}\right) = \mathbf{B}\cdot\nabla\left(\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right) - B^\zeta\mathbf{e}_\zeta\cdot\nabla\left(\frac{B^2}{\mathcal{J}B^\theta B^\zeta B^2}\right) \tag{4.6.32}$$

$$= \mathbf{B}\cdot\nabla\left(\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right) - B^\zeta\mathbf{e}_\zeta\cdot\nabla\left(\frac{\mathcal{J}}{\Psi'_P\Psi'_T}\right) \tag{4.6.33}$$

Similarly we can construct a $\mathbf{B}\cdot$ term off of the $B^\zeta\hat{\mathbf{e}}_\zeta$ term.

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla\mathcal{V}}{B^2}\right) = -\mathbf{B}\cdot\nabla\left(\frac{B_\theta}{\mathcal{J}B^\zeta B^2}\right) + B^\theta\hat{\mathbf{e}}_\theta\cdot\nabla\left(\frac{B_\theta}{\mathcal{J}B^\zeta B^2}\right) + B^\theta\hat{\mathbf{e}}_\theta\cdot\nabla\left(\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right) \tag{4.6.34}$$

$$= -\mathbf{B}\cdot\nabla\left(\frac{B_\theta}{\mathcal{J}B^\zeta B^2}\right) + B^\theta\mathbf{e}_\theta\cdot\nabla\left(\frac{B_\zeta B^\zeta + B_\theta B^\theta}{\mathcal{J}B^\theta B^\zeta B^2}\right) \tag{4.6.35}$$

Once again, we can use that $B^2 = B_\theta B^\theta + B_\zeta B^\zeta$ (due to $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_{ij}$) with our identities for $\mathcal{J}B^\theta = \Psi'_P(\mathcal{V})$ and $\mathcal{J}B^\zeta = \Psi'_T(\mathcal{V})$ so that we can write

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla\mathcal{V}}{B^2}\right) = -\mathbf{B}\cdot\nabla\left(\frac{B_\theta}{\mathcal{J}B^\zeta B^2}\right) + B^\theta\mathbf{e}_\theta\cdot\nabla\left(\frac{B^2}{\mathcal{J}B^\theta B^\zeta B^2}\right) \tag{4.6.36}$$

$$= -\mathbf{B}\cdot\nabla\left(\frac{B_\theta}{\mathcal{J}B^\zeta B^2}\right) + B^\theta\mathbf{e}_\theta\cdot\nabla\left(\frac{\mathcal{J}}{\Psi'_P\Psi'_T}\right) \tag{4.6.37}$$

Now we can take $\frac{\alpha_2}{\alpha_1+\alpha_2}$(4.6.33) and add this to $\frac{\alpha_1}{\alpha_1+\alpha_2}$(4.6.37) for another expression for $\boldsymbol{\nabla}\cdot(\mathbf{B}\times\frac{\nabla\mathcal{V}}{B^2})$ given by

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla\mathcal{V}}{B^2}\right) &= \frac{\alpha_2}{\alpha_1+\alpha_2}\mathbf{B}\cdot\nabla\left(\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right) - \frac{\alpha_2}{\alpha_1+\alpha_2}B^\zeta\mathbf{e}_\zeta\cdot\nabla\left(\frac{\mathcal{J}}{\Psi'_P\Psi'_T}\right) \\ &\quad - \frac{\alpha_1}{\alpha_1+\alpha_2}\mathbf{B}\cdot\nabla\left(\frac{B_\theta}{\mathcal{J}B^\zeta B^2}\right) + \frac{\alpha_1}{\alpha_1+\alpha_2}B^\theta\mathbf{e}_\theta\cdot\nabla\left(\frac{\mathcal{J}}{\Psi'_P\Psi'_T}\right)\end{aligned} \tag{4.6.38}$$

$$= \frac{\mathbf{B}}{\alpha_1+\alpha_2}\cdot\nabla\left[\frac{\alpha_2 B_\zeta}{\mathcal{J}B^\theta B^2} - \frac{\alpha_1 B_\theta}{\mathcal{J}B^\zeta B^2}\right] + \nabla\left(\frac{\mathcal{J}}{\Psi'_P\Psi'_T}\right)\cdot\frac{\alpha_1 B^\theta\mathbf{e}_\theta - \alpha_2 B^\zeta\mathbf{e}_\zeta}{\alpha_1+\alpha_2} \tag{4.6.39}$$

We can note that

$$\nabla\left(\frac{\mathcal{J}}{\Psi'_P\Psi'_T}\right) = -\frac{\mathcal{J}(\Psi'_P\Psi'_T)'}{(\Psi'_P\Psi'_T)^2}\nabla\mathcal{V} + \frac{\nabla\mathcal{J}}{\Psi'_P\Psi'_T} \tag{4.6.40}$$

so that the $\nabla\mathcal{V}$ term vanishes when dotted into $\mathbf{e}_\theta$ or $\mathbf{e}_\zeta$ and we in fact have

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla\mathcal{V}}{B^2}\right) = \frac{\mathbf{B}}{\alpha_1+\alpha_2}\cdot\nabla\left[\frac{\alpha_2 B_\zeta}{\mathcal{J}B^\theta B^2} - \frac{\alpha_1 B_\theta}{\mathcal{J}B^\zeta B^2}\right] + \left(\frac{\nabla\mathcal{J}}{\Psi'_P\Psi'_T}\right)\cdot\frac{\alpha_1 B^\theta\mathbf{e}_\theta - \alpha_2 B^\zeta\mathbf{e}_\zeta}{\alpha_1+\alpha_2} \tag{4.6.41}$$

Thus, when we substitute we find

$$\mathbf{B}\cdot\nabla\sigma = -P'\boldsymbol{\nabla}\cdot\left[\frac{\mathbf{B}\times\nabla\mathcal{V}}{B^2}\right] \tag{4.6.42}$$

$$\mathbf{B}\cdot\nabla\sigma = -P'\left[\frac{\mathbf{B}}{\alpha_1+\alpha_2}\cdot\nabla\left[\frac{\alpha_2 B_\zeta}{\mathcal{J}B^\theta B^2}-\frac{\alpha_1 B_\theta}{\mathcal{J}B^\zeta B^2}\right]+\left(\frac{\nabla\mathcal{J}}{\Psi_P'\Psi_T'}\right)\cdot\frac{\alpha_1 B^\theta\mathbf{e}_\theta-\alpha_2 B^\zeta\mathbf{e}_\zeta}{\alpha_1+\alpha_2}\right] \tag{4.6.43}$$

$$\mathbf{B}\cdot\nabla\sigma = \left[\frac{\mathbf{B}}{\alpha_1+\alpha_2}\cdot\nabla\left[\frac{-P'\alpha_2 B_\zeta}{\mathcal{J}B^\theta B^2}-\frac{-P'\alpha_1 B_\theta}{\mathcal{J}B^\zeta B^2}\right]+-P'\left(\frac{\nabla\mathcal{J}}{\Psi_P'\Psi_T'}\right)\cdot\frac{\alpha_1 B^\theta\mathbf{e}_\theta-\alpha_2 B^\zeta\mathbf{e}_\zeta}{\alpha_1+\alpha_2}\right] \tag{4.6.44}$$

$$\tag{4.6.45}$$

and so

$$\mathbf{B}\cdot\nabla\left[\sigma-\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{-P'B_\zeta}{\mathcal{J}B^\theta B^2}-\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{-P'B_\theta}{\mathcal{J}B^\zeta B^2}\right)\right] = -\frac{P'}{\Psi_P'\Psi_T'}\left[\frac{\alpha_1}{\alpha_1+\alpha_2}B^\theta\mathbf{e}_\theta-\frac{\alpha_2}{\alpha_2}B^\zeta\mathbf{e}_\zeta\right]\cdot\nabla\mathcal{J} \tag{4.6.46}$$

$$\mathbf{B}\cdot\nabla\left[\sigma-P'\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{B_\theta}{\mathcal{J}B^\zeta B^2}-\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{B_\zeta}{\mathcal{J}B^\theta B^2}\right)\right] = -\frac{P'}{\Psi_P'\Psi_T'}\left[\frac{\alpha_1}{\alpha_1+\alpha_2}B^\theta\mathbf{e}_\theta-\frac{\alpha_2}{\alpha_2}B^\zeta\mathbf{e}_\zeta\right]\cdot\nabla\mathcal{J} \tag{4.6.47}$$

Here α_1 and α_2 are weight factors free except for $\alpha_1+\alpha_2\neq 0$.

If we take Hamada coordinates then $\mathcal{J}=1\Rightarrow\nabla\mathcal{J}=0$ and so we find

$$\mathbf{J}_\parallel = \sigma\mathbf{B} = P'\left[\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{B_\theta}{B^\zeta}-\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{B_\zeta}{B^\theta}\right]\frac{\mathbf{B}}{B^2}+\gamma(\mathcal{V})\mathbf{B} \tag{4.6.48}$$

where $\gamma(\mathcal{V})$ is the integrating factor. We can write this instead as

$$\mathbf{J}_\parallel = P'\frac{B_\theta}{B^\zeta}\frac{\mathbf{B}}{B^2}+\gamma_1(\mathcal{V})\mathbf{B} \tag{4.6.49}$$

$$= P'\frac{B_\theta}{B_\zeta}\frac{\mathbf{B}}{B^2}-P'\frac{\alpha_2}{\alpha_1+\alpha_2}\left[\frac{B_\theta}{B^\zeta B^2}+\frac{B_\zeta}{B^\theta B^2}\right]\mathbf{B}+\gamma(\mathcal{V})\mathbf{B} \tag{4.6.50}$$

so that it is simple to see

$$\gamma_1(\mathcal{V}) = \gamma(\mathcal{V})+\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{B_\theta B^\theta+B_\zeta B^\zeta}{B^\zeta B^\theta B^2}\mathbf{B} \tag{4.6.51}$$

$$= \gamma(\mathcal{V})+\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{1}{B^\zeta B^\theta}\mathbf{B} \tag{4.6.52}$$

Note that the toroidal current contained within a flux surface can be written as

$$I_T = \int_0^{\mathcal{V}}\mathrm{d}\mathcal{V}'\int_0^{2\pi}\mathrm{d}\theta\;\mathbf{J}\cdot\nabla\zeta \tag{4.6.53}$$

It is not actually clear that this implies

$$I_T'(\mathcal{V}) = J^\zeta(\mathcal{V}) \tag{4.6.54}$$

because it is not obvious that $I_T = I_T(\mathcal{V})$ or $\int\mathrm{d}\theta\;\mathbf{J}\cdot\nabla\zeta = J^\zeta(\mathcal{V})$ is a true statement (where the factor of 2π has clearly been suppressed).

A better way of seeing the truth of this statement is using that

$$I_T = \int_{S_T} \mathrm{d}S \; \frac{\nabla\zeta}{|\nabla\zeta|} \cdot \mathbf{B} = \int_{S_T} \mathrm{d}S \; \mathbf{n} \cdot \mathbf{B} \tag{4.6.55}$$

We can use that

$$\mu_0 I_{\text{enc}} = \int_S \mathrm{d}S \; \mathbf{n} \cdot \mathbf{B} \tag{4.6.56}$$

It is clear from this that $I_{\text{enc}} \equiv I_T$ for our situation and that I_{enc} can only depend on the flux surface you are on (it can have no θ or ζ dependence in our case). Thus $I_T = I_T(\mathcal{V})$ is proven. Now, we have

$$I_T(\mathcal{V}) = \int_0^{\mathcal{V}} \mathrm{d}\mathcal{V}' \int_0^{2\pi} \mathrm{d}\theta \; \mathbf{J} \cdot \nabla\zeta \tag{4.6.57}$$

So if we take $\frac{\partial}{\partial \mathcal{V}}$ we easily see

$$I_T'(\mathcal{V}) = \int_0^{2\pi} \mathrm{d}\theta \; \mathbf{J} \cdot \nabla\zeta \tag{4.6.58}$$

Unless $J^\zeta = \mathbf{J} \cdot \nabla\zeta$ is independent of θ the only thing we can take from this is that $J^\zeta = J^\zeta(\mathcal{V}, \theta)$. Luckily, we know that in Hamada coordinates we have that B^θ and B^ζ are flux coordinates and so using

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{4.6.59}$$

$$J^\theta B^\zeta - J^\zeta B^\theta = p' \tag{4.6.60}$$

We use that $\boldsymbol{\nabla} \cdot \mathbf{J} = 0 \Rightarrow \frac{\partial J^\theta}{\partial\theta} + \frac{\partial J^\zeta}{\partial\zeta} = 0$ and so

$$\frac{\partial}{\partial\theta}\left[\frac{p'}{B^\zeta} - J^\zeta \frac{B^\theta}{B^\zeta}\right] + \frac{\partial J^\zeta}{\partial\zeta} = 0 \tag{4.6.61}$$

$$\left[\frac{B^\theta}{B^\zeta}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\zeta}\right] J^\zeta = 0 \tag{4.6.62}$$

$$\left[B^\theta \frac{\partial}{\partial\theta} + B^\zeta \frac{\partial}{\partial\zeta}\right] J^\zeta = 0 \tag{4.6.63}$$

$$\mathbf{B} \cdot \nabla J^\zeta = 0 \tag{4.6.64}$$

so that $J^\zeta = J^\zeta(\mathcal{V})$ is indeed a flux surface function.

Chapter 5

MHD Instabilities in Heliotrons