As stated in **Jackson 5.28**,

Show that the mutual inductance of two circular coaxial loops in a homogeneous medium of permeability μ is

$$M_{12} = \mu\sqrt{ab}\left[\left(\frac{2}{k} - k\right)K(k) - \frac{2}{k}E(k)\right] \tag{1}$$

where

$$k^2 = \frac{4ab}{(a+b)^2 + d^2} \tag{2}$$

and a, b, are the radii of the loops, d is the distance between their centers, and K and E are the complete elliptic integrals of the First and Second Kind, respectively.

Find the limiting value when $d \ll a, b$ and $a \cong b$.

Solution:

1 Full Integral

We begin by looking at the integral

$$M_{12} = \frac{\mu}{4\pi}\oint_{\mathrm{C}_1}\oint_{\mathrm{C}_2}\frac{\mathrm{d}\boldsymbol{\ell}_1 \cdot \mathrm{d}\boldsymbol{\ell}_2}{|\mathbf{x}_1 - \mathbf{x}_2|} \tag{1.1}$$

where $\mathrm{d}\boldsymbol{\ell}_i$ points along the current direction (for convenience we will assume that they point counterclockwise as you go around the circles).

Let our setup look as in Figure 1 and 2.

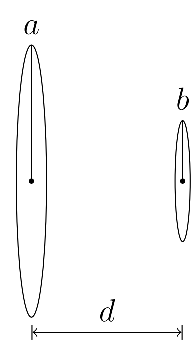

Figure 1: Setup of two coaxial rings of current from a side view.

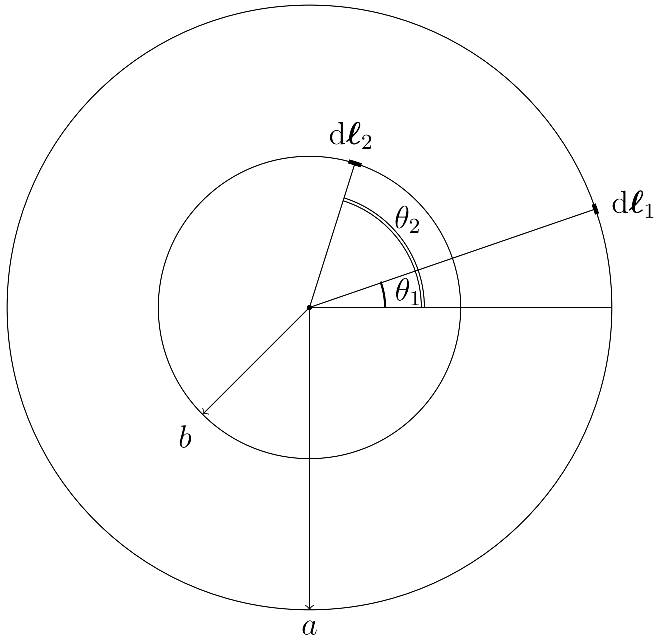

Figure 2: Setup of two coaxial rings of current when looking down their shared axis.

Now here are some useful trigonometric identies I will use.

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2 \tag{1.2}$$
$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \sin\theta_2 \cos\theta_1 \tag{1.3}$$
$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \tag{1.4}$$
$$\cos\theta = 1 - 2\sin^2(\theta/2) = 2\cos^2(\theta/2) - 1 \tag{1.5}$$
$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2). \tag{1.6}$$

Now we need to be careful, as I will do this in cylindrical coordinates, but because of the subtleties of the double integral, it is easiest to do do cartesian unit vectors with cylindrical components.

Hence

$$d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2 = a\, d\theta_1 \hat{\boldsymbol{\theta}}_1 \cdot b\, d\theta_2 \hat{\boldsymbol{\theta}}_2 \tag{1.7}$$
$$= (a\, d\theta_1)\left[-\sin\theta_1 \hat{\mathbf{x}} + \cos\theta_1 \hat{\mathbf{y}}\right] \cdot \{(b\, d\theta_2\left[-\sin\theta_2 \hat{\mathbf{x}} + \cos\theta_2 \hat{\mathbf{y}}\right])\} \tag{1.8}$$
$$= ab\, d\theta_1\, d\theta_2 \left[\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2\right] \tag{1.9}$$
$$d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2 \overset{(1.2)}{=} ab\, d\theta_1\, d\theta_2 \cos(\theta_1 - \theta_2) = ab\, d\theta_1\, d\theta_2 \cos(\theta_2 - \theta_1). \tag{1.10}$$

Also

$$|\mathbf{x}_1 - \mathbf{x}_2| = |[a\cos\theta_1 \hat{\mathbf{x}} + a\sin\theta_1 \hat{\mathbf{x}} + 0\hat{\mathbf{z}}] - [b\cos\theta_2 \hat{\mathbf{x}} + b\sin\theta_2 \hat{\mathbf{y}} + d\hat{\mathbf{z}}]| \tag{1.11}$$
$$= \sqrt{(a\cos\theta_1 - b\cos\theta_2)^2 + (a\sin\theta_1 - b\sin\theta_2)^2 + (0-d)^2} \tag{1.12}$$
$$= \sqrt{a^2 + b^2 + d^2 - 2ab\left[\cos\theta_2 \cos\theta_1 - \sin\theta_2 \sin\theta_1\right]} \tag{1.13}$$
$$|\mathbf{x}_1 - \mathbf{x}_2| \overset{(1.2)}{=} \sqrt{a^2 + b^2 + d^2 - 2ab\cos(\theta_1 - \theta_2)} = \sqrt{a^2 + b^2 + d^2 - 2ab\cos(\theta_2 - \theta_1)}. \tag{1.14}$$

So we may rewrite our integrals as

$$M_{12} = \frac{\mu}{4\pi} \oint_{C_1} \oint_{C_2} \frac{ab\, d\theta_1\, d\theta_2 \cos(\theta_2 - \theta_1)}{\sqrt{a^2 + b^2 + d^2 - 2ab\cos(\theta_2 - \theta_1)}} \tag{1.15}$$

$$\stackrel{(1.10)\ \&\ (1.14)}{=} \frac{ab\mu}{4\pi} \oint_{C_1} d\theta_1 \oint_{C_2} \frac{d\theta_2 \cos(\theta_2 - \theta_1)}{\sqrt{a^2 + b^2 + d^2 - 2ab\cos(\theta_2 - \theta_1)}}. \tag{1.16}$$

Now in the $d\theta_2$ integral, we have that θ_1 is a constant so we may define new angles, $\epsilon = (\theta_2 - \theta_1) + \pi$ and $\gamma = (\theta_2 - \theta_1)$ with $d\gamma = d\epsilon = d\theta_1$ and $2\phi = \epsilon$ with $2\, d\phi = d\gamma$ such that

$$\cos(\theta_2 - \theta_1) = \cos\gamma = \cos(\epsilon - \pi) \Rightarrow \cos\left(\frac{\epsilon - \pi}{2}\right) = \sin\left(\frac{\epsilon}{2}\right) = \sin\phi. \tag{1.17}$$

So now let's calculate an arbitrary integral of the correct form,

$$\oint_{C_1} d\gamma\, \frac{\cos\gamma}{\sqrt{\alpha - \beta\cos\gamma}} \stackrel{(1.5)}{=} \oint_{C_1} d\gamma\, \frac{2\cos^2\left(\frac{\gamma}{2}\right) - 1}{\sqrt{\alpha - \beta[2\cos^2\left(\frac{\gamma}{2}\right) - 1]}} \tag{1.18}$$

$$= \oint_{C_1} d\epsilon\, \frac{2\cos^2\left(\frac{\epsilon}{2} - \frac{\pi}{2}\right) - 1}{\sqrt{\alpha - \beta[2\cos^2\left(\frac{\epsilon}{2} - \frac{\pi}{2}\right) - 1]}} = \int_0^{2\pi} d\epsilon\, \frac{2\sin^2\left(\frac{\epsilon}{2}\right) - 1}{\sqrt{\alpha + \beta - 2\beta\sin^2\left(\frac{\epsilon}{2}\right)}} \tag{1.19}$$

$$= 2\int_0^{\pi} d\phi\, \frac{2\sin^2\phi - 1}{\sqrt{\alpha + \beta - 2\beta\sin^2\phi}} = \frac{2}{\sqrt{\alpha+\beta}}\left[2\underbrace{\int_0^{\pi} d\phi\, \frac{\sin^2\phi}{\sqrt{1 - \frac{2\beta}{\alpha+\beta}\sin^2\phi}}}_{\text{I2}} - \int_0^{\pi} d\phi\, \frac{1}{\sqrt{1 - \frac{2\beta}{\alpha+\beta}\sin^2\phi}}\right] \tag{1.20}$$

$$= \frac{2}{\sqrt{\alpha+\beta}}\left[2(\text{I2}) - F\left(\pi, \sqrt{\frac{2\beta}{\alpha+\beta}}\right)\right]. \tag{1.21}$$

Where we have $F(\varphi, k)$, the Lagrange Incomplete Elliptic Function of the First Kind, and $E(\varphi, k)$, the Lagrange Incomplete Elliptic Function of the Second Kind, defined as

$$F(\varphi, k) = \int_0^{\varphi} d\theta\, \frac{1}{\sqrt{1 - k^2\sin^2\theta}} \tag{1.22}$$

$$E(\varphi, k) = \int_0^{\varphi} d\theta\, \sqrt{1 - k^2\sin^2\theta}. \tag{1.23}$$

Now I2 can be calculated using the fact that

$$\frac{\kappa x}{\sqrt{1-\kappa x}} = \frac{\kappa x - 1}{\sqrt{1-\kappa x}} + \frac{1}{\sqrt{1-\kappa x}} = \frac{1}{\sqrt{1-\kappa x}} - \frac{1-\kappa x}{\sqrt{1-\kappa x}} = \frac{1}{\sqrt{1-\kappa x}} - \sqrt{1-\kappa x}. \tag{1.24}$$

So seeing that $\kappa \leftrightarrow \frac{2\beta}{\beta+\alpha}$ we can rewrite I2 as

$$\mathrm{I2} = \frac{\alpha+\beta}{2\beta}\left[\int_0^\pi \mathrm{d}\phi\, \frac{\frac{2\beta}{\alpha+\beta}\sin^2\phi}{\sqrt{1-\frac{2\beta}{\alpha+\beta}\sin^2\phi}}\right] \tag{1.25}$$

$$\mathrm{I2} \overset{(1.24)}{=} \frac{\alpha+\beta}{2\beta}\left[\int_0^\pi \mathrm{d}\phi\, \frac{1}{\sqrt{1-\frac{2\beta}{\alpha+\beta}\sin^2\phi}} - \int_0^\pi \mathrm{d}\phi\, \sqrt{1-\frac{2\beta}{\alpha+\beta}\sin^2\phi}\right] \tag{1.26}$$

$$= \frac{\alpha+\beta}{2\beta}\left[F\left(\pi, \sqrt{\frac{2\beta}{\alpha+\beta}}\right) - E\left(\pi, \sqrt{\frac{2\beta}{\alpha+\beta}}\right)\right]. \tag{1.27}$$

Reminding ourselves that

$$K(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \mathrm{d}\theta\, \frac{1}{\sqrt{1-k^2\sin^2\theta}} \tag{1.28}$$

$$E(k) = E\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \mathrm{d}\theta\, \sqrt{1-k^2\sin^2\theta}. \tag{1.29}$$

Now using these definitions, one may easily find using

$$\sin(-\theta \pm \pi) \overset{(1.3)}{=} \sin\theta \tag{1.30a}$$

$$\varphi = -\theta + \pi \;\Rightarrow\; \mathrm{d}\varphi = -\,\mathrm{d}\theta \tag{1.30b}$$

that

$$F(\pi, k) = \int_0^\pi \mathrm{d}\theta\, \frac{1}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\pi/2} \mathrm{d}\theta\, \frac{1}{\sqrt{1-k^2\sin^2\theta}} + \int_{\pi/2}^\pi \mathrm{d}\theta\, \frac{1}{\sqrt{1-k^2\sin^2\theta}} \tag{1.31}$$

$$\overset{(1.28)\,\&\,(1.30b)}{=} K(k) + \int_{+\pi/2}^0 -\mathrm{d}\varphi\, \frac{1}{\sqrt{1-k^2\sin^2(-\varphi-\pi)}} \overset{(1.30a)}{=} K(k) + \int_0^{\pi/2} \mathrm{d}\varphi\, \frac{1}{\sqrt{1-k^2\sin^2\varphi}} \tag{1.32}$$

$$\boxed{F(\pi, k) = 2K(k)} \tag{1.33}$$

$$E(\pi, k) = \int_0^\pi \mathrm{d}\theta\, \sqrt{1-k^2\sin^2\theta} = \int_0^{\pi/2} \mathrm{d}\theta\, \sqrt{1-k^2\sin^2\theta} + \int_{\pi/2}^\pi \mathrm{d}\theta\, \sqrt{1-k^2\sin^2\theta} \tag{1.34}$$

$$\overset{(1.29)\,\&\,(1.30b)}{=} E(k) + \int_{+\pi/2}^0 -\mathrm{d}\varphi\, \sqrt{1-k^2\sin^2(-\varphi-\pi)} \overset{(1.30a)}{=} E(k) + \int_0^{\pi/2} \mathrm{d}\varphi\, \sqrt{1-k^2\sin^2\varphi} \tag{1.35}$$

$$\boxed{E(\pi, k) = 2E(k).} \tag{1.36}$$

We write that

$$\mathrm{I2} = \frac{\alpha+\beta}{\beta}\left[K\left(\sqrt{\frac{2\beta}{\alpha+\beta}}\right) - E\left(\sqrt{\frac{2\beta}{\alpha+\beta}}\right)\right] \tag{1.37}$$

$$\Rightarrow$$

$$(1.21) = \frac{2}{\sqrt{\alpha+\beta}}\left\{2\frac{\alpha+\beta}{\beta}\left[K\left(\sqrt{\frac{2\beta}{\alpha+\beta}}\right) - E\left(\sqrt{\frac{2\beta}{\alpha+\beta}}\right)\right] - 2K\left(\sqrt{\frac{2\beta}{\alpha+\beta}}\right)\right\} \tag{1.38}$$

Now we have for our case that $\alpha = a^2 + b^2 + d^2$ and $\beta = 2ab$ so that

$$\begin{aligned} \frac{2\beta}{\alpha+\beta} &= \frac{4ab}{a^2+b^2+d^2+2ab} = \frac{4ab}{(a+b)^2+d^2} = k^2 \\ \sqrt{\alpha+\beta} &= \frac{\sqrt{2\beta}}{k} = \frac{2\sqrt{ab}}{k}. \end{aligned} \tag{1.39}$$

Therefore we have (noting there is no θ_1 dependence and so we just get a 2π from that integral)

$$M_{12} = \frac{ab\mu}{4\pi} \oint_{C_1} d\theta_1 \oint_{C_2} \frac{d\theta_2 \cos(\theta_2 - \theta_1)}{\sqrt{a^2+b^2+d^2-2ab\cos(\theta_2-\theta_1)}} \tag{1.40}$$

$$\overset{(1.38)}{=} \frac{ab\mu}{4\pi} \oint_{C_1} d\theta_1 \frac{2}{\sqrt{\alpha+\beta}} \left\{ 2\frac{\alpha+\beta}{\beta}\left[K(k) - E(k)\right] - 2K(k) \right\} \tag{1.41}$$

$$\overset{(1.39)}{=} \frac{ab\mu}{4\pi} 2\pi \frac{2k}{2\sqrt{ab}} \left\{ \frac{4}{k^2}\left[K(k) - E(k)\right] - 2K(k) \right\} \tag{1.42}$$

$$= \frac{\mu\sqrt{ab}}{2} \left[\left(\frac{4k}{k^2} - 2k \right) K(k) - 4kE(k) \right] \tag{1.43}$$

$$M_{12} = \mu\sqrt{ab} \left[\left(\frac{2}{k} - k \right) K(k) - \frac{2}{k} E(k) \right]. \tag{1.44}$$

Hence we get (1.44) as is required by Jackson (1).

2 Approximation of Integral

Now we need to do an approximation for when $d \ll a, b$, and $a \cong b$.

2.1 Method Due to Maxwell

This method is inspired by Maxwell's original approach in article 703 of *A Treatise on Electricity and Magnetism, Vol. 2* by Maxwell.

In this case, let $a > b$ with $a = b + c$ and we have that $\rho = \sqrt{(a-b)^2 + d^2} = \sqrt{c^2 + d^2}$ is the closest that the two circles come together and that this quantity ρ is small.

Now we need to find the magnetic field in coming from this ring and affecting the inner ring at b. We know from elementary magnetostatics (from the Biot-Savart Law) that the field from a tiny bit of ring $d\boldsymbol{\ell}$ gives

$$dB = \frac{d\boldsymbol{\ell} \times \mathbf{r}}{|\mathbf{r}|^3} = \frac{d\ell \sin\theta}{r^2} \tag{2.1}$$

where the $\sin\theta$ is coming from the cross product and so θ is the angle between the piece $d\boldsymbol{\ell}$ (for convenience let it point counterclockwise relative to the circle) and the position $\mathcal{P}$ we're calculating the field at.

Looking at Figure 3 helps to see this.

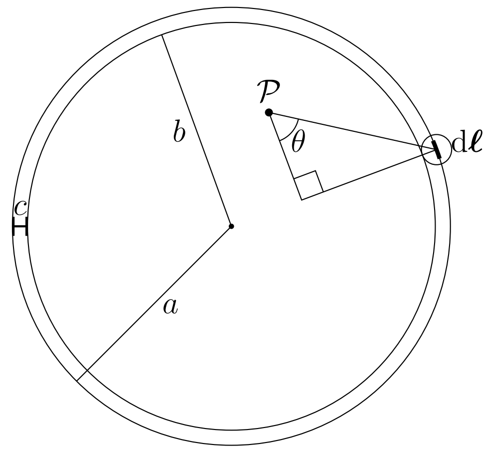

Figure 3: Picture for approximation. We exclude the small region around $\mathrm{d}\boldsymbol{\ell}$ of radius c.

Now we need to find how far to integrate out in r. We exclude a small circle around $\mathrm{d}\ell$ and so we will need to go from $c/\sin\theta$ to $2a\sin\theta$ as we will see below. So, suggestively, let's call these limits from r_1 to r_2, respectively.

To see this, we need only look at how far in Figure 3 we need to integrate over (which is the entire region of the circle with radius b or less).

To see how to get these limits let's rotate the circle so that differential element $\mathrm{d}\ell$ has its center lying on the x-axis. Then let's put the origin at the center of the circle centered around $\mathrm{d}\ell$ (i.e. the circle with diameter $2c$). So in this coordinate system the circle of radius b has its center left of the origin by $(b+c)$ units on the x-axis. Now the equation for the bigger circle of radius b in these coordinates is

$$(x+(b+c))^2+y^2=b^2 \tag{2.2}$$

$$\overbrace{x^2+y^2}^{r^2}+2(b+c)x+(b+c)^2=b^2 \tag{2.3}$$

$$r^2+2(b+c)x+c^2+2bc=0. \tag{2.4}$$

Now if we look at the angle that x is defined by, let's call it α, which is measured from the x-axis and its relationship to θ as we defined it. Remembering that θ is defined from the y-axis, we see that $\theta=\alpha-\pi/2$. Figure 4 should help see this relationship. Now as

$$\cos(\alpha)=\cos(\theta+\pi/2)\overset{(1.2)}{=}-\sin(\theta) \tag{2.5}$$

we have our equation in terms of θ.

Hence we need only to find where (2.4) is 0 to find the distances from the center of the differential element $\mathrm{d}\ell$ with a line connecting to the point $\mathcal{P}$ which lies in the circle of radius b. The distance is, of course, called r here. So in terms of θ we find

$$r^2+2(b+c)x+c^2+2bc=0 \tag{2.6}$$

$$r^2+2(b+c)r\cos\alpha+c^2+2bc=0 \tag{2.7}$$

$$r^2+2(b+c)\cos(\theta+\pi/2)r+c^2+2bc=0 \tag{2.8}$$

$$\overset{(2.5)}{\Rightarrow}$$
$$r^2 - 2(b+c)\sin\theta\ r + c^2 + 2bc = 0. \tag{2.9}$$

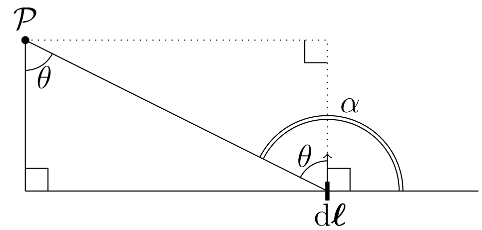

Figure 4: Relationship between α and θ.

The solution for r for (2.9) (calling each r_1 and r_2)

$$r_1 = (b+c)\sin\theta - \sqrt{(b+c)^2\sin^2\theta - (c^2+2bc)} \tag{2.10}$$
$$r_2 = (b+c)\sin\theta + \sqrt{(b+c)^2\sin^2\theta - (c^2+2bc)}, \tag{2.11}$$

which in the small c limit become

$$r_1 = (b+c)\sin\theta - (b+c)\sin\theta\sqrt{1 - \frac{c^2+2bc}{(b+c)^2\sin^2\theta}} \tag{2.12}$$
$$\approx (b+c)\sin\theta - (b+c)\sin\theta\left(1 - \frac{c(2b+c)}{2(b+c)^2\sin^2\theta}\right) = \frac{c(2b+c)}{2(b+c)\sin\theta} \overset{b\gg c}{\approx} \frac{2bc}{2b\sin\theta} \tag{2.13}$$
$$\boxed{r_1 \approx \frac{c}{\sin\theta}} \tag{2.14}$$
$$r_2 = (b+c)\sin\theta + (b+c)\sin\theta\sqrt{1 - \frac{c^2+2bc}{(b+c)^2\sin^2\theta}} \tag{2.15}$$
$$\approx (b+c)\sin\theta + (b+c)\sin\theta\left(1 - \frac{c(2b+c)}{2(b+c)^2\sin^2\theta}\right) = 2(b+c)\sin\theta - \frac{c}{\sin\theta} \overset{b\gg c}{\approx} 2b\sin\theta \tag{2.16}$$
$$\boxed{r_2 = 2b\sin\theta} \tag{2.17}$$

which yield the two limits promised. Now we see that we need to have θ run from $\theta_1 \approx 0$ to $\theta_2 \approx \pi$, where we can use the approximations as long as we are careful. In fact, it would be from θ_1 to $\pi - \theta_1$ by symmetry.

$$M_{bc} = \frac{\mu}{4\pi}\int_0^\pi \int_{c/\sin\theta}^{2b\sin\theta} \frac{\mathrm{d}\ell\sin\theta}{r^2} r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\mu}{4\pi}\mathrm{d}\ell\int_0^\pi \sin\theta\ln\left(\frac{2b\sin^2\theta}{c}\right)\mathrm{d}r \tag{2.18}$$

To see how to do this integral look to the at section 3.

So then

$$\frac{\mathrm{d}\ell\mu}{4\pi}\int_0^\pi \sin\theta\ln\left(\frac{2b\sin^2\theta}{c}\right)\mathrm{d}r \overset{(3.8)}{=} \frac{\mathrm{d}\ell\mu}{4\pi}2\left(\ln\left(\frac{4(2b)}{c}\right) - 2\right) = \frac{\mathrm{d}\ell}{2\pi}\left(\ln\left(\frac{8b}{c}\right) - 2\right). \tag{2.19}$$

Hence, integrating over $\mathrm{d}\ell$ with $\int_0^{2\pi} R\,\mathrm{d}\ell = 2\pi R$ where R is the radius of the circle that $\mathrm{d}\ell$ goes around which is for $a \cong b$, just about b and we get for our case

$$M_{bc} = \mu b\left[\ln\left(\frac{8b}{c}\right) - 2\right]. \tag{2.20}$$

Now for the region between b and a we see that we can treat this as two straight wires because if $b \cong a$ then there is a small radius of curvature and if we subtract out the inductance from the inner region which we have just calculated, then it is almost the exact same as two wires. So

$$M_{ba} - M_{bc} = \mu b\left[\ln\left(\frac{c}{\rho}\right)\right] \tag{2.21}$$

and so combining (2.20) and (2.21) we get

$$M_{ba} = M_{12} = \mu b\left(\ln\left(\frac{8b}{\rho}\right) - 2\right). \tag{2.22}$$

(note that $\rho = \sqrt{c^2 + d^2} \approx d$, as $c \ll 1$)

2.2 Asymptotic Expression for Elliptic Integrals

Alternatively, we may use the approximation that for $k \to 1$ with $k' = \sqrt{1-k^2}$ with $\psi(m)$ being the digamma function as is stated in the DLMF 19.12 at http://dlmf.nist.gov/19.12, we find

$$k^2 = \frac{4ab}{(a+b)^2 + d^2} = 1 - k'^2 \tag{2.23}$$

$$\Rightarrow$$

$$k'^2 = \left(1 - \frac{4ab}{(a+b)^2+d^2}\right) = \left(\frac{(a+b)^2 + d^2 - 4ab}{(a+b)^2+d^2}\right) = \frac{a^2+b^2+2ab-4ab+d^2}{(a+b)^2+d^2} \tag{2.24}$$

$$= \frac{a^2+b^2-2ab+d^2}{(a+b)^2+d^2} = \frac{\overbrace{(a-b)^2}^{c^2}+d^2}{(a+b)^2+d^2} = \frac{\overbrace{c^2+d^2}^{\rho^2}}{(a+b)^2+d^2} = \frac{\rho^2}{(a+b)^2+d^2} \tag{2.25}$$

$$k'^2 \overset{(a+b)^2 \gg d^2}{\approx} \frac{\rho^2}{(a+b)^2} \overset{a\cong b}{\approx} \frac{\rho^2}{4b^2} \tag{2.26}$$

$$k' \approx \frac{\rho}{2b}. \tag{2.27}$$

To lowest order we find that

$$K(k) \approx \ln\left(\frac{1}{k'}\right) + \underbrace{d(0)}_{\psi(1)-\psi(1/2)=\ln 4} \tag{2.28}$$

$$E(k) \approx 1 \tag{2.29}$$

and so (1.44) becomes

$$M_{12} \approx \mu \underbrace{\sqrt{ab}}_{\substack{\approx b \\ \approx a}} [(2-1)K(k) - 2E(k)] \approx \mu b \left[\ln\left(\frac{1}{k'}\right) + \ln 4 - 2 \right] \tag{2.30}$$

$$\approx \mu b \left(\ln\left(\frac{4}{k'}\right) - 2 \right) \overset{(2.27)}{\approx} \mu b \left(\ln\left(\frac{4}{\frac{\rho}{2b}}\right) - 2 \right) \tag{2.31}$$

$$M_{12} \approx \mu b \left(\ln\left(\frac{8b}{\rho}\right) - 2 \right) \tag{2.32}$$

(note that $\rho = \sqrt{c^2 + d^2} \approx d$, as $c \ll 1$) as expected.

3 Log Integral

The integral of interest is

$$\int_0^\pi \sin\theta \ln\left(B \sin^2\theta\right) \,\mathrm{d}\theta \tag{3.1}$$

so using $u = \cos\theta \Rightarrow \mathrm{d}u = -\sin\theta \,\mathrm{d}\theta$

$$\int_{-1}^{1} \ln\left(B(1-u^2)\right) \mathrm{d}u = \int_{-1}^{1} \left[\ln B + \ln(1-u^2)\right] \,\mathrm{d}u \tag{3.2}$$

$$= 2\ln B + \int_{-1}^{1} \ln\left((1+u)(1-u)\right) \mathrm{d}u = 2\ln B + \int_{-1}^{1} \ln(1+u)\,\mathrm{d}u + \int_{-1}^{1} \ln(1-u)\,\mathrm{d}u \tag{3.3}$$

now using $x = 1 + u \Rightarrow \mathrm{d}x = \mathrm{d}u$ and $z = 1 - u \Rightarrow \mathrm{d}z = -\mathrm{d}u$ we find

$$= 2\ln B + \int_0^2 \ln x \,\mathrm{d}x - \int_2^0 \ln z \,\mathrm{d}z = 2\ln B + \int_0^2 \ln x \,\mathrm{d}x + \int_0^2 \ln z \,\mathrm{d}z \tag{3.4}$$

$$= 2\left(\ln B + \int_0^2 \ln x \,\mathrm{d}x \right) \tag{3.5}$$

Now integrating by parts with $u = \ln x,\ \mathrm{d}u = \frac{\mathrm{d}x}{x}$ and $\mathrm{d}v = \mathrm{d}x$ so $v = x$ we find

$$\int \ln x \,\mathrm{d}x = x\ln(x) - \int x \frac{\mathrm{d}x}{x} = x(\ln x - 1) \tag{3.6}$$

Hence

$$= 2\left(\ln B + [x(\ln x - 1)]_0^2\right) = 2\left(\ln B + 2(\ln 2 - 1)\right) \tag{3.7}$$

$$= 2\left(\ln B + \ln 4 - 2\right) = 2\left(\ln(4B) - 2\right). \tag{3.8}$$

4 Energy Method

An alternative method is to use that

$$\Delta W = M_{12} I_1 I_2 = 2\left(\frac{1}{2} \int \mathbf{J}_1 \cdot \mathbf{A}_2 \,\mathrm{d}^3 x \right) \tag{4.1}$$

with the factor of 2 coming from the two contributions to ΔW from the rings.

We then can note that

$$\mathbf{J}_1 = \frac{I_1}{r}\delta(\theta - \arctan\left[\frac{b}{d}\right])\delta(r - \sqrt{d^2+b^2})\hat{\boldsymbol{\varphi}} \tag{4.2}$$

$$A_{2\varphi} = \frac{I_2 a\mu}{\pi\sqrt{a^2+r^2+2ar\sin\theta}}\left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right] \tag{4.3}$$

from Jackson (5.33) and Jackson (5.37) with

$$k^2 = \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta} \tag{4.4}$$

And so

$$\begin{aligned}
\Delta W &= \mu\int_0^{\pi} \mathrm{d}\theta\ \sin\theta\int_0^{2\pi}\mathrm{d}\varphi\ \int_0^{\infty} r^{\not 2}\,\mathrm{d}r\frac{I_1}{\not r}\delta(\theta-\arctan\left[\frac{b}{d}\right])\delta(r-\sqrt{b^2+d^2})A_{2\varphi} && (4.5)\\
&= 2I_1\pi\mu\int_0^{\pi}\mathrm{d}\theta\ \sin\theta\int_0^{\infty}\mathrm{d}r\ r\delta(\theta-\arctan\left[\frac{b}{d}\right])\frac{\delta(r-\sqrt{b^2+d^2})}{k^2}\frac{I_2a[(2-k^2)K(k)-2E(k)]}{\pi\sqrt{a^2+r^2+2ar\sin\theta}} && (4.6)\\
&= \frac{2I_1I_2\not\pi a\mu\sqrt{b^2+d^2}[(2-k^2)K(k)-2E(k)]\sin\left[\arctan\left(\frac{b}{d}\right)\right]}{\not\pi\sqrt{a^2+b^2+d^2+2a\sqrt{b^2+d^2}\sin\left[\arctan\left(\frac{b}{d}\right)\right]}} && (4.7)
\end{aligned}$$

We then use

$$k^2 = \frac{4a\sqrt{b^2+d^2}\sin\left[\arctan\left(\frac{b}{d}\right)\right]}{a^2+b^2+d^2+2a\sqrt{b^2+d^2}\sin\left[\arctan\left(\frac{b}{d}\right)\right]} \tag{4.8}$$

$$\sin\left[\arctan\left(\frac{b}{d}\right)\right] = \frac{b}{\sqrt{b^2+d^2}} \tag{4.9}$$

$$k^2 = \frac{4ab}{a^2+b^2+d^2+2ab} = \frac{4ab}{(a+b)^2+d^2} \tag{4.10}$$

Therefore,

$$\begin{aligned}
M_{12} &= \frac{\mu ab[(2-k^2)K(k)-2E(k)]}{k^2\sqrt{a^2+2ab+b^2+d^2}} = \frac{2\mu\frac{k}{2}\sqrt{ab}[(2-k^2)K(k)-2E(k)]}{k^2} && (4.11)\\
&= \mu\sqrt{ab}\left[\left(\frac{2}{k}-k\right)K(k)-\frac{2}{k}E(k)\right] && (4.12)
\end{aligned}$$

as desired.