Collisional Transport in Magnetized Plasmas Notes

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Contents

Chapter 1

Introduction

We get the difference in momentum by finding the impulse **I**, which is defined as the change in momentum $\mathbf{I} = \Delta p \ I = m\Delta v = \int_a^b \mathbf{F}(t)\,\mathrm{d}t$. we get the y component by just looking at the Coulomb force,

$$m\Delta v_{\,y} = \int_{-\infty}^{\infty} \mathrm{d}t \ F_y(t) = \int_{-\infty}^{\infty} \mathrm{d}t \ F_C(t)\sin[\theta(t)] \tag{1.1}$$

where θ is the angle given by 1.1.

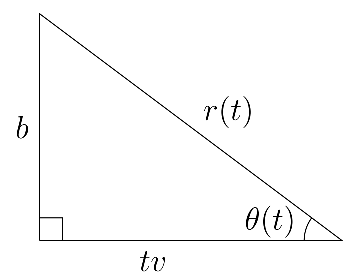

Figure 1.1: Schematic yielding relations for θ due to the right triangle formed. This assumes little deviation so that b is approximately constant, and that the particle continues at a constant velocity v for it's entire time. (Also that the test and source particle are vertically aligned at $t = 0$).

So we find with $r(t) = \sqrt{b^2 + t^2v^2}$ that

$$m\Delta v_{\,y} = \int_{-\infty}^{\infty} \mathrm{d}t \ \frac{e_i e}{4\pi\epsilon_0 r(t)^2}\frac{b}{r(t)} = \int_{-\infty}^{\infty} \mathrm{d}t \ \frac{be_i e}{4\pi\epsilon_0\left[b^2 + t^2v^2\right]^{3/2}} \tag{1.2}$$

Let's now calculate this explicitly. First let's choose $u = tv/b$ with $\mathrm{d}u = \frac{v}{b}\,\mathrm{d}t$ so that

$$m\Delta v_{\,y} = \frac{be_i e}{4\pi\epsilon_0}\int_{-\infty}^{\infty} \mathrm{d}u \ \frac{b}{vb^3}\frac{1}{\left[1+u^2\right]^{3/2}} = \frac{e_i e}{4\pi\epsilon_0 vb}\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{\left[1+u^2\right]^{3/2}} \tag{1.3}$$

Use $u = \tan\theta$ so $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$ and

$$\begin{aligned} m\Delta v_{\,y} &= \frac{e_i e}{4\pi\epsilon vb}\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta \ \frac{\sec^2\theta}{\left[\sec^2\theta\right]^{3/2}} = \frac{e_i e}{4\pi\epsilon vb}\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta \ \cos\theta = \frac{e_i e}{4\pi\epsilon vb}\sin(\theta)\big|_{-\pi/2}^{\pi/2} \\ &= 2\frac{e_i e}{4\pi\epsilon vb}\sin\left(\frac{\pi}{2}\right) = \frac{e_i e}{2\pi\epsilon vb} \end{aligned} \tag{1.4}$$

just as advertised.

1.4 Random Walk Estimate of Bohm-Diffusion

From the Lorentz equation for an electron

$$m_e \dot{\mathbf{v}} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.5}$$

We take $\cdot \times \mathbf{B}$ and find (using $\mathbf{v} \times \mathbf{B} = \mathbf{v}_\perp \times \mathbf{B}$, with $\parallel$ referenced to $\mathbf{B}$)

$$m_e \dot{\mathbf{v}} \times \mathbf{B} = -e\mathbf{E} \times \mathbf{B} + e\mathbf{B} \times (\mathbf{v}_\perp \times \mathbf{B}) = -e\mathbf{E} \times \mathbf{B} + e\left[\mathbf{v}_\perp B^2 - \cancel{\mathbf{B}\mathbf{v}_\perp \cdot \mathbf{B}}\right] \tag{1.6}$$

$$m_e \dot{\mathbf{v}} \times \mathbf{B} = -e\mathbf{E} \times \mathbf{B} + e\mathbf{v}_\perp B^2 \tag{1.7}$$

$$\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{m_e \dot{\mathbf{v}} \times \mathbf{B}}{eB^2} \tag{1.8}$$

as indicated. Note that if we hadn't switched $\mathbf{v} \to \mathbf{v}_\perp$ we would have had

$$\mathbf{B} \times (\mathbf{v} \times \mathbf{B}) = \mathbf{v}B^2 - \mathbf{B}Bv_\parallel = \mathbf{v}_\perp B^2 + \mathbf{B}Bv_\parallel - \mathbf{B}Bv_\parallel = \mathbf{v}_\perp B^2 \tag{1.9}$$

which is a bit trickier to see, as we need to use $\mathbf{v} = \mathbf{B}v_\parallel / B + \mathbf{v}_\perp$.

1.5 Exercises Chapter 1

1.5.1 Dominant Diffusivities

Above which temperature does the Bohm diffusivity exceed the (a) classical (b) neoclassical heat diffusivity across a magnetic field of 1 T? Assume $n = 1 \times 10^{20}\,\mathrm{m}^{-3}$, $\epsilon = 0.1$, and $B/B_p = 10$.

Solution:

The classical diffusivity is given by

$$\xi_i^{\mathrm{C}} = \frac{\rho_i^2}{2\tau_{ii}} \tag{1.10}$$

$$\tau_{ii} = \frac{12\pi^{3/2}}{\sqrt{2}} \frac{\sqrt{m_i} T_i^{3/2} \epsilon_0^2}{n_i Z^4 e^4 \ln \Lambda} \tag{1.11}$$

The neoclassical diffusivity is given by

$$\xi_i^{\mathrm{NC}} = \sqrt{2\epsilon} \frac{\rho_{pi}^2}{\tau_{ii}} = \sqrt{2\epsilon} \left(\frac{B}{B_p}\right)^2 \frac{\rho_i^2}{\tau_{ii}} = \sqrt{\frac{\epsilon}{2}} \left(\frac{B}{B_p}\right)^2 \xi_i^{\mathrm{C}} \tag{1.12}$$

while for Bohm diffusivity we simply have

$$D_{\mathrm{Bohm}} = \frac{T}{16eB} \tag{1.13}$$

So if we take the ratio of classical to Bohm, and set it less than 1, we can solve for the T where D_{Bohm} begins dominating,

$$\frac{\xi_i^{\text{C}}}{D_{\text{Bohm}}} = \frac{16eB\rho_i^2}{2T\tau_{ii}} = \frac{8eB\frac{2Tm_i}{e^2B^2}}{12\pi^{3/2}\sqrt{m_i}T^{5/2}\epsilon_0^2}\sqrt{2}n_iZ^4e^4\ln\Lambda = \frac{4\sqrt{2m_i}n_iZ^4e^3\ln\Lambda}{3\pi^{3/2}BT^{3/2}\epsilon_0^2} < 1 \tag{1.14}$$

$$T > \left(\frac{4\sqrt{2m_i}n_iZ^4e^3\ln\Lambda}{3\pi^{3/2}B\epsilon_0^2}\right)^{2/3} \tag{1.15}$$

We note in passing that for $T >$ than the right hand side we have D_{Bohm} dominating, as expected.

Now, the T is actualy going to be k_BT, and the right has units of Joules when we put everything in in SI. Plugging in values

$$T = \left(\frac{4\sqrt{2(1.67\times10^{-27}\,\text{kg})}(1\times10^{20}\,\text{m}^{-3})(1.60\times10^{-19}\,\text{C})^3(15)}{3\pi^{3/2}(1\,\text{T})(8.854\times10^{-12}\,\text{F/m})^2}\right)^{2/3} \approx \left(1.08\times10^{-27}\,\text{J}^{3/2}\right)^{2/3} \tag{1.16}$$

$$\approx 1.06\times10^{-18}\,\text{J} \approx 6.63\,\text{eV} \tag{1.17}$$

Now we use

$$\begin{aligned}\frac{\xi_i^{\text{NC}}}{D_{\text{Bohm}}} &= \sqrt{\frac{\epsilon}{2}}\left(\frac{B}{B_p}\right)^2\frac{16eB\rho_i^2}{2T\tau_{ii}} = \sqrt{\frac{\epsilon}{2}}\left(\frac{B}{B_p}\right)^2\frac{8eB\frac{2Tm_i}{e^2B^2}}{12\pi^{3/2}\sqrt{m_i}T^{5/2}\epsilon_0^2}\sqrt{2}n_iZ^4e^4\ln\Lambda \\ &= \sqrt{\frac{\epsilon}{2}}\left(\frac{B}{B_p}\right)^2\frac{4\sqrt{2m_i}n_iZ^4e^3\ln\Lambda}{3\pi^{3/2}BT^{3/2}\epsilon_0^2} < 1\end{aligned} \tag{1.18}$$

$$\begin{aligned}T &> \left(\sqrt{\frac{\epsilon}{2}}\left(\frac{B}{B_p}\right)^2\right)^{2/3}\left(\frac{4\sqrt{2m_i}n_iZ^4e^3\ln\Lambda}{3\pi^{3/2}B\epsilon_0^2}\right)^{2/3} \\ &\approx (22.36)^{2/3}(1.06\times10^{-18}\,\text{J}) \approx (7.9)(1.06\times10^{-18}\,\text{J}) \approx 8.374\times10^{-18}\,\text{J} \approx 52.4\,\text{eV}\end{aligned} \tag{1.19}$$

So we find $D_{\text{Bohm}} > \xi_i^{\text{C}}$ for $T > 6\,\text{eV}$ and $D_{\text{Bohm}} > \xi_i^{\text{NC}}$ for $T > 50\,\text{eV}$, as given in the book.

1.5.2 Derive the Rutherford Formula

Referring to Figures 1.1 and 1.5 in the book, derive the Rutherford formula

$$\tan\frac{\alpha}{2} = \frac{Ze^2}{4\pi\epsilon m_e v^2 b} \tag{1.20}$$

relating the deflection angle α in an electron-ion collision to the impact parameter b. Hint: Note that angular momentum is conserved because the electrostatic Coulomb interaction is spherically symmetric, so that $r^2\dot{\theta} = \text{const} = bv$.

Solution:

We see that for the electron very far away, that θ_0 is the angle between the dashed line and u, so that the component of v in the u direction will be $v_u = v\hat{\mathbf{x}}\cdot(\cos(\pi-\theta_0)\hat{\mathbf{x}}+\sin(\pi-\theta_0)\hat{\mathbf{y}}) = -v\cos\theta_0$. Because u is symmetric, we now that when the electron is long past the ion it will have precisely the opposite velocity. Because this is a small angle, we will say that the component along u is almost always $\cos\theta$. Thus, the change in momentum is

$$\begin{aligned}\Delta p &= 2m_e v\cos\theta_0 = \int_{-\infty}^{\infty} \mathrm{d}t\ F(t)\cos\theta = \int_{-\theta_0}^{\theta_0} \mathrm{d}\theta\ \frac{F(r)\cos\theta}{\dot{\theta}} \\ &= \int_{-\theta_0}^{\theta_0} \mathrm{d}\theta\ \frac{Ze^2\cos\theta}{4\pi\epsilon_0 r^2\frac{bv}{r^2}} = \frac{Ze^2}{4\pi\epsilon_0 bv}\int_{-\theta_0}^{\theta_0}\cos\theta \mathrm{d}\theta\ = \frac{Ze^2\sin\theta_0}{2\pi\epsilon_0 bv}\end{aligned} \tag{1.21}$$

Now we use that $\theta_0 = \alpha + \beta$ where 2β is the angle that u bisects when you trace the original particle path's asymptote and the reverse particle path's asymptote. We then see that $2\beta + \alpha = \pi$ and so we get $\alpha = 2\theta_0 - \pi$. If this is so, then

$$\tan{(\alpha/2)} = \tan(\theta_0 - \pi/2) = -\cot\theta_0 \tag{1.22}$$

Now, we haven't been careful about the signs of things, and if one were, we'd eliminate the $-$ sign, and so we find

$$2m_e v\cos\theta_0 = \frac{Ze^2\sin\theta_0}{2\pi\epsilon_0 bv} \tag{1.23}$$

$$\cot\theta_0 = \frac{Ze^2}{2\pi\epsilon_0 bv} \tag{1.24}$$

$$\tan\frac{\alpha}{2} = \frac{Ze^2}{2\pi\epsilon_0 bv} \tag{1.25}$$

as desired.

Chapter 2

Kinetic and Fluid Descriptions of a Plasma

2.2 Fluid Equations

Note that we have my less desirable definition for divergence of a tensor,

$$\boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\Pi}} = \partial_k \Pi_{jk} \tag{2.1}$$

so that in some cases I will have to be careful of the order I do things in integrals.

2.3 Exercises Chapter 2

2.3.1 Momentum and Energy Equations

Derive the momentum equation (HS-2.16) and the energy equation (HS-2.17) from the conservation laws (HS-2.9)-(HS-2.11).

$$m_a n_a \frac{\mathrm{d}\mathbf{V}_a}{\mathrm{d}t}\bigg|_a = -\nabla p_a - \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\pi}}_a + e_a n_a(\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \mathbf{R}_a \tag{HS-2.16}$$

$$\frac{3}{2} n_a \frac{\mathrm{d}T_a}{\mathrm{d}t}\bigg|_a + p_a \boldsymbol{\nabla} \cdot \mathbf{V}_a = -\boldsymbol{\nabla} \cdot \mathbf{q}_a + \overleftrightarrow{\boldsymbol{\pi}}_a : \nabla \mathbf{V}_a + Q_a \tag{HS-2.17}$$

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{V}) = 0 \tag{HS-2.9}$$

$$\frac{\partial}{\partial t}(mn\mathbf{V}) + \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\Pi}} = ne(\mathbf{E} + \mathbf{V} \times \mathbf{B}) + \int \mathrm{d}^3 v \; m\mathbf{v}\, C(f) \tag{HS-2.10}$$

$$\frac{\partial}{\partial t}\left(\frac{3nT}{2} + \frac{mnV^2}{2}\right) + \boldsymbol{\nabla} \cdot \mathbf{Q} = en\mathbf{E} \cdot \mathbf{V} + \int \mathrm{d}^3 v \; \frac{mv^2}{2} C(f) \tag{HS-2.11}$$

$$\overleftrightarrow{\boldsymbol{\Pi}} = p\mathbb{1} + \overleftrightarrow{\boldsymbol{\pi}} + mn\mathbf{V}\mathbf{V} \tag{2.2}$$

$$\mathbf{Q} = \mathbf{q} + \frac{5p\mathbf{V}}{2} + \overleftrightarrow{\boldsymbol{\pi}} \cdot \mathbf{V} + \frac{mnV^2}{2}\mathbf{V} \tag{2.3}$$

Solution:

First let's get (HS-2.16). We first use from (HS-2.13) that

$$\nabla\cdot\overleftrightarrow{\boldsymbol{\pi}} = \nabla p + \nabla\cdot\overleftrightarrow{\boldsymbol{\pi}} + \nabla\cdot(mn\mathbf{V}\mathbf{V}) \tag{2.4}$$

$$\nabla\cdot(mn\mathbf{V}\mathbf{V}) = mn\mathbf{V}\nabla\cdot\mathbf{V} + mn\mathbf{V}\cdot\nabla\mathbf{V} + m\mathbf{V}\mathbf{V}\cdot\nabla n = m\mathbf{V}\left(n\nabla\cdot\mathbf{V} + \nabla n\cdot\mathbf{V}\right) + mn\mathbf{V}\cdot\nabla\mathbf{V} \tag{2.5}$$

or, more clearly perhaps,

$$\partial_k(nV_jV_k) = nV_j\partial_kV_k + nV_k\partial_kV_j + V_jV_k\partial_kn \tag{2.6}$$

note number density conservation can be written as

$$\frac{\partial n}{\partial t} + \nabla n\cdot\mathbf{V} + n\nabla\cdot\mathbf{V} = 0 \tag{2.7}$$

So then, we find

$$\nabla\cdot(mn\mathbf{V}\mathbf{V}) = m\mathbf{V}\frac{\partial n}{\partial t} - mn\mathbf{V}\cdot\nabla\mathbf{V} \tag{2.8}$$

We can then plug this into (HS-2.10) using that the integral is defined to be $\mathbf{R}$, and see

$$\frac{\partial}{\partial t}(mn\mathbf{V}) + \nabla p + \nabla\cdot\overleftrightarrow{\boldsymbol{\pi}} + m\mathbf{V}\frac{\partial n}{\partial t} - mn\mathbf{V}\cdot\nabla\mathbf{V} = ne(\mathbf{E}+\mathbf{V}\times\mathbf{B}) + \mathbf{R} \tag{2.9}$$

$$mn\frac{\partial\mathbf{V}}{\partial t} + \cancel{m\mathbf{V}\frac{\partial n}{\partial t}} + \nabla p + \nabla\cdot\overleftrightarrow{\boldsymbol{\pi}} - \cancel{m\mathbf{V}\frac{\partial n}{\partial t}} + mn\mathbf{V}\cdot\nabla\mathbf{V} = ne(\mathbf{E}+\mathbf{V}\times\mathbf{B}) + \mathbf{R} \tag{2.10}$$

$$mn\frac{\partial\mathbf{V}}{\partial t} + mn\mathbf{V}\cdot\nabla\mathbf{V} = -\nabla p - \nabla\cdot\overleftrightarrow{\boldsymbol{\pi}} + ne(\mathbf{E}+\mathbf{V}\times\mathbf{B}) + \mathbf{R} \tag{2.11}$$

Now, we use that

$$\left.\frac{\mathrm{d}q}{\mathrm{d}t}\right|_a = \frac{\partial q}{\partial t} + \mathbf{V}\cdot\nabla q \tag{2.12}$$

and reference all to this frame, so that

$$mn\left(\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = -\nabla p - \nabla\cdot\overleftrightarrow{\boldsymbol{\pi}} + ne(\mathbf{E}+\mathbf{V}\times\mathbf{B}) + \mathbf{R} \tag{2.13}$$

$$mn\left.\frac{\mathrm{d}\mathbf{V}_a}{\mathrm{d}t}\right|_a = -\nabla p_a - \nabla\cdot\overleftrightarrow{\boldsymbol{\pi}}_a + ne(\mathbf{E}+\mathbf{V}_a\times\mathbf{B}) + \mathbf{R}_a \tag{2.14}$$

That was the simple one. Now let's get the energy equation.

First let's expand $\nabla\cdot\mathbf{Q}$ out

$$\nabla\cdot\mathbf{Q} = \nabla\cdot\mathbf{q} + \frac{5}{2}\nabla\cdot(nT\mathbf{V}) + \nabla\cdot(\overleftrightarrow{\boldsymbol{\pi}}\cdot\mathbf{V}) + \nabla\cdot(\frac{mnV^2}{2}\mathbf{V}) \tag{2.15}$$

Let's handle this term by term (we use that $\overleftrightarrow{\boldsymbol{\pi}} = \overleftrightarrow{\boldsymbol{\pi}}^{\intercal}$, or $\pi_{jk} = \pi_{kj}$.

$$\boldsymbol{\nabla}\cdot(nT\mathbf{V}) = T\,\nabla n\cdot\mathbf{V} + n\,\nabla T\cdot\mathbf{V} + nT\boldsymbol{\nabla}\cdot\mathbf{V} \tag{2.16}$$
$$\boldsymbol{\nabla}\cdot(\overleftrightarrow{\boldsymbol{\pi}}\cdot\mathbf{V}) = \mathbf{V}\cdot\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}^{\intercal} + \overleftrightarrow{\boldsymbol{\pi}}^{\intercal}:\nabla\mathbf{V} = \mathbf{V}\cdot\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}} + \overleftrightarrow{\boldsymbol{\pi}}:\nabla\mathbf{V} \tag{2.17}$$
$$\boldsymbol{\nabla}\cdot(nV^2\mathbf{V}) = 2nV\,\nabla V\cdot\mathbf{V} + V^2\,\nabla n\cdot\mathbf{V} + nV^2\boldsymbol{\nabla}\cdot\mathbf{V} \tag{2.18}$$

Wher the tensor relationship is easily seen from

$$\partial_j(\pi_{jk}V_k) = V_k\partial_j\pi_{jk} + \pi_{jk}\partial_jV_k = V_k\partial_j\pi_{kj} + \pi_{kj}\partial_jV_k \tag{2.19}$$

We can also break up the time derivative terms and the defintion for the collision integral,

$$\frac{\partial(nT)}{\partial t} = n\frac{\partial T}{\partial t} + T\frac{\partial n}{\partial t}\frac{\partial(nV^2)}{\partial t} \qquad = 2nV\frac{\partial V}{\partial t} + V^2\frac{\partial n}{\partial t}\int \mathrm{d}^3v\,\frac{mv^2}{2}\,C(f) = Q + \mathbf{R}\cdot\mathbf{V} \tag{2.20}$$

So that altogether, the energy conservation law states

$$\begin{aligned}\frac{3n}{2}\frac{\partial T}{\partial t} &+ \frac{3T}{2}\frac{\partial n}{\partial t} + mnV\frac{\partial V}{\partial t} + \frac{mV^2}{2}\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{q}\\ &+ \frac{5T}{2}\,\nabla n\cdot\mathbf{V} + \frac{5n}{2}\,\nabla T\cdot\mathbf{V} + \frac{5nT}{2}\boldsymbol{\nabla}\cdot\mathbf{V} + \mathbf{V}\cdot\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}} + \overleftrightarrow{\boldsymbol{\pi}}:\nabla\mathbf{V}\\ &+ mnV\,\nabla V\cdot\mathbf{V} + \frac{mV^2}{2}\,\nabla n\cdot\mathbf{V} + \frac{mnV^2}{2}\boldsymbol{\nabla}\cdot\mathbf{V} = en\mathbf{E}\cdot\mathbf{V} + Q + \mathbf{R}\cdot\mathbf{V}\end{aligned} \tag{2.21}$$

We can change some quantities into full derivatives right now, (I will suppress the a subscript for now)

$$\begin{aligned}\frac{3n}{2}\frac{\mathrm{d}T}{\mathrm{d}t} &+ \frac{3T}{2}\frac{\mathrm{d}n}{\mathrm{d}t} + mnV\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{mV^2}{2}\frac{\mathrm{d}n}{\mathrm{d}t} + \boldsymbol{\nabla}\cdot\mathbf{q}\\ &+ T\,\nabla n\cdot\mathbf{V} + n\,\nabla T\cdot\mathbf{V} + \frac{5nT}{2}\boldsymbol{\nabla}\cdot\mathbf{V} + \mathbf{V}\cdot\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}} + \overleftrightarrow{\boldsymbol{\pi}}:\nabla\mathbf{V}\\ &+ \frac{mnV^2}{2}\boldsymbol{\nabla}\cdot\mathbf{V} = en\mathbf{E}\cdot\mathbf{V} + Q + \mathbf{R}\cdot\mathbf{V}\end{aligned} \tag{2.22}$$

If we take $\mathbf{V}\cdot(2.11)$, we would get

$$\mathbf{V}\cdot mn\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} + \mathbf{V}\cdot\,\nabla p + \mathbf{V}\cdot\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}} = ne\mathbf{E}\cdot\mathbf{V} + \mathbf{R}\cdot\mathbf{V} \tag{2.23}$$

subtracting this from our previous equation yields some cancellations, especially when we recognize $T\,\nabla n\cdot\mathbf{V} + n\,\nabla T\cdot\mathbf{V} = \nabla p\cdot\mathbf{V}$,

$$\begin{aligned}\frac{3n}{2}\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{3T}{2}\frac{\mathrm{d}n}{\mathrm{d}t} + mnV\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{mV^2}{2}\frac{\mathrm{d}n}{\mathrm{d}t} + \boldsymbol{\nabla}\cdot\mathbf{q} + \frac{5nT}{2}\boldsymbol{\nabla}\cdot\mathbf{V} + \overleftrightarrow{\boldsymbol{\pi}}:\nabla\mathbf{V}\\ + \frac{mnV^2}{2}\boldsymbol{\nabla}\cdot\mathbf{V} - mn\mathbf{V}\cdot\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = Q\end{aligned} \tag{2.24}$$

We can now collect like terms

$$\begin{aligned}\frac{3n}{2}\frac{\mathrm{d}T}{\mathrm{d}t} &+ \frac{3T}{2}\left(\frac{\mathrm{d}n}{\mathrm{d}t} + n\boldsymbol{\nabla}\cdot\mathbf{V}\right) + \frac{mV^2}{2}\left(\frac{\mathrm{d}n}{\mathrm{d}t} + n\boldsymbol{\nabla}\cdot\mathbf{V}\right) + \boldsymbol{\nabla}\cdot\mathbf{q} + nT\boldsymbol{\nabla}\cdot\mathbf{V} + \overleftrightarrow{\boldsymbol{\pi}}:\nabla\mathbf{V}\\ &+ mn\frac{\mathrm{d}V^2}{\mathrm{d}t} - mn\frac{\mathrm{d}V^2}{\mathrm{d}t} = Q\end{aligned} \tag{2.25}$$

We use number continuity which says

$$\frac{\partial n}{\partial t} + \nabla n \cdot \mathbf{V} + n\boldsymbol{\nabla}\cdot\mathbf{V} = \frac{\mathrm{d}n}{\mathrm{d}t} + n\boldsymbol{\nabla}\cdot\mathbf{V} = 0 \tag{2.26}$$

to eliminate those terms, use $nT = p$, as well, and we are left with

$$\frac{3n}{2}\frac{\mathrm{d}T}{\mathrm{d}t} + p\boldsymbol{\nabla}\cdot\mathbf{V} + \boldsymbol{\nabla}\cdot\mathbf{q} + \overleftrightarrow{\boldsymbol{\pi}} : \nabla\mathbf{V} = Q \tag{2.27}$$

Which, can be written (adding back the subscript a on all terms to remind ourselves that the full derivatives are convective derivatives)

$$\left.\frac{3n_a}{2}\frac{\mathrm{d}T_a}{\mathrm{d}t}\right|_a + p_a\boldsymbol{\nabla}\cdot\mathbf{V}_a = -\boldsymbol{\nabla}\cdot\mathbf{q}_a - \overleftrightarrow{\boldsymbol{\pi}}_a : \nabla\mathbf{V}_a + Q_a \tag{2.28}$$

Chapter 3

The Collision Operator

3.9 Exercises Chapter 3

3.9.1 Calculate Some Frequencies

Calculate the deflection frequency (HS-3.45) and the parallel velocity diffusion frequency (HS-3.47) from their definitions (HS-3.38) and (HS-3.39) by using the relation (HS-3.44).

$$\nu_D^{ab}(v) = -\frac{2L^{ab}}{v^3}\psi_b'(v) \tag{HS-3.38}$$

$$\nu_\parallel^{ab} = -2L^{ab}\frac{\psi_b''(v)}{v^2} \tag{HS-3.39}$$

$$\frac{1}{v^2}\frac{\mathrm{d}}{\mathrm{d}v}\left(v^2\frac{\mathrm{d}\psi_b}{\mathrm{d}v}\right) = \varphi_b(v) \tag{HS-3.44}$$

$$\nu_D^{ab}(v) = \widehat{\nu}_{ab}\frac{\phi(x_b) - G(x_b)}{x_a^3} \tag{HS-3.45}$$

$$\nu_\parallel^{ab}(v) = 2\widehat{\nu}_{ab}\frac{G(x_b)}{x_a^3} \tag{HS-3.47}$$

$$\widehat{\nu}_{ab} = \frac{n_b e_a^2 e_b^2 \ln\Lambda}{4\pi\epsilon_0^2 m_a^2 v_{\mathrm{th}_a}^2} \tag{HS-3.48}$$

$$G(x) = \frac{\phi(x) - x\phi'(x)}{2x^2} \tag{HS-3.43}$$

$$L^{ab} = \left(\frac{e_a e_b}{m_a \epsilon_0}\right)^2 \ln\Lambda \tag{3.1}$$

Solution:

We first need to get ψ_b' from ϕ_b. We see that (HS-3.44) implies

$$\psi_b' = v^{-2}\int \mathrm{d}v\ v^2\varphi_b \tag{3.2}$$

For a Maxwellian, we have (remember that $x_b = v/v_{\mathrm{th}_b}$)

$$\varphi_b'(v) = \frac{m_b n_b}{4\pi T_b} G(x_b) \tag{3.3}$$

$$\varphi_b = \frac{v_{\mathrm{th}_b} m_b n_b}{4\pi T_b} \int \mathrm{d}x_b \; G(x_b) \tag{3.4}$$

If we look at the integral, we see

$$\begin{aligned}\int \mathrm{d}x \; G(x) &= \int \mathrm{d}x \; \frac{\phi(x) - x\phi'(x)}{2x^2} = \int \mathrm{d}x \; \frac{\phi}{2x^2} + \int \mathrm{d}x \; \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2x}\right)\phi(x) - \frac{\phi(x)}{2x} \\ &= \int \mathrm{d}x \; \left[\frac{\phi}{2x^2} - \frac{\phi}{2x^2}\right] - \frac{\phi}{2x} = -\frac{\phi}{2x}\end{aligned} \tag{3.5}$$

so

$$\varphi_b = -\frac{v_{\mathrm{th}_b} m_b n_b}{4\pi T_b} \frac{\phi(x_b)}{2x_b} \tag{3.6}$$

So (with $2T_b/m_b = v_{\mathrm{th}_b}^2$)

$$\psi_b' = v^{-2} \int \mathrm{d}v \; v^2 \varphi_b = -\frac{v_{\mathrm{th}_b}^3}{v^2} \int \mathrm{d}x_b \; x_b^2 \frac{v_{\mathrm{th}_b} m_b n_b}{4\pi T_b} \frac{\phi(x_b)}{2x_b} = -\frac{n_b v_{\mathrm{th}_b}^2}{4\pi v^2} \int \mathrm{d}x_b \; x_b \phi(x_b) \tag{3.7}$$

$$= -\frac{n_b v_{\mathrm{th}_b}^2}{4\pi v^2} \left[\frac{x_b^2 \phi(x_b)}{2} - \int \mathrm{d}x_b \; \frac{x_b^2}{2} \phi'(x_b)\right] \tag{3.8}$$

We can now use

$$\phi'(x_b) = \frac{2}{\sqrt{\pi}} e^{-x_b^2} \tag{3.9}$$

So that

$$\int \mathrm{d}x_b \; \frac{x_b^2}{\sqrt{\pi}} e^{-x_b^2} = \int \mathrm{d}x_b \; \left(\frac{x_b^2}{\sqrt{\pi}(-2x_b)} \frac{\mathrm{d}e^{-x_b^2}}{\mathrm{d}x_b}\right) = \frac{-x_b}{2\sqrt{\pi}} e^{-x_b^2} - \int \mathrm{d}x_b \; \frac{-e^{-x_b^2}}{2\sqrt{\pi}} \tag{3.10}$$

$$= \frac{-x_b \phi'(x_b)}{4} + \int \mathrm{d}x_b \; \frac{\phi'(x_b)}{4} = \frac{-x_b \phi'(x_b) + \phi(x_b)}{4} = \frac{x_b^2 G(x)}{2} \tag{3.11}$$

And so, we get

$$\psi_b' = -\frac{n_b v_{\mathrm{th}_b}^2}{4\pi v^2} \left[\frac{x_b^2 \phi(x_b)}{2} - \frac{x_b^2 G(x_b)}{2}\right] = -\frac{n_b}{8\pi}[\phi(x_b) - G(x_b)] \tag{3.12}$$

And so

$$\nu_D^{ab} = -\frac{2L^{ab}}{v^3} \psi_b'(v) = -\frac{2e_a^2 e_b^2 \ln\Lambda}{m_a^2 \epsilon_0^2 v^3} \left[-\frac{n_b}{8\pi}[\phi(x_b) - G(x_b)]\right] = \frac{e_a^2 e_b^2 \ln\Lambda}{4\pi\epsilon_0^2 m_a^2 v^3} \left(\phi(x_b) - G(x_b)\right) \tag{3.13}$$

$$\nu_D^{ab} = \widehat{\nu}_{ab} \frac{\phi(x_b) - G(x_b)}{x_a^3} \tag{3.14}$$

as promised.

Now we use

$$\psi_b'' = \frac{-n_b}{8\pi}\frac{\mathrm{d}}{\mathrm{d}v}\left[\phi(x_b) - G(x_b)\right] = \frac{-n_b}{8\pi v_{\mathrm{th}_b}}\left(\phi'(x_b) - G'(x_b)\right) \tag{3.15}$$

We use

$$\begin{aligned}\phi'(x_b) - G'(x_b) &= \phi'(x_b) - \frac{2x_b^2[\phi'(x_b) - \phi'(x_b) - x_b\phi''(x_b)] - [\phi(x_b) - x\phi'(x_b)]4x_b}{4x_b^4} \\ &= \phi'(x_b) + \frac{\phi''(x_b)}{2x_b} + \frac{2G(x_b)}{x_b}\end{aligned} \tag{3.16}$$

We use that $\phi''(x_b) = -2x_b\phi'(x_b)$ so that

$$\phi'(x_b) - G'(x_b) = \phi'(x_b) - \phi'(x_b) + \frac{2G(x_b)}{x_b} = \frac{2G(x_b)}{x_b} \tag{3.17}$$

Therefore

$$\psi_b'' = -\frac{n_b}{4\pi v_{\mathrm{th}_b} x_b}G(x_b) = -\frac{n_b}{4\pi v}G(x_b) \tag{3.18}$$

And so

$$\nu_\parallel^{ab} = -2L^{ab}\frac{\psi_b''(v)}{v^2} = 2\frac{e_a^2 e_b^2 \ln\Lambda}{m_a^2\epsilon_0^2 v^2}\frac{n_b}{4\pi v}G(x_b) = 2\widehat{\nu}_{ab}\frac{v_{\mathrm{th}_a}^3 G(x_b)}{v^3} = 2\widehat{\nu}_{ab}\frac{G(x_b)}{x_a^3} \tag{3.19}$$

3.9.2 Pressure Anisotropy

For viscosity calculations, it is useful to know at what rate the pressure ansisotropy

$$p_{a\parallel} - p_{a\perp} = \int \mathrm{d}^3v\; m_a f_a \left(v_\parallel^2 - \frac{v_\perp^2}{2}\right) \tag{3.20}$$

decays by collisions with a Maxwellian background species f_{b0}. Calculate this rate along the same lines as used for the friction force and momentum exchange in Section 3.7.

Solution:

We begin by looking at the collision integral with appropriate weighting,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_{a,\parallel} - p_{a,\perp}\right)\Big|_{\text{collisions}} = \int \mathrm{d}^3v\; m_a \left(v_\parallel^2 - \frac{v_\perp^2}{2}\right) C_{ab}(f_a, f_{b0}) \tag{3.21}$$

we use identities and see (remembering that pre-collision there was no perpendicular velocity and there was a $v_\parallel$)

$$\int \mathrm{d}^3v\; m_a \left(v_\parallel^2 - \frac{v_\perp^2}{2}\right) C_{ab}(f_a, f_{b0}) = \int \mathrm{d}^3v\; m_a \left(\frac{\left\langle (v_\parallel + \Delta v_{\;\parallel})^2 - v_\parallel^2\right\rangle}{\Delta t} - \frac{\langle \Delta v_{\;\perp}^{\,2}\rangle}{2\Delta t}\right) f_a \tag{3.22}$$

$$= \int \mathrm{d}^3v\; m_a \left(\frac{\langle 2v_\parallel \Delta v_{\;\parallel}\rangle}{\Delta t} + \frac{\left\langle \Delta v_{\;\parallel}^{\,2}\right\rangle}{\Delta t} - \frac{\langle \Delta v_{\;\perp}^{\,2}\rangle}{2\Delta t}\right) f_a \tag{3.23}$$

$$= \int \mathrm{d}^3v\; m_a v^2 \left(-2\nu_s^{ab} + \nu_\parallel^{ab} - \nu_D^{ab}\right) f_a \tag{3.24}$$

$$= -\int \mathrm{d}^3v\; m_a v^2 \nu_T^{ab} f_a \tag{3.25}$$

with

$$\nu_T^{ab} = 2\nu_s^{ab} + \nu_D^{ab} - \nu_\parallel^{ab} = \nu_E^{ab} + 3\nu_D^{ab} \tag{3.26}$$

since

$$\nu_E^{ab} = 2\nu_s^{ab} - 2\nu_D^{ab} - \nu_\parallel^{ab} \tag{3.27}$$

3.9.3 Entropy

The entropy s per particle of a species with the distribution function f and density n is defined by

$$s \equiv \frac{-1}{n} \int \mathrm{d}^3 v \; f \ln f \tag{3.28}$$

3.9.3.1 Maxwellian Entropy

Show that $s = \frac{3}{2} \ln(p/n^{5/3}) + \text{const.}$ for a Maxwellian species, as claimed at the end of Section 2.2.

Solution:

We simply use that

$$f = \frac{n}{\pi^{3/2} v_{\text{th}}^3} e^{-v^2/v_{\text{th}}^2} \tag{3.29}$$

$$\ln f = \ln \left(\frac{n}{\pi^{3/2} v_{\text{th}}^3} \right) - \frac{v^2}{v_{\text{th}}^2} \equiv A - \frac{v^2}{v_{\text{th}}^2} \tag{3.30}$$

$$x \equiv \frac{v}{v_{\text{th}}} \tag{3.31}$$

Thus,

$$\begin{aligned} s &= -\frac{1}{n} \int \mathrm{d}^3 v \; f \ln f = -\frac{1}{n} \int_0^\infty \mathrm{d}v \; 4\pi v^2 f \ln f = -\frac{4}{\sqrt{\pi} v_{\text{th}}^3} \int_0^\infty \mathrm{d}v \; v^2 e^{-v^2/v_{\text{th}}^2} \left[\ln \left(\frac{n}{\pi^{3/2} v_{\text{th}}^3} - \frac{v^2}{v_{\text{th}}^2} \right) \right] \\ &= -\frac{4}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x \; x^2 e^{-x^2} \left[\ln \left(A - x^2 \right) \right] = -\frac{4}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{4} \ln A - \frac{3\sqrt{\pi}}{8} \right] = -\ln A + \frac{3}{2} \end{aligned} \tag{3.32}$$

Thus, (using $\ln(ab) = \ln(a) + \ln(b)$ multiple times)

$$s = \ln \left(\frac{v_{\text{th}}^3}{n} \right) + \text{const.} = \ln \left(\frac{T^{3/2}}{n} \right) + \text{const.} = \ln \left(\frac{p^{3/2}}{n^{5/2}} \right) + \text{const.} = \ln \left(\left[\frac{p}{n^{5/3}} \right]^{3/2} \right) + \text{const.} \tag{3.33}$$

$$s = \frac{3}{2} \ln \left(\frac{p}{n^{5/3}} \right) + \text{const.} \tag{3.34}$$

as desired.

3.9.3.2 Find the rate of change of entropy

Calculate the rate of change in the total entropy $S = \int \mathrm{d}^3x\ ns$ from the kinetic equation (HS-2.1).

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \nabla f_a + \frac{e_a}{m_a} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_a}{\partial \mathbf{v}} = C_a(f_a) \tag{HS-2.1}$$

Solution:

We have

$$ns = -\int \mathrm{d}^3v\ f \ln f \tag{3.35}$$

and so

$$S = \int \mathrm{d}^3x\ ns = -\int \mathrm{d}^3x \int \mathrm{d}^3v\ f \ln f \tag{3.36}$$

Thus,

$$\frac{\partial S}{\partial t} = -\int \mathrm{d}^3x \int \mathrm{d}^3v \left[\frac{\partial f}{\partial t} \ln f + f \frac{1}{f}\frac{\partial f}{\partial t}\right] = -\int \mathrm{d}^3x \int \mathrm{d}^3v\ \frac{\partial f}{\partial t} (1 + \ln f) \tag{3.37}$$

Let's take the first term

$$-\int \mathrm{d}^3x \int \mathrm{d}^3v\ \frac{\partial f}{\partial t} = -\int \mathrm{d}^3x\ \frac{\partial}{\partial t} \int \mathrm{d}^3v\ f = -\int \mathrm{d}^3x\ \frac{\partial n}{\partial t} \tag{3.38}$$

If there are no sources or sinks of particles in the entire volume of interest, then this clearly vanishes by conservation of particle density. Thus, we find

$$\frac{\partial S}{\partial t} = -\int \mathrm{d}^3x \int \mathrm{d}^3v\ \frac{\partial f}{\partial t} \ln f \tag{3.39}$$

We use that

$$\frac{\partial f}{\partial t} = -\boldsymbol{\nabla} \cdot (f\mathbf{v}) - \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a} f) + C(f) \tag{3.40}$$

where $\mathbf{a} = \frac{e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B})$. We have used that $\nabla f \cdot \mathbf{v} = \boldsymbol{\nabla} \cdot (f\mathbf{v})$ and $\mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot (f\mathbf{a})$. Thus,

$$\frac{\partial S}{\partial t} = \int \mathrm{d}^3x \int \mathrm{d}^3v \left[\boldsymbol{\nabla} \cdot (f\mathbf{v}) + \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a} f) - C(f)\right] \ln f \tag{3.41}$$

Let's take the first term,

$$\int \mathrm{d}^3x \int \mathrm{d}^3v\ \ln f \boldsymbol{\nabla} \cdot (f\mathbf{v}) = \int \mathrm{d}^3v \int \mathrm{d}^3x\ [\boldsymbol{\nabla} \cdot (f\mathbf{v} \ln f) - \nabla(\ln f) \cdot f\mathbf{v}] \tag{3.42}$$

$$= \int \mathrm{d}^3v \int \mathrm{d}^3x\ [\boldsymbol{\nabla} \cdot (f\mathbf{v} \ln f) - \nabla f \cdot \mathbf{v}] = \int \mathrm{d}^3v \int \mathrm{d}^3x\ [\boldsymbol{\nabla} \cdot (f\mathbf{v} \ln f) - \boldsymbol{\nabla} \cdot (f\mathbf{v})] \tag{3.43}$$

$$= \int \mathrm{d}^3v \int \mathrm{d}^2x\ \mathbf{n} \cdot [(f\mathbf{v} \ln f) - (f\mathbf{v})] = \int \mathrm{d}^3v\ 0 = 0 \tag{3.44}$$

where we applied Gauss's law and used that at a surface approaching $r = \infty$, we must have the quantities vanish.

Similarly, for the second term

$$\int \mathrm{d}^3x \int \mathrm{d}^3v \ \ln f \frac{\partial}{\partial \mathbf{v}} \cdot (f\mathbf{a}) = \int \mathrm{d}^3x \int \mathrm{d}^3v \ \left[\frac{\partial}{\partial \mathbf{v}} \cdot (f\mathbf{a}\ln f) - \frac{\partial(\ln f)}{\partial \mathbf{v}} \cdot f\mathbf{a}\right] \tag{3.45}$$

$$= \int \mathrm{d}^3x \int \mathrm{d}^3x \ \left[\frac{\partial}{\partial \mathbf{v}} \cdot (f\mathbf{a}\ln f) - \frac{\partial f}{\partial \mathbf{v}} \cdot \mathbf{a}\right] = \int \mathrm{d}^3v \int \mathrm{d}^3x \ \left[\frac{\partial}{\partial \mathbf{v}} \cdot (f\mathbf{a}\ln f) - \frac{\partial}{\partial \mathbf{v}} \cdot (f\mathbf{a})\right] \tag{3.46}$$

$$= \int \mathrm{d}^3x \int \mathrm{d}^2v \ \mathbf{n} \cdot [(f\mathbf{a}\ln f) - (f\mathbf{a})] = \int \mathrm{d}^3x \ 0 = 0 \tag{3.47}$$

where we applied Gauss's law and used that at a surface approaching $v = \infty$, we must have the quantities vanish.

Thus, we are left with

$$\frac{\partial S}{\partial t} = -\int \mathrm{d}^3x \int \mathrm{d}^3v \ \ln f \, C(f) \tag{3.48}$$

and entropy production is due only to the collision term.

3.9.3.3 Self-Collisions Increase Entropy

Show that self-collisions increase the entropy for a distribution function close to equilibrium.

Solution:

If we are near equilibrium, then we can write for f_0 a Maxwellian background

$$f = f_0(1 + \widetilde{f}) \tag{3.49}$$

$$\ln f = \ln f_0 + \widetilde{f} \tag{3.50}$$

So we know that (locally)

$$\frac{\partial(ns)}{\partial t} = -\int \mathrm{d}^3v \ C(f)\ln f = -\int \mathrm{d}^3v \ C^l(f)(\ln f_0 + \widetilde{f}) = -\int \mathrm{d}^3v \ C(f)\ln f_0 - \int \mathrm{d}^3v \ C(f)\widetilde{f} \tag{3.51}$$

The first integral is the entropy production due to a Maxwellian, and so is zero. Thus, we have using $C(f) = C^l(f_0\widetilde{f})$ that

$$\frac{\partial(ns)}{\partial t} = -\int \mathrm{d}^3v \ C^l(f_0\widetilde{f})\widetilde{f} = -S[\widetilde{f}, \widetilde{f}] \tag{3.52}$$

Now, we have that

$$S[\widetilde{f}, \widetilde{f}] = \frac{-L}{16\pi} \int \mathrm{d}^3v \int \mathrm{d}^3v' \ f_0 f_0' U_{kl} \left(\frac{\partial \widetilde{f}'}{\partial v_l'} - \frac{\partial \widetilde{f}}{\partial v_l}\right)\left(\frac{\partial \widetilde{f}'}{\partial v_k'} - \frac{\partial \widetilde{f}}{\partial v_k}\right) \tag{3.53}$$

We call the integrand

$$U_{kl}a_ka_l = \frac{u^2\delta_{kl} - u_ku_l}{u^3}a_ka_l = \frac{u^2a_ka_k - u_ka_ku_la_l}{u^3} = \frac{u^2a^2 - (\mathbf{u}\cdot\mathbf{a})^2}{u^3} \tag{3.54}$$

Now, we, know by the inequality

$$(\mathbf{u}\cdot\mathbf{a})^2 \leq u^2a^2 \tag{3.55}$$

then that (given $u^3 > 0$)

$$u^3U_{kl}a_ka_l = u^2a^2 - (\mathbf{u}\cdot\mathbf{a})^2 \geq 0 \tag{3.56}$$

So that $S[\widetilde{f}, \widetilde{f}] \leq 0$. This implies that $-S[\widetilde{f}, \widetilde{f}] \geq 0$, so that entropy production is always positive.

3.9.3.4 Boltzmann's H-theorem

Show that the fully nonlinearized self-collision operator increases entropy by a procedure similar to that leading to (HS-3.61).

Solution:

We use

$$C_{ab}(f_a, f_b) = \frac{-e_a^2e_b^2\ln\Lambda}{8\pi\epsilon_0^2m_a}\frac{\partial}{\partial v_k}\int \mathrm{d}^3v'\; U_{kl}\left[\frac{f_a}{m_b}\frac{\partial f_b'}{\partial v_l'} - \frac{f_b'}{m_a}\frac{\partial f_a}{\partial v_l}\right] \tag{HS-3.22}$$

So that for same collisions

$$C_{aa}(f_a, f_a) = \frac{-e_a^4\ln\Lambda}{8\pi\epsilon_0^2m_a^2}\frac{\partial}{\partial v_k}\int \mathrm{d}^3v'\; U_{kl}\left[f_a\frac{\partial f_a'}{\partial v_l'} - f_a'\frac{\partial f_a}{\partial v_l}\right] \tag{3.57}$$

Thus,

$$\int \mathrm{d}^3v\; C_{aa}\ln f_a = \frac{-e_a^4\ln\Lambda}{8\pi\epsilon_0^2m_a^2}\int \mathrm{d}^3v\;\; \ln f_a\frac{\partial}{\partial v_k}\int \mathrm{d}^3v'\; U_{kl}\left[f_a\frac{\partial f_a'}{\partial v_l'} - f_a'\frac{\partial f_a}{\partial v_l}\right] \tag{3.58}$$

$$= \frac{-e_a^4\ln\Lambda}{8\pi\epsilon_0^2m_a^2}\int \mathrm{d}^3v\int \mathrm{d}^3v'\;\left(\frac{\partial}{\partial v_k}\left\{\ln f_aU_{kl}\left[f_a\frac{\partial f_a'}{\partial v_l'} - f_a'\frac{\partial f_a}{\partial v_l}\right]\right\} - \frac{\partial \ln f_a}{\partial v_k}U_{kl}\left[f_a\frac{\partial f_a'}{\partial v_l'} - f_a'\frac{\partial f_a}{\partial v_l}\right]\right) \tag{3.59}$$

The term in the curly braces will clearly vanish by using Gauss's law. Thus, we have

$$\int \mathrm{d}^3v\; C_{aa}\ln f_a = \frac{e_a^4\ln\Lambda}{8\pi\epsilon_0^2m_a^2}\int \mathrm{d}^3v\int \mathrm{d}^3v'\;\frac{\partial \ln f_a}{\partial v_k}U_{kl}\left[f_a\frac{\partial f_a'}{\partial v_l'} - f_a'\frac{\partial f_a}{\partial v_l}\right] \tag{3.60}$$

$$= \frac{e_a^4\ln\Lambda}{8\pi\epsilon_0^2m_a^2}\int \mathrm{d}^3v\int \mathrm{d}^3v'\;\frac{\partial \ln f_a'}{\partial v_k'}U_{kl}\left[f_a'\frac{\partial f_a}{\partial v_l} - f_a\frac{\partial f_a'}{\partial v_l'}\right] \tag{3.61}$$

where in the second expression we swapped $v \rightarrow v'$ and $v' \rightarrow v$, recognizing that U_{kl} is unchanged under this swapping.

Thus, symmetrizing by adding these two expressions we see that we get

$$\int \mathrm{d}^3v\ C_{aa} \ln f_a = \frac{e_a^4 \ln\Lambda}{16\pi\epsilon_0^2 m_a^2} \int \mathrm{d}^3v \int \mathrm{d}^3v'\ U_{kl} \left(\frac{\partial \ln f_a'}{\partial v_k'} - \frac{\partial \ln f_a}{\partial v_k} \right) \left[f_a' \frac{\partial f_a}{\partial v_l} - f_a \frac{\partial f_a'}{\partial v_l'} \right] \tag{3.62}$$

$$= \frac{e_a^4 \ln\Lambda}{16\pi\epsilon_0^2 m_a^2} \int \mathrm{d}^3v \int \mathrm{d}^3v'\ U_{kl} f_a f_a' \left(\frac{\partial \ln f_a'}{\partial v_k'} - \frac{\partial \ln f_a}{\partial v_k} \right) \left[\frac{\partial \ln f_a}{\partial v_l} - \frac{\partial \ln f_a'}{\partial v_l'} \right] \tag{3.63}$$

$$= -\frac{e_a^4 \ln\Lambda}{16\pi\epsilon_0^2 m_a^2} \int \mathrm{d}^3v \int \mathrm{d}^3v'\ U_{kl} f_a f_a' \left(\frac{\partial \ln f_a'}{\partial v_k'} - \frac{\partial \ln f_a}{\partial v_k} \right) \left[\frac{\partial \ln f_a'}{\partial v_l'} - \frac{\partial \ln f_a}{\partial v_l} \right] \tag{3.64}$$

We call the integrand (excluding the positive definite $f_a f_a'$)

$$U_{kl} a_k a_l = \frac{u^2 \delta_{kl} - u_k u_l}{u^3} a_k a_l = \frac{u^2 a_k a_k - u_k a_k u_l a_l}{u^3} = \frac{u^2 a^2 - (\mathbf{u} \cdot \mathbf{a})^2}{u^3} \tag{3.65}$$

Now, we, know by the inequality

$$(\mathbf{u} \cdot \mathbf{a})^2 \leq u^2 a^2 \tag{3.66}$$

then that (given $u^3 > 0$)

$$u^3 U_{kl} a_k a_l = u^2 a^2 - (\mathbf{u} \cdot \mathbf{a})^2 \geq 0 \tag{3.67}$$

Thus, the integrand is positive definite, and so the expression is always negative, indicating entropy production because $\frac{\partial(ns)}{\partial t} = -\int \mathrm{d}^3v\ C_{aa}(f_a, f_a) \ln f_a$.

3.9.3.5 Local Maxwellian

Show that the entropy production from self-collisions vanishes only when the distribution function is locally Maxwellian.

Solution:

The only way for the expression to vanish, is when

$$U_{kl} a_k a_l = 0 \tag{3.68}$$

where

$$a_k = \frac{\partial \ln f_a'}{\partial v_k'} - \frac{\partial \ln f_a}{\partial v_k} \tag{3.69}$$

This implies that $a_k a_l$ is "perpendicular" to U_{kl}. That is we require

$$u^2 a^2 = (\mathbf{u} \cdot \mathbf{a})^2 \tag{3.70}$$

which implies that $\mathbf{a}$ points in the exact same (parallel or antiparallel) direction as $\mathbf{u}$. Thus,

$$\frac{\partial \ln f_a'}{\partial v_k'} - \frac{\partial \ln f_a}{\partial v_k} \propto u_i \propto (v_k - v_k') \tag{3.71}$$

This means that we can have (where V is a constant)

$$\frac{\partial \ln f'_a}{\partial v'_k} = \alpha v'_k + V \tag{3.72}$$

$$\frac{\partial \ln f_a}{\partial v_k} = \alpha v_k + V \tag{3.73}$$

Integrating, we see (β is another constant)

$$\ln f'_a = \alpha (v'_k)^2/2 + V v'_k + \beta \tag{3.74}$$

$$\ln f_a = \alpha v_k^2/2 + V v_k + \beta \tag{3.75}$$

so that

$$f'_a = e^{\alpha (v'_k)^2/2 + V v'_k + \beta} = C e^{-\kappa (v'_k - V)^2} \tag{3.76}$$

$$f_a = e^{\alpha v_k^2/2 + V V_k + \beta} = C e^{-\kappa (v_k - V)^2} \tag{3.77}$$

where κ and C are constants. This implies f_a is in the form of a Maxwellian.

Chapter 4

Plasma Fluid Equations

4.2 Lorentz Plasma

We look at

$$C(f_e) + \Omega_e(\mathbf{b}\times\mathbf{v})\cdot\frac{\partial f_e}{\partial \mathbf{v}} = \mathbf{v}\cdot\nabla f_e - \frac{e\mathbf{E}}{m_e}\cdot\frac{\partial f_e}{\partial \mathbf{v}} + \frac{\partial f_e}{\partial t} \tag{HS-4.7}$$

where $\mathbf{b} = \mathbf{B}/B$.

If we use that $E_\parallel \sim T/(eL_\parallel)$, and that $v_\parallel \sim v_{\text{th}_e}$ with $v_{\text{th}_e}/\nu_e = \lambda$, then we see

$$\frac{(\mathbf{v}\cdot\nabla f_e)_\parallel}{C_e(f_e)} \sim \frac{v_\parallel \nabla_\parallel f_e}{C_e(f_e)} \sim \frac{f_e v_{\text{th}_e}/L_\parallel}{\nu_e f_e} \sim \frac{\lambda}{L_\parallel} \tag{4.1}$$

and

$$\frac{(\frac{e\mathbf{E}}{m_e}\cdot\frac{\partial f_e}{\partial \mathbf{v}})_\parallel}{C_e(f_e)} \sim \frac{\frac{eE_\parallel}{m_e}\frac{f_e}{v_\parallel}}{\nu_e f_e} \sim \frac{\frac{e}{m_e v_{\text{th}_e}}\frac{T}{eL_\parallel}}{\nu_e} \sim \frac{\frac{v_{\text{th}_e}}{L_\parallel}}{\nu_e} \sim \frac{\lambda}{L_\parallel} \tag{4.2}$$

Similarly, if we compare the perpendicular portions to the Ω_e (using $v_\perp/\Omega_e = \rho$ and $v_\perp \sim v_{\text{th}_e}$), term we find

$$\frac{(\mathbf{v}\cdot\nabla f_e)_\perp}{\Omega_e(\mathbf{b}\times\mathbf{v})\cdot\frac{\partial f_e}{\partial \mathbf{v}}} \sim \frac{\mathbf{v}_\perp\cdot\nabla_\perp f_e}{\Omega_e v_\perp \frac{f_e}{v_\perp}} \sim \frac{v_\perp \frac{f_e}{L_\perp}}{\Omega_e f_e} \sim \frac{\rho}{L_\perp} \tag{4.3}$$

and

$$\frac{(\frac{e\mathbf{E}}{m_e}\cdot\frac{\partial f_e}{\partial \mathbf{v}})_\perp}{\Omega_e(\mathbf{b}\times\mathbf{v})\cdot\frac{\partial f_e}{\partial \mathbf{v}}} \sim \frac{\frac{eE_\perp}{m_e}\frac{f_e}{v_\perp}}{\Omega_e v_\perp \frac{f_e}{v_\perp}} \sim \frac{\frac{eT}{eL_\perp m_e}\frac{f_e}{v_\perp}}{\Omega_e f_e} \sim \frac{\frac{v_{\text{th}_e}^2}{v_\perp L_\perp}}{\Omega_e} \sim \frac{\rho}{L_\perp} \tag{4.4}$$

with $E_\perp \sim eT/L_\perp$.

Note that

$$\int \mathrm{d}^3v\ \ln f_{e0}\,\frac{\partial f_{e0}}{\partial\varphi} = \cancel{\int \mathrm{d}^3v\ \frac{\partial}{\partial\varphi}(f_{e0}\ln f_{e0})} - \int \mathrm{d}^3v\ f_{e0}\frac{\partial \ln f_{e0}}{\partial\varphi} = \int \mathrm{d}^3v\ \frac{f_{e0}}{f_{e0}}\frac{\partial f_{e0}}{\partial\varphi} = 0 \tag{4.5}$$

where we use that

$$\int \mathrm{d}^3v \, \frac{\partial g}{\partial \varphi} = \int \mathrm{d}^2v \, \int_0^T \mathrm{d}\varphi \, \frac{\partial g}{\partial \varphi} = \int \mathrm{d}^2v \, \left[g(\varphi = T) - g(\varphi = 0)\right] = 0 \tag{4.6}$$

(with T being a period in φ, so usually $T = 2\pi$) so that because g is periodic in φ with period T then the integral vanishes.

Let's derive the expansion using the tag ϵ to keep track of small quantities. We then write the kinetic equation as

$$C_e(f_e) + \Omega_e(\mathbf{b} \times \mathbf{v}) \cdot \frac{\partial f_e}{\partial \mathbf{v}} = \mathbf{v} \cdot \nabla f_e - \frac{e\mathbf{E}}{m_e} \cdot \frac{\partial f_e}{\partial \mathbf{v}} + \frac{\partial f_e}{\partial t} \tag{4.7}$$

$$C_e(f_e) + \Omega_e(\mathbf{b} \times \mathbf{v}) \cdot \frac{\partial f_e}{\partial \mathbf{v}} = \epsilon \mathbf{v} \cdot \nabla f_e - \epsilon \frac{e\mathbf{E}}{m_e} \cdot \frac{\partial f_e}{\partial \mathbf{v}} + \epsilon^2 \frac{\partial f_e}{\partial t} \tag{4.8}$$

We use the expansion

$$f_e = f_{e0} + \epsilon f_{e1} + \epsilon^2 f_{e2} + \cdots \tag{4.9}$$

Thus, to zeroth order we find

$$C_e(f_{e0}) + \Omega_e(\mathbf{b} \times \mathbf{v}) \cdot \frac{\partial f_{e0}}{\partial \mathbf{v}} = 0 \tag{4.10}$$

if we put $\mathbf{b}$ in the z-direction, we would find

$$(\hat{\mathbf{z}} \times \mathbf{v}) \cdot \frac{\partial f_{e0}}{\partial \mathbf{v}} = (v_x \hat{\mathbf{y}} - v_y \hat{\mathbf{x}}) \cdot \left(\hat{\mathbf{x}} \frac{\partial f_{e0}}{\partial v_x} + \hat{\mathbf{y}} \frac{\partial f_{e0}}{\partial v_y} \right) = v_x \frac{\partial f_{e0}}{\partial v_y} - v_y \frac{\partial f_{e0}}{\partial v_x} \tag{4.11}$$

If we introduce spherical coordinates

$$v_x = v \sin\theta \cos\varphi \tag{4.12}$$

$$v_y = v \sin\theta \sin\varphi \tag{4.13}$$

$$v_z = v \cos\theta \tag{4.14}$$

$$\tan\varphi = \frac{v_y}{v_x} \tag{4.15}$$

$$\sec^2\varphi \, \mathrm{d}\varphi = \frac{v_x \, \mathrm{d}v_y - v_y \, \mathrm{d}v_x}{v_x^2} \tag{4.16}$$

$$\mathrm{d}\varphi = \frac{v_x^2}{v_x^2 + v_y^2} \frac{v_x \, \mathrm{d}v_y - v_y \, \mathrm{d}v_x}{v_x^2} = \frac{v_x}{v_x^2 + v_y^2} \, \mathrm{d}v_y - \frac{v_y}{v_x^2 + v_y^2} \, \mathrm{d}v_y \tag{4.17}$$

$$\mathrm{d}\varphi = \frac{v \sin\theta \cos\varphi}{v^2 \sin^2\theta} \, \mathrm{d}v_y - \frac{v \sin\theta \sin\varphi}{v^2 \sin^2\theta} \, \mathrm{d}v_y = -\frac{\sin\varphi}{v \sin\theta} \, \mathrm{d}x + \frac{\cos\varphi}{v \sin\theta} \, \mathrm{d}v_y \tag{4.18}$$

Thus,

$$\frac{\partial f_{e0}}{\partial v_y} = \frac{\partial f_{e0}}{\partial \varphi} \frac{\partial \varphi}{\partial v_y} = \frac{\partial f_{e0}}{\partial \varphi} \frac{\cos\varphi}{v \sin\theta} \tag{4.19}$$

$$\frac{\partial f_{e0}}{\partial v_x} = \frac{\partial f_{e0}}{\partial \varphi} \frac{\partial \varphi}{\partial v_x} = -\frac{\partial f_{e0}}{\partial \varphi} \frac{-\sin\varphi}{v \sin\theta} \tag{4.20}$$

and so

$$v_x \frac{\partial f_{e0}}{\partial v_y} - v_y \frac{\partial f_{e0}}{\partial v_x} = \left(\frac{v \sin\theta \cos^2\varphi}{v \sin\theta} + \frac{v \sin\theta \sin^2\varphi}{v\sin\theta} \right) \frac{\partial f_{e0}}{\partial \varphi} = \frac{\partial f_{e0}}{\partial \varphi} \tag{4.21}$$

So that we do recover

$$C_e(f_{e0}) + \Omega_e \frac{\partial f_{e0}}{\partial \varphi} = 0 \tag{4.22}$$

which implies f_{e0} is a Maxwellian as described in the book.

To next order in ϵ we would find

$$C_e(f_{e1}) + \Omega_e(\mathbf{b} \times \mathbf{v}) \cdot \frac{\partial f_{e1}}{\partial \mathbf{v}} = \mathbf{v} \cdot \nabla f_{e0} - \frac{e\mathbf{E}}{m_e} \cdot \frac{\partial f_{e0}}{\partial \mathbf{v}} \tag{4.23}$$

We can state that since $f_{e0} = \frac{n}{\pi^{3/2} v_{\text{th}_e}^3} e^{-v^2/v_{\text{th}_e}^2} = \frac{n m_e^{3/2}}{(2\pi T)^{3/2}} e^{-m_e v^2/(2T)}$ with $T = T_e$, so that

$$\begin{aligned} \nabla f_{e0} =& \ \nabla n \left(\frac{m_e}{2\pi T}\right)^{3/2} e^{-m_e v^2/(2T)} - \frac{3n}{2T}\left(\frac{m_e}{2\pi T}\right)^{3/2} \nabla T e^{-m_e v^2/(2T)} + \frac{n m_e v^2}{2T^2} \nabla T \left(\frac{m_e}{2\pi T}\right)^{3/2} e^{-m_e v^2/(2T)} \\ =& \left(\nabla n + \frac{n m_e v^2}{2T^2} \nabla T - \frac{3n}{2T} \nabla T \right) \left(\frac{m_e}{2\pi T}\right)^{3/2} e^{-m_e v^2/(2T)} \end{aligned} \tag{4.24}$$

$$\frac{\partial f_{e0}}{\partial \mathbf{v}} = \frac{\partial v}{\partial \mathbf{v}} \frac{\partial f_{e0}}{\partial v} = \hat{\mathbf{v}} \frac{n}{\pi^{3/2} v_{\text{th}_e}^3} \frac{-2v}{v_{\text{th}_e}^2} e^{-v^2/v_{\text{th}_e}^2} = -\hat{\mathbf{v}} \frac{2nv}{\pi^{3/2}} \left(\frac{m_e}{2T}\right)^{5/2} e^{-m_e v^2/(2T)} = -\mathbf{v} \frac{2n}{\pi^{3/2}} \left(\frac{m_e}{2T}\right)^{5/2} e^{-m_e v^2/(2T)} \tag{4.25}$$

So we can write (using $\nabla \ln p = \nabla \ln T + \nabla \ln n$)

$$\mathbf{v} \cdot \nabla f_{e0} - \frac{e\mathbf{E}}{m_e} \cdot \frac{\partial f_{e0}}{\partial \mathbf{v}} = \mathbf{v} \cdot \left(\nabla n + \frac{n m_e v^2}{2T^2} \nabla T - \frac{3n}{2T} \nabla T + \frac{e\mathbf{E}}{m_e} \frac{n m_e}{T} \right) \left(\frac{m_e}{2\pi T}\right)^{3/2} e^{-m_e v^2/(2T)} \tag{4.26}$$

$$= \mathbf{v} \cdot \left(\nabla \ln n + \frac{m_e v^2}{2T} \nabla \ln T - \frac{3}{2} \nabla \ln T + \frac{e\mathbf{E}}{T} \right) f_{e0} \tag{4.27}$$

$$= \mathbf{v} \cdot \left(\nabla \ln p + \frac{m_e v^2}{2T} \nabla \ln T - \frac{5}{2} \nabla \ln T + \frac{e\mathbf{E}}{T} \right) f_{e0} \tag{4.28}$$

$$= \mathbf{v} \cdot \left(\left(\nabla \ln p + \frac{e\mathbf{E}}{T} \right) + \left(\frac{m_e v^2}{2T} - \frac{5}{2} \right) \nabla \ln T \right) f_{e0} \tag{4.29}$$

So that if we define

$$\mathbf{Q} = \left[\mathbf{A}_1 + \left(\frac{m_e v^2}{2T} - \frac{5}{2} \right) \mathbf{A}_2 \right] f_{e0} \tag{4.30}$$

$$\mathbf{A}_1 = \nabla \ln p + \frac{e\mathbf{E}}{T} \tag{4.31}$$

$$\mathbf{A}_2 = \nabla \ln T \tag{4.32}$$

Thus, we recover

$$C_e(f_{e1}) + \Omega_e(\mathbf{b} \times \mathbf{v}) \cdot \frac{\partial f_{e1}}{\partial \mathbf{v}} = \mathbf{v} \cdot \mathbf{Q}(v) \tag{4.33}$$

Let's now show that $\mathcal{L}(\mathbf{v}) = -\mathbf{v}$.

$$\mathcal{L}(\mathbf{v}) = \frac{1}{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathbf{v}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\mathbf{v}}{\partial\varphi^2}\right] \tag{4.34}$$

If we think of $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$ with $v^2 = v_x^2 + v_y^2 + v_z^2$, so that

$$v_x = v\sin\theta\cos\varphi \tag{4.35}$$
$$v_y = v\sin\theta\sin\varphi \tag{4.36}$$
$$v_z = v\cos\theta \tag{4.37}$$
$$\frac{\partial v_x}{\partial\theta} = v\cos\theta\cos\varphi = v_z\cos\varphi \tag{4.38}$$
$$\frac{\partial v_y}{\partial\theta} = v\cos\theta\sin\varphi = v_z\sin\varphi \tag{4.39}$$
$$\frac{\partial v_z}{\partial\theta} = -v\sin\theta \tag{4.40}$$
$$\tag{4.41}$$

So that

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_x}{\partial\theta}\right) = \frac{\partial}{\partial\theta}\left(v\sin\theta\cos\theta\cos\varphi\right) = \frac{v}{2}\cos\varphi\frac{\partial}{\partial\theta}\left(\sin(2\theta)\right) = v\cos\varphi\cos(2\theta) \tag{4.42}$$
$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_x}{\partial\theta}\right) = \frac{\partial}{\partial\theta}\left(v\sin\theta\cos\theta\sin\varphi\right) = \frac{v}{2}\sin\varphi\frac{\partial}{\partial\theta}\left(\sin(2\theta)\right) = v\sin\varphi\cos(2\theta) \tag{4.43}$$
$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_z}{\partial\theta}\right) = \frac{\partial}{\partial\theta}\left(-v\sin\theta\sin\theta\right) = -v\frac{\partial}{\partial\theta}\left(\sin^2(\theta)\right) = -2v\sin\theta\cos\theta \tag{4.44}$$

and

$$\frac{\partial^2 v_x}{\partial\varphi^2} = \frac{\partial}{\partial\varphi}(-v\sin\theta\sin\varphi) = -v\sin\theta\cos\varphi \tag{4.45}$$
$$\frac{\partial^2 v_y}{\partial\varphi^2} = \frac{\partial}{\partial\varphi}(v\sin\theta\cos\varphi) = -v\sin\theta\sin\varphi \tag{4.46}$$
$$\frac{\partial^2 v_z}{\partial\varphi^2} = 0 \tag{4.47}$$

So that altogether for the v_x we find

$$\begin{aligned}\mathcal{L}(v_x) &= \frac{v\cos\varphi\cos(2\theta)}{2\sin\theta} + \frac{-v\sin\theta\cos\varphi}{2\sin^2\theta} = \frac{v\cos\varphi}{2}\left(\frac{\cos^2\theta}{\sin\theta} - \sin\theta - \frac{1}{\sin\theta}\right)\\ &= \frac{v\cos\varphi\sin\theta}{2}\left(\cot^2\theta - 1 - \operatorname{sech}^2\theta\right) = \frac{v_x}{2}\left((\operatorname{sech}^2\theta - 1) - 1 - \operatorname{sech}^2\theta\right) = -v_x\end{aligned} \tag{4.48}$$

$$\begin{aligned}\mathcal{L}(v_y) &= \frac{v\sin\varphi\cos(2\theta)}{2\sin\theta} + \frac{-v\sin\theta\sin\varphi}{2\sin^2\theta} = \frac{v\sin\varphi}{2}\left(\frac{\cos^2\theta}{\sin\theta} - \sin\theta - \frac{1}{\sin\theta}\right)\\ &= \frac{v\sin\varphi\sin\theta}{2}\left(\cot^2\theta - 1 - \operatorname{sech}^2\theta\right) = \frac{v_y}{2}\left((\operatorname{sech}^2\theta - 1) - 1 - \operatorname{sech}^2\theta\right) = -v_y\end{aligned} \tag{4.49}$$

$$\mathcal{L}(v_z) = \frac{-2v\sin\theta\cos\theta}{2\sin\theta} = -v\cos\theta = -v_z \tag{4.50}$$

So that we find

$$\mathcal{L}(\mathbf{v}) = -\mathbf{v} \tag{4.51}$$

If we now take the Ansatz $f_{e1} = \mathbf{v} \cdot \mathbf{F}(v)$, with $C_e \to \nu_{ei}(v)\mathcal{L}$ we see that

$$C_e(\mathbf{v} \cdot \mathbf{F}(v)) + \Omega_e(\mathbf{b} \times \mathbf{v}) \cdot \frac{\partial}{\partial \mathbf{v}} (\mathbf{v} \cdot \mathbf{F}(v)) = \mathbf{v} \cdot \mathbf{Q}(v) \tag{4.52}$$

$$\nu_{ei}(v)\mathcal{L}(\mathbf{v}) \cdot \mathbf{F}(v) + \Omega_e(\mathbf{b} \times \mathbf{v}) \cdot \left[\frac{\partial \mathbf{F}(v)}{\partial \mathbf{v}} \cdot \mathbf{v} + \frac{\partial \mathbf{v}}{\partial \mathbf{v}} \cdot \mathbf{F}(v)\right] = \mathbf{v} \cdot \mathbf{Q}(v) \tag{4.53}$$

$$- \nu_{ei}(v)\mathbf{v} \cdot \mathbf{F}(v) + \Omega_e(\mathbf{b} \times \mathbf{v}) \cdot \left[\cancel{\hat{\mathbf{v}} \left(\frac{\partial \mathbf{F}(v)}{\partial v} \cdot \mathbf{v}\right)} + \mathbf{F}(v)\right] = \mathbf{v} \cdot \mathbf{Q}(v) \tag{4.54}$$

$$- \nu_{ei}(v)\mathbf{v} \cdot \mathbf{F}(v) + \Omega_e(\mathbf{b} \times \mathbf{v}) \cdot \mathbf{F}(v) = \mathbf{v} \cdot \mathbf{Q}(v) \tag{4.55}$$

with $\frac{\partial \mathbf{F}}{\partial \mathbf{v}} = \frac{\partial v}{\partial \mathbf{v}} \frac{\partial \mathbf{F}}{\partial v} = \hat{\mathbf{v}} \frac{\partial \mathbf{F}}{\partial \mathbf{v}}$.

We should note that $\mathbf{F}(v)$ is rather ambiguous notation. In this case, it means that in Cartesian form, that every component of $\mathbf{F}$ is dependent only on v. Clearly, this is not coordinate independent. If we move into spherical or cylindrical coordinates, then $\mathbf{F}$ is no longer simply a function of v.

Note we could also have gone down the route of

$$\frac{\partial}{\partial \mathbf{v}}(\mathbf{v} \cdot \mathbf{F}(v)) = \mathbf{v} \cdot \frac{\partial \mathbf{F}(v)}{\partial \mathbf{v}} + \mathbf{F}(v) \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{v}} + \mathbf{v} \times \left(\frac{\partial}{\partial \mathbf{v}} \times \mathbf{F}(v)\right) + \mathbf{F}(v) \times \left(\frac{\partial}{\partial \mathbf{v}} \times \mathbf{v}\right) \tag{4.56}$$

$$= v\frac{\partial \mathbf{F}(v)}{\partial v} + \mathbf{F}(v) + \mathbf{v} \times \left(\frac{\partial}{\partial \mathbf{v}} \times \mathbf{F}(v)\right) \tag{4.57}$$

where I have used

$$\mathbf{v} \cdot \frac{\partial \mathbf{F}(v)}{\partial \mathbf{v}} = \mathbf{v} \cdot \frac{\partial v}{\partial \mathbf{v}} \frac{\partial \mathbf{F}(v)}{\partial v} = \mathbf{v} \cdot \hat{\mathbf{v}} \frac{\partial \mathbf{F}(v)}{\partial v} = v\frac{\partial \mathbf{F}(v)}{\partial v} \tag{4.58}$$

$$\mathbf{F} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{v}} = \mathbf{F} \cdot \mathbb{1} = \mathbf{F} \tag{4.59}$$

$$\frac{\partial}{\partial \mathbf{v}} \times \mathbf{v} = \mathbf{0} \tag{4.60}$$

$$\mathbf{b} \times \mathbf{v} = v \cos\theta \hat{\boldsymbol{\varphi}} \tag{4.61}$$

So if we look at the curl term, we would find in spherical coordinates that

$$\mathbf{v} \times \left(\frac{\partial}{\partial \mathbf{v}} \times \mathbf{F}(v)\right) = \hat{\mathbf{v}} \times \left(\left[\frac{1}{\sin\theta} \frac{\partial F_v}{\partial \varphi} - \frac{\partial}{\partial v}(vF_\varphi)\right] \hat{\boldsymbol{\theta}} + \left(\frac{\partial}{\partial v}(vF_\theta) - \frac{\partial F_v}{\partial \theta}\right) \hat{\boldsymbol{\varphi}}\right) \tag{4.62}$$

where we would need to transform

$$\mathbf{F}(v) = F_{v_x}(v)\hat{\mathbf{x}} + F_{v_y}\hat{\mathbf{y}} + F_{v_z}\hat{\mathbf{z}} = F_v(v,\theta,\varphi)\hat{\mathbf{v}} + F_\theta(v,\theta,\varphi)\hat{\boldsymbol{\theta}} + F_\varphi(v,\theta,\varphi)\hat{\boldsymbol{\varphi}} \tag{4.63}$$

$$F_v = F_{v_x} \sin\theta \cos\varphi + F_{v_y} \sin\theta \sin\varphi + F_{v_z} \cos\theta \tag{4.64}$$

$$F_\theta = F_{v_x} \cos\theta \cos\varphi + F_{v_y} \cos\theta \sin\varphi - F_{v_z} \sin\theta \tag{4.65}$$

$$F_\varphi = -F_{v_x} \sin\varphi + F_{v_y} \cos\varphi \tag{4.66}$$

So that this method requires a bit more effort, as can be seen.

So, using that we need only look at the $\mathbf{v} \times \hat{\boldsymbol{\theta}}$ component, we'd find

$$\begin{aligned}
&\left(F_{v_y}\cos\varphi - F_{v_x}\sin\varphi\right) - \left(-\sin\varphi\frac{\partial}{\partial v}(vF_{v_x}) + \cos\varphi\frac{\partial}{\partial v}(vF_{v_y})\right) \\
&= \cos\varphi\left(F_{v_y} - v\frac{\partial F_{v_y}}{\partial v} - F_y\right) - \sin\varphi\left(F_{v_x} - v\frac{\partial F_{v_x}}{\partial v} - F_{v_x}\right) \\
&= -v\frac{\partial}{\partial v}\left(\sin\varphi F_{v_x} - \cos\varphi F_{v_y}\right) = -v\frac{\partial F_\varphi}{\partial v}
\end{aligned} \tag{4.67}$$

So that we find

$$\begin{aligned}
&\hat{\boldsymbol{\varphi}}\cdot\left[v\frac{\partial \mathbf{F}}{\partial v} + \mathbf{F} + \mathbf{v}\times\left(\frac{\partial}{\partial \mathbf{v}}\times\mathbf{F}\right)\right] \\
&= v\frac{\partial F_\varphi}{\partial v} + F_\varphi - v\frac{\partial F_\varphi}{\partial v} = F_\varphi
\end{aligned} \tag{4.68}$$

exactly as before.

So we have as our equation

$$-\nu_{ei}\mathbf{v}\cdot\mathbf{F} + \Omega_e(\mathbf{b}\times\mathbf{v})\cdot\mathbf{F} = \mathbf{v}\cdot\mathbf{Q} \tag{4.69}$$

$$-\nu_{ei}\mathbf{v}\cdot\mathbf{F} + \Omega_e(\mathbf{F}\times\mathbf{b})\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{Q} \tag{4.70}$$

$$\mathbf{v}\cdot(-\nu_{ei}\mathbf{F} + \Omega_e(\mathbf{F}\times\mathbf{b}) - \mathbf{Q}) = 0 \tag{4.71}$$

and so (since it's true for arbitrary $\mathbf{v}$)

$$-\nu_{ei}\mathbf{F} + \Omega_e(\mathbf{F}\times\mathbf{b}) = \mathbf{Q} \tag{4.72}$$

$$\nu_{ei}\mathbf{F} + \Omega_e(\mathbf{b}\times\mathbf{F}) = -\mathbf{Q} \tag{4.73}$$

$$\tag{4.74}$$

taking $\mathbf{b}\cdot$ the equation, we see

$$F_\parallel = -\frac{Q_\parallel}{\nu_{ei}} \tag{4.75}$$

If we take $\mathbf{b}\times$ the equation, we find

$$-\nu_{ei}\mathbf{F}_\vee + \Omega_e\mathbf{F}_\perp = \mathbf{Q}_\vee \tag{4.76}$$

with the definition

$$\mathbf{A}_\vee = \mathbf{b}\times\mathbf{A} \tag{4.77}$$

$$\mathbf{A}_\perp = -\mathbf{b}\times(\mathbf{b}\times\mathbf{A}) = -\mathbf{b}\times\mathbf{A}_\vee \tag{4.78}$$

$$\mathbf{b}\times\mathbf{A}_\perp = \mathbf{A}_\vee \tag{4.79}$$

$$\mathbf{A} = \mathbf{A}_\parallel + \mathbf{A}_\perp \tag{4.80}$$

with $\mathbf{A}_\parallel = \mathbf{A}\cdot\mathbf{b}\mathbf{b}$.

So that if we take $-\mathbf{b}\times\mathbf{b}\times$ the equations we find

$$-\nu_{ei}\mathbf{F}_\perp - \Omega_e\mathbf{F}_\vee = \mathbf{Q}_\perp \tag{4.81}$$

So that adding together our equations, we find

$$\begin{aligned}
(-\nu_{ei}\mathbf{F}_\vee + \Omega_e\mathbf{F}_\perp) + (-\nu_{ei}\mathbf{F}_\perp - \Omega_e\mathbf{F}_\vee) &= \mathbf{Q}_\vee + \mathbf{Q}_\perp && (4.82)\\
\mathbf{F}_\perp(\Omega_e - \nu_{ei}) - \mathbf{F}_\vee(\Omega_e + \nu_{ei}) &= \mathbf{Q}_\vee + \mathbf{Q}_\perp && (4.83)\\
\mathbf{F}_\perp\frac{(\Omega_e - \nu_{ei})}{\Omega_e + \nu_{ei}} - \mathbf{F}_\vee &= \frac{\mathbf{Q}_\vee + \mathbf{Q}_\perp}{\Omega_e + \nu_{ei}} && (4.84)
\end{aligned}$$

If we were to subtract we'd find

$$\begin{aligned}
(-\nu_{ei}\mathbf{F}_\vee + \Omega_e\mathbf{F}_\perp) - (-\nu_{ei}\mathbf{F}_\perp - \Omega_e\mathbf{F}_\vee) &= \mathbf{Q}_\vee - \mathbf{Q}_\perp && (4.85)\\
\mathbf{F}_\perp(\Omega_e + \nu_{ei}) + \mathbf{F}_\vee(\Omega_e - \nu_{ei}) &= \mathbf{Q}_\vee - \mathbf{Q}_\perp && (4.86)\\
\mathbf{F}_\perp\frac{(\Omega_e + \nu_{ei})}{\Omega_e - \nu_{ei}} + \mathbf{F}_\vee &= \frac{\mathbf{Q}_\vee - \mathbf{Q}_\perp}{\Omega_e - \nu_{ei}} && (4.87)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{F}_\perp\left(\frac{\Omega_e - \nu_{ei}}{\Omega_e + \nu_{ei}} + \frac{\Omega_e + \nu_{ei}}{\Omega_e - \nu_{ei}}\right) &= \mathbf{Q}_\vee\left(\frac{1}{\Omega_e + \nu_{ei}} + \frac{1}{\Omega_e - \nu_{ei}}\right) + \mathbf{Q}_\perp\left(\frac{1}{\Omega_e + \nu_{ei}} - \frac{1}{\Omega_e - \nu_{ei}}\right) && (4.88)\\
\mathbf{F}_\perp\left(\frac{2(\Omega_e^2 + \nu_{ei}^2)}{\Omega_e^2 - \nu_{ei}^2}\right) &= \mathbf{Q}_\vee\left(\frac{2\Omega_e}{\Omega_e^2 - \nu_{ei}^2}\right) + \mathbf{Q}_\perp\left(\frac{-2\nu_{ei}}{\Omega_e^2 - \nu_{ei}^2}\right) && (4.89)\\
\mathbf{F}_\perp &= \mathbf{Q}_\vee\left(\frac{\Omega_e}{\Omega_e^2 + \nu_{ei}^2}\right) + \mathbf{Q}_\perp\left(\frac{-\nu_{ei}}{\Omega_e^2 + \nu_{ei}^2}\right) && (4.90)
\end{aligned}$$

and therefore

$$\mathbf{F}_\vee = \mathbf{F}_\perp\frac{\Omega_e - \nu_{ei}}{\Omega_e + \nu_{ei}} - \frac{\mathbf{Q}_\vee + \mathbf{Q}_\perp}{\Omega_e + \nu_{ei}} = \mathbf{Q}_\vee\left(\frac{-\nu_{ei}}{\Omega_e^2 + \nu_{ei}^2}\right) + \mathbf{Q}_\perp\left(\frac{-\Omega_e}{\Omega_e^2 + \nu_{ei}^2}\right) \tag{4.91}$$

which is here simply for completeness. Note that

$$\begin{aligned}
\mathbf{v}_\vee\cdot\mathbf{Q} &= \mathbf{b}\times\mathbf{v}\cdot\mathbf{Q} = -\mathbf{b}\times\mathbf{Q}\cdot\mathbf{v} = -\mathbf{Q}_\vee\cdot\mathbf{v} = -\mathbf{v}\cdot\mathbf{Q}_\vee && (4.92)\\
\mathbf{v}_\perp\cdot\mathbf{Q} &= \mathbf{v}\cdot\mathbf{Q}_\perp && (4.93)\\
\mathbf{A} &= \mathbf{A}_\parallel + \mathbf{A}_\perp && (4.94)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{F} &= (\mathbf{v}_\parallel + \mathbf{v}_\perp)\cdot(\mathbf{F}_\parallel + \mathbf{F}_\perp) = \mathbf{v}_\parallel\cdot\mathbf{F}_\parallel + (\mathbf{v}_\parallel + \mathbf{v}_\perp)\cdot\mathbf{F}_\perp\\
&= \mathbf{v}_\parallel\cdot\mathbf{F}_\parallel + \mathbf{v}\cdot\mathbf{F}_\perp\\
&= \mathbf{v}_\parallel\cdot\frac{-\mathbf{Q}_\parallel}{\nu_{ei}} + \mathbf{v}\cdot\left(\mathbf{Q}_\vee\frac{\Omega_e}{\Omega_e^2 + \nu_{ei}^2} + \mathbf{Q}_\perp\frac{-\nu_{ei}}{\Omega_e^2 + \nu_{ei}^2}\right)\\
&= -\mathbf{v}_\parallel\cdot\frac{\mathbf{Q}_\parallel}{\nu_{ei}} - \mathbf{v}_\vee\cdot\mathbf{Q}\frac{\Omega_e}{\Omega_e^2 + \nu_{ei}^2} - \mathbf{v}_\perp\cdot\mathbf{Q}\frac{\nu_{ei}}{\Omega_e^2 + \nu_{ei}^2}\\
&= -\left(\frac{\mathbf{v}_\parallel}{\nu_{ei}} + \frac{\nu_{ei}\mathbf{v}_\perp + \Omega_e\mathbf{v}_\vee}{\nu_{ei}^2 + \Omega_e^2}\right)\cdot\mathbf{Q} = -\left(\frac{\mathbf{v}_\parallel}{\nu_{ei}} + \frac{\nu_{ei}\mathbf{v}_\perp + \Omega_e\mathbf{b}\times\mathbf{v}}{\nu_{ei}^2 + \Omega_e^2}\right)\cdot\mathbf{Q}
\end{aligned} \tag{4.95}$$

We then see that we have something slightly different than the given

$$f_{e1} = -\left(\frac{\mathbf{v}_{\parallel}}{\nu_{ei}} + \frac{\nu_{ei}\mathbf{v}_{\perp} - \Omega_e \mathbf{b}\times\mathbf{v}}{\nu_{ei}^2 + \Omega_e^2}\right)\cdot\mathbf{Q} \tag{4.96}$$

We can prove that this is a typo on the part of Helander and Sigmar. Unfortunately, the differing sign on the cross component appears to give results agreeing with Braginskii, implying that they have done something incorrectly.

Take $\mathbf{b} = \hat{\mathbf{z}}$ and do everything component by component in (x, y, z). First we note that

$$-\nu_{ei}F_x\hat{\mathbf{x}} - \nu_{ei}F_y\hat{\mathbf{y}} - \nu_{ei}F_z\hat{\mathbf{z}} + \Omega_e\left(-F_x\hat{\mathbf{y}} + F_y\hat{\mathbf{x}}\right) = Q_x\hat{\mathbf{x}} + Q_y\hat{\mathbf{y}} + Q_z\hat{\mathbf{z}} \tag{4.97}$$

The components then say

$$-\nu_{ei}F_z = Q_z \tag{4.98}$$
$$-\nu_{ei}F_y - \Omega_e F_x = Q_y \tag{4.99}$$
$$-\nu_{ei}F_x + \Omega_e F_y = Q_x \tag{4.100}$$

Isolate F_x in the last two equations

$$\frac{-\nu_{ei}}{\Omega_e}F_y - F_x = \frac{Q_y}{\Omega_e} \tag{4.101}$$
$$F_x - \frac{\Omega_e}{\nu_{ei}}F_y = -\frac{Q_x}{\nu_{ei}} \tag{4.102}$$

Solve for F_y by adding these two equations

$$-F_y\left(\frac{\nu_{ei}}{\Omega_e} + \frac{\Omega_e}{\nu_{ei}}\right) = \frac{Q_y}{\Omega_e} - \frac{Q_x}{\nu_{ei}} \tag{4.103}$$
$$F_y\left(\frac{\nu_{ei}^2 + \Omega_e^2}{\nu_{ei}\Omega_e}\right) = -\frac{Q_y}{\Omega_e} + \frac{Q_x}{\nu_{ei}} \tag{4.104}$$
$$F_y = Q_x\frac{\Omega_e}{\nu_{ei}^2 + \Omega_e^2} + Q_y\frac{-\nu_{ei}}{\nu_{ei}^2 + \Omega_e^2} \tag{4.105}$$

Backsolve for F_x

$$F_x = \frac{-Q_x}{\nu_{ei}} + \frac{\Omega_e}{\nu_{ei}}F_y = -Q_x\frac{1}{\nu_{ei}} - Q_y\frac{\Omega_e}{\nu_{ei}^2 + \Omega_e^2} + Q_x\frac{\Omega_e^2}{\nu_{ei}(\nu_{ei}^2 + \Omega_e^2)} \tag{4.106}$$
$$F_x = Q_x\frac{-\nu_{ei}^2 - \Omega_e^2 + \Omega_e^2}{\nu_{ei}(\nu_{ei}^2 + \Omega_e^2)} - Q_y\frac{\Omega_e}{\nu_{ei}^2 + \Omega_e^2} = Q_x\frac{-\nu_{ei}}{\nu_{ei}^2 + \Omega_e^2} + Q_y\frac{-\Omega_e}{\nu_{ei}^2 + \Omega_e^2} \tag{4.107}$$

Thus, we find

$$\mathbf{v}\cdot\mathbf{F} = v_xF_x + v_yF_y + v_zF_z \tag{4.108}$$
$$= v_xQ_x\frac{-\nu_{ei}}{\nu_{ei}^2 + \Omega_e^2} + v_xQ_y\frac{-\Omega_e}{\nu_{ei}^2 + \Omega_e^2} + v_yQ_x\frac{\Omega_e}{\nu_{ei}^2 + \Omega_e^2} - v_yQ_y\frac{\nu_{ei}}{\nu_{ei}^2 + \Omega_e^2} + v_z\frac{-Q_z}{\nu_{ei}} \tag{4.109}$$
$$= \frac{\Omega_e}{\nu_{ei}^2 + \Omega_e^2}\left(v_yQ_x - v_xQ_y\right) - \frac{\nu_{ei}}{\nu_{ei}^2 + \Omega_e^2}\left(v_xQ_x + v_yQ_y\right) + v_z\frac{-Q_z}{\nu_{ei}} \tag{4.110}$$

Note that with $\mathbf{b} = \hat{\mathbf{z}}$ that

$$(\mathbf{b} \times \mathbf{v}) \cdot \mathbf{Q} = (v_x \hat{\mathbf{y}} - v_y \hat{\mathbf{x}}) \cdot \mathbf{Q} = v_x Q_y - v_y Q_x \tag{4.111}$$

$$\mathbf{v}_\perp \cdot \mathbf{Q} = -\mathbf{b} \times (\mathbf{b} \times \mathbf{v}) \cdot \mathbf{Q} = (v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}) \cdot \mathbf{Q} = v_x Q_x + v_y Q_y \tag{4.112}$$

and so

$$\mathbf{v} \cdot \mathbf{F} = \frac{\Omega_e}{\nu_{ei}^2 + \Omega_e^2} (-\mathbf{b} \times \mathbf{v} \cdot \mathbf{Q}) - \frac{\nu_{ei}}{\nu_{ei}^2 + \Omega_e^2} (\mathbf{v}_\perp \cdot \mathbf{Q}) - \frac{\mathbf{v}_\parallel}{\nu_{ei}} \cdot \mathbf{Q} \tag{4.113}$$

$$= -\left(\frac{\mathbf{v}_\parallel}{\nu_{ei}} + \frac{\Omega_e \mathbf{b} \times \mathbf{v} + \nu_{ei} \mathbf{v}_\perp}{\nu_{ei}^2 + \Omega_e^2} \right) \cdot \mathbf{Q} \tag{4.114}$$

exactly matching the value we calculated above in the same manner.

Now, the book takes the idea that

$$f_{e1} = f_\parallel + f_\wedge + f_\perp \tag{4.115}$$

with

$$f_\parallel = \frac{-v_\parallel Q_\parallel}{\nu_{ei}} \tag{4.116}$$

$$f_\wedge = \frac{\Omega_e \mathbf{b} \times \mathbf{v} \cdot \mathbf{Q}}{\nu_{ei}^2 + \Omega_e^2} \tag{4.117}$$

$$f_\perp = -\frac{\mathbf{v}_\perp \cdot \mathbf{Q}}{\nu_{ei}^2 + \Omega_e^2} \tag{4.118}$$

(of course, we would actually expect $f_\wedge$ to be the negative of this, but this gives an incorrect sign for some reason.) so that

$$n_e \mathbf{u}_\parallel = \int \mathrm{d}^3 v \; f_\parallel \mathbf{v} \tag{4.119}$$

$$n_e \mathbf{u}_\wedge = \int \mathrm{d}^3 v \; f_\wedge \mathbf{v} \tag{4.120}$$

$$n_e \mathbf{u}_\perp = \int \mathrm{d}^3 v \; f_\perp \mathbf{v} \tag{4.121}$$

Let's do this component by component. Now let's convert into a spherical coordinate system. We then have

$$\int_{-\infty}^{\infty} \mathrm{d}v_x \int_{-\infty}^{\infty} \mathrm{d}v_y \int_{-\infty}^{\infty} \mathrm{d}v_z \to \int_0^\infty \mathrm{d}v \int_0^{2\pi} \mathrm{d}\varphi \int_0^\pi \mathrm{d}\theta \; v^2 \sin\theta \tag{4.122}$$

If we take $\zeta = \cos\theta$ with $\cos\theta = \frac{v_\parallel}{v}$ then $\mathrm{d}\zeta = -\sin\theta \, \mathrm{d}\theta$ and so we get

$$\int_0^\infty \mathrm{d}v \int_0^{2\pi} \mathrm{d}\varphi \int_0^\pi \mathrm{d}\theta \; v^2 \sin\theta \to \int_0^\infty \mathrm{d}v \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^1 \mathrm{d}\zeta \; v^2 \tag{4.123}$$

where we note that there will be no dependence on φ in our problem, as we have aligned $\mathbf{b} = \mathbf{z}$ and there is no azimuthal variation. Note that this is completely general, as we can align our v_x,

v_y, and v_z at any point where there is a magnetic field such that v_z is pointing in the direction of the magnetic field (because we are going to integrate over v). We could as well call it $\hat{\mathbf{v}}_z$ rather than $\hat{\mathbf{z}}$ to emphasize it isn't important in space, but only in velocity space.

Thus

$$\int_0^\infty \mathrm{d}v \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^1 \mathrm{d}\zeta\ v^2 \to \int_0^\infty \mathrm{d}v \int_{-1}^1 \mathrm{d}\zeta\ 2\pi v^2 \tag{4.124}$$

if we have no φ dependence. Thus, we would see (using $\Omega_e \gg \nu_{ei}$)

$$\left(\frac{\mathbf{v}_\parallel}{\nu_{ei}} + \frac{\mathbf{b}\times\mathbf{v}}{\Omega_e} + \frac{\nu_{ei}\mathbf{v}_\perp}{\Omega_e^2}\right)\cdot\mathbf{Q} = \frac{vQ_\parallel\zeta}{\nu_{ei}} + \frac{v\sin\theta\,(Q_x\sin\varphi - Q_y\cos\varphi)}{\Omega_e} + \frac{\nu_{ei}v\sin\theta(Q_x\cos\varphi + Q_y\sin\varphi)}{\Omega_e^2} \tag{4.125}$$

And so, eliminating the dependences on φ because they will go to zero,

$$n_e u_\parallel = -\int_0^\infty \mathrm{d}v \int_{-1}^1 \mathrm{d}\zeta\ 2\pi v^2 f_{e0}\left(\frac{v_\parallel Q_\parallel}{\nu_{ei}}\right)\mathbf{b}\cdot\mathbf{v} \tag{4.126}$$

and find for the $\mathbf{v}_\parallel\cdot\mathbf{Q}$ part of the integral that (using $\nu_{ei} = \frac{3\sqrt{\pi}}{4\tau_{ei}}\frac{v_{\mathrm{th}_e}^3}{v^3}$)

$$-\int_0^\infty \mathrm{d}v \int_{-1}^1 \mathrm{d}\zeta\ 2\pi v^2\left(\frac{v\zeta}{\nu_{ei}}\left[A_{1,\parallel} + \left(\frac{m_e v^2}{2T_e} - \frac{5}{2}\right)A_{2,\parallel}\right]\right)v\zeta\frac{n_e}{\pi^{3/2}v_{\mathrm{th}_e}^3}e^{-v^2/v_{\mathrm{th}_e}^2} \tag{4.127}$$

$$= -\frac{8\pi\tau_{ei}}{3\sqrt{\pi}v_{\mathrm{th}_e}^3}\int_0^\infty \mathrm{d}v \int_{-1}^1 \mathrm{d}\zeta\ v^7\zeta^2\left[A_{1,\parallel} + \left(\frac{m_e v^2}{2T_e} - \frac{5}{2}\right)A_{2,\parallel}\right]\frac{n_e}{\pi^{3/2}v_{\mathrm{th}_e}^3}e^{-v^2/v_{\mathrm{th}_e}^2} \tag{4.128}$$

$$= -\frac{2}{3}\frac{8n_e v_{\mathrm{th}_e}^2\tau_{ei}}{3\pi}\int_0^\infty \mathrm{d}x\ x^7\left[A_{1,\parallel} + \left(\frac{m_e v_{\mathrm{th}_e}^2 x^2}{2T_e} - \frac{5}{2}\right)A_{2,\parallel}\right]e^{-x^2} \tag{4.129}$$

$$= -\frac{16n_e v_{\mathrm{th}_e}^2\tau_{ei}}{9\pi}\left[3A_{1,\parallel} + \left(12\frac{m_e v_{\mathrm{th}_e}^2}{2T_e} - 3\frac{5}{2}\right)A_{2,\parallel}\right] \tag{4.130}$$

$$= -\frac{32n_e T_e\tau_{ei}}{9\pi m_e}\left[3A_{1,\parallel} + \left(12 - \frac{15}{2}\right)A_{2,\parallel}\right] \tag{4.131}$$

$$= -\frac{32n_e T_e\tau_{ei}}{9\pi m_e}\left[3\left(\frac{\nabla_\parallel p_e}{n_e T_e} + \frac{eE_\parallel}{T_e}\right) + \left(\frac{9}{2}\right)\frac{\nabla_\parallel T_e}{T_e}\right] \tag{4.132}$$

$$= -\frac{32}{3\pi}\frac{\tau_{ei}}{m_e}\left[\nabla_\parallel p_e + n_e E_\parallel + \frac{3}{2}n_e\,\nabla_\parallel T_e\right] \tag{4.133}$$

which matches the required calculation.

Our above calculation matches this.

As an aside, let's remind ourselves of some notation in our case. Given a vector $\mathbf{Q}$ and a vector

$\mathbf{T}$, we can write

$$\mathbf{Q} = Q_\parallel \mathbf{T}_\parallel + Q_\wedge \mathbf{T}_\wedge + Q_\perp \mathbf{T}_\perp \tag{4.134}$$

$$\mathbf{T}_\parallel = \mathbf{b} \tag{4.135}$$

$$\mathbf{T}_\wedge = \mathbf{b} \times \mathbf{T} \tag{4.136}$$

$$\mathbf{T}_\perp = -\mathbf{b} \times (\mathbf{b} \times \mathbf{T}) \tag{4.137}$$

$$Q_\parallel = \frac{\mathbf{Q} \cdot \mathbf{T}_\parallel}{\mathbf{T}_\parallel \cdot \mathbf{T}_\parallel} \tag{4.138}$$

$$Q_\wedge = \frac{\mathbf{Q} \cdot \mathbf{T}_\wedge}{\mathbf{T}_\wedge \cdot \mathbf{T}_\wedge} \tag{4.139}$$

$$Q_\perp = \frac{\mathbf{Q} \cdot \mathbf{T}_\perp}{\mathbf{T}_\perp \cdot \mathbf{T}_\perp} \tag{4.140}$$

for some vector $\mathbf{T}$. Thus, when we write $\mathbf{u}_\wedge$ this is what we mean. We are writing it so that it comes out as terms that are $\mathbf{T}_\wedge$ for (perhaps) multiple $\mathbf{T}$.

Now, let's work on the cross component.

$$n_e \mathbf{u}_\wedge = -\int_0^\infty \mathrm{d}v \ \int_{-1}^1 \mathrm{d}\zeta \ \int_0^{2\pi} \mathrm{d}\varphi \ v^2 v^2 f_{e0} (1-\zeta^2) \frac{Q_x \sin\varphi - Q_y \cos\varphi}{\Omega_e} (\cos\varphi \hat{\mathbf{x}} + \sin\varphi \hat{\mathbf{y}} + \zeta \hat{\mathbf{z}}) \tag{4.141}$$

Clearly the $\hat{\mathbf{z}}$ term will cancel out because it is odd in ζ. Now let's focus on $\hat{\mathbf{x}}$. Clearly the $\sin\varphi\cos\varphi$ term will be odd in φ and so cancel. Thus

$$\begin{aligned}
&\hat{\mathbf{x}} \int_0^\infty \mathrm{d}v \ \int_{-1}^1 \mathrm{d}\zeta \ \int_0^{2\pi} \mathrm{d}\varphi \ v^4 f_{e0} (1-\zeta^2) \frac{Q_y \cos^2\varphi}{\Omega_e} \\
&= \hat{\mathbf{x}} \int_0^\infty \mathrm{d}v \ v^4 f_{e0} \frac{4\pi}{3\Omega_e} Q_y = \hat{\mathbf{x}} \int_0^\infty \mathrm{d}v \ v^4 f_{e0} \frac{4\pi}{3\Omega_e} \left[A_{1,y} + \left(\frac{v^2}{v_{\mathrm{th}_e}^2} - \frac{5}{2} \right) A_{2,y} \right] \\
&= \hat{\mathbf{x}} \frac{4 n_e v_{\mathrm{th}_e}^2}{3\sqrt{\pi}\Omega_e} \int_0^\infty \mathrm{d}x \ x^4 \left[A_{1,y} + \left(x^2 - \frac{5}{2} \right) A_{2,y} \right] = \hat{\mathbf{x}} \frac{8 n_e T_e}{3 m_e \sqrt{\pi} \Omega_e} \left[\frac{3\sqrt{\pi}}{8} A_{1,y} + (0) A_{2,y} \right] \\
&= \hat{\mathbf{x}} \frac{n_e T_e}{m_e \Omega_e} A_{1,y}
\end{aligned} \tag{4.142}$$

Similarly for the $\hat{\mathbf{y}}$ component we find

$$\begin{aligned}
&-\hat{\mathbf{y}} \int_0^\infty \mathrm{d}v \ \int_{-1}^1 \mathrm{d}\zeta \ \int_0^{2\pi} \mathrm{d}\varphi \ v^4 f_{e0} (1-\zeta^2) \frac{Q_x \sin^2\varphi}{\Omega_e} \\
&= -\hat{\mathbf{y}} \int_0^\infty \mathrm{d}v \ v^4 f_{e0} \frac{4\pi}{3\Omega_e} Q_x = -\hat{\mathbf{y}} \int_0^\infty \mathrm{d}v \ v^4 f_{e0} \frac{4\pi}{3\Omega_e} \left[A_{1,x} + \left(\frac{v^2}{v_{\mathrm{th}_e}^2} - \frac{5}{2} \right) A_{2,x} \right] \\
&= -\hat{\mathbf{y}} \frac{4 n_e v_{\mathrm{th}_e}^2}{3\sqrt{\pi}\Omega_e} \int_0^\infty \mathrm{d}x \ x^4 \left[A_{1,x} + \left(x^2 - \frac{5}{2} \right) A_{2,x} \right] = -\hat{\mathbf{y}} \frac{8 n_e T_e}{3 m_e \sqrt{\pi} \Omega_e} \left[\frac{3\sqrt{\pi}}{8} A_{1,x} + (0) A_{2,x} \right] \\
&= -\hat{\mathbf{y}} \frac{n_e T_e}{m_e \Omega_e} A_{1,x}
\end{aligned} \tag{4.143}$$

Using

$$\frac{n_e T_e}{m_e} \left[\frac{\nabla_i p_e}{p_e} + \frac{e \mathbf{E}_i}{T_e} \right] = \frac{\nabla_i p_e}{m_e} + n_e e \mathbf{E}_i \tag{4.144}$$

We also use that $\mathbf{b}\times\boldsymbol{\nabla}p_e = \nabla_x p_e\hat{\mathbf{y}} - \nabla_y p_e\hat{\mathbf{x}}$ and $\hat{\mathbf{b}}\times\mathbf{E} = E_x\hat{\mathbf{y}} - E_y\hat{\mathbf{x}}$ so that

$$n_e\mathbf{u}_\wedge = \frac{\nabla p_e\times\mathbf{b}}{\Omega_e m_e} + \frac{n_e e}{m_e\Omega_e}\mathbf{E}\times\mathbf{b} = \frac{\nabla p_e\times\mathbf{b}}{m_e\Omega_e} - n_e\frac{\mathbf{E}\times\mathbf{B}}{B^2} = -\left(\frac{\mathbf{b}\times\nabla p_e}{m_e\Omega_e} + n_e\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right) \tag{4.145}$$

where I have used that $\frac{eB}{m_e\Omega_e} = \frac{eB}{m_e\frac{-eB}{m_e}} = -1$. Note we get the opposite sign to the book, because the book defined $f_\wedge$ as the negative to mine. It appears that the definition given in the book corresponds to the correct value, though.

Similarly, we can recover $n_e\mathbf{u}_\perp$ with $f_\perp$.

Let's also calculate $\mathbf{q}_\wedge$

$$\mathbf{q}_\wedge = -\int_0^\infty \mathrm{d}v\int_{-1}^1\mathrm{d}\zeta\int_0^{2\pi}\mathrm{d}\varphi\; v^2v^2 f_{e0}(1-\zeta^2)T_e\left(\frac{v^2}{v_{\mathrm{th}_e}^2}-\frac{5}{2}\right)\frac{Q_x\sin\varphi - Q_y\cos\varphi}{\Omega_e}\left(\cos\varphi\hat{\mathbf{x}}+\sin\varphi\hat{\mathbf{y}}+\zeta\hat{\mathbf{z}}\right) \tag{4.146}$$

$$\tag{4.147}$$

Once again, the $\hat{\mathbf{z}}$ component will cancel. The $\hat{\mathbf{x}}$ component yields

$$\hat{\mathbf{x}}\frac{4\pi T_e}{3\Omega_e}\int_0^\infty \mathrm{d}v\; v^4\left(\frac{v^2}{v_{\mathrm{th}_e}^2}-\frac{5}{2}\right)f_{e0}\left[A_{1,y}+\left(\frac{v^2}{v_{\mathrm{th}_e}^2}-\frac{5}{2}\right)A_{2,y}\right] \tag{4.148}$$

$$= \hat{\mathbf{x}}\frac{4\pi n T_e v_{\mathrm{th}_e}^2}{3\pi^{3/2}\Omega_e}\int_0^\infty \mathrm{d}x\; x^4\left(x^2-\frac{5}{2}\right)\left[A_{1,y}+\left(x^2-\frac{5}{2}\right)A_{2,y}\right] \tag{4.149}$$

$$= \hat{\mathbf{x}}\frac{4nT_e v_{\mathrm{th}_e}^2}{3\sqrt{\pi}\Omega_e}A_{2,y}\frac{15\sqrt{\pi}}{16} = \frac{5nT_e v_{\mathrm{th}_e}^2}{4\Omega_e}A_{2,y}\hat{\mathbf{x}} \tag{4.150}$$

And similarly for the $\hat{\mathbf{y}}$ we find

$$-\hat{\mathbf{y}}\frac{4\pi T_e}{3\Omega_e}\int_0^\infty \mathrm{d}v\; v^4\left(\frac{v^2}{v_{\mathrm{th}_e}^2}-\frac{5}{2}\right)f_{e0}\left[A_{1,x}+\left(\frac{v^2}{v_{\mathrm{th}_e}^2}-\frac{5}{2}\right)A_{2,x}\right] \tag{4.151}$$

$$= -\hat{\mathbf{y}}\frac{4\pi n T_e v_{\mathrm{th}_e}^2}{3\pi^{3/2}\Omega_e}\int_0^\infty \mathrm{d}x\; x^4\left(x^2-\frac{5}{2}\right)\left[A_{1,x}+\left(x^2-\frac{5}{2}\right)A_{2,x}\right] \tag{4.152}$$

$$= -\hat{\mathbf{y}}\frac{4nT_e v_{\mathrm{th}_e}^2}{3\sqrt{\pi}\Omega_e}A_{2,x}\frac{15\sqrt{\pi}}{16} = -\frac{5nT_e v_{\mathrm{th}_e}^2}{4\Omega_e}A_{2,x}\hat{\mathbf{y}} \tag{4.153}$$

Thus, using $\mathbf{b}\times\mathbf{A} = A_x\hat{\mathbf{y}} - A_y\hat{\mathbf{x}}$,

$$\mathbf{q}_\wedge = -\frac{5nT_e v_{\mathrm{th}_e}^2}{4\Omega_e}\hat{\mathbf{b}}\times\mathbf{A}_2 = -\frac{5nT_e^2}{2m_e\Omega_e}\hat{\mathbf{b}}\times\frac{\nabla T_e}{T_e} = -\frac{5nT_e}{2m_e\Omega_e}\hat{\mathbf{b}}\times\nabla T_e \tag{4.154}$$

Let's note that

$$f_\wedge = -\frac{\mathbf{b}\times\mathbf{v}\cdot\mathbf{Q}}{\Omega_e} \tag{4.155}$$

Note that

$$\nabla f_{e0} = \left[\nabla\ln p + \left(\frac{v^2}{v_{\mathrm{th}_e}^2}-\frac{5}{2}\right)\nabla\ln T\right]f_{e0} \tag{4.156}$$

whereas

$$\mathbf{Q} = \left[\nabla \ln p + \frac{e\mathbf{E}}{T} + \left(\frac{v^2}{v_{\text{th}_e}^2} - \frac{5}{2} \right) \nabla \ln T \right] \tag{4.157}$$

so that the claim that $f_\wedge = -\frac{\mathbf{b}\times\mathbf{v}\cdot\nabla f_{e0}}{\Omega_e}$ is the same as

$$\mathbf{b} \times \mathbf{v} \cdot \frac{e\mathbf{E}}{T} = 0 \tag{4.158}$$

This must not be what Helander actually means. What Helander must mean is that the diamagnetic velocity given by

$$-\frac{\mathbf{b} \times \nabla p_e}{m_e n_e \Omega_e} \tag{4.159}$$

and the diamagnetic heat flux (so purposely excluding the $\mathbf{E} \times \mathbf{B}$ drift) will give f_D the drift part of $f_\wedge$. If we say that the $\mathbf{E} \times \mathbf{B}$ drift is excluded then $f_D = -\boldsymbol{\rho} \cdot \nabla f_{e0}$ with $\boldsymbol{\rho} = \mathbf{b} \times \mathbf{v}/\Omega_e$.

$$f_{e0} + f_D = (1 - \boldsymbol{\rho} \cdot \nabla) f_{e0}(\mathbf{r}) \tag{4.160}$$

Now let's Taylor expand around $\mathbf{R} = \mathbf{r} - \boldsymbol{\rho}$

$$f_{e0}(\mathbf{r}) = f_{e0}(\mathbf{R}) + \boldsymbol{\rho} \cdot \nabla f_{e0}|_{\mathbf{r}=\mathbf{R}} + \cdots \tag{4.161}$$

So that to this order we find near $\mathbf{r} = \mathbf{R}$ (so that $\nabla f_{e0}|_{\mathbf{r}=\mathbf{R}} \approx \nabla f_{e0}(\mathbf{r})$)

$$f_{e0}(\mathbf{r}) - \boldsymbol{\rho} \cdot \nabla f_{e0} \approx f_{e0}(\mathbf{R}) \tag{4.162}$$

4.3 Onsager Symmetry and a Variational Principle

Let's show that $\delta Q = \delta R$. First

$$R = 2P[\varphi] - S[\varphi, \varphi] \tag{4.163}$$

$$Q = \frac{P^2[\varphi]}{S[\varphi, \varphi]} \tag{4.164}$$

so that

$$\delta R = 2P\delta P - \delta S \tag{4.165}$$

$$\delta Q = \frac{(2P\delta P)S - P^2\delta S}{S^2} = \frac{P}{S} 2\delta P - \frac{P^2}{S^2}\delta S = \frac{P}{S}\left(2\delta P - \frac{P}{S}\delta S\right) \tag{4.166}$$

But $P/S = 1$ and so

$$\delta Q = 2\delta P - \delta S = \delta R \tag{4.167}$$

4.4 Spitzer Conductivity

Let's look at how to get f_{e1} from

$$\left(\nu_D^{ee} + \nu_D^{ei}\right) \mathcal{L} f_{e1} = -(u_* + \nu_D^{ee} u) \frac{m_e v_\parallel}{T_e} f_{e0} \tag{4.168}$$

$$u_* = \frac{eE_\parallel}{m_e} \tag{4.169}$$

So that

$$\mathcal{L} f_{e1} = -\frac{u_* + \nu_D^{ee} u}{\nu_D^{ee} + \nu_D^{ei}} \frac{m_e v_\parallel}{T_e} f_{e0} \tag{4.170}$$

Given that $\mathcal{L}(\mathbf{v}) = -\mathbf{v}$ for $\mathbf{v} = v\hat{\mathbf{e}}$ for some direction $\hat{\mathbf{e}}$ then assuming $f_{e1} = \mathbf{v} \cdot \mathbf{F}(v)$ again we see that

$$-f_{e1} = -\mathbf{v} \cdot \mathbf{F}(v) = -\frac{u_* + \nu_D^{ee} u}{\nu_D^{ee} + \nu_D^{ei}} \frac{m_e v_\parallel}{T_e} f_{e0} \tag{4.171}$$

$$f_{e1} = \frac{u_* + \nu_D^{ee} u}{\nu_D^{ee} + \nu_D^{ei}} \frac{m_e v_\parallel}{T_e} f_{e0} \tag{4.172}$$

as promised.

So that using momentum conservation

$$\int \mathrm{d}^3 v \; v_\parallel C_{ee}(f_{e1}) = \int \mathrm{d}^3 v \; \nu_D^{ee} \left(-\frac{\nu_D^{ee} u + u_*}{\nu_D^{ee} + \nu_D^{ei}} + u \right) \frac{m_e v_\parallel^2}{T_e} f_{e0} \tag{4.173}$$

from

$$C_{ee}(f_{e1}) = \nu_{DD}^{ee} \mathcal{L} f_{e1} + \frac{m_e \nu_D^{ee} u}{T_e} v_\parallel f_{e0} \tag{4.174}$$

If we define

$$\{F(v)\} = \int \mathrm{d}^3 v \; F \frac{m v_\parallel^2}{nT} f_M = \frac{8}{3\sqrt{\pi}} \int_0^\infty \mathrm{d}x \; F(x) e^{-x^2} x^4 \tag{4.175}$$

we can find (using that $v_\parallel^2 \, \mathrm{d}^3 v = 4\pi v^4 \zeta^2 \, \mathrm{d}v \, \mathrm{d}\zeta$ with $\zeta = \frac{v_\parallel}{v}$)

$$\int \mathrm{d}^3 v \; \nu_D^{ee} \left(-\frac{\nu_D^{ee} u + u_*}{\nu_D^{ee} + \nu_D^{ei}} + u \right) \frac{m_e v_\parallel^2}{T_e} f_{e0} = n \left\{ \nu_D^{ee} \left(-\frac{\nu_D^{ee} u + u_*}{\nu_D^{ee} + \nu_D^{ei}} + u \right) \right\} = 0 \tag{4.176}$$

so that

$$\{\nu_D^{ee}\} u - \left\{ \nu_D^{ee} \frac{\nu_D^{ee}}{\nu_D^{ee} + \nu_D^{ei}} \right\} u = \left\{ \frac{\nu_D^{ee}}{\nu_D^{ee} + \nu_D^{ei}} \right\} u_* \tag{4.177}$$

$$\left\{ \frac{\nu_D^{ee} \left(\nu_D^{ee} + \nu_D^{ei}\right)}{\nu_D^{ee} + \nu_D^{ei}} - \frac{(\nu_D^{ee})^2}{\nu_D^{ee} + \nu_D^{ei}} \right\} u = \left\{ \frac{\nu_D^{ee}}{\nu_D^{ee} + \nu_D^{ei}} \right\} u_* \tag{4.178}$$

$$\left\{ \frac{\nu_D^{ee} \nu_D^{ei}}{\nu_D^{ee} + \nu_D^{ei}} \right\} u = \left\{ \frac{\nu_D^{ee}}{\nu_D^{ee} + \nu_D^{ei}} \right\} u_* \tag{4.179}$$

$$u = \frac{\left\{ \frac{\nu_D^{ee}}{\nu_D^{ee} + \nu_D^{ei}} \right\}}{\left\{ \frac{\nu_D^{ee} \nu_D^{ei}}{\nu_D^{ee} + \nu_D^{ei}} \right\}} u_* \tag{4.180}$$

just as we had assumed.

Thus,

$$j_\parallel = -e\int \mathrm{d}^3v\ f_{e1}v_\parallel = -e\int \mathrm{d}^3v\ \frac{\nu_D^{ee}u + u_*}{\nu_D^{ee}+\nu_D^{ei}}\frac{m_e v_\parallel^2}{T_e} = -n_e e\left\{\frac{\nu_D^{ee}u + u_*}{\nu_D^{ee}+\nu_D^{ei}}\right\} \tag{4.181}$$

$$= -n_e e\left\{\frac{\nu_D^{ee}}{\nu_D^{ee}+\nu_D^{ei}}\frac{\left\{\frac{\nu_D^{ee}}{\nu_D^{ee}+\nu_D^{ei}}\right\}}{\left\{\frac{\nu_D^{ee}\nu_D^{ei}}{\nu_D^{ee}+\nu_D^{ei}}\right\}}u_* + \frac{1}{\nu_D^{ee}+\nu_D^{ei}}u_*\right\} \tag{4.182}$$

$$= \left\{\frac{\nu_D^{ee}}{\nu_D^{ee}+\nu_D^{ei}}\frac{\left\{\frac{\nu_D^{ee}}{\nu_D^{ee}+\nu_D^{ei}}\right\}}{\left\{\frac{\nu_D^{ee}\nu_D^{ei}}{\nu_D^{ee}+\nu_D^{ei}}\right\}} + \frac{1}{\nu_D^{ee}+\nu_D^{ei}}\right\}\left(-n_e e\frac{-eE_\parallel}{m_e}\right) \tag{4.183}$$

$$= \left(\left\{\frac{\nu_D^{ee}}{\nu_D^{ee}+\nu_D^{ei}}\right\}\frac{\left\{\frac{\nu_D^{ee}}{\nu_D^{ee}+\nu_D^{ei}}\right\}}{\left\{\frac{\nu_D^{ee}\nu_D^{ei}}{\nu_D^{ee}+\nu_D^{ei}}\right\}} + \left\{\frac{1}{\nu_D^{ee}+\nu_D^{ei}}\right\}\right)\frac{n_e e^2 E_\parallel}{m_e} \tag{4.184}$$

and so all the results follow.

4.5 Expansion in Orthogonal Polynomials

Let's show that the Sonine polynomials (i.e., associated Laguerre polynomials) have the generating function given

$$\frac{e^{-xy/(1-y)}}{(1-y)^{m+1}} = \sum_{j=0}^{\infty} y^j L_j^{(m)}(x) \tag{4.185}$$

$$L_j^{(m)}(x) = \frac{1}{j!}\frac{e^x}{x^m}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left(x^{j+m}e^{-x}\right) \tag{4.186}$$

This is by no means obvious. Let's use induction. We know that the power series can be obtained by Taylor series, The simplest method of obtaining the generating function is to use Cauchy's integral formula (so generalize to complex numbers z). Let's use the more common t instead of y and so

$$\frac{e^{-zt/(1-t)}}{(1-t)^{m+1}} = \sum_{j=0}^{\infty} t^j L_j^{(m)}(z) \tag{4.187}$$

$$L_j^{(m)}(z) = \frac{1}{j!}\frac{e^z}{z^m}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left(z^{j+m}e^{-z}\right) \tag{4.188}$$

Then Cauchy's integral formula states

$$\frac{\mathrm{d}^n f(z)}{\mathrm{d}z^n} = \frac{n!}{2\pi i}\oint_C \mathrm{d}\zeta\ \frac{f(\zeta)}{(\zeta - z)^{n+1}} \tag{4.189}$$

So for $L_j^{(m)}$ we see that

$$L_j^{(m)}(z) = \frac{1}{j!}\frac{e^z}{z^m}\frac{j!}{2\pi i}\oint_C \frac{\zeta^{j+m}e^{-\zeta}}{(\zeta-z)^{j+1}} = \frac{1}{2\pi i}\oint_C \mathrm{d}\zeta\ \frac{\zeta^m e^{z-\zeta}}{z^m\left(1-\frac{z}{\zeta}\right)^j(\zeta-z)} \tag{4.190}$$

Take $t = 1 - \frac{z}{\zeta}$ so that $\mathrm{d}t = \frac{z\,\mathrm{d}\zeta}{\zeta^2}$, then $\zeta = \frac{z}{1-t}$ so $\mathrm{d}t = \frac{(1-t)^2}{z}\,\mathrm{d}\zeta$ and $\zeta t = \zeta - z$ so $z - \zeta = -\zeta t = -\frac{zt}{1-t}$. Thus

$$L_j^m(z) = \frac{1}{2\pi i}\oint_C \mathrm{d}t\ \frac{z}{(1-t)^2}\frac{\left(\frac{z}{1-t}\right)^m e^{-zt/(1-t)}}{z^m t^j \frac{zt}{1-t}} = \frac{1}{2\pi i}\oint_C \mathrm{d}t\ \frac{e^{-zt/(1-t)}}{(1-t)^{m+1}t^{j+1}} \tag{4.191}$$

We use the Cauchy Contour integral formula in the form

$$\frac{\mathrm{d}^n f(0)}{\mathrm{d}z^n} = \frac{n!}{2\pi i}\oint_C \mathrm{d}\zeta\ \frac{f(\zeta)}{(\zeta)^{n+1}} \tag{4.192}$$

noting that $f(\zeta) = e^{-zt/(1-t)}/(1-t)^{m+1}$. So that

$$L_j^{(m)}(z) = \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\frac{e^{-zt/(1-t)}}{(1-t)^{m+1}}\right)_{t=0} \tag{4.193}$$

This is what we wanted as, we must have

$$\frac{e^{-zt/(1-t)}}{(1-t)^{m+1}} = \sum_{j=0}^{\infty}\frac{t^j}{j!}\left[\frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\frac{e^{-zt/(1-t)}}{(1-t)}\right)\right]_{t=0} = \sum_{j=0}^{\infty} t^j L_j^{(m)}(z) \tag{4.194}$$

as desired.

4.6 Braginskii's Equations

Let's take the kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\nabla f_a + \frac{e_a}{m_a}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}} = C_a(f_a) \tag{4.195}$$

and use $v_a' = v - V_a$ to find

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \frac{\partial \mathbf{V}_a}{\partial t}\cdot\frac{\partial}{\partial \mathbf{v}_a'} \tag{4.196}$$

$$\frac{\partial}{\partial \mathbf{r}} \to \frac{\partial}{\partial \mathbf{r}} - \frac{\partial \mathbf{V}_a}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{v}_a'} = \nabla - \nabla\mathbf{V}_a\cdot\frac{\partial}{\partial \mathbf{v}_a'} \tag{4.197}$$

$$\frac{\partial}{\partial v} \to \frac{\partial}{\partial \mathbf{v}_a'} \tag{4.198}$$

so

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\nabla f_a + \frac{e_a}{m_a}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}} = C_a(f_a) \tag{4.199}$$

$$\begin{aligned}\frac{\partial f_a}{\partial t} - \frac{\partial \mathbf{V}_a}{\partial t}\cdot\frac{\partial f_a}{\partial \mathbf{v}_a'} + (\mathbf{v}_a' + \mathbf{V}_a)\cdot\left(\nabla f_a - \nabla\mathbf{V}_a\cdot\frac{\partial f_a}{\partial \mathbf{v}_a'}\right) + \frac{e_a}{m_a}\mathbf{E}\cdot\frac{\partial f_a}{\partial \mathbf{v}_a'}\\ + \frac{e_a}{m_a}(\mathbf{v}_a'\times\mathbf{B} + \mathbf{V}_a\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}_a'} = C_a(f_a)\end{aligned} \tag{4.200}$$

If we use $\mathbf{E}' = \mathbf{E} = \mathbf{V}_a\times\mathbf{B}$ and $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{V}_a\cdot\nabla$, then

$$\frac{\mathrm{d}f_a}{\mathrm{d}t} + \mathbf{v}_a'\cdot\nabla f_a + \left(\mathbf{E}' + \mathbf{v}_a'\times\mathbf{B} - \frac{\partial \mathbf{V}_a}{\partial t} - \mathbf{v}_a'\cdot\nabla\mathbf{V}_a - \mathbf{V}_a\cdot\nabla\mathbf{V}_a\right)\cdot\frac{\partial f_a}{\partial \mathbf{v}_a'} = C_a(f_a) \tag{4.201}$$

$$\frac{\mathrm{d}f_a}{\mathrm{d}t} + \mathbf{v}_a'\cdot\nabla f_a + \left(\mathbf{E}' + \mathbf{v}_a'\times\mathbf{B} - \frac{\mathrm{d}\mathbf{V}_a}{\mathrm{d}t} - \mathbf{v}_a'\cdot\nabla\mathbf{V}_a\right)\cdot\frac{\partial f_a}{\partial \mathbf{v}_a'} = C_a(f_a) \tag{4.202}$$

where, for clarity, $\mathbf{v}_a'\cdot\nabla\mathbf{V}_a\cdot\frac{\partial f_a}{\partial \mathbf{v}_a'} = v_{a,j}'\frac{\partial V_{a,k}}{\partial x_j}\frac{\partial f_a}{\partial v_{a,k}'}$ is what is meant. This matches Helander.

4.7 Diamagnetic Flows

Let's use $\mathbf{B}\times$ on (HS-2.16)

$$\mathbf{B}\times m_a n_a \frac{\mathrm{d}\mathbf{V}_a}{\mathrm{d}t}\bigg|_a = \mathbf{B}\times\Big(-\nabla p_a - \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_a + e_a n_a \mathbf{E} + \mathbf{R}_a\Big) + e_a n_a B^2 \mathbf{V}_{a,\perp} \tag{4.203}$$

Isolating $\mathbf{V}_{a,\perp}$ and suppressing the a subscript,

$$\mathbf{V}_\perp = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mathbf{b}\times\Big(\nabla p + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}} - \mathbf{R} + mn\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\Big)}{enB} \tag{4.204}$$

$$\mathbf{V}_\perp = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mathbf{b}\times\Big(\nabla p_a + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_a - \mathbf{R}_a + m_a n_a\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\Big)}{mn\Omega} \tag{4.205}$$

4.8 Chapter 4 Exercises

4.8.1 Hydrogen Plasma Resistivity

Calculate the resistivity $\eta = 1/\sigma_\parallel$ of a hydrogen plasma and compare with copper, $\eta_{\mathrm{Cu}} = 1.7\times10^{-8}\,\Omega\mathrm{m}$

Solution:

We use that

$$\sigma_\parallel = \frac{L_{11} n_e e^2 \tau_{ei}}{m_e} \tag{4.206}$$

and for a hydrogen plasma, we require

$$\frac{L_{11}}{32/(3\pi)} = 0.58 \Rightarrow L_{11}\approx 2.0 \tag{4.207}$$

and

$$\tau_{ei}\approx 3.44\times10^{11}\frac{T_e^{3/2}}{n_e Z\ln\Lambda} \tag{4.208}$$

with T_e in eV and n_e in m^3. So that

$$\nu_{ei}\approx 2.91\times10^{-12} n_e T_e^{-3/2} Z\ln\Lambda \tag{4.209}$$

Therefore,

$$\begin{aligned}\eta &= \frac{m_e}{n_e e^2 L_{11}}\nu_{ei} = 2.91\times10^{-12}\frac{m_e \cancel{n_e} T_e^{-3/2} Z\ln\Lambda}{\cancel{n_e} e^2 L_{11}} \\ &\approx \frac{(9.1\times10^{-31}\,\mathrm{kg})(2.91\times10^{-12}\,\mathrm{s})}{(1.60\times10^{-19}\,\mathrm{C})^2(2.00)}\ln\Lambda T_e^{-3/2} \approx 5.2\times10^{-5}\ln\Lambda T_e^{-3/2}\Omega-\mathrm{m}\end{aligned} \tag{4.210}$$

with T_e measured in eV and the entire expression in ohm-meters. If we choose $\ln\Lambda\approx15$ then we get $7.8\times10^{-4}T_e^{-3/2}\Omega-\mathrm{m}$.

Thus, for the plasma to have the same resistivity as copper, we require

$$7.8 \times 10^{-4} T_e^{-3/2} = 1.7 \times 10^{-8} \tag{4.211}$$

$$T_e^{-3/2} = \frac{1.7 \times 10^{-8}}{7.8 \times 10^{-4}} \tag{4.212}$$

$$T_e = \left(\frac{7.8 \times 10^{-4}}{1.7 \times 10^{-8}}\right)^{2/3} = 1280 \tag{4.213}$$

So we need about a 1.3 keV plasma to get similar resistivity as copper for parallel resistivity.

Helander gives the rougly similar

$$\eta = (5.3 \times 10^{-5}\,\Omega - \mathrm{m}) T_e^{-3/2} \ln \Lambda \tag{4.214}$$

which would give 1290 or about 1.3 keV as the correct placement. These both are qualitatively similar, so at about a keV we have a resistivity on the same order of magnitude as copper.

We could also go from the original formula and have

$$\tau_{ei} = \frac{12\pi^{3/2}\sqrt{m_e} T_e^{3/2} \epsilon_0^2}{\sqrt{2} n_i Z^2 e^4 \ln \Lambda} \tag{4.215}$$

so that (using $n_i = n_e$)

$$\sigma_{\|} = \frac{L_{11} n_e e^2}{m_e} \frac{12\pi^{3/2}\sqrt{m_e} T_e^{3/2} \epsilon_0^2}{\sqrt{2} n_i Z^2 e^4 \ln \Lambda} = \frac{12 L_{11} \pi^{3/2} \epsilon_0^2 T_e^{3/2}}{\sqrt{2 m_e} Z^2 e^2 \ln \Lambda} \tag{4.216}$$

Thus,

$$\eta = \frac{\sqrt{2 m_e} Z^2 e^2 \ln \Lambda}{12\pi^{3/2} \epsilon_0^2 e^{3/2} (T_e/1\,\mathrm{eV})^{3/2}} = \frac{\sqrt{2 m_e e} Z^2 \ln \Lambda}{12 L_{11} \pi^{3/2} \epsilon_0^2 (T_e/1\,\mathrm{eV})^{3/2}} \tag{4.217}$$

Putting in the numbers yields (for T_e in eV and $L_{11} = 1$)

$$\eta = 1.03 \times 10^{-4} \ln \Lambda T_e^{-3/2} \tag{4.218}$$

which matches the plasma formulary.

4.8.2 Verify Branginskii's Energy Transfer Rate

Verify Braginskii's result (HS-4.49) by calculating the rate of energy transfer between Maxwellian ions and electrons of different temperatures, using the ion-electron collision operator (HS-3.56).

$$C_{ie}(f_i) = \frac{\mathbf{R}_{ei}}{m_i n_i} \cdot \frac{\partial f_i}{\partial \mathbf{v}} + \frac{m_e n_e}{m_i n_i \tau_{ei}} \frac{\partial}{\partial \mathbf{v}} \cdot \left[(\mathbf{v} - \mathbf{V}_i) f_i + \frac{T_e}{m_i} \frac{\partial f_i}{\partial \mathbf{v}}\right] \tag{HS-3.56}$$

$$Q_i = -Q_e - \mathbf{R}_e \cdot \mathbf{u} = \frac{3 n_e m_e}{m_i \tau_e} (T_e - T_i) \tag{HS-4.49}$$

Solution:

We use that

$$Q_i = \int \mathrm{d}^3v \; \frac{m_i v'^2}{2} C_{ie}(f_i) \tag{4.219}$$

presumably $f_i = f_{i0}$, a Maxwellian background to lowest order. Let it be $f_{i0} = \frac{n}{\pi^{3/2} v_{\text{th}_i}^3} e^{-(\mathbf{v}-\mathbf{V}_i)^2/v_{\text{th}_i}^2}$. Thus

$$Q_i = \int \mathrm{d}^3v \; \frac{m_i v'^2}{2} \left[\frac{\mathbf{R}_{ei}}{m_i n_i} \cdot \frac{\partial f_{i0}}{\partial \mathbf{v}} + \frac{m_e n_e}{m_i n_i \tau_{ei}} \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ (\mathbf{v} - \mathbf{V}_i) f_{i0} + \frac{T_e}{m_i} \frac{\partial f_{i0}}{\partial \mathbf{v}} \right\} \right] \tag{4.220}$$

with $\mathbf{u} = \mathbf{v} - \mathbf{V}_i$ so

$$\frac{\partial f_{i0}}{\partial \mathbf{v}} = \frac{\partial f_{i0}}{\partial \mathbf{u}} = \hat{\mathbf{u}} \frac{\partial f_{i0}}{\partial u} = -\mathbf{u} \frac{2 f_{i0}}{v_{\text{th}_i}^2} = (\mathbf{V}_i - \mathbf{v}) \frac{m_i}{T_i} f_{i0} \tag{4.221}$$

We see that $\mathbf{R}_{ei} \cdot \mathbf{v}$ term will vanish because the dot product will provide a sin or cos term that will vanish in the integral. Thus, we have

$$Q_i = \int \mathrm{d}^3v \; \frac{m_i v'^2}{2} \frac{m_e n_e}{m_i n_i \tau_{ei}} \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ \left(1 - \frac{T_e}{T_i} \right) (\mathbf{v} - \mathbf{V}_i) f_{i0} \right\} \tag{4.222}$$

So only the divergence term is necessary to calculate. Note we can use integration by parts with

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{m_i v'^2}{2} \mathbf{A} \right) = \frac{\partial}{\partial \mathbf{v}} \left(\frac{m_i v'^2}{2} \right) \cdot \mathbf{A} + \frac{m_i v'^2}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{A} \tag{4.223}$$

and the divergence theorem will eliminate the full divergence, thus with $\mathbf{v}' = \mathbf{v} - \mathbf{V}_i$

$$\frac{m_i}{2} \frac{\partial (v')^2}{\partial \mathbf{v}} = \frac{m_i}{2} \frac{\partial (v')^2}{\partial \mathbf{v}'} = m_i v' \frac{\partial v'}{\partial \mathbf{v}'} = m_i \mathbf{v}' = m_i (\mathbf{v} - \mathbf{V}_i) \tag{4.224}$$

So we have

$$\begin{aligned} Q_i &= -\frac{m_e}{m_i \tau_{ei}} \int \mathrm{d}^3v \; m_i (\mathbf{v} - \mathbf{V}_i) \cdot \left((\mathbf{v} - \mathbf{V}_i) \left(1 - \frac{T_e}{T_i} \right) \right) f_{i0} = -\frac{m_e}{m_i \tau_{ei}} \left(1 - \frac{T_e}{T_i} \right) \int \mathrm{d}^3v \; m_i v'^2 \\ &= -\frac{3 m_e}{m_i \tau_{ei}} (T_i - T_e) = \frac{3 m_e}{m_i \tau_{ei}} (T_e - T_i) \end{aligned} \tag{4.225}$$

4.8.3 Transport Matrix and Thermodynamics

Demonstrate that the transport matrix L_{jk} introduced in Section 4.3 must be positive definite in order to comply with the second law of thermodynamics.

Solution:

We have

$$L_{12} = L_{21} = -\frac{m_e}{n_e T_e \tau_{ei}} S[h_1, h_2] = -\frac{m_e}{n_e T_e \tau_{ei}} S[h_2, h_1] \tag{4.226}$$

$$S[h_1, h_2] = \int \mathrm{d}^3 v \ \frac{h_1}{f_{e0}} C_e^l(h_2) \tag{4.227}$$

$$L_{jk} = -\frac{m_e}{n_e T_e \tau_{ei}} S[h_j, h_k] \tag{4.228}$$

$$C_e^l(h_1) = v_\| f_{e0} \tag{4.229}$$

$$C_e^l(h_2) = \left(\frac{mv^2}{2T} - \frac{5}{2}\right) v_\| f_{e0} \tag{4.230}$$

We want to show that $x_j L_{jk} x_k > 0$ for all nonzero x_j.

Remember that we have proven that $-S[f_{e1}, f_{e1}]$ is the entropy production rate. We have $f_{e1} = A_1 h_1 + A_2 h_2$. Thus we require

$$-S[A_1 h_1 + A_2 h_2, A_1 h_1 + A_2 h_2] > 0 \tag{4.231}$$

$$S[A_1 h_1, A_1 h_1 + A_2 h_2] + S[A_2 h_2, A_1 h_1 + A_2 h_2] < 0 \tag{4.232}$$

$$S[A_1 h_1, A_1 h_1] + S[A_1 h_1, A_2 h_2] + S[A_2 h_2, A_1 h_1] + S[A_2 h_2, A_2 h_2] < 0 \tag{4.233}$$

$$A_1 S[h_1, h_1] A_1 + A_1 S[h_1, h_2] A_2 + A_2 S[h_2, h_1] A_1 + A_2 S[h_2, h_2] A_2 < 0 \tag{4.234}$$

$$\sum_{j,k=1}^{2} A_j S[h_j, j_k] A_k < 0 \tag{4.235}$$

$$\sum_{j,k=1}^{2} A_j \left(-\frac{m_e}{n_e T_e \tau_{ei}} L_{jk}\right) A_k < 0 \tag{4.236}$$

$$\sum_{j,k=1}^{2} A_j L_{jk} A_k > 0 \tag{4.237}$$

Thus, we must have L_{jk} be positive definite to be consistent with thermodynamics.

4.8.4 Ion Thermal Conductivity in Hydrogen Plasma

Calculate the ion thermal conductivity in a pure hydrogen plasma by the method given in Section 4.5.

Solution:

We use (HS-4.31)

$$\sum_{b,k} \frac{m_a}{\tau_{ab} T_a} \left(M_{ab}^{jk} u_{ak} + N_{ab}^{jk} u_{bk}\right) = A_{a1} \delta_{j0} - \frac{5}{2} A_{a2} \delta_{j1} \tag{4.238}$$

We use that $C_{ie} \ll C_{ii}$ so that we can ignore C_{ie}. Then we just have

$$\sum_k \frac{m_i}{\tau_{ii} T_i} \left(M_{ii}^{jk} u_{ik} + N_{ii}^{jk} u_{ik} \right) = A_{i1}\delta_{j0} - \frac{5}{2} A_{i2}\delta_{j1} \tag{4.239}$$

$$\sum_k \frac{m_i}{\tau_{ii} T_i} \left(M_{ii}^{jk} + N_{ii}^{jk} \right) u_{ik} = A_{i1}\delta_{j0} - \frac{5}{2} A_{i2}\delta_{j1} \tag{4.240}$$

We could try $j = 0$ but we know that $M_{ab}^{0k} = -N_{ab}^{0k}$, so that $j = 0$ will never yield any information. Thus, we go to the next order and take $j = 1$ with $k = 1$ as the smallest possible system. We use

$$M_{ab}^{11} = -\frac{\frac{13}{4} + 4x_{ab}^2 + \frac{15}{2}x_{ab}^4}{(1 + x_{ab}^2)^{5/2}} \tag{4.241}$$

$$N_{ab}^{11} = \frac{27}{4} \frac{T_a}{T_b} \frac{x_{ab}^2}{(1 + x_{ab}^2)^{5/2}} \tag{4.242}$$

$$x_{ab} = \frac{v_{\text{th}_b}}{v_{\text{th}_a}} \tag{4.243}$$

So that for us

$$M_{ii}^{11} = -\frac{\frac{13}{4} + 4 + \frac{15}{2}}{2^{5/2}} = -\frac{59}{(4)2^{5/2}} \tag{4.244}$$

$$N_{ii}^{11} = \frac{27}{4(2^{5/2})} \tag{4.245}$$

$$M_{ii}^{11} + N_{ii}^{11} = -\frac{32}{16\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2} \tag{4.246}$$

Thus, we find

$$-\sqrt{2}\frac{m_i}{\tau_{ii} T_i} u_{i1} = \frac{5}{2} A_{i2} \tag{4.247}$$

$$\sqrt{2}\frac{m_i}{\tau_{ii} T_i} \frac{2q_{i,\|}}{5p_i} = \frac{5}{2} \nabla \ln T_i \tag{4.248}$$

We want

$$q_{i,\|} = -\kappa_{i,\|} \nabla T_i \tag{4.249}$$

so that we see we get

$$q_{i,\|} = \frac{25}{4} \frac{T_i \tau_{ii}}{m_i} \frac{n_i T_i}{\sqrt{2} T_i} \nabla T_i = \frac{25}{4\sqrt{2}} \frac{\tau_{ii} n_i T_i}{m_i} \nabla T_i \approx 4.42 \frac{\tau_{ii} p_i}{m_i} \nabla T_i \tag{4.250}$$

If we use $\tau_i = \sqrt{2}\tau_{ii}$, we'd find

$$q_{i,\|} = \frac{25}{4\sqrt{2}} \frac{\tau_i p_i}{\sqrt{2} m_i} \nabla T_i = \frac{25}{8} \frac{\tau_i p_i}{m_i} \nabla T_i = 3.125 \frac{\tau_i p_i}{m_i} \nabla T_i \tag{4.251}$$

Chapter 5

Transport in a Cylindrical Plasma

Nothing was confusing to me.

Chapter 6

Particle Motion

6.1 Equations of Motion

The usual Lagrangian/Hamiltonian dynamics expected from classical mechanics.

6.2 Nearly Periodic Motion

Clear.

6.3 Guiding Center Motion

With

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\rho} \tag{6.1}$$
$$\mathbf{r} = \mathbf{R} + (\rho\cos\theta\hat{\mathbf{x}} + \rho\sin\theta\hat{\mathbf{y}}) \tag{6.2}$$
$$\mathbf{r} = \left(R_\perp\cos\theta\hat{\mathbf{x}} + R_\perp\sin\theta\hat{\mathbf{y}} + R_\parallel\hat{\mathbf{z}}\right) + (\rho\cos\theta\hat{\mathbf{x}} + \rho\sin\theta\hat{\mathbf{y}}) \tag{6.3}$$
$$\mathbf{b} = \hat{\mathbf{z}} \tag{6.4}$$
$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\boldsymbol{\rho}} \tag{6.5}$$
$$\dot{\mathbf{R}} = \dot{R}\hat{\mathbf{z}} \tag{6.6}$$
$$\dot{\boldsymbol{\rho}} = \left(\dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta\right)\hat{\mathbf{x}} + \left(\dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta\right)\hat{\mathbf{y}} \tag{6.7}$$

with $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ a local coordinate system with $\hat{\mathbf{z}}$ in the magnetic field direction.

Then

$$|\dot{\mathbf{r}}|^2 = |\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}|^2 = \dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}}\cdot\dot{\boldsymbol{\rho}} + \dot{\boldsymbol{\rho}}^2 \tag{6.8}$$

Breaking this up, we find

$$|\dot{\mathbf{r}}|^2 = (\mathbf{b}\cdot\dot{\mathbf{R}})^2 + \dot{\rho}^2\cos^2\theta - 2\rho\dot{\rho}\sin\theta\cos\theta + \rho^2\dot{\theta}^2\sin^2\theta + \dot{\rho}^2\sin^2\theta + 2\rho\dot{\rho}\sin\theta\cos\theta + \rho^2\dot{\theta}^2\cos^2\theta \tag{6.9}$$
$$= (\mathbf{b}\cdot\dot{\mathbf{R}})^2 + \dot{\rho}^2 + \rho^2\dot{\theta}^2 \tag{6.10}$$

Presumably $\dot{\rho}$ is very small, especially in comparison to the other terms, thus, we do indeed recover

$$|\dot{\mathbf{r}}|^2 = (\mathbf{b} \cdot \dot{\mathbf{R}})^2 + (\rho\dot{\theta})^2 \tag{6.11}$$

6.4 Other Adiabatic Invariants

Note that $\mathbf{A} \cdot \dot{\mathbf{R}} = \alpha\dot{\mathbf{R}} \cdot \nabla\beta = \alpha(\dot{\beta} - \frac{\partial\beta}{\partial t})$ comes from

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \nabla\beta \cdot \dot{\mathbf{R}} + \frac{\partial\beta}{\partial t} \tag{6.12}$$

where the allowance of magnetic field temporal change is in $\frac{\partial\beta}{\partial t}$.

6.6 Chapter 6 Exercises

6.6.1 Derive Hamilton's Equations

Derive Hamilton's equations of motion (HS-6.6).

$$\dot{\mathbf{r}} = \frac{\partial\mathcal{H}}{\partial\mathbf{p}} \tag{6.13}$$
$$\dot{\mathbf{p}} = -\frac{\partial\mathcal{H}}{\partial\mathbf{r}} \tag{6.14}$$

Solution:

We have

$$\mathrm{d}\mathcal{H} = \mathrm{d}\mathbf{p} \cdot \dot{\mathbf{r}} + \mathbf{p} \cdot \mathrm{d}\dot{\mathbf{r}} - \mathrm{d}\mathcal{L} \tag{6.15}$$
$$\tag{6.16}$$

We use that $\mathcal{L} = \mathcal{L}(\dot{\mathbf{r}}, \mathbf{r})$, so that

$$\mathrm{d}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}} \cdot \mathrm{d}\dot{\mathbf{r}} + \frac{\partial\mathcal{L}}{\partial\mathbf{r}} \cdot \mathrm{d}\mathbf{r} \tag{6.17}$$

If we use that $\partial\mathcal{L}/\partial\dot{\mathbf{r}} = \mathbf{p}$ and $\partial\mathcal{L}/\partial\mathbf{r} = \dot{\mathbf{p}}$ [this follows from $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) = \frac{\partial\mathcal{L}}{\partial q}$], then

$$\mathrm{d}\mathcal{L} = \mathbf{p} \cdot \mathrm{d}\dot{\mathbf{r}} + \dot{\mathbf{p}} \cdot \mathrm{d}\mathbf{r} \tag{6.18}$$

Thus,

$$\mathrm{d}\mathcal{H} = \mathrm{d}\mathbf{p} \cdot \dot{\mathbf{r}} + \mathbf{p} \cdot \mathrm{d}\dot{\mathbf{r}} - [\mathbf{p} \cdot \mathrm{d}\dot{\mathbf{r}} + \dot{\mathbf{p}} \cdot \mathrm{d}\mathbf{r}] \tag{6.19}$$
$$\mathrm{d}\mathcal{H} = \dot{\mathbf{r}} \cdot \mathrm{d}\mathbf{p} - \dot{\mathbf{p}} \cdot \mathrm{d}\mathbf{r} \tag{6.20}$$

So that we recover Hamilton's equations.

6.6.2 Hamiltonian Guiding Center Coordinates

Derive Hamiltonian guiding center equations of motion in a stationary magnetic field described by the coordinates (α, β, s).

Solution:

We have the Lagrangian given by

$$\mathcal{L} = \frac{m\dot{s}^2}{2} + Ze\alpha\left(\dot{\beta} - \frac{\partial \beta}{\partial t}\right) - \mu B - Ze\Phi \tag{6.21}$$

Then we use

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = 0 \tag{6.22}$$
$$p_\beta = \frac{\partial \mathcal{L}}{\partial \dot{\beta}} = Ze\alpha \tag{6.23}$$
$$p_s = \frac{\partial \mathcal{L}}{\partial \dot{s}} = m\dot{s} \tag{6.24}$$

So $\mathcal{H} = \mathbf{p} \cdot \dot{\mathbf{q}} - \mathcal{L}$ with $\mathbf{q} = (\alpha, \beta, s)$, so The $p_\alpha = 0$ implies that $\frac{\partial \mathcal{L}}{\partial \alpha} = 0$ and so

$$Ze\left(\dot{\beta} - \frac{\partial \beta}{\partial t}\right) - \mu\frac{\partial B}{\partial \alpha} - Ze\frac{\partial \Phi}{\partial \alpha} = 0 \tag{6.25}$$
$$Ze\left(\dot{\beta} - \frac{\partial \beta}{\partial t}\right) = \mu\frac{\partial B}{\partial \alpha} + Ze\frac{\partial \Phi}{\partial \alpha} \tag{6.26}$$

$$\mathcal{H} = p_\beta\dot{\beta} + p_s\frac{p_s^2}{2m} - \left[\frac{m\dot{s}^2}{2} + Ze\alpha\left(\dot{\beta} - \frac{\partial \beta}{\partial t}\right) - \mu B - Ze\Phi\right] \tag{6.27}$$
$$\mathcal{H} = p_\beta\dot{\beta} + \frac{p_s^2}{m} - \left[\frac{p_s^2}{2m} + p_\beta\left(\dot{\beta} - \frac{\partial \beta}{\partial t}\right) - \mu B - Ze\Phi\right] \tag{6.28}$$
$$\mathcal{H} = \frac{p_s^2}{2m} + p_\beta\frac{\partial \beta}{\partial t} + \mu B + Ze\Phi \tag{6.29}$$

Now, we use a stationary magnetic field so $\frac{\partial \beta}{\partial t} = 0$ and we have

$$\mathcal{H} = \frac{p_s^2}{2m} + \mu B + Ze\Phi \tag{6.30}$$

So we have

$$\frac{\partial \mathcal{H}}{\partial \mathbf{p}} = \dot{\mathbf{r}} \tag{6.31}$$
$$\frac{\partial \mathcal{H}}{\partial \mathbf{r}} = -\dot{\mathbf{p}} \tag{6.32}$$

become (since $p_\alpha = 0$ it is unclear that there is any meaning in considering the $\frac{\partial \mathcal{H}}{\partial p_\alpha}$ derivative, and because $p_\beta = Ze\alpha$, we have to realize that B and Φ are functions of p_β)

$$\frac{\partial \mathcal{H}}{\partial p_\beta} = \frac{1}{Ze}\frac{\partial \mathcal{H}}{\partial \alpha} = \frac{\mu}{Ze}\frac{\partial B}{\partial \alpha} + \frac{\partial \Phi}{\partial \alpha} = \dot{\beta} \tag{6.33}$$

$$\frac{\partial \mathcal{H}}{\partial p_s} = \frac{p_s}{m} = \dot{s} \tag{6.34}$$

$$\frac{\partial \mathcal{H}}{\partial \alpha} = \mu\frac{\partial B}{\partial \alpha} + Ze\frac{\partial \Phi}{\partial \alpha} = -\dot{p}_\alpha = 0 \tag{6.35}$$

$$\frac{\partial \mathcal{H}}{\partial \beta} = \mu\frac{\partial B}{\partial \beta} + Ze\frac{\partial \Phi}{\partial \beta} = -\dot{p}_\beta = -Ze\dot{\alpha} \tag{6.36}$$

$$\frac{\partial \mathcal{H}}{\partial s} = \mu\frac{\partial B}{\partial s} + Ze\frac{\partial \Phi}{\partial s} = -\dot{p}_s \tag{6.37}$$

with $\dot{p}_\beta = Ze\dot{\alpha}$ we can see

$$\mu\frac{\partial B}{\partial \beta} + Ze\frac{\partial \Phi}{\partial \beta} = -Ze\dot{\alpha} \tag{6.38}$$

$$-\dot{\alpha} = \frac{\mu}{Ze}\frac{\partial B}{\partial \beta} + \frac{\partial \Phi}{\partial \beta} \tag{6.39}$$

Note that Helander misses m in the $\dot{s}$ equation. So, the reduced equations should be

$$-\dot{\alpha} = \frac{\mu}{Ze}\frac{\partial B}{\partial \beta} + \frac{\partial \Phi}{\partial \beta} \tag{6.40}$$

$$\mu\frac{\partial B}{\partial \alpha} = -Ze\frac{\partial \Phi}{\partial \alpha} \Leftrightarrow \dot{p}_\alpha = 0 \tag{6.41}$$

$$\dot{\beta} = \frac{\mu}{Ze}\frac{\partial B}{\partial \alpha} + \frac{\partial \Phi}{\partial \alpha} = 0 \tag{6.42}$$

$$\dot{p}_s = -\mu\frac{\partial B}{\partial s} - Ze\frac{\partial \Phi}{\partial s} \tag{6.43}$$

$$\dot{s} = \frac{p_s}{m} \tag{6.44}$$

6.6.3 Analytical Mechanics with Previous Hamiltonian

6.6.3.1 Bounce Frequency

Show that the angular frequency of bounce motion executed by a particle trapped in a magnetic field is

$$\omega_b = \left(\frac{\partial \mathcal{H}}{\partial J}\right)_{\alpha,\beta,\mu} \tag{6.45}$$

Solution:

We will transform to action-angle coordinates, i.e., $(s, p_s, \beta, p_\beta) \to (J, \theta, \beta, p_\beta)$ with $\mathcal{H} \to \mathcal{K}$. We have

$$J = \oint p_s \, \mathrm{d}s \tag{6.46}$$

Now, this J is taken along a constant $\mathcal{H}$ orbit. Thus, J can depend on other constants of motion (such as β and $\mathcal{H}$, but it does not depend on the actual motion of the particle, and so should be a constant of motion. Because $\mathcal{H}(s, p_s, \dots) = \mathcal{K}(J, \theta, \dots)$ the new Hamilton equations for our new coordinates are

$$\left(\frac{\partial \mathcal{K}}{\partial \theta}\right)_{\alpha,\beta,\mu} = -\frac{\mathrm{d}J}{\mathrm{d}t} \tag{6.47}$$

$$\left(\frac{\partial \mathcal{K}}{\partial J}\right)_{\alpha,\beta,\mu} = \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{6.48}$$

Now because J is independent of time, we know $\frac{\mathrm{d}J}{\mathrm{d}t} = 0$ and so

$$\mathcal{K}(J, \theta, \dots) = \mathcal{K}(J, \dots) \tag{6.49}$$

(that is, we have eliminated the dependence of θ in $\mathcal{K}$). Thus,

$$\left(\frac{\partial \mathcal{K}}{\partial J}\right)_{\alpha,\beta,\mu} = \frac{\mathrm{d}\theta}{\mathrm{d}t} \equiv \omega_b(J, \mathcal{H}, \beta, \mu) \tag{6.50}$$

where the ω_b must be a constant of motion because $\mathcal{K} \equiv \mathcal{H}$ is a constant of motion and so is J.

This is the same since we can write $\mathcal{H}(J, \dots) = \mathcal{K}(J, \dots)$ and recover the formula given by Helander.

Note this works because we have found that $\mathcal{H}$ and β are constants of motion.

6.6.3.2 Toroidal Precession

Show that the drift across the magnetic field is described by

$$\dot{\alpha} = \frac{\omega_b}{Ze}\left(\frac{\partial J}{\partial \beta}\right)_{\alpha,\mathcal{H},\mu} \tag{6.51}$$

In view of (HS-6.27), this implies that the toroidal precession frequency of trapped particles in a tokamak is

$$\omega_\varphi = \frac{\omega_b}{Ze}\left(\frac{\partial J}{\partial \psi}\right)_{\mathcal{H},\mu} \tag{6.52}$$

$$\mathbf{B} = \nabla(\phi - q\theta) \times \nabla\psi \tag{HS-6.27}$$

Solution:

We use our previous coordinate system, using the fact that $J = J(\beta, \mathcal{H})$ is a constant of motion, but there is no time or θ dependence in J. Note that we have $\mathcal{K} = \mathcal{H}(J, \beta)$ only. Then we have

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}\beta} = \frac{\partial \mathcal{K}}{\partial \beta} + \frac{\partial \mathcal{K}}{\partial J}\frac{\mathrm{d}J}{\mathrm{d}\beta} = \frac{\partial \mathcal{K}}{\partial \beta} + \frac{\partial \mathcal{K}}{\partial J}\frac{\partial J}{\partial \beta} + \underbrace{\frac{\partial \mathcal{K}}{\partial J}\frac{\partial J}{\partial \mathcal{K}}}_{=1}\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}\beta} \tag{6.53}$$

$$\frac{\partial \mathcal{K}}{\partial \beta} + \frac{\partial \mathcal{K}}{\partial J}\frac{\partial J}{\partial \beta} = 0 \tag{6.54}$$

so that we see that the above is true by virtue of the total derivative of $\mathcal{K} = \mathcal{H}$ canceling.

So that

$$\dot{p}_\beta = -\frac{\partial \mathcal{K}}{\partial \beta} = \frac{\partial \mathcal{K}}{\partial J}\frac{\partial J}{\partial \beta} \equiv \omega_b \frac{\partial J}{\partial \beta} \tag{6.55}$$

and we then use $\dot{p}_\beta = Ze\dot{\alpha}$ and recover the result.

6.6.4 Second Equality

Demonstrate the second equality in (HS-6.30).

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{z}} + \dot{z}_k \left(\frac{\partial f}{\partial z_k}\right)_t = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{w}} + \dot{w}_k \left(\frac{\partial f}{\partial w_k}\right)_t \tag{HS-6.30}$$

Solution:

This is actually trickier than Helander seems to admit. We use [with $\mathbf{z} = \mathbf{z}(\mathbf{w}, t)$ assumed to be known via $\mathbf{w} = \mathbf{w}(\mathbf{z}, t)$]

$$\left(\frac{\partial f}{\partial t}\right)_{\mathbf{z}} = \left(\frac{\partial f}{\partial w_k}\right)_t \left(\frac{\partial w_k}{\partial t}\right)_{\mathbf{z}} + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{w}} \tag{6.56}$$

$$\frac{\mathrm{d}z_k}{\mathrm{d}t} = \left(\frac{\partial z_k}{\partial w_j}\right)_t \frac{\mathrm{d}w_j}{\mathrm{d}t} + \left(\frac{\partial z_k}{\partial t}\right)_{\mathbf{w}} \tag{6.57}$$

$$\frac{\mathrm{d}w_k}{\mathrm{d}t} = \left(\frac{\partial w_k}{\partial z_j}\right)_t \frac{\mathrm{d}z_j}{\mathrm{d}t} + \left(\frac{\partial w_k}{\partial t}\right)_{\mathbf{z}} \tag{6.58}$$

So that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{z}} + \dot{z}_k \left(\frac{\partial f}{\partial z_k}\right)_t = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{z}} + \frac{\mathrm{d}z_k}{\mathrm{d}t} \left(\frac{\partial f}{\partial z_k}\right)_t \tag{6.59}$$

$$= \left(\frac{\partial f}{\partial w_k}\right)_t \left(\frac{\partial w_k}{\partial t}\right)_{\mathbf{z}} + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{w}} + \left[\left(\frac{\partial z_k}{\partial w_j}\right)_t \frac{\mathrm{d}w_j}{\mathrm{d}t} + \left(\frac{\partial z_k}{\partial t}\right)_{\mathbf{w}}\right] \left(\frac{\partial f}{\partial w_i}\right)_t \left(\frac{\partial w_i}{\partial z_k}\right)_t \tag{6.60}$$

$$= \left(\frac{\partial f}{\partial w_k}\right)_t \left(\frac{\partial w_k}{\partial t}\right)_{\mathbf{z}} + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{w}} + \frac{\mathrm{d}w_j}{\mathrm{d}t} \left(\frac{\partial f}{\partial w_i}\right)_t \left(\frac{\partial z_k}{\partial w_j}\right)_t \left(\frac{\partial w_i}{\partial z_k}\right)_t + \left(\frac{\partial f}{\partial w_i}\right)_t \left(\frac{\partial w_i}{\partial z_k}\right)_t \left(\frac{\partial z_k}{\partial t}\right)_{\mathbf{w}} \tag{6.61}$$

$$= \left(\frac{\partial f}{\partial w_k}\right)_t \left[\left(\frac{\partial w_k}{\partial t}\right)_{\mathbf{z}} + \left(\frac{\partial w_k}{\partial z_i}\right)_t \left(\frac{\partial z_i}{\partial t}\right)_{\mathbf{w}}\right] + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{w}} + \frac{\mathrm{d}w_j}{\mathrm{d}t} \left(\frac{\partial f}{\partial w_i}\right)_t \left(\frac{\partial z_k}{\partial w_j}\right)_t \left(\frac{\partial w_i}{\partial z_k}\right)_t \tag{6.62}$$

$$= \left(\frac{\partial f}{\partial w_k}\right)_t \left[\left(\frac{\partial w_k}{\partial t}\right)_{\mathbf{z}} + \left(\frac{\partial w_k}{\partial z_i}\right)_t \left(\frac{\partial z_i}{\partial t}\right)_{\mathbf{w}}\right] + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{w}} + \frac{\mathrm{d}w_j}{\mathrm{d}t} \left(\frac{\partial f}{\partial w_i}\right)_t \delta_{ij} \tag{6.63}$$

$$= \left(\frac{\partial f}{\partial w_k}\right)_t \left[\left(\frac{\partial w_k}{\partial t}\right)_{\mathbf{z}} + \left(\frac{\partial w_k}{\partial z_i}\right)_t \left(\frac{\partial z_i}{\partial t}\right)_{\mathbf{w}}\right] + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{w}} + \frac{\mathrm{d}w_j}{\mathrm{d}t} \left(\frac{\partial f}{\partial w_j}\right)_t \tag{6.64}$$

$$= \left(\frac{\partial f}{\partial w_k}\right)_t \left[\left(\frac{\partial w_k}{\partial t}\right)_{\mathbf{z}} + \left(\frac{\partial w_k}{\partial z_i}\right)_t \left(\frac{\partial z_i}{\partial t}\right)_{\mathbf{w}}\right] + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{w}} + \dot{w}_k \left(\frac{\partial f}{\partial w_k}\right)_t \tag{6.65}$$

We now need to use

$$\left(\frac{\partial w_k}{\partial t}\right)_{\mathbf{z}} + \left(\frac{\partial w_k}{\partial z_i}\right)_t \left(\frac{\partial z_i}{\partial t}\right)_{\mathbf{w}} = \frac{\mathrm{d}w_k}{\mathrm{d}t} - \left(\frac{\partial w_k}{\partial z_j}\right)_t \frac{\mathrm{d}z_j}{\mathrm{d}t} + \left(\frac{\partial w_k}{\partial z_j}\right)_t \left(\frac{\partial z_j}{\partial t}\right)_{\mathbf{w}} \tag{6.66}$$

$$= \frac{\mathrm{d}w_k}{\mathrm{d}t} + \left(\frac{\partial w_k}{\partial z_j}\right)_t \left[\left(\frac{\partial z_j}{\partial t}\right)_{\mathbf{w}} - \frac{\mathrm{d}z_j}{\mathrm{d}t}\right] \tag{6.67}$$

$$= \frac{\mathrm{d}w_k}{\mathrm{d}t} + \left(\frac{\partial w_k}{\partial z_j}\right)_t \left[-\left(\frac{\partial z_j}{\partial w_i}\right)_t \frac{\mathrm{d}w_i}{\mathrm{d}t}\right] \tag{6.68}$$

$$= \frac{\mathrm{d}w_k}{\mathrm{d}t} - \delta_{ik}\frac{\mathrm{d}w_i}{\mathrm{d}t} = 0 \tag{6.69}$$

So that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left(\frac{\partial f}{\partial w_k}\right)_t \left[\left(\frac{\partial w_k}{\partial t}\right)_{\mathbf{z}} + \left(\frac{\partial w_k}{\partial z_i}\right)_t \left(\frac{\partial z_i}{\partial t}\right)_{\mathbf{w}}\right] + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{w}} + \dot{w}_k \left(\frac{\partial f}{\partial w_k}\right)_t \tag{6.70}$$

$$= \left(\frac{\partial f}{\partial t}\right)_{\mathbf{w}} + \dot{w}_k \left(\frac{\partial f}{\partial w_k}\right)_t \tag{6.71}$$

as desired.

Chapter 7

Toroidal Plasmas

7.1 Magnetic Field

It would be simpler to say we have a right-handed coordinate system (R, z, φ), rather than that we have a coordinate system (R, φ, z) with $\hat{\mathbf{R}} \times \hat{\mathbf{z}} = \hat{\boldsymbol{\varphi}}$. Then note that $\hat{\boldsymbol{\varphi}}$ is the opposite direction to conventional cylindrical coordinates.

Given

$$\mathbf{B} = \hat{\mathbf{R}} B_R + \hat{\boldsymbol{\varphi}} B_\varphi + \hat{\mathbf{z}} B_z = \boldsymbol{\nabla} \times \mathbf{A} \tag{7.1}$$

$$\mathbf{A} = \hat{\mathbf{R}} A_R + \hat{\boldsymbol{\varphi}} A_\varphi + \hat{\mathbf{z}} A_z \tag{7.2}$$

we want the poloidal magnetic field $\mathbf{B}_p$ given by

$$\mathbf{B}_p = \hat{\mathbf{R}} B_R + \hat{\mathbf{z}} B_z \tag{7.3}$$

One way of finding this from $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$ is to write $\mathbf{A}$ in its covariant form,

$$\mathbf{A} = \widetilde{A}_R \, \nabla R + \widetilde{A}_\varphi \, \nabla \varphi + \widetilde{A}_z \, \nabla z \tag{7.4}$$

$$\widetilde{A}_i = \widetilde{A}_i(R, z, \varphi) \tag{7.5}$$

$$\nabla \widetilde{A}_i = \frac{\partial \widetilde{A}_i}{\partial R} \, \nabla R + \frac{\partial \widetilde{A}_i}{\partial z} \, \nabla z + \frac{\partial \widetilde{A}_i}{\partial \varphi} \, \nabla \varphi \tag{7.6}$$

Then

$$\boldsymbol{\nabla} \times \mathbf{A} = \nabla \widetilde{A}_R \times \nabla R + \cancel{\widetilde{A}_r \boldsymbol{\nabla} \times \nabla R} + \nabla \widetilde{A}_\varphi \times \nabla \varphi + \cancel{\widetilde{A}_\varphi \boldsymbol{\nabla} \times \nabla \varphi} + \nabla \widetilde{A}_z \times \nabla z + \cancel{\widetilde{A}_z \boldsymbol{\nabla} \times \nabla z} \tag{7.7}$$

$$\begin{aligned} &= \frac{\partial \widetilde{A}_R}{\partial z} \nabla z \times \nabla R + \frac{\partial \widetilde{A}_R}{\partial \varphi} \nabla \varphi \times \nabla R + \frac{\partial \widetilde{A}_\varphi}{\partial R} \nabla R \times \nabla \varphi + \frac{\partial \widetilde{A}_\varphi}{\partial z} \nabla z \times \nabla \varphi \\ &\quad + \frac{\partial \widetilde{A}_z}{\partial R} \nabla R \times \nabla z + \frac{\partial \widetilde{A}_z}{\partial \varphi} \nabla \varphi \times \nabla z \end{aligned} \tag{7.8}$$

We can use that

$$\nabla R = \hat{\mathbf{R}} \tag{7.9}$$
$$\nabla z = \hat{\mathbf{z}} \tag{7.10}$$
$$\nabla \varphi = \frac{1}{R}\hat{\boldsymbol{\varphi}} \tag{7.11}$$

which implies that

$$\widetilde{A}_R = A_R \tag{7.12}$$
$$\widetilde{A}_z = A_z \tag{7.13}$$
$$\widetilde{A}_\varphi = RA_\varphi \tag{7.14}$$

So that

$$\boldsymbol{\nabla}\times\mathbf{A} = -\frac{\partial\widetilde{A}_r}{\partial z}\hat{\boldsymbol{\varphi}} + \frac{1}{R}\frac{\partial\widetilde{A}_R}{\partial\varphi}\hat{\mathbf{z}} - \frac{1}{R}\frac{\partial\widetilde{A}_\varphi}{\partial R}\hat{\mathbf{z}} + \frac{1}{R}\frac{\partial\widetilde{A}_\varphi}{\partial z}\hat{\mathbf{R}} + \frac{\partial\widetilde{A}_z}{\partial R}\hat{\boldsymbol{\varphi}} - \frac{\partial\widetilde{A}_z}{\partial\varphi}\hat{\mathbf{R}} \tag{7.15}$$
$$= \hat{\mathbf{R}}\left(\frac{1}{R}\frac{\partial(RA_\varphi)}{\partial z} - \frac{\partial A_z}{\partial\varphi}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_z}{\partial R} - \frac{\partial A_R}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{1}{R}\frac{\partial A_R}{\partial\varphi} - \frac{1}{R}\frac{\partial(RA_\varphi)}{\partial R}\right) \tag{7.16}$$
$$= \hat{\mathbf{R}}\left(\frac{\partial A_\varphi}{\partial z} - \frac{\partial A_z}{\partial\varphi}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_z}{\partial R} - \frac{\partial A_R}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{1}{R}\frac{\partial A_R}{\partial\varphi} - \frac{1}{R}\frac{\partial(RA_\varphi)}{\partial R}\right) \tag{7.17}$$

Because we have an axisymmetric magnetic field, the $\frac{\partial}{\partial\varphi}$ terms vanish, and so

$$\boldsymbol{\nabla}\times\mathbf{A} = \hat{\mathbf{R}}\frac{\partial A_\varphi}{\partial z} + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_z}{\partial R} - \frac{\partial A_R}{\partial z}\right) - \hat{\mathbf{z}}\frac{1}{R}\frac{\partial(RA_\varphi)}{\partial R} \tag{7.18}$$

So, indeed, we find

$$\mathbf{B}_p = \hat{\mathbf{R}}\frac{\partial A_\varphi}{\partial z} - \hat{\mathbf{z}}\frac{1}{R}\frac{\partial(RA_\varphi)}{\partial R} \tag{7.19}$$

If we introduce $\psi = \psi(R,z) = -RA_\varphi(R,z)$, then we can write

$$\mathbf{B}_p = -\hat{\mathbf{R}}\frac{1}{R}\frac{\partial\psi}{\partial z} + \hat{\mathbf{z}}\frac{1}{R}\frac{\partial\psi}{\partial R} \tag{7.20}$$

And see that with $\nabla\psi = \frac{\partial\psi}{\partial R}\hat{\mathbf{R}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$ so that

$$\nabla\varphi\times\nabla\psi = \frac{1}{R}\hat{\boldsymbol{\varphi}}\times\left(\frac{\partial\psi}{\partial R}\hat{\mathbf{R}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}\right) = \hat{\mathbf{z}}\frac{1}{R}\frac{\partial\psi}{\partial R} - \hat{\mathbf{R}}\frac{1}{R}\frac{\partial\psi}{\partial z} \tag{7.21}$$

and so

$$\mathbf{B}_p = \hat{\mathbf{R}}\frac{\partial A_\varphi}{\partial z} - \hat{\mathbf{z}}\frac{1}{R}\frac{\partial(RA_\varphi)}{\partial R} = \nabla\varphi\times\nabla\psi \tag{7.22}$$

If we define

$$\mathbf{B}_t = \hat{\boldsymbol{\varphi}}B_\varphi \equiv I(R,z)\,\nabla\varphi \tag{7.23}$$

then

$$\mu_0 \mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla} \times (I\,\nabla\varphi + \nabla\varphi \times \nabla\psi) \tag{7.24}$$

$$= \nabla I \times \nabla\varphi + I\cancel{\boldsymbol{\nabla} \times \nabla\varphi} + \nabla\varphi\,\nabla^2\psi + \nabla\psi \cdot \nabla\nabla\varphi - \nabla\varphi \cdot \nabla\nabla\psi - \cancel{\nabla\psi\,\nabla^2\varphi} \tag{7.25}$$

where I have used that $\nabla^2\varphi = 0$ (rather clearly). Note that in cylindrical coordinates, we have

$$\begin{aligned} \mathbf{A} \cdot \nabla\mathbf{B} = &\left(A_R \frac{\partial B_R}{\partial R} + \frac{A_\varphi}{R}\frac{\partial B_R}{\partial \varphi} + A_z \frac{\partial B_R}{\partial z} - \frac{A_\varphi B_\varphi}{R} \right) \hat{\mathbf{R}} \\ &+ \left(A_R \frac{\partial B_\varphi}{\partial R} + \frac{A_\varphi}{R}\frac{\partial B_\varphi}{\partial \varphi} + A_z \frac{\partial B_\varphi}{\partial z} + \frac{A_\varphi B_R}{R} \right) \hat{\boldsymbol{\varphi}} \\ &+ \left(A_R \frac{\partial B_z}{\partial R} + \frac{A_\varphi}{R}\frac{\partial B_z}{\partial \varphi} + A_z \frac{\partial B_z}{\partial z} \right) \hat{\mathbf{z}} \end{aligned} \tag{7.26}$$

This says that

$$\nabla\psi \cdot \nabla\nabla\varphi = \frac{-1}{R^2}\frac{\partial\psi}{\partial R}\hat{\boldsymbol{\varphi}} = \frac{-1}{R}\frac{\partial\psi}{\partial R}\,\nabla\varphi \tag{7.27}$$

$$\nabla\varphi \cdot \nabla\nabla\psi = \frac{1}{R^2}\frac{\partial\psi}{\partial R}\hat{\boldsymbol{\varphi}} = \frac{1}{R}\frac{\partial\psi}{\partial R}\,\nabla\varphi \tag{7.28}$$

So that

$$\mu_0 \mathbf{J} = \nabla I \times \nabla\varphi + \nabla^2\psi\,\nabla\varphi - \frac{1}{R}\frac{\partial\psi}{\partial R}\,\nabla\varphi - \frac{1}{R}\frac{\partial\psi}{\partial R}\,\nabla\varphi \tag{7.29}$$

$$= \nabla I \times \nabla\varphi + \left[\nabla^2\psi - \frac{2}{R}\frac{\partial\psi}{\partial R} \right] \nabla\varphi \tag{7.30}$$

We note that Helander is in fact missing the term $\nabla\varphi \cdot \nabla\psi$ in his expression. Let us also note that

$$\begin{aligned} \Delta^*\psi &= \nabla^2\psi - \frac{2}{R}\frac{\partial\Psi}{\partial R} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R} \right) + \cancel{\frac{1}{R}\frac{\partial}{\partial\varphi}\left(\frac{1}{R}\frac{\partial\psi}{\partial\varphi} \right)} + \frac{\partial^2\psi}{\partial z^2} - \frac{2}{R}\frac{\partial\psi}{\partial R} \\ &= \frac{\partial^2\psi}{\partial R^2} + \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial z^2} - \frac{2}{R}\frac{\partial\psi}{\partial R} \\ &= \frac{\partial^2\psi}{\partial R^2} - \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial z^2} \\ &= R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R} \right) + \frac{\partial^2\psi}{\partial z^2} \\ &= R^2 \boldsymbol{\nabla} \cdot \left(\frac{\nabla\psi}{R^2} \right) \end{aligned} \tag{7.31}$$

In any case, it is clear that

$$\mu_0 \mathbf{J}_p = \nabla I \times \nabla\varphi \tag{7.32}$$

from our expression. Thus,

$$\mu_0 \mathbf{J} \cdot \nabla\psi = (\,\nabla I \times \nabla\varphi + g(R,z)\,\nabla\varphi) \cdot \nabla\psi = 0 \tag{7.33}$$

Note that $\nabla\varphi \cdot \nabla\psi = 0$ because ψ has no φ dependence. Thus,

$$\nabla I \times \nabla\varphi \cdot \nabla\psi = -\nabla I \times \nabla\psi \cdot \nabla\varphi = 0 \tag{7.34}$$

Now because $\nabla I \times \nabla\psi$ is only in $\nabla\varphi$ this implies

$$\nabla I \times \nabla\psi = \mathbf{0} \tag{7.35}$$

This implies that $\nabla I = \frac{\partial I}{\partial\psi} \nabla\psi$ so that I is function only of ψ. Finally, we get

$$\mathbf{B} = I(\psi)\,\nabla\varphi + \nabla\varphi \times \nabla\psi \tag{7.36}$$

aso our expression for an axisymmetric magnetic field.

Force balance requires

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{7.37}$$

Using our $\mathbf{B}$ and dotting this into $\nabla\psi$ we find

$$\mathbf{J} \times (I\,\nabla\varphi + \nabla\varphi \times \nabla\psi) \cdot \nabla\psi = \nabla\psi \cdot \nabla p \tag{7.38}$$

$$(I\mathbf{J} \times \nabla\varphi + \mathbf{J} \times (\nabla\varphi \times \nabla\psi)) \cdot \nabla\psi = \nabla\psi \cdot \nabla p \tag{7.39}$$

$$(I\mathbf{J} \times \nabla\varphi + \nabla\varphi(\mathbf{J} \cdot \nabla\psi) - \nabla\psi(\mathbf{J} \cdot \nabla\varphi)) \cdot \nabla\psi = \nabla\psi \cdot \nabla p \tag{7.40}$$

Because $\mathbf{J} \cdot \nabla\psi = 0$ we find

$$(I\mathbf{J} \times \nabla\varphi - \nabla\psi(\mathbf{J} \cdot \nabla\varphi)) \cdot \nabla\psi = \nabla\psi \cdot \nabla p \tag{7.41}$$

Using that $\mu_0 \mathbf{J}_p = \nabla I \times \nabla\varphi$, we find

$$-|\nabla\psi|^2 \mathbf{J} \cdot \nabla\varphi = \nabla\psi \cdot \nabla p - \frac{I}{\mu_0}(\nabla I \times \nabla\varphi) \times \nabla\varphi \tag{7.42}$$

$$\mathbf{J} \cdot \nabla\varphi = \frac{\nabla\psi \cdot \nabla p + \frac{I}{\mu_0} \nabla I |\nabla\varphi|^2 - I\,\nabla\varphi\cancel{(\nabla I \cdot \nabla\varphi)}}{-|\nabla\psi|^2} \tag{7.43}$$

$$\mathbf{J} \cdot \nabla\varphi = -\frac{\nabla\psi \cdot \nabla p + \frac{I}{\mu_0 R^2} \nabla I}{|\nabla\psi|^2} \tag{7.44}$$

If we use that $\nabla p = \frac{\partial p}{\partial\psi} \nabla\psi = p' \nabla\psi$ and $\nabla I = \frac{\partial I}{\partial\psi} \nabla\psi = I' \nabla\psi$, we then find

$$\mathbf{J} \cdot \nabla\varphi = -\frac{(p' + \frac{II'}{\mu_0 R^2})|\nabla\psi|^2}{|\nabla\psi|^2} \tag{7.45}$$

$$\mathbf{J} \cdot \nabla\varphi = -\left(p' + \frac{II'}{\mu_0 R^2}\right) \tag{7.46}$$

7.2 Guiding Center Orbits in Tokamaks

Note that in a high-aspect-ratio tokamak we can use

$$\mathbf{v}_d = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{v_\perp^2}{2\Omega}\mathbf{b} \times \nabla \ln B + \frac{v_\parallel^2}{\Omega}\mathbf{b} \times \boldsymbol{\kappa} \tag{HS-6.20}$$

$$\boldsymbol{\kappa} \approx \nabla_\perp \ln B \tag{HS-6.23}$$

$$\mathbf{v}_d = \frac{v_\perp^2 + 2v_\parallel^2}{2\Omega}\mathbf{b} \times \nabla \ln B \tag{7.47}$$

for our case because $\mathbf{B} \approx B_\varphi \hat{\boldsymbol{\varphi}}$ so that $\nabla_\perp = \nabla$. We have $B_\varphi = \frac{B_{\varphi 0}}{R}$, so that

$$\mathbf{v}_d = \frac{v_\perp^2 + 2v_\parallel^2}{2\Omega_\varphi} \boldsymbol{\varphi} \times \nabla \ln\left(\frac{1}{R}\right) = \frac{v_\perp^2 + 2v_\parallel^2}{2\Omega_\varphi} \boldsymbol{\varphi} \times \frac{-\nabla R}{R} = \frac{v_\perp^2 + 2v_\parallel^2}{2\Omega_\varphi R} \hat{\mathbf{z}} \tag{7.48}$$

as given in (HS-7.24), because $\nabla R = \hat{\mathbf{R}}$, and $\Omega = \Omega_\varphi$ in this limit. Notice that $v^2 + v_\parallel^2 = v_\parallel^2 + v_\perp^2 + v_\parallel^2 = 2v_\parallel^2 + v_\perp^2$ to yield this exactly.

7.5 Chapter 7 Exercises

7.5.1 Poloidal Beta

The poloidal beta can be defined in different ways; one is

$$\beta_p(\psi) \equiv \frac{8\pi}{\mu_0 I_p^2} \int \mathrm{d}S \ p \tag{7.49}$$

where the integral is taken over the area inside the flux surface ψ in a poloidal cross section of the torus, and I_p is the total toroidal plasma current inside this flux surface.

7.5.1.1 Large aspect ratio

Show that in the large aspect-ratio tokamak with circular cross-section that

$$\beta_p(\psi) = \frac{2\mu_0 \langle p \rangle}{B_\theta(a)^2} \tag{7.50}$$

where $\langle p \rangle$ is the average pressure inside r.

Solution:

In the large aspect-ratio, circular cross section, we have that $\psi = r$, and $(R, z, \varphi) \to (r, \theta, Z)$, so that

$$I_p = \int \mathrm{d}S \ J_\varphi = \int_0^{2\pi} \mathrm{d}\theta \int_0^r \mathrm{d}r' \ r' J_Z(r') \tag{7.51}$$

$$\int \mathrm{d}S \ p = \int_0^{2\pi} \mathrm{d}\theta \int_0^r \mathrm{d}r' \ r' \cdot p(r') \tag{7.52}$$

We can use that in the large aspect ratio, we go from

$$\mu_0 J_Z = \hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{r}\left(\frac{\partial(rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial \theta}\right) = \frac{1}{r}\frac{\partial(rB_\theta)}{\partial r} \tag{7.53}$$

So

$$I_p = 2\pi \int_0^r \mathrm{d}r' \ \frac{r'}{\mu_0 r'} \frac{\partial(r' B_\theta)}{\partial r'} = \frac{2\pi r B_\theta}{\mu_0} \tag{7.54}$$

Rewritten, this says

$$B_\theta = \frac{\mu_0 I_p}{2\pi r} \tag{7.55}$$

So we have

$$\beta_p(\psi) = \frac{8\pi}{\mu_0 \left(\frac{2\pi r B_\theta(r)}{\mu_0}\right)^2} 2\pi \int_0^r \mathrm{d}r'\ r' p(r') = \frac{4\mu_0}{r^2 B_\theta(r)^2} \int_0^r \mathrm{d}r'\ r' p(r') \tag{7.56}$$

If we use

$$\langle p \rangle = \frac{\int_0^r \mathrm{d}r'\ r' p}{\int_0^r \mathrm{d}r'\ r'} = \frac{2\int_0^r \mathrm{d}r'\ r' p}{r^2} \tag{7.57}$$

Then we have

$$\beta_p(r) = \frac{2\mu_0 \langle p \rangle}{B_\theta(r)^2} \tag{7.58}$$

We note that this expression is the actual answer, and that if $r = a$ we get the expression given.

7.5.1.2 Equilibrium

Derive the equilibrium condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(p + \frac{B_\varphi^2}{2\mu_0}\right) + \frac{B_\theta}{\mu_0 r}\frac{\mathrm{d}(rB_\theta)}{\mathrm{d}r} = 0 \tag{7.59}$$

Solution:

We start from

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{7.60}$$

We have that $\nabla p = p'(r)\hat{\mathbf{r}}$, so that

$$J_\theta B_Z - J_Z B_\theta = p' \tag{7.61}$$

We can then use that

$$\mu_0 J_Z = \frac{1}{r}\frac{\partial(rB_\theta)}{\partial r} \tag{7.62}$$

$$\mu_0 J_\theta = -\frac{\partial B_Z}{\partial r} \tag{7.63}$$

So we find (using $B_Z = B_\varphi$)

$$-\frac{\partial B_\varphi}{\partial r}\frac{B_\varphi}{\mu_0} - \frac{B_\theta}{r\mu_0}\frac{\partial(rB_\theta)}{\partial r} = \frac{\partial p}{\partial r} \tag{7.64}$$

$$\frac{\partial}{\partial r}\left(\frac{B_\varphi^2}{2\mu_0}\right) + \frac{B_\theta}{r}\frac{\partial(rB_\theta)}{\partial r} = -\frac{\partial p}{\partial r} \tag{7.65}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(p + \frac{B_\varphi^2}{2\mu_0}\right) + \frac{B_\theta}{r}\frac{\mathrm{d}(rB_\theta)}{\mathrm{d}r} = 0 \tag{7.66}$$

where we have used that everything only has an r dependence.

7.5.1.3 Poloidal Beta Relation

Show that

$$\beta_p = 1 + \frac{1}{(rB_\theta)^2}\int_0^r \mathrm{d}r'\ r'^2 \frac{\mathrm{d}B_\varphi^2}{\mathrm{d}r'} \tag{7.67}$$

and note the significance of this result.

Solution:

We have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(p + \frac{B_\varphi^2}{2\mu_0}\right) + \frac{B_\theta}{r\mu_0}\frac{\mathrm{d}(rB_\theta)}{\mathrm{d}r} = 0 \tag{7.68}$$

from before. If we take $\int_0^r \mathrm{d}r'\ r'^2$, then the first term can be integrated by parts and

$$\int_0^r \mathrm{d}r'\ r'^2 \frac{\mathrm{d}p}{\mathrm{d}r'} = \left[pr'^2\right]_{r=0}^{r'=r} - 2\int \mathrm{d}r'\ r'p = r^2 p(r) - r^2\langle p\rangle = r^2 p(r) - \beta_p \frac{r^2 B_\theta(r)^2}{2\mu_0} \tag{7.69}$$

The second term becomes

$$\int_0^r \mathrm{d}r'\ \frac{r'^2}{2\mu_0}\frac{\mathrm{d}B_\varphi^2}{\mathrm{d}r'} \tag{7.70}$$

and the last term becomes

$$\int_0^r \mathrm{d}r'\ r'^2 \frac{B_\theta}{\mu_0 r'}\frac{\mathrm{d}(r'B_\theta)}{\mathrm{d}r'} = \int_0^r \mathrm{d}r'\ r'^2 \frac{1}{2\mu_0 r'^2}\frac{\mathrm{d}(r'B_\theta)^2}{\mathrm{d}r'} = \frac{r^2 B_\theta(r)^2}{2\mu_0} \tag{7.71}$$

So we find

$$r^2 p(r) - \beta_p \frac{r^2 B_\theta(r)^2}{2\mu_0} + \int_0^r \mathrm{d}r'\ \frac{r'^2}{2\mu_0}\frac{\mathrm{d}B_\varphi^2}{\mathrm{d}r'} + \frac{r^2 B_\theta(r)^2}{2\mu_0} = 0 \tag{7.72}$$

$$\beta_p(r) = \frac{2\mu_0 p(r)}{B_\theta(r)^2} + 1 + \frac{1}{r^2 B_\theta(r)^2}\int_0^r \mathrm{d}r'\ r'^2 \frac{\mathrm{d}B_\varphi^2}{\mathrm{d}r'} \tag{7.73}$$

Note that if we choose $r = a$ where $p(a) = 0$ we recover the desired result for the edge β_p. This implies that $\beta_p \approx 1$ for B_φ^2 a constant. Thus, we expect poloidal beta to be almost 1 for large aspect-ratio tokamaks. If $\beta_p > 1$ this says that $\frac{\mathrm{d}B_\varphi^2}{\mathrm{d}r} > 0$ and so B_φ is weakest at the magnetic axis. This means the J_θ set up by the plasma is opposing B_φ and we have a diamagnetic plasma. If $\beta_p < 1$ this says that $\frac{\mathrm{d}B_\varphi^2}{\mathrm{d}r} < 0$ and so B_φ is strongest at the magnetic axis. This means the J_θ set up by the plasma is strengthening B_φ and we have a paramagnetic plasma.

7.5.2 Shafranov Shift

Show that at the plasma boundary, $r = a$, the radial derivative of the Shafranov shift is equal to

$$\left.\frac{\mathrm{d}\Delta_s}{\mathrm{d}r}\right|_{r=a} = \epsilon\left(\beta_p + \frac{l_i}{2}\right) \tag{7.74}$$

and that (HS-7.14) satisfies

$$\Lambda(a) = \beta_p + \frac{l_i}{2} - 1 \tag{7.75}$$

where

$$l_i = \frac{2}{a^2 B_p(a)^2}\int_0^a \mathrm{d}r\; rB_p(r)^2 \tag{7.76}$$

is the internal inductance of the plasma.

$$\Lambda = -1 - \frac{R_c'}{r'} = \frac{1}{\epsilon}\frac{\mathrm{d}\Delta_s}{\mathrm{d}r} - 1 = \mathcal{O}(1) \tag{HS-7.14}$$

Solution:

We begin by looking at (HS-7.17)

$$\frac{\mathrm{d}\Delta_s}{\mathrm{d}r} = \frac{q(r)^2}{r^3}\int_0^r \mathrm{d}r'\;\frac{r'}{R_0}\left(\frac{r'^2}{q(r')^2} - \frac{2\mu_0 R_0^2 r'}{B_0^2}\frac{\mathrm{d}p}{\mathrm{d}r'}\right) \tag{HS-7.17}$$

At $r = a$, we clearly get

$$\left.\frac{\mathrm{d}\Delta_s}{\mathrm{d}r}\right|_{r=a} = \frac{q(a)^2}{a^3}\int_0^a \mathrm{d}r'\;\frac{r'}{R_0}\left(\frac{r'^2}{q(r')^2} - \frac{2\mu_0 R_0^2 r'}{B_0^2}\frac{\mathrm{d}p}{\mathrm{d}r'}\right) \tag{7.77}$$

Let's first deal with the second part of the integral by integrating by parts

$$-\int_0^a \mathrm{d}r'\;\frac{2\mu_0 R_0 r'^2}{B_0^2}\frac{\mathrm{d}p}{\mathrm{d}r'} = \cancel{\left[-\frac{2\mu_0 R_0 r'^2}{B_0^2}p\right]_{r'=0}^{r'=a}} + \frac{4\mu_0 R_0}{B_0^2}\int_0^a \mathrm{d}r'\; r'p \tag{7.78}$$

$$= \frac{4\mu_0 R_0}{B_0^2}\frac{a^2}{2}\langle p\rangle_a = \frac{4\mu_0 R_0}{B_0^2}\frac{a^2}{2}\frac{B_\theta(a)^2\beta_p(a)}{2\mu_0} = \frac{R_0 B_\theta(a)^2 a^2\beta_p(a)}{B_0^2} \tag{7.79}$$

So that

$$\frac{q(a)^2}{a^3}\frac{R_0 B_\theta(a)^2 a^2 \beta_p(a)}{B_0^2} = \frac{a^2}{a^3 R_0^2}\frac{B_0^2}{B_\theta(a)^2}\frac{R_0 B_\theta(a)^2 a^2\beta_p(a)}{B_0^2} = \frac{a}{R_0}\beta_p(a) = \epsilon\beta_p(a) \tag{7.80}$$

The first term in the integral is

$$\frac{q(a)^2}{a^3}\int_0^a \mathrm{d}r'\;\frac{r'^3}{R_0}\frac{R_0^2 B_\theta(r')^2}{r'^2 B_0^2} = \frac{a^2}{a^3 R_0^2}\frac{B_0^2}{B_\theta(a)^2}\int_0^a \mathrm{d}r'\; r'\frac{R_0 B_\theta(r')^2}{B_0^2} = \frac{1}{R_0 a B_\theta(a)^2}\int_0^a \mathrm{d}r'\; r'B_\theta(r)^2 \tag{7.81}$$

$$= \frac{\epsilon}{2}\frac{2}{a^2 B_\theta(a)^2}\int_0^a \mathrm{d}r'\; r'B_\theta(r)^2 = \frac{\epsilon}{2}\frac{2}{a^2 B_p(a)^2}\int_0^a \mathrm{d}r'\; r'B_p(r)^2 = \epsilon\frac{l_i}{2} \tag{7.82}$$

So that we find

$$\left.\frac{\mathrm{d}\Delta_s}{\mathrm{d}r}\right|_{r=a} = \epsilon\left(\frac{l_i}{2} + \beta_p(a)\right) \tag{7.83}$$

so that clearly

$$\Lambda(a) = \frac{1}{\epsilon}\left.\frac{\mathrm{d}\Delta_s}{\mathrm{d}r}\right|_{r=a} - 1 = \frac{l_i}{2} + \beta_p(a) - 1 \tag{7.84}$$

as desired.

7.5.3 Tokamak Safety Factor

7.5.3.1 Expression

Show that the tokamak safety factor can be expressed as

$$q(\psi) = \frac{I(\psi)V'(\psi)}{4\pi^2}\left\langle\frac{1}{R^2}\right\rangle \tag{7.85}$$

where $V(\psi)$ is the volume inside the flux surface ψ.

Solution:

We use that

$$\mathbf{B} = I(\psi)\,\nabla\varphi + \nabla\varphi\times\nabla\psi \tag{7.86}$$

$$q(\psi) = \frac{\langle\mathbf{B}\cdot\nabla\varphi\rangle}{\langle\mathbf{B}\cdot\nabla\theta\rangle} \tag{7.87}$$

so that

$$\mathbf{B}\cdot\nabla\varphi = I(\psi)|\nabla\varphi|^2 = \frac{I(\psi)}{R^2} \tag{7.88}$$

So

$$\langle\mathbf{B}\cdot\nabla\varphi\rangle = \left\langle\frac{I(\psi)}{R^2}\right\rangle = I(\psi)\left\langle\frac{1}{R^2}\right\rangle \tag{7.89}$$

because $I(\psi)$ is a flux function it is constant and can be taken out of the flux-surface average.

Now we use

$$\frac{V'(\psi)}{2\pi} = \int_0^{2\pi}\frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta} \tag{7.90}$$

from (with $1/\sqrt{g} = \mathbf{B}\cdot\nabla\theta$)

$$V(\psi) = \int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\psi}\mathrm{d}\psi'\,\sqrt{g} = 2\pi\int_0^{\psi}\mathrm{d}\psi'\int_0^{2\pi}\frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta} \tag{7.91}$$

And so

$$\langle \mathbf{B}\cdot\nabla\theta\rangle = \frac{\oint \mathrm{d}\theta\, \dfrac{\mathbf{B}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\theta}}{\oint \dfrac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta}} = \frac{\oint \mathrm{d}\theta}{\oint \dfrac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta}} = \frac{2\pi}{\frac{V'(\psi)}{2\pi}} = \frac{4\pi^2}{V'(\psi)} \tag{7.92}$$

Thus,

$$q(\psi) = \frac{I(\psi)\left\langle \dfrac{1}{R^2}\right\rangle}{\frac{4\pi^2}{V'}} = \frac{I(\psi)V'(\psi)}{4\pi^2}\left\langle \frac{1}{R^2}\right\rangle \tag{7.93}$$

as desired.

7.5.3.2 Circular Cross Section Equilibrium

For an equilibrium with circular cross section and large aspect ratio, q is given by (HS-7.16). How is this formula modified if the cross section is elliptical?

$$q(r) = \frac{rB_0}{RB_{\theta 0}(r)} \tag{7.94}$$

Solution:

We can use our previous formula. We will have $\left\langle \frac{1}{R^2}\right\rangle \approx \frac{1}{R^2}$ in the high-aspect-ratio and $I(\psi) = RB_\varphi$ still, but the area for an ellipse is $A = \pi ab$ where a and b are the major and minor radii of the ellipse. Now, we can write $\psi(r)$ is constant on an ellipse given by (k is the vertical elongation)

$$r^2 = \frac{z^2}{k^2} + x^2 \tag{7.95}$$

So then

$$A = \pi k r^2 \tag{7.96}$$

and since the volume is $A(2\pi R)$ we have $V = 2\pi^2 k r^2 R$. Since $V'(\psi) = \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\psi} = 4\pi^2 k r R \frac{\mathrm{d}r}{\mathrm{d}\psi}$ we have

$$q(r) = \frac{I(\psi)V'(\psi)}{4\pi^2 R^2} = \frac{(RB_\varphi)4\pi^2 k r R\frac{\mathrm{d}r}{\mathrm{d}\psi}}{4\pi^2 R^2} = \frac{rkB_\varphi}{\frac{\mathrm{d}\psi}{\mathrm{d}r}} \tag{7.97}$$

We have $\psi = -RA_\varphi$ so that $\psi' = -R\frac{\partial A_\varphi}{\partial r} = RB_\theta(r)$, so that

$$q(r) = \frac{rkB_\varphi}{RB_\theta} = kq_{\text{circular}}(r) \tag{7.98}$$

7.5.4 Bounce Time

Evaluate the bounce time (HS-7.26) for trapped particles.

$$\tau_b = \frac{qR}{v\sqrt{2\epsilon\lambda}} \oint \frac{\mathrm{d}\theta}{\sigma\sqrt{k^2 - \sin^2(\theta/2)}} \tag{7.99}$$

Solution:

For trapped particles we have to consider $\sigma = \pm 1$. We can use that

$$F(\phi, k) = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{7.100}$$

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2\theta}\,\mathrm{d}\theta \tag{7.101}$$

are the incomplete elliptic integrals of the first and second kind, respectively. Note $K(k) = E(\pi/2, k)$. Therefore, for a trapped particle turning at $\theta = \theta_b$ we have

$$I \equiv 4\int_0^{\theta_b} \frac{\mathrm{d}\theta}{\sqrt{k^2 - \sin^2(\theta/2)}} = \oint \frac{\mathrm{d}\theta}{\sigma\sqrt{k^2 - \sin^2(\theta/2)}} \tag{7.102}$$

(the 4 comes from doubling due to each sign of σ, and then doubling because $-\theta_b$ to 0 is symmetric to 0 to θ_b, or alternatively using σ and realizing we go from $-\theta_b$ to θ_b, and then from θ_b to $-\theta_b$ for a full orbit so you get four times from 0 to θ_b because σ changes sign for θ_b to $-\theta_b$). And so

$$I = \frac{4}{k}\int_0^{\theta_b} \frac{\mathrm{d}\theta}{\sqrt{1 - \alpha^2 \sin^2(\theta/2)}} = \frac{8}{k}\int_0^{\theta_b/2} \frac{\mathrm{d}\chi}{\sqrt{1 - \alpha^2 \sin^2\chi}} = \frac{8}{k}F(\theta_b/2, \alpha) \tag{7.103}$$

where $\alpha = \frac{1}{k}$, and $\chi = \theta/2$. However, we can apparently do better, using $k = \sin(\theta_b/2)$ so let's choose $\alpha \sin\chi = \sin\zeta$ so at $\chi = \theta_b/2$ we have $\sin\zeta_b = \sin(\theta_b/2)/\sin(\theta_b/2) = 1$ and $\zeta_b = \arcsin(1) = \pi/2$. We also have $\alpha\cos\chi\,\mathrm{d}\chi = \cos\zeta\,\mathrm{d}\zeta$

$$\int_0^{\theta_b/2} \frac{\mathrm{d}\chi}{\sqrt{1 - \alpha^2\sin^2\xi}} = \int_0^{\pi/2} \mathrm{d}\zeta\,\frac{\cos\zeta}{\alpha\cos\chi\sqrt{1 - \alpha^2\sin^2\chi}} = \int_0^{\pi/2}\mathrm{d}\zeta\,\frac{\sqrt{1 - \sin^2\zeta}}{\alpha\sqrt{(1 - \alpha^2\sin^2\chi)(1 - \sin^2\chi)}} \tag{7.104}$$

$$= \frac{1}{\alpha}\int_0^{\pi/2}\mathrm{d}\zeta\,\frac{\sqrt{1 - \sin^2\zeta}}{\sqrt{(1 - \sin^2\zeta)(1 - k^2\sin^2\zeta)}} = \frac{1}{\alpha}\int_0^{\pi/2}\frac{\mathrm{d}\zeta}{\sqrt{1 - k^2\sin^2\zeta}} \tag{7.105}$$

Thus,

$$I = \frac{8}{k\alpha}\int_0^{\pi/2}\frac{\mathrm{d}\zeta}{\sqrt{1 - k^2\sin^2\zeta}} = 8K(k) \tag{7.106}$$

So we find

$$\tau_b = \frac{qR}{v\sqrt{2\epsilon\lambda}}8K(k) = \frac{8qR}{v\sqrt{2\epsilon}}K(k) \tag{7.107}$$

where $\lambda = 1 + \mathcal{O}(\epsilon)$ is used for trapped particles.

7.5.5 Toroidal Precession Frequency, Large-Aspect-Ratio Tokamak

As discussed in Section 6.4, trapped orbits drift slowly across the magnetic field. Without the drift a trapped particle would remain on the same field line all the time, but due to the magnetic drift the orbit does not quite close on itself, so that the field line label $\varphi_0 = \varphi - q\theta$ varies in time. This implies that trapped orbits undergo slow toroidal precession. Calculate the precession frequency in a large-aspect-ratio tokamak with circular cross section.

Solution:

We begin by calculating $\dot{\varphi}_0$ using that this should be the drift dotted into the direction of φ_0, that is

$$\dot{\varphi}_0 = \mathbf{v}_d \cdot \nabla(\varphi - q\theta) \tag{7.108}$$

We have for the large-aspect ratio that this is dominated by ($\mathbf{v}_d$ is in $\hat{\mathbf{z}}$ while $\nabla\varphi$ is clearly perpendicular to this direction)

$$\mathbf{v}_d \cdot \nabla(q\theta) \tag{7.109}$$

If we break this into components, we use

$$\nabla(q\theta) = \frac{\partial(q\theta)}{\partial r}\hat{\mathbf{r}} + q\frac{\partial\theta}{\partial\theta}\nabla\theta + \frac{q}{R}\frac{\partial\theta}{\partial\varphi}\hat{\boldsymbol{\varphi}} \tag{7.110}$$

We use

$$\nabla z = \hat{\mathbf{z}} = \sin\theta\,\nabla r + r\cos\theta\,\nabla\theta \tag{7.111}$$

$$|\,\nabla\theta|^2 = r^{-2} \tag{7.112}$$

So that using $\mathbf{v}_d = -\frac{v^2+v_\parallel^2}{2\Omega_\varphi R}\hat{\mathbf{z}}$, so that

$$\dot{\varphi}_0 = \frac{v^2+v_\parallel^2}{2\Omega_\varphi R}\left(\sin\theta\frac{\partial(q\theta)}{\partial r} + rq\cos\theta|\,\nabla\theta|^2\right) = \frac{v^2+v_\parallel^2}{2\Omega_\varphi R}\left(\theta\frac{\partial q}{\partial r}\sin\theta + \frac{q\cos\theta}{r}\right) \tag{7.113}$$

To get the precession now, we average over θ with the bounce average.

The bounce average is devined by

$$\langle f\rangle_b = \frac{1}{\tau_b}\oint f\frac{\mathrm{d}\theta}{\dot{\theta}} = \frac{1}{\tau_b}\oint \frac{qRf}{\sigma v\sqrt{2\epsilon\lambda}\sqrt{k^2-\sin^2(\theta/2)}}\,\mathrm{d}\theta \tag{7.114}$$

so (once again using $\lambda = 1 + \mathcal{O}(\epsilon)$ for trapped particles)

$$\langle\theta\sin\theta\rangle_b = \frac{4qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\theta_b}\frac{\theta\sin\theta}{\sqrt{k^2-\sin^2(\theta/2)}}\,\mathrm{d}\theta \tag{7.115}$$

Let's take $u = \theta/2$

$$\langle\theta\sin\theta\rangle_b = \frac{16qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\theta_b/2}\frac{u\sin(2u)}{\sqrt{k^2-\sin^2 u}}\mathrm{d}u \tag{7.116}$$

We can use previous trick of $\sin u = k \sin\zeta$, $\cos u \,\mathrm{d}u = k\cos\zeta\,\mathrm{d}\zeta$ with $k = \sin(\theta_b/2)$ so that

$$\langle\theta\sin\theta\rangle_b = \frac{16qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\pi/2}\frac{2u(\sin u)(\cos u)k\cos\zeta}{\cos u\sqrt{k^2-\sin^2 u}}\,\mathrm{d}\zeta = \frac{32qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\pi/2}\frac{ku\sin\zeta\cos\zeta}{\sqrt{1-\sin^2\zeta}}\,\mathrm{d}\zeta \quad (7.117)$$

$$= \frac{32qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\pi/2} ku\sin\zeta\,\mathrm{d}\zeta = \frac{32qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\pi/2} ku\frac{\mathrm{d}}{\mathrm{d}\zeta}(-\cos\zeta)\,\mathrm{d}\zeta \quad (7.118)$$

$$= \frac{32qkR}{\sqrt{2\epsilon}v\tau_b}\left([-u\cos\zeta]_0^{\pi/2} + \int_0^{\pi/2}\frac{\mathrm{d}u}{\mathrm{d}\zeta}\cos\zeta\,\mathrm{d}\zeta\right) = \frac{32qkR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\pi/2}\frac{\mathrm{d}u}{\mathrm{d}\zeta}\cos\zeta\,\mathrm{d}\zeta \quad (7.119)$$

because $\cos\frac{\pi}{2} = 0$ and $u(\zeta = 0) = 0$. so that

$$\langle\theta\sin\theta\rangle_b = \frac{32qkR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\pi/2}\frac{k\cos\zeta}{\cos u}\cos\zeta\,\mathrm{d}\zeta = \frac{32qkR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\pi/2}\frac{k\cos^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\,\mathrm{d}\zeta \quad (7.120)$$

$$= \frac{32qk^2R}{\sqrt{2\epsilon}v\tau_b}\left[\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\zeta}}\,\mathrm{d}\zeta - \int_0^{\pi/2}\frac{\sin^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\,\mathrm{d}\zeta\right] \quad (7.121)$$

Let's look at

$$\int_0^{\pi/2}\frac{k^2\sin^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\,\mathrm{d}\zeta = \int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\zeta}}\,\mathrm{d}\zeta - \int_0^{\pi/2}\frac{1-k^2\sin^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\,\mathrm{d}\zeta \quad (7.122)$$

$$= K(k) - \int_0^{\pi/2}\sqrt{1-k^2\sin^2\zeta}\,\mathrm{d}\zeta = K(k) - E(k) \quad (7.123)$$

Thus,

$$\langle\theta\sin\theta\rangle_b = \frac{32qR}{\sqrt{2\epsilon}v\tau_b}\left(k^2K(k) - [K(k) - E(k)]\right) \quad (7.124)$$

If we use

$$\tau_b = \frac{8qR}{v\sqrt{2\epsilon}}K(k) \quad (7.125)$$

we find

$$\langle\theta\sin\theta\rangle_b = \frac{32qR}{\sqrt{2\epsilon}v}\frac{v\sqrt{2\epsilon}}{8qRK(k)}\left(E(k) + K(k)(k^2-1)\right) \quad (7.126)$$

$$= 4\left(\frac{E(k)}{K(k)} + k^2 - 1\right) \quad (7.127)$$

Now let's do

$$\langle\cos\theta\rangle_b = \frac{4qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\theta_b}\frac{\cos\theta}{\sqrt{k^2-\sin^2(\theta/2)}}\,\mathrm{d}\theta \quad (7.128)$$

Let's take $u = \theta/2$

$$\langle\cos\theta\rangle_b = \frac{8qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\theta_b/2}\frac{\cos(2u)}{\sqrt{k^2-\sin^2 u}}\mathrm{d}u \quad (7.129)$$

$$= \frac{8qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\theta_b/2}\frac{1-2\sin^2 u}{\sqrt{k^2-\sin^2 u}}\mathrm{d}u \quad (7.130)$$

We can use previous trick of $\sin u = k\sin\zeta$, $\cos u\,\mathrm{d}u = k\cos\zeta\,\mathrm{d}\zeta$ with $k = \sin(\theta_b/2)$ so that

$$\langle\cos\theta\rangle_b = \frac{8qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\theta_b/2}\frac{1-2k^2\sin^2\zeta}{k\sqrt{1-\sin^2\zeta}}\left(\frac{k\cos\zeta}{\cos u}\right)\mathrm{d}\zeta = \frac{8qR}{\sqrt{2\epsilon}v\tau_b}\int_0^{\theta_b/2}\frac{1-2k^2\sin^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\mathrm{d}\zeta \quad (7.131)$$

Which, from our previous result indicates

$$\langle\cos\theta\rangle_b = \frac{8qR}{\sqrt{2\epsilon}v\tau_b}\left[K(k) - 2\left[K(k)-E(k)\right]\right] = \frac{8qR}{\sqrt{2\epsilon}v\tau_b}\left[2E(k)-K(k)\right] \quad (7.132)$$

Using the definition of τ_b we then find

$$\langle\cos\theta\rangle_b = \frac{2E(k)}{K(k)} - 1 \quad (7.133)$$

So we finally find (ignoring $v_\parallel^2$, which we will see is justified by smallness of ϵ)

$$\dot{\varphi}_0 = \frac{v^2}{2\Omega_\varphi R}\left(4\frac{\mathrm{d}q}{\mathrm{d}r}\left[\frac{E(k)}{K(k)} + k^2 - 1\right] + \frac{q}{r}\left[\frac{2E(k)}{K(k)} - 1\right]\right) \quad (7.134)$$

If we pull out an r and q we find with $\frac{\mathrm{d}q}{\mathrm{d}r} = q'$

$$\dot{\varphi}_0 = \frac{qv^2}{\Omega_\varphi rR}\left[\frac{E(k)}{K(k)} - \frac{1}{2} + \frac{2rq'}{q}\left(\frac{E(k)}{K(k)} + k^2 - 1\right)\right] \quad (7.135)$$

where there must be a typo in the book's answer.

Note the book also ignores the $v_\parallel^2$ which I have not calculated, but because $v_\parallel = v_\parallel(\theta)$, it would need to be averaged, giving extra terms [which will be $\mathcal{O}(\epsilon)$].

For example,

$$\left\langle v_\parallel^2\theta\sin\theta\right\rangle_b = \frac{4\sqrt{2\epsilon}v}{qR\tau_b}\int_0^{\theta_b}\theta\sin\theta\sqrt{k^2-\sin^2(\theta/2)}\,\mathrm{d}\theta \quad (7.136)$$

$$\left\langle v_\parallel^2\cos\theta\right\rangle_b = \frac{4\sqrt{2\epsilon}v}{qR\tau_b}\int_0^{\theta_b}\cos\theta\sqrt{k^2-\sin^2(\theta/2)}\,\mathrm{d}\theta \quad (7.137)$$

For fun, let's calculate these, as they should be simple to calculate. First let's do the $\theta\sin\theta$ term again. Use $u = \theta/2$ to find

$$\left\langle v_\parallel^2\theta\sin\theta\right\rangle_b = \frac{16\sqrt{2\epsilon}v}{qR\tau_b}\int_0^{\theta_b/2}2\sin u\cos u\sqrt{k^2-\sin^2 u}\,\mathrm{d}u \quad (7.138)$$

Then use $\sin u = k\sin\zeta$ with $\cos u\,\mathrm{d}u = k\cos\zeta\,\mathrm{d}\zeta$, $\theta_b/2 \to \pi/2$ in the limit and

$$\left\langle v_\parallel^2\theta\sin\theta\right\rangle_b = \frac{32\sqrt{2\epsilon}v}{qR\tau_b}\int_0^{\pi/2}\sin u\cos u\sqrt{k^2-\sin^2 u}\frac{k\cos\zeta}{\cos u}\,\mathrm{d}\zeta \quad (7.139)$$

$$= \frac{32\sqrt{2\epsilon}}{qRv\tau_b}\int_0^{\pi/2}(k\sin\zeta)k\sqrt{1-\sin^2\zeta}(k\cos\zeta)\,\mathrm{d}\zeta \quad (7.140)$$

$$= \frac{32k^3\sqrt{2\epsilon}v}{qR\tau_b}\int_0^{\pi/2}\sin\zeta\cos^2\zeta\,\mathrm{d}\zeta \quad (7.141)$$

Take $w = \cos\zeta$ so $\mathrm{d}w = -\sin\zeta\,\mathrm{d}\zeta$ and we find

$$\left\langle v_\parallel^2 \theta \sin\theta \right\rangle_b = \frac{32k^3\sqrt{2\epsilon}v}{qR\tau_b}\int_0^1 w^2\,\mathrm{d}w = \frac{32k^3\sqrt{2\epsilon}v}{qR\tau_b}\frac{2}{3} = \frac{64k^3\sqrt{2\epsilon}v}{3qR}\frac{v\sqrt{2\epsilon}}{8qRK(k)} = \frac{16\epsilon v^2}{3q^2R^2K(k)} \tag{7.142}$$

Note that this is $\mathcal{O}(\epsilon)$ so that one would suspect it is much smaller than the contribution from v^2. And for the $\cos\theta$ term we find, using $u = \theta/2$,

$$\left\langle v_\parallel^2 \cos\theta \right\rangle_b = \frac{4\sqrt{2\epsilon}v}{qR\tau_b}\int_0^{\theta_b} \cos\theta\sqrt{k^2 - \sin^2(\theta/2)}\,\mathrm{d}\theta \tag{7.143}$$

$$= \frac{8\sqrt{2\epsilon}v}{qR\tau_b}\int_0^{\theta_b/2} \cos(2u)\sqrt{k^2 - \sin^2 u}\,\mathrm{d}u \tag{7.144}$$

$$= \frac{8\sqrt{2\epsilon}v}{qR\tau_b}\int_0^{\theta_b/2} \left(1 - 2\sin^2 u\right)\sqrt{k^2 - \sin^2 u}\,\mathrm{d}u \tag{7.145}$$

We then use $\sin u = k\sin\zeta$ with $\cos u\,\mathrm{d}u = k\cos\zeta\,\mathrm{d}\zeta$, $\theta_b/2 \to \pi/2$ in the limit and

$$\left\langle v_\parallel^2 \cos\theta \right\rangle_b = \frac{8\sqrt{2\epsilon}v}{qR\tau_b}\int_0^{\pi/2} (1 - 2k^2\sin^2\zeta)k\sqrt{1 - \sin^2\zeta}\frac{k\cos\zeta}{\cos u}\,\mathrm{d}\zeta \tag{7.146}$$

$$= \frac{8\sqrt{2\epsilon}v}{qR\tau_b}\int_0^{\pi/2} (1 - 2k^2\sin^2\zeta)k\sqrt{1 - \sin^2\zeta}\frac{k\cos\zeta}{\sqrt{1 - k^2\sin^2\zeta}}\,\mathrm{d}\zeta \tag{7.147}$$

$$= \frac{8\sqrt{2\epsilon}vk^2}{qR\tau_b}\int_0^{\pi/2} (1 - k^2\sin^2\zeta - k^2\sin^2\zeta)\frac{\cos^2\zeta}{\sqrt{1 - k^2\sin^2\zeta}}\,\mathrm{d}\zeta \tag{7.148}$$

$$= \frac{8\sqrt{2\epsilon}vk^2}{qR\tau_b}\int_0^{\pi/2} \left[\frac{1 - k^2\sin^2\zeta}{\sqrt{1 - k^2\sin^2\zeta}}\cos^2\zeta - \frac{k^2\sin^2\zeta\cos^2\zeta}{\sqrt{1 - k^2\sin^2\zeta}}\right]\mathrm{d}\zeta \tag{7.149}$$

$$= \frac{8\sqrt{2\epsilon}vk^2}{qR\tau_b}\int_0^{\pi/2} \left[\sqrt{1 - k^2\sin^2\zeta} - \frac{k^2\sin^2\zeta}{\sqrt{1 - k^2\sin^2\zeta}}\right]\cos^2\zeta\,\mathrm{d}\zeta \tag{7.150}$$

Let's expand $\cos^2\zeta = (1 + \cos(2\zeta))/2$ and look at the $\cos(2\zeta)$ part on the first term in the integral.

$$\int_0^{\pi/2} \sqrt{1 - k^2\sin^2\zeta}\frac{\cos(2\zeta)}{2}\,\mathrm{d}\zeta = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\zeta}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\zeta}\sin(2\zeta)\,\mathrm{d}\zeta \tag{7.151}$$

$$= \left[\sqrt{1 - k^2\sin^2\zeta}\frac{\sin(2\zeta)}{4}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{-k^2\sin\zeta\cos\zeta\sin(2\zeta)}{4\sqrt{1 - k^2\sin^2\zeta}}\,\mathrm{d}\zeta \tag{7.152}$$

$$= \int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{2\sqrt{1 - k^2\sin^2\zeta}}\,\mathrm{d}\zeta \tag{7.153}$$

Thus,

$$\left\langle v_\parallel^2 \cos\theta \right\rangle_b = \frac{8\sqrt{2\epsilon}vk^2}{qR\tau_b}\left[\frac{E(k)}{2} + \int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{2\sqrt{1 - k^2\sin^2\zeta}}\,\mathrm{d}\zeta - \int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{\sqrt{1 - k^2\sin^2\zeta}}\,\mathrm{d}\zeta\right] \tag{7.154}$$

$$= \frac{4\sqrt{2\epsilon}vk^2}{qR\tau_b}\left[E(k) - \int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{\sqrt{1 - k^2\sin^2\zeta}}\,\mathrm{d}\zeta\right] \tag{7.155}$$

Let's look at (using $\sin^2\zeta\cos^2\zeta = \sin^2\zeta - \sin^4\zeta$)

$$\int_0^{\pi/2} \mathrm{d}\zeta\ \frac{k^2\sin^4\zeta}{\sqrt{1-k^2\sin^2\zeta}} = \int_0^{\pi/2} \mathrm{d}\zeta\ \sin^2\zeta \frac{1-(1-k^2\sin^2\zeta)}{\sqrt{1-k^2\sin^2\zeta}} \tag{7.156}$$

$$= \int_0^{\pi/2} \mathrm{d}\zeta\ \frac{\sin^2\zeta}{\sqrt{1-k^2\sin^2\zeta}} - \int_0^{\pi/2} \mathrm{d}\zeta\ \sin^2\zeta\sqrt{1-k^2\sin^2\zeta} \tag{7.157}$$

$$= \frac{K(k)-E(k)}{k^2} - \int_0^{\pi/2} \sin^2\zeta\sqrt{1-k^2\sin^2\zeta} \tag{7.158}$$

Now we use $\sin^2\zeta = (1-\cos(2\zeta))/2$ and use our previous result

$$= \frac{K(k)-E(k)}{k^2} - \int_0^{\pi/2} \sin^2\zeta\sqrt{1-k^2\sin^2\zeta} \tag{7.159}$$

$$= \frac{K(k)-E(k)}{k^2} - \frac{1}{2}E(k) + \int_0^{\pi/2} \mathrm{d}\zeta\ \frac{\cos(2\zeta)}{2}\sqrt{1-k^2\sin^2\zeta} \tag{7.160}$$

$$= \frac{K(k)-E(k)}{k^2} - \frac{1}{2}E(k) + \int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{2\sqrt{1-k^2\sin^2\zeta}}\mathrm{d}\zeta \tag{7.161}$$

$$\tag{7.162}$$

Thus, we find

$$\int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\mathrm{d}\zeta = \int_0^{\pi/2} \frac{k^2\sin^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\mathrm{d}\zeta - \int_0^{\pi/2} \frac{k^2\sin^4\zeta}{\sqrt{1-k^2\sin^2\zeta}}\mathrm{d}\zeta \tag{7.163}$$

$$= K(k) - E(k) - \left[\frac{K(k)-E(k)}{k^2} - \frac{1}{2}E(k) + \frac{1}{2}\int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\mathrm{d}\zeta\right] \tag{7.164}$$

So that

$$\frac{3}{2}\int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\mathrm{d}\zeta = K(k) - E(k) - \frac{K(k)-E(k)}{k^2} + \frac{E(k)}{2} \tag{7.165}$$

$$\int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\mathrm{d}\zeta = \frac{2}{3}\left[K(k) - \frac{E(k)}{2} + \frac{E(k)-K(k)}{k^2}\right] \tag{7.166}$$

$$\int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\mathrm{d}\zeta = K(k)\left(\frac{2}{3} - \frac{2}{3k^2}\right) + E(k)\left(\frac{2}{3k^2} - \frac{1}{3}\right) \tag{7.167}$$

Thus,

$$\left\langle v_\parallel^2\cos\theta\right\rangle_b = \frac{4\sqrt{2\epsilon}vk^2}{qR\tau_b}\left[E(k) - \int_0^{\pi/2} \frac{k^2\sin^2\zeta\cos^2\zeta}{\sqrt{1-k^2\sin^2\zeta}}\mathrm{d}\zeta\right] \tag{7.168}$$

$$= \frac{4\sqrt{2\epsilon}vk^2}{qR\tau_b}\left[E(k) - \left\{K(k)\left(\frac{2}{3} - \frac{2}{3k^2}\right) + E(k)\left(\frac{2}{3k^2} - \frac{1}{3}\right)\right\}\right] \tag{7.169}$$

$$= \frac{4\sqrt{2\epsilon}vk^2}{qR\tau_b}\left(E(k)\left[\frac{4}{3} - \frac{2}{3k^2}\right] + K(k)\left[\frac{2}{3k^2} - \frac{2}{3}\right]\right) \tag{7.170}$$

If we plug in τ_b we find

$$\frac{4\sqrt{2\epsilon}vk^2}{qR}\frac{v\sqrt{2\epsilon}}{8qRK(k)} = \frac{\epsilon v^2 k^2}{q^2R^2K(k)} \tag{7.171}$$

and so finally

$$\left\langle v_\parallel^2 \cos\theta \right\rangle_b = \frac{\epsilon v^2 k^2}{q^2R^2K(k)} \left(E(k)\left[\frac{4}{3} - \frac{2}{3k^2}\right] + K(k)\left[\frac{2}{3k^2} - \frac{2}{3}\right]\right) \tag{7.172}$$

$$\left\langle v_\parallel^2 \cos\theta \right\rangle_b = \frac{\epsilon v^2 k^2}{q^2R^2} \left(\frac{E(k)}{K(k)}\left[\frac{4}{3} - \frac{2}{3k^2}\right] + \left[\frac{2}{3k^2} - \frac{2}{3}\right]\right) \tag{7.173}$$

$$\left\langle v_\parallel^2 \cos\theta \right\rangle_b = \frac{\epsilon v^2}{q^2R^2} \left(\frac{E(k)}{K(k)}\left[\frac{4k^2}{3} - \frac{2}{3}\right] + \frac{2}{3}\left[1 - k^2\right]\right) \tag{7.174}$$

where once again, the prescence of ϵ implies the result is quite a bit smaller than the v^2 component.

Chapter 8

Transport in Toroidal Plasmas

8.1 Transport Ordering

check.

8.2 Collisionality

check.

8.3 Distribution Function

We find

$$\int \mathrm{d}^3v\ (\ln f_0)v_\parallel\ \nabla_\parallel f_0 = \sum_\sigma \frac{2\pi B}{m^2 B_0}\int \mathrm{d}\mathcal{E}\int \mathrm{d}\lambda\ \frac{\mathcal{E}}{v_\parallel}(\ln f_0)v_\parallel\ \nabla_\parallel f_0 = \sum_\sigma \frac{2\pi B}{m^2 B_0}\int \mathrm{d}\mathcal{E}\int \mathrm{d}\lambda\ \mathcal{E}(\ln f_0)\ \nabla_\parallel f_0 \tag{8.1}$$

We then use that

$$\ln f_0\ \nabla_\parallel f_0 = \nabla_\parallel(f_0 \ln f_0) - f_0\ \nabla_\parallel \ln f_0 = \nabla_\parallel(f_0 \ln f_0) - \nabla_\parallel f_0 = \nabla_\parallel(f_0[\ln f_0 - 1]) \tag{8.2}$$

with $\nabla_\parallel = \mathbf{b}\cdot\nabla$, so that

$$\int \mathrm{d}^3v\ (\ln f_0)v_\parallel\ \nabla_\parallel f_0 = \sum_\sigma \frac{2\pi B}{m^2 B_0}\int \mathrm{d}\mathcal{E}\int \mathrm{d}\lambda\ \mathcal{E}\mathbf{b}\cdot\nabla[f_0(\ln f_0 - 1)] \tag{8.3}$$

$$= \sum_\sigma \frac{2\pi}{m^2 B_0}\mathbf{B}\cdot\int \mathrm{d}\mathcal{E}\int \mathrm{d}\lambda\ \mathcal{E}\,\nabla[f_0(\ln f_0 - 1)] \tag{8.4}$$

$$= \sum_\sigma \frac{2\pi}{m^2 B_0}\mathbf{B}\cdot\nabla\int \mathrm{d}\mathcal{E}\int \mathrm{d}\lambda\ \mathcal{E}f_0(\ln f_0 - 1) \tag{8.5}$$

And since $\langle \mathbf{B}\cdot\nabla g\rangle = 0$ for any function g the above will vanish.

Now, for (using $\mathbf{B}\cdot\nabla\psi = 0$ and $\mathbf{J}\cdot\nabla\psi = 0$)

$$\dot{\mathbf{R}}\cdot\nabla\psi = \left(\cancel{v_\parallel \mathbf{b}} + \mathbf{v}_d\right)\cdot\nabla\psi = \frac{v_\parallel}{B_\parallel}\boldsymbol{\nabla}\times\left(\frac{v_\parallel \mathbf{B}}{\Omega}\right)\cdot\nabla\psi \tag{8.6}$$

$$= \left[v_\parallel \nabla\left(\frac{v_\parallel}{\Omega}\right)\times\mathbf{b} + \cancel{\frac{\mu_0 v_\parallel^2}{\Omega B}\mathbf{J}}\right]\cdot\nabla\psi \tag{8.7}$$

$$= v_\parallel \nabla\left(\frac{v_\parallel}{\Omega}\right)\times\mathbf{b}\cdot\nabla\psi = v_\parallel \mathbf{b}\times\nabla\psi\cdot\nabla\left(\frac{v_\parallel}{\Omega}\right) \tag{8.8}$$

as required.

One uses

$$\frac{\mathbf{B}\times\nabla\psi}{B^2} = \frac{\mathbf{b}\times\nabla\psi}{B} = \frac{I}{B}\mathbf{b} - R\hat{\boldsymbol{\varphi}} \tag{8.9}$$

with $R\,\nabla\psi = \hat{\boldsymbol{\varphi}}$. This comes from

$$\mathbf{B}\times\nabla\psi = (I\,\nabla\varphi + \nabla\varphi\times\nabla\psi)\times\nabla\psi = I\,\nabla\varphi\times\nabla\psi + \cancel{\nabla\psi(\nabla\varphi\cdot\nabla\psi)} - \nabla\varphi|\nabla\psi|^2 \tag{8.10}$$

with

$$B^2 = I|\nabla\varphi|^2 + |\nabla\varphi\times\nabla\psi|^2 = I^2R^{-2} + R^{-2}|\hat{\boldsymbol{\varphi}}\times\nabla\psi|^2 = R^{-2}(I^2 + |\nabla\psi|^2) \tag{8.11}$$

$$\hat{\mathbf{b}} = \frac{\mathbf{B}}{B} = \frac{IR\,\nabla\varphi + R\,\nabla\varphi\times\nabla\psi}{\sqrt{I^2+|\nabla\psi|^2}} = \frac{I\hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\varphi}}\times\nabla\psi}{\sqrt{I^2+|\nabla\psi|^2}} \tag{8.12}$$

where I have used that $\nabla\varphi\cdot\nabla\psi = 0$. So

$$\mathbf{B}\times\nabla\psi = \frac{I}{R}\hat{\boldsymbol{\varphi}}\times\nabla\psi - \frac{\hat{\boldsymbol{\varphi}}}{R}|\nabla\psi|^2 \tag{8.13}$$

$$\frac{\mathbf{B}\times\nabla\psi}{B^2} = \frac{IR^2}{R(I^2+|\nabla\psi|^2)}\hat{\boldsymbol{\varphi}}\times\nabla\psi - \frac{R^2\hat{\boldsymbol{\varphi}}}{R(I^2+|\nabla\psi|^2)}|\nabla\psi|^2 = \frac{IR}{I^2+|\nabla\psi|^2}\hat{\boldsymbol{\varphi}}\times\nabla\psi - \frac{R\hat{\boldsymbol{\varphi}}}{I^2+|\nabla\psi|^2}|\nabla\psi|^2 \tag{8.14}$$

$$= \frac{IR}{\sqrt{I^2+|\nabla\psi|^2}}\left(\frac{\hat{\boldsymbol{\varphi}}\times\nabla\psi + I^2\hat{\boldsymbol{\varphi}}}{\sqrt{I^2+|\nabla\psi|^2}}\right) - \frac{I^2R\hat{\boldsymbol{\varphi}}}{I^2+|\nabla\psi|^2} - \frac{R|\nabla\psi|^2\hat{\boldsymbol{\varphi}}}{I^2+|\nabla\psi|^2} \tag{8.15}$$

$$= \frac{I}{B}\mathbf{b} - R\frac{I^2+|\nabla\psi|^2}{I^2+|\nabla\psi|^2}\hat{\boldsymbol{\varphi}} = \frac{I}{B}\mathbf{b} - R\hat{\boldsymbol{\varphi}} \tag{8.16}$$

Thus,

$$\dot{\mathbf{R}}\cdot\nabla\psi = v_\parallel(\mathbf{b}\times\nabla\psi)\cdot\nabla\left(\frac{v_\parallel}{\Omega}\right) = v_\parallel\left(I\hat{\mathbf{b}} - RB\hat{\boldsymbol{\varphi}}\right)\cdot\nabla\left(\frac{v_\parallel}{\Omega}\right) \tag{8.17}$$

$$= Iv_\parallel\mathbf{b}\cdot\nabla\left(\frac{v_\parallel}{\Omega}\right) = v_\parallel\nabla_\parallel\left(\frac{v_\parallel}{\Omega}\right) \tag{8.18}$$

where we've used there is no φ variation of the quantities in $v_\parallel/\Omega$.

8.4 Current

check.

8.5 Parallel Particle and Heat Fluxes

check.

8.6 Flow Across Flux Surfaces

Let's prove

$$\mathbf{\Gamma} \cdot \nabla\psi = R\hat{\boldsymbol{\varphi}} \cdot (n\mathbf{V} \times \mathbf{B}) = R\hat{\boldsymbol{\varphi}} \cdot (\mathbf{\Gamma} \times \mathbf{B}) \tag{8.19}$$

with $\mathbf{B} = I\,\nabla\varphi + \nabla\varphi \times \nabla\psi$.

$$\mathbf{\Gamma} \times \mathbf{B} = I\mathbf{\Gamma} \times \nabla\varphi + \mathbf{\Gamma} \times \nabla\varphi \times \nabla\psi \tag{8.20}$$

$$= I\mathbf{\Gamma} \times \nabla\varphi + \nabla\varphi(\mathbf{\Gamma} \cdot \nabla\psi) - \nabla\psi(\nabla\varphi \cdot \mathbf{\Gamma}) = \frac{\hat{\boldsymbol{\varphi}}}{R}(\mathbf{\Gamma} \cdot \nabla\psi) + \frac{I}{R}\mathbf{\Gamma} \times \hat{\boldsymbol{\varphi}} - \frac{\nabla\psi}{R}(\hat{\boldsymbol{\varphi}} \cdot \mathbf{\Gamma}) \tag{8.21}$$

$$R\hat{\boldsymbol{\varphi}} \cdot (\mathbf{\Gamma} \times \mathbf{B}) = \mathbf{\Gamma} \cdot \nabla\psi + \cancel{I\hat{\boldsymbol{\varphi}} \cdot (\mathbf{\Gamma} \times \hat{\boldsymbol{\varphi}})} - \cancel{\nabla\psi \cdot \hat{\boldsymbol{\varphi}}}(\hat{\boldsymbol{\varphi}} \cdot \mathbf{\Gamma}) = \mathbf{\Gamma} \cdot \nabla\psi \tag{8.22}$$

as required.

8.7 Chapter 8 Exercises

8.7.1 Neglected Inertia is Small

Demonstrate that the neglected inertia term in (HS-8.27) is small in comparison with the friction.

$$\langle e_a\mathbf{\Gamma}_a \cdot \nabla\psi\rangle = \left\langle R\hat{\boldsymbol{\varphi}} \cdot \frac{\partial}{\partial t}(m_a n_a \mathbf{V}_a)\right\rangle + \left\langle R\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{\Pi}}\right\rangle - \left\langle n_a e_a R E_\varphi^{(A)}\right\rangle - \langle RF_{a\varphi}\rangle \tag{HS-8.27}$$

Solution:

We have

$$R\frac{\partial}{\partial t}(m_a n_a V_a) \sim R\omega m n \delta^2 \nu_{\mathrm{coll}} V \tag{8.23}$$

$$RF_{a\varphi} \sim RmnV\nu_{\mathrm{coll}} \tag{8.24}$$

$$\frac{R\frac{\partial}{\partial t}(m_a n_a V_a)}{RF_{a\varphi}} \sim \frac{RmnV\delta^2\nu_{\mathrm{coll}}}{RmnV\nu_{\mathrm{coll}}} = \delta^2 \ll 1 \tag{8.25}$$

Where I have used (HS-8.2), $\frac{\partial}{\partial t} \sim \delta^2\nu$.

8.7.2 Neglected Viscosity is Small

Show that the viscosity term, which was neglected in (HS-8.27), is small by going through the following steps.

8.7.2.1 Identity 1

Show that $R\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\pi}} = \boldsymbol{\nabla} \cdot (R\hat{\boldsymbol{\varphi}} \cdot \overleftrightarrow{\boldsymbol{\pi}})$.

Solution:

We use (with $R\hat{\boldsymbol{\varphi}} = \mathbf{f}$)

$$R\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\pi}} = f_i \partial_j \pi_{ij} = \partial_j (f_i \pi_{ij}) - \pi_{ij} \partial_j (f_i) \tag{8.26}$$

Thus

$$R\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\pi}} = \boldsymbol{\nabla} \cdot (R\hat{\boldsymbol{\varphi}} \cdot \overleftrightarrow{\boldsymbol{\pi}}) - \overleftrightarrow{\boldsymbol{\pi}} : \nabla(R\hat{\boldsymbol{\varphi}}) \tag{8.27}$$

We can use that

$$\nabla(R\hat{\boldsymbol{\varphi}}) = \nabla R \hat{\boldsymbol{\varphi}} - R\,\nabla\hat{\boldsymbol{\varphi}} \tag{8.28}$$

We use $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} - \cos\varphi\hat{\mathbf{y}} = \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} - \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{y}}$ and $\hat{\mathbf{R}} = \nabla R = \cos\varphi\hat{\mathbf{x}} - \sin\varphi\hat{\mathbf{y}} = \frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}}{x^2+y^2}$ with $\tan\varphi = -\frac{y}{x}$ so that

$$\begin{aligned}
\nabla\hat{\boldsymbol{\varphi}} &= \nabla\left(\frac{y}{\sqrt{x^2+y^2}}\right)\hat{\mathbf{x}} - \nabla\left(\frac{x}{\sqrt{x^2+y^2}}\right)\hat{\mathbf{y}} && (8.29)\\
&= \left(\frac{-yx}{\sqrt{x^2+y^2}^3}\hat{\mathbf{x}} + \frac{\sqrt{x^2+y^2} - \frac{y^2}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}^4}\hat{\mathbf{y}}\right)\hat{\mathbf{x}} - \left(\frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}^4}\hat{\mathbf{x}} - \frac{yx}{\sqrt{x^2+y^2}^3}\hat{\mathbf{y}}\right)\hat{\mathbf{y}} && (8.30)\\
&= \left(\frac{-yx}{\sqrt{x^2+y^2}^3}\hat{\mathbf{x}} + \frac{x^2}{\sqrt{x^2+y^2}^3}\hat{\mathbf{y}}\right)\hat{\mathbf{x}} - \left(\frac{y^2}{\sqrt{x^2+y^2}^3}\hat{\mathbf{x}} - \frac{yx}{\sqrt{x^2+y^2}^3}\hat{\mathbf{y}}\right)\hat{\mathbf{y}} && (8.31)\\
&= -x\left(\frac{-\sin\varphi}{R}\hat{\mathbf{x}} - \frac{\cos\varphi}{R}\hat{\mathbf{y}}\right)\hat{\mathbf{x}} - y\left(\frac{-\sin\varphi}{R}\hat{\mathbf{x}} - \frac{\cos\varphi}{R}\hat{\mathbf{y}}\right)\hat{\mathbf{y}} && (8.32)\\
&= -\frac{\hat{\boldsymbol{\varphi}}}{R}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) = -\hat{\boldsymbol{\varphi}}\hat{\mathbf{R}} && (8.33)
\end{aligned}$$

So that we find

$$\nabla(R\hat{\boldsymbol{\varphi}}) = \hat{\mathbf{R}}\hat{\boldsymbol{\varphi}} - \hat{\boldsymbol{\varphi}}\hat{\mathbf{R}} \tag{8.34}$$

Thus,

$$\overleftrightarrow{\boldsymbol{\pi}} : \nabla(R\hat{\boldsymbol{\varphi}}) = \overleftrightarrow{\boldsymbol{\pi}} : \hat{\mathbf{R}}\hat{\boldsymbol{\varphi}} - \overleftrightarrow{\boldsymbol{\pi}} : \hat{\boldsymbol{\varphi}}\hat{\mathbf{R}} \tag{8.35}$$

we use that $\overleftrightarrow{\boldsymbol{\pi}}^{\intercal} = \overleftrightarrow{\boldsymbol{\pi}}$ so that

$$\overleftrightarrow{\boldsymbol{\pi}} : \hat{\boldsymbol{\varphi}}\hat{\mathbf{R}} = \overleftrightarrow{\boldsymbol{\pi}}^{\intercal} : \hat{\mathbf{R}}\hat{\boldsymbol{\varphi}} = \overleftrightarrow{\boldsymbol{\pi}} : \hat{\mathbf{R}}\hat{\boldsymbol{\varphi}} \tag{8.36}$$

so that

$$\overleftrightarrow{\boldsymbol{\pi}} : \nabla(R\hat{\boldsymbol{\varphi}}) = \overleftrightarrow{\boldsymbol{\pi}} : \hat{\mathbf{R}}\hat{\boldsymbol{\varphi}} - \overleftrightarrow{\boldsymbol{\pi}} : \hat{\boldsymbol{\varphi}}\hat{\mathbf{R}} = \overleftrightarrow{\boldsymbol{\pi}} : \hat{\mathbf{R}}\hat{\boldsymbol{\varphi}} - \overleftrightarrow{\boldsymbol{\pi}} : \hat{\mathbf{R}}\hat{\boldsymbol{\varphi}} = 0 \tag{8.37}$$

Thus

$$R\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\nabla}\overleftrightarrow{\boldsymbol{\pi}} = \boldsymbol{\nabla} \cdot (R\hat{\boldsymbol{\varphi}} \cdot \overleftrightarrow{\boldsymbol{\pi}}) \tag{8.38}$$

8.7.2.2 Identity 2

Verify that for any vector $\mathbf{A}$

$$\langle \boldsymbol{\nabla} \cdot \mathbf{A} \rangle = \oint \frac{\partial}{\partial \psi} \left(\sqrt{g} \mathbf{A} \cdot \nabla \psi \right) \mathrm{d}\theta \Big/ \oint \sqrt{g} \, \mathrm{d}\theta = \frac{1}{V'} \frac{\partial}{\partial \psi} \left(V' \left\langle \mathbf{A} \cdot \nabla \psi \right\rangle \right) \tag{8.39}$$

where $V' \equiv 2\pi \oint \sqrt{g} \, \mathrm{d}\theta$ and $\sqrt{g} = 1/\mathbf{B} \cdot \nabla\theta$.

Solution:
Let's begin with

$$\langle \boldsymbol{\nabla} \cdot \mathbf{A} \rangle = \frac{\oint \mathrm{d}\theta \, \sqrt{g} \boldsymbol{\nabla} \cdot \mathbf{A}}{\oint \mathrm{d}\theta \, \sqrt{g}} = \frac{2\pi}{V'} \oint \mathrm{d}\theta \, \sqrt{g} \boldsymbol{\nabla} \cdot \mathbf{A} \tag{8.40}$$

We use

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial \psi} \left(\sqrt{g} A^{\psi} \right) + \frac{\partial}{\partial \theta} \left(\sqrt{g} A^{\theta} \right) + \frac{\partial}{\partial \varphi} \left(\sqrt{g} A^{\varphi} \right) \right] \tag{8.41}$$

We then use

$$A^{\psi} = \mathbf{A} \cdot \nabla \psi \tag{8.42}$$

$$A^{\theta} = \mathbf{A} \cdot \nabla \theta \tag{8.43}$$

$$A^{\varphi} = \mathbf{A} \cdot \nabla \varphi \tag{8.44}$$

Thus,

$$\oint \mathrm{d}\theta \, \sqrt{g} \boldsymbol{\nabla} \cdot \mathbf{A} = \oint \mathrm{d}\theta \, \left[\frac{\partial}{\partial \psi} \left(\sqrt{g} \mathbf{A} \cdot \nabla \psi \right) + \frac{\partial}{\partial \theta} \left(\sqrt{g} \mathbf{A} \cdot \nabla \theta \right) + \frac{\partial}{\partial \varphi} \left(\sqrt{g} \mathbf{A} \cdot \nabla \varphi \right) \right] \tag{8.45}$$

The last term is

$$\frac{\partial}{\partial \varphi} \oint \mathrm{d}\theta \, \sqrt{g} \mathbf{A} \cdot \nabla \varphi = \frac{\partial}{\partial \varphi} \left(\left\langle \mathbf{A} \cdot \nabla \varphi \right\rangle \oint \mathrm{d}\theta \, \sqrt{g} \right) = \frac{\partial}{\partial \varphi} \left(\frac{V'}{2\pi} \left\langle \mathbf{A} \cdot \nabla \varphi \right\rangle \right) \tag{8.46}$$

Similarly,

$$\frac{\partial}{\partial \psi} \oint \mathrm{d}\theta \, \sqrt{g} \mathbf{A} \cdot \nabla \psi = \frac{\partial}{\partial \psi} \left(\left\langle \mathbf{A} \cdot \nabla \psi \right\rangle \oint \mathrm{d}\theta \, \sqrt{g} \right) = \frac{\partial}{\partial \psi} \left(\frac{V'}{2\pi} \left\langle \mathbf{A} \cdot \nabla \psi \right\rangle \right) \tag{8.47}$$

For the A^{θ} term we have

$$\oint \mathrm{d}\theta \, \frac{\partial}{\partial \theta} \left(\sqrt{g} \mathbf{A} \cdot \nabla \theta \right) = 0 \tag{8.48}$$

since it is a closed integral and $\sqrt{g}\mathbf{A} \cdot \nabla\theta$ is single valued. Now we need to note that we have imposed axisymmetry in our definition of the flux average. If we hadn't, we would have $\oint \mathrm{d}\theta \; \rightarrow \oint \mathrm{d}\theta \; \oint \mathrm{d}\varphi$ and so the $\frac{\partial}{\partial \varphi}$ term would cancel out as well. Thus, we have the $\frac{\partial}{\partial \varphi}$ term cancel as well and so for any vector $\mathbf{A}$ we find

$$\langle \boldsymbol{\nabla} \cdot \mathbf{A} \rangle = \frac{2\pi}{V'} \frac{\partial}{\partial \psi} \left(\frac{V'}{2\pi} \left\langle \mathbf{A} \cdot \nabla \psi \right\rangle \right) = \frac{1}{V'} \frac{\partial}{\partial \psi} \left(V' \left\langle \mathbf{A} \cdot \nabla \psi \right\rangle \right) \tag{8.49}$$

as desired.

8.7.2.3 Viscosity Off-Diagonal Terms

By inspecting (HS-4.68), show that the off-diagonal elements in the viscosity tensor are $\mathcal{O}(\delta^2 p)$.

$$\overleftrightarrow{\mathbf{K}}(\overleftrightarrow{\mathbf{P}}) \equiv \overleftrightarrow{\mathbf{S}}/\Omega \tag{HS-4.68}$$

Solution:

We use $\overleftrightarrow{\mathbf{S}} = p\overleftrightarrow{\mathbf{W}} + \nu\overleftrightarrow{\boldsymbol{\pi}}$ and that

$$\overleftrightarrow{\mathbf{K}}(\overleftrightarrow{\mathbf{P}}) = \overleftrightarrow{\mathbf{P}} \times \mathbf{b} + (\overleftrightarrow{\mathbf{P}} \times \mathbf{b})^{\intercal} \tag{8.50}$$

We use that $\overleftrightarrow{\mathbf{K}}(\overleftrightarrow{\mathbf{P}}) = \overleftrightarrow{\mathbf{K}}(\overleftrightarrow{\boldsymbol{\pi}})$ and so we then see

$$\overleftrightarrow{\mathbf{K}}(\overleftrightarrow{\boldsymbol{\pi}}) = \begin{bmatrix} 2\pi_{xy} & \pi_{yy} - \pi_{xx} & \pi_{yz} \\ \pi_{yy} - \pi_{xx} & -2\pi_{xy} & -\pi_{xz} \\ \pi_{yz} & -\pi_{xz} & 0 \end{bmatrix} \tag{8.51}$$

while

$$\overleftrightarrow{\mathbf{S}} = \begin{bmatrix} pW_{xx} + \nu\pi_{xx} & pW_{xy} + \nu\pi_{xy} & pW_{xz} + \nu\pi_{xz} \\ pW_{yx} + \nu\pi_{yx} & pW_{yy} + \nu\pi_{yy} & pW_{yz} + \nu\pi_{yz} \\ pW_{zx} + \nu\pi_{zx} & pW_{zy} + \nu\pi_{zy} & pW_{zz} + \nu\pi_{zz} \end{bmatrix} = \begin{bmatrix} pW_{xx} + \nu\pi_{xx} & pW_{xy} + \nu\pi_{xy} & pW_{xz} + \nu\pi_{xz} \\ pW_{xy} + \nu\pi_{xy} & pW_{yy} + \nu\pi_{yy} & pW_{yz} + \nu\pi_{yz} \\ pW_{xz} + \nu\pi_{xz} & pW_{yz} + \nu\pi_{yz} & pW_{zz} + \nu\pi_{zz} \end{bmatrix} \tag{8.52}$$

Because $\overleftrightarrow{\mathbf{W}} = \nabla V + (\nabla V)^{\intercal} - \frac{2}{3}\boldsymbol{\nabla} \cdot \mathbf{V}$ we can use that $|\nabla V| \sim \delta v_{\text{th}}/L$ Thus, we get $|\mathbf{S}| \sim p\delta v_{\text{th}}/(L\Omega) \sim p\delta v_{\text{th}}\rho/(Lv_{\text{th}}) \sim p\delta^2$.

Note that we have used $\nu/\Omega \ll 1$ to eliminate couplings between π_{yz} and $\nu\pi_{xz}$, for example, to get this estimate.

8.7.2.4 Compare

Use this information to compare viscosity with friction.

Solution:

We are comparing

$$\frac{\left\langle R\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\pi}} \right\rangle}{\langle RF_{a\varphi} \rangle} = \frac{\left\langle \boldsymbol{\nabla} \cdot (R\hat{\boldsymbol{\varphi}} \cdot \overleftrightarrow{\boldsymbol{\pi}} \right\rangle}{\langle RF_{a\varphi} \rangle} = \frac{\frac{1}{V'}\frac{\partial}{\partial\psi}\left(V'\left\langle R\hat{\boldsymbol{\varphi}} \cdot \overleftrightarrow{\boldsymbol{\pi}} \cdot \nabla\psi \right\rangle\right)}{\langle RF_{a\varphi} \rangle} \tag{8.53}$$

We note that $R\hat{\boldsymbol{\varphi}} \cdot \overleftrightarrow{\boldsymbol{\pi}} \cdot \nabla\psi$ is an off diagonal element so is of order $R\delta^2 p/L$, thus

$$\frac{\left\langle R\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\pi}} \right\rangle}{\langle RF_{a\varphi} \rangle} \sim \frac{R\delta^2 p/L}{Rmn\nu_{\text{coll}}\delta v_{\text{th}}} \sim \frac{T\delta}{m\nu_{\text{coll}}v_{\text{th}}} \sim \frac{v_{\text{th}}^2\delta}{Lv_{\text{th}}\nu_{\text{coll}}} = \frac{\delta v_{\text{th}}}{L\nu_{\text{coll}}} \sim \delta\frac{\rho\Omega}{L\nu_{\text{coll}}} \sim \frac{\delta^2}{\Delta} \sim \delta \ll 1 \tag{8.54}$$

8.7.3 Neoclassical Flux

Prove (HS-8.29) from (HS-8.31).

$$\langle \mathbf{\Gamma}_a \cdot \nabla\psi \rangle^{\text{neo}} \equiv -I \left\langle \frac{F_{a\|} + n_a e_a E_\|^{(A)}}{e_a B} \right\rangle \tag{HS-8.29}$$

$$\langle \mathbf{\Gamma}_a \cdot \nabla\psi \rangle^{\text{neo}} = \left\langle \int \mathrm{d}^3 v \; f_a \mathbf{v}_d \cdot \nabla\psi \right\rangle \tag{HS-8.31}$$

Solution:

We use

$$\mathbf{v}_d \cdot \nabla\psi = I v_\| \nabla_\| \left(\frac{v_\|}{\Omega}\right) \tag{8.55}$$

$$\mathrm{d}^3 v = \sum_\sigma \frac{\pi B v^3}{|v_\||B_0} \,\mathrm{d}v\,\mathrm{d}\lambda \tag{8.56}$$

Thus,

$$\int \mathrm{d}^3 v \; f_a \mathbf{v}_d \cdot \nabla\psi = \sum_\sigma \int \mathrm{d}\lambda \mathrm{d}v \; f_a \frac{\pi B v^3}{|v_\||B_0} I v_\| \nabla_\| \left(\frac{v_\|}{\Omega_a}\right) = \sum_\sigma \sigma \int \mathrm{d}\lambda \mathrm{d}v \; f_a \frac{I\pi B v^3}{B_0} \nabla_\| \left(\frac{v_\|}{\Omega_a}\right) \tag{8.57}$$

So

$$\langle \mathbf{\Gamma}_a \cdot \nabla\psi \rangle^{\text{neo}} = \frac{\oint \frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta} \sum_\sigma \sigma \int \mathrm{d}\lambda \mathrm{d}v \; f_a \frac{I\pi v^3}{B_0} \mathbf{B}\cdot\nabla\left(\frac{v_\|}{\Omega_a}\right)}{\oint \frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta}} \tag{8.58}$$

We can use that $\mathbf{B}\cdot\nabla = B^\theta \frac{\partial}{\partial\theta}$ due to $\mathbf{B}\cdot\nabla\psi = \mathbf{B}\cdot\nabla\varphi = 0$ by definition and axisymmetry, respectively. Note this also implies $B^\theta = B$. Thus

$$\frac{\mathbf{B}\cdot\nabla(f)}{\mathbf{B}\cdot\nabla\theta} = \frac{B^\theta \frac{\partial f}{\partial\theta}}{B^\theta} = \frac{\partial f}{\partial\theta} \tag{8.59}$$

So

$$\langle \mathbf{\Gamma}_a \cdot \nabla\psi \rangle^{\text{neo}} = \frac{\sum_\sigma \sigma \int \mathrm{d}\lambda \mathrm{d}v \; \frac{\pi v^3}{B_0} \oint \mathrm{d}\theta \; I f_a \frac{\partial}{\partial\theta}\left(\frac{v_\|}{\Omega_a}\right)}{\oint \frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta}} \tag{8.60}$$

$$= \frac{\sum_\sigma \sigma \int \mathrm{d}\lambda \mathrm{d}v \; \frac{\pi v^3}{B_0} \oint \mathrm{d}\theta \; I \left[\cancel{\frac{\partial}{\partial\theta}\left(f_a \frac{v_\|}{\Omega_a}\right)} - \frac{v_\|}{\Omega_a}\frac{\partial f_a}{\partial\theta}\right]}{\oint \frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta}} \tag{8.61}$$

We use that $\sigma v_\parallel = \frac{v_\parallel}{|v_\parallel|} v_\parallel = |v_\parallel|$ with I is independent of θ and so

$$\langle \mathbf{\Gamma}_a \cdot \nabla\psi \rangle^{\text{neo}} = \frac{-\sum_\sigma \int \mathrm{d}\lambda \mathrm{d}v \ \frac{\pi v^3}{B_0} \oint \mathrm{d}\theta \ I \frac{|v_\parallel|}{\Omega_a} \frac{\partial f_a}{\partial \theta}}{\oint \frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta}} = \frac{-\oint \mathrm{d}\theta \ \sum_\sigma \int \mathrm{d}\lambda \mathrm{d}v \ \frac{\pi B v^3}{|v_\parallel| B_0} \frac{I |v_\parallel|^2}{B\Omega_a} \frac{\partial f_a}{\partial \theta}}{\oint \frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta}} \tag{8.62}$$

$$= \frac{-\oint \mathrm{d}\theta \ \int \mathrm{d}^3v \ \frac{I|v_\parallel|^2}{B\Omega_a} \frac{\partial f_a}{\partial\theta}}{\oint \frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta}} = \frac{-\oint \frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta} \int \mathrm{d}^3v \ \frac{I v_\parallel^2}{\Omega_a} \nabla_\parallel f_a}{\oint \frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta}} \tag{8.63}$$

$$= -\left\langle \frac{I}{\Omega_a} \int \mathrm{d}^3v \ v_\parallel^2 \nabla_\parallel f_a \right\rangle \tag{8.64}$$

From the kinetic equation

$$v_\parallel \nabla_\parallel \left(f_{a1} + \frac{I v_\parallel}{\Omega_a} \frac{\partial f_{a0}}{\partial \psi} \right) - \frac{e_a E_\parallel^{(A)}}{T_a} v_\parallel f_{a0} = C_a(f_{a1}) \tag{8.65}$$

we can substitute using $f_a = f_{a0} + f_{a1}$, and note that f_{a0} should have no θ dependence, as it is Maxwellian, thus $\nabla_\parallel f_{a0} = 0$ (this comes from $v_\parallel \nabla_\parallel f_0 = C(f_0) = 0$ since the collision operator annihilates Maxwellians). We thus have

$$\langle \mathbf{\Gamma}_a \cdot \nabla\psi \rangle^{\text{neo}} = -\left\langle \frac{I}{\Omega_a} \int \mathrm{d}^3v \ v_\parallel^2 \nabla_\parallel f_{a1} \right\rangle = -\left\langle \frac{I}{\Omega_a} \int \mathrm{d}^3v \ v_\parallel \left[-v_\parallel \nabla_\parallel \left(\frac{I v_\parallel}{\Omega_a} \frac{\partial f_{a0}}{\partial\psi} \right) + v_\parallel f_{a0} \frac{e_a E_\parallel^{(A)}}{T_a} + C_a(f_{a1}) \right] \right\rangle \tag{8.66}$$

$$= -I \left\langle \int \frac{\mathrm{d}^3 v}{e_a B} \left[f_{a0} \frac{m_a v_\parallel^2 e_a E_\parallel^{(A)}}{T_a} + m_a v_\parallel C_a(f_{a1}) \right] \right\rangle + \left\langle \frac{I}{\Omega_a} \int \mathrm{d}^3v \ v_\parallel^2 \nabla_\parallel \left(\frac{I v_\parallel}{\Omega_a} \frac{\partial f_{a0}}{\partial \psi} \right) \right\rangle \tag{8.67}$$

$$= -I \left\langle \frac{F_{a\parallel} + n_a e_a E_\parallel^{(A)}}{e_a B} \right\rangle + \left\langle \frac{I}{\Omega_a} \int \mathrm{d}^3v \ v_\parallel^2 \nabla_\parallel \left(\frac{I v_\parallel}{\Omega_a} \frac{\partial f_{a0}}{\partial \psi} \right) \right\rangle \tag{8.68}$$

The second term will vanish because $\nabla_\parallel (v_\parallel \frac{I}{\Omega} \frac{\partial f_{a0}}{\partial \psi}) = \frac{I}{\Omega} \frac{\partial f_{a0}}{\partial\psi} \nabla_\parallel v_\parallel$ thus

$$\left\langle \frac{I}{\Omega} \int \mathrm{d}^3v \ v_{\parallel^2} \nabla_\parallel \left(\frac{I v_\parallel}{\Omega_a} \frac{\partial f_{a0}}{\partial\psi} \right) \right\rangle = \left\langle \frac{I}{\Omega} \int \mathrm{d}^3v \ \nabla_\parallel \left(\frac{I v_\parallel^3}{3\Omega_a} \frac{\partial f_{a0}}{\partial \psi} \right) \right\rangle = 0 \tag{8.69}$$

since $\nabla_\parallel = \frac{\partial}{\partial\theta}$. Thus, we do find

$$\langle \mathbf{\Gamma}_a \cdot \nabla\psi \rangle^{\text{neo}} = -I \left\langle \frac{F_{a\parallel} + n_a e_a E_\parallel^{(A)}}{e_a B} \right\rangle \tag{8.70}$$

as is required.

8.7.4 Time Derivative Identity

Prove (HS-8.37).

$$J(t) = \int_{U(t)} A(\mathbf{r}, t)\, \mathrm{d}V$$

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \int_{U(t)} \frac{\partial A}{\partial t}\, \mathrm{d}V + \int_{\partial U(t)} A\mathbf{u} \cdot \mathrm{d}\mathbf{S} \qquad \text{(HS-8.37)}$$

Solution:

Take the definition of a derivative

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{J(t + \Delta t) - J(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{U(t+\Delta t)} A(\mathbf{r}, t + \Delta t)\, \mathrm{d}V - \int_{U(t)} A(\mathbf{r}, t)\, \mathrm{d}V \right] \qquad (8.71)$$

The integral over the volume $U(t + \Delta t)$ should be

$$\int_{U(t+\Delta t)} A(\mathbf{r}, t + \Delta t)\, \mathrm{d}V \approx \int_{U(t)} A(\mathbf{r}, t + \Delta t)\, \mathrm{d}V + \int_{\partial U(t)} A(\mathbf{r}, t)(\Delta t)\mathbf{u} \cdot \mathrm{d}\mathbf{S} + \mathcal{O}((\Delta t)^2) \qquad (8.72)$$

where $\mathbf{u}$ is the velocity of the boundary region and $\mathbf{S}$ points normal tot he boundary (so this adds up the small amount of volume added in the time from t to $t + \Delta t$ for the volume.

Thus,

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \lim_{\Delta t \to \infty} \left[\int_{U(t)} \frac{A(\mathbf{r}, t + \Delta t) - A(\mathbf{r}, t)}{\Delta t}\, \mathrm{d}V + \int_{\partial U(t)} A(\mathbf{r}, t)\mathbf{u} \cdot \mathrm{d}\mathbf{S} \right] \qquad (8.73)$$

$$= \int_{U(t)} \frac{\partial A(\mathbf{r}, t)}{\partial t}\, \mathrm{d}V + \int_{\partial(U(t)} A(\mathbf{r}, t)\mathbf{u} \cdot \mathrm{d}\mathbf{S} \qquad (8.74)$$

as required.

Chapter 9

Transport in the Pfirsch-Schlüter Regime

9.3 Chapter 9 Exercises

9.3.1 Average of Inverse Square Identity

Show that

$$\left\langle \frac{1}{B^2} \right\rangle - \frac{1}{\langle B^2 \rangle} \geq 0 \tag{9.1}$$

Solution:

Consider

$$\left\langle \left(\frac{1}{B} - \frac{B}{\langle B^2 \rangle} \right)^2 \right\rangle \geq 0 \tag{9.2}$$

by definition. Expanding,

$$\left\langle \frac{1}{B^2} - \frac{2}{\langle B^2 \rangle} + \frac{B^2}{\langle B^2 \rangle^2} \right\rangle \geq 0 \tag{9.3}$$

$$\left\langle \frac{1}{B^2} \right\rangle - \frac{2}{\langle B^2 \rangle} + \frac{\langle B^2 \rangle}{\langle B^2 \rangle^2} \geq 0 \tag{9.4}$$

$$\left\langle \frac{1}{B^2} \right\rangle - \frac{2}{\langle B^2 \rangle} + \frac{1}{\langle B^2 \rangle} \geq 0 \tag{9.5}$$

$$\left\langle \frac{1}{B^2} \right\rangle - \frac{1}{\langle B^2 \rangle} \geq 0 \tag{9.6}$$

and so we have proven the desired result.

Chapter 10

Transport in the Plateau Regime

10.3 Chapter 10 Exercises

10.3.1 The Resonant Distribution Function

10.3.1.1 Solve

Solve (HS-10.9) ignoring the second term on the left.

$$\eta \frac{\partial h_a}{\partial \theta} - \frac{\epsilon}{\widehat{\nu}^{2/3}} \sin\theta \frac{\partial h_a}{\partial \eta} - \frac{\partial^2 h_a}{\partial \eta^2} = \frac{s_a}{\widehat{\nu}^{1/3}} \sin\theta \tag{HS-10.9}$$

The given solution

$$h_a = \frac{s_a}{\widehat{\nu}^{1/3}} \int_0^\infty \mathrm{d}\tau \; e^{-\tau^3/3} \sin\left(\theta - \eta\tau\right) \tag{HS-10.10}$$

Solution:

Let's assume separability so that $h_a(\eta,\theta) = E(\eta)T(\theta)$ and look at the homogeneous equation first (set the right hand side to $g(\theta) \equiv s_a \sin\theta/\widehat{\nu}^{1/3}$). We would find

$$\eta E(\eta) \frac{\partial T}{\partial \theta} - T \frac{\partial^2 E}{\partial \eta^2} = 0 \tag{10.1}$$

$$\frac{1}{T}\frac{\partial T}{\partial \theta} = \frac{1}{\eta E(\eta)} \frac{\partial^2 E}{\partial \eta^2} \tag{10.2}$$

Note that the second line says that each side of the equation must be a constant. Let's set this constant to λ. Then

$$T(\theta)' = \lambda T(\theta) \tag{10.3}$$

$$T(\theta) = C_1 e^{\lambda\theta} \tag{10.4}$$

We also find

$$E(\eta)'' = \lambda \eta E(\eta) \tag{10.5}$$

This defines the solution of

$$E(\eta) = C_2\,\mathrm{AiryAi}\left(\lambda^{1/3}\eta\right) \tag{10.6}$$

from AiryAi(x) being the solution to

$$F(x)'' = xF(x) \tag{10.7}$$

$$\mathrm{AiryAi}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\,\mathrm{d}t \tag{10.8}$$

Thus, we find the solution to our homogeneous problem is

$$h_a(\eta,\theta) = C_3 e^{\lambda\theta}\int_0^\infty \mathrm{d}t\ \cos\left(\frac{t^3}{3} + \lambda^{1/3}\eta t\right) \tag{10.9}$$

We can note that

$$\frac{\partial^2 h_a(\eta,\theta)}{\partial\eta^2} = -C_3 e^{\lambda\theta}\lambda^{2/3}\int_0^\infty \mathrm{d}t\ t^2\cos\left(\frac{t^3}{3} + \lambda^{1/3}\eta t\right) \tag{10.10}$$

Let's recognize if we choose to view this as a complex equation we can write it as ($\Im(F) = h_a$)

$$\eta\frac{\partial F}{\partial\theta} - \frac{\partial^2 F}{\partial\eta^2} = \frac{s_a}{\widehat{\nu}^{1/3}}e^{i\theta} \tag{10.11}$$

If we further choose $F = f(\eta)e^{i\theta}$ we find

$$i\eta f e^{i\theta} - e^{i\theta}\frac{\partial^2 f}{\partial\eta^2} = \frac{s_a}{\widehat{\nu}^{1/3}}e^{i\theta} \tag{10.12}$$

$$i\eta f - \frac{\partial^2 f}{\partial\eta^2} = \frac{s_a}{\widehat{\nu}^{1/3}} \tag{10.13}$$

$$\tag{10.14}$$

because $s_a/\widehat{\nu}^{1/3}$ is a constant with respect to η define $H = f\widehat{\nu}^{1/3}/s_a$ and the equation becomes

$$i\eta H - \frac{\partial^2 H}{\partial\eta^2} = 1 \tag{10.15}$$

We then see that if we try

$$H(\eta) = \int_0^\infty \mathrm{d}\tau\ e^{-i\eta\tau-\tau^3/3} \tag{10.16}$$

We find

$$i\eta\int_0^\infty \mathrm{d}\tau\ e^{-i\eta\tau-\tau^3/3} + \int_0^\infty \mathrm{d}\tau\ \tau^2 e^{-i\eta\tau-\tau^3/3} = \int_0^\infty \mathrm{d}\tau\ \left(i\eta+\tau^2\right)e^{-i\eta\tau-\tau^3/3} \tag{10.17}$$

$$= \int_0^\infty \mathrm{d}\tau\ -\frac{\partial}{\partial\tau}e^{-i\eta\tau-\tau^3/3} = \left[-e^{i\eta\tau-\tau^3/3}\right]_0^\infty = 1 \tag{10.18}$$

and so satisfies the above equation. So we then get

$$h_a = \Im\left(\frac{s_a}{\widehat{\nu}^{1/3}} H(\eta) e^{i\theta}\right) = \Im\left(\frac{s_a}{\widehat{\nu}^{1/3}} e^{i\theta} \int_0^\infty \mathrm{d}\tau\ e^{-i\eta\tau - \tau^3/3}\right) \tag{10.19}$$

$$= \Im\left(\frac{s_a}{\widehat{\nu}^{1/3}} \int_0^\infty \mathrm{d}\tau\ e^{i\theta - i\eta\tau} e^{-\tau^3/3}\right) \tag{10.20}$$

$$= \frac{s_a}{\widehat{\nu}^{1/3}} \int_0^\infty \mathrm{d}\tau\ e^{-\tau^3/3} \sin(\theta - \eta\tau) \tag{10.21}$$

as described.

10.3.1.2 Limit

Show that in the limit $\widehat{\nu} \to 0$, the part of h_a that is even in $\xi = \eta\widehat{\nu}^{1/3}$,

$$h_a^{\text{even}} = \frac{s_a}{\widehat{\nu}^{1/3}} \sin\theta \int_0^\infty \mathrm{d}\tau\ e^{-\tau^3/3} \cos(\eta\tau) \tag{10.22}$$

approaches a delta function, $h_a^{\text{even}}(\xi) \to \pi s_a \delta(\xi) \sin\theta$.

Solution:

We see that this yields

$$h_a^{\text{even}} = \frac{s_a}{\widehat{\nu}^{1/3}} \sin\theta \int_0^\infty \mathrm{d}\tau\ e^{-\tau^3/3} \cos(\xi\widehat{\nu}^{-1/3}\tau) \tag{10.23}$$

Let's call $\nu \equiv \widehat{\nu}^{1/3}$ for convenience. We need to show

$$\lim_{\nu\to 0} \int_{-1}^{1} \mathrm{d}\xi\ f(\xi) \int_0^\infty \mathrm{d}\tau\ \frac{e^{-\tau^3/3}\cos\left(\frac{\xi\tau}{\nu}\right)}{\nu} \propto f(0) \tag{10.24}$$

We assume this integral converges and so can switch the order of integration and the order of the limit and take $\mu = \frac{\xi}{\nu}$ to find

$$\int_0^\infty \mathrm{d}\tau\ \lim_{\nu\to 0} \int_{-1/\nu}^{1/\nu} \mathrm{d}\mu\ f(\nu\mu) e^{-\tau^3/3} \cos(\tau\mu) = \int_0^\infty \mathrm{d}\tau\ \lim_{\nu\to 0} \int_{-1/\nu}^{1/\nu} \mathrm{d}\mu\ f(0) e^{-\tau^3/3} \cos(\tau\mu) \tag{10.25}$$

$$= f(0) \int_0^\infty \mathrm{d}\tau\ \lim_{\nu\to 0} \int_{-1}^{1} \mathrm{d}\xi\ \frac{e^{-\tau^3/3}\cos\left(\frac{\tau\xi}{\nu}\right)}{\nu} = f(0) \lim_{\nu\to 0} \int_0^\infty \mathrm{d}\tau\ \frac{e^{-\tau^3/3}}{\nu}\left[\frac{2\sin(\tau/\nu)}{\frac{\tau}{\nu}}\right] \tag{10.26}$$

$$= 2f(0) \lim_{\nu\to 0} \int_{0/\nu}^\infty \mathrm{d}t\ e^{-\nu^3 t^3/3} \frac{\sin(t)}{t} = 2f(0) \lim_{\nu\to 0} \int_0^\infty \mathrm{d}t\ \frac{\sin(t)}{t} = \lim_{\nu\to 0} f(0)\pi = \pi f(0) \tag{10.27}$$

where I have used $t = \frac{\tau}{\nu}$ and $\int_0^\infty \mathrm{d}t\ \frac{\sin(t)}{t} = \pi$. Thus,

$$h_a^{\text{even}} = \frac{s_a}{\widehat{\nu}^{1/3}} \sin\theta \int_0^\infty \mathrm{d}\tau\ e^{-\tau^3/3} \cos(\eta\tau) = s_a \pi (\sin\theta) \delta(\xi) \tag{10.28}$$

as desired. The fastest way to see the $\sin(t)/t$ identity is to remember Cauchy's integral formula for poles (using that $\sin(t)/t$ is even) so that

$$\int_{-\infty}^\infty \mathrm{d}t\ \frac{\sin(t)}{t} = \int_C \mathrm{d}t\ \frac{e^{it} - e^{-it}}{2it} = \left(\frac{2\pi i}{2i} e^{-i(0)}\right) = \pi \tag{10.29}$$

where I have closed the contour in the necessary way (small circle around the origin and large circle in the upper plane).

Chapter 11

Transport in the Banana Regime

11.6 Chapter 11 Exercises

11.6.1 Fraction of Trapped Particles

Calculate the effective fraction of trapped particles (HS-11.24) in a large-aspect-ratio tokamak with circular cross section.

$$f_t = 1 - \frac{3}{4}\int_0^{\lambda_c} \frac{\lambda\,\mathrm{d}\lambda}{\langle 1-\lambda/h\rangle} \simeq 1.46\sqrt{\epsilon} \tag{HS-11.24}$$

11.6.1.1 Flux Surface Average

First, evaluate the flux-surface average $\langle \xi \rangle = \sigma \left\langle \sqrt{1-\lambda(1-\epsilon\cos\theta)} \right\rangle$ for circulating particles appearing in the denominator of (HS-11.24). Express the result in terms of the elliptic integral

$$E(k) = \int_0^{\pi/2} \mathrm{d}x\ \sqrt{1-k^2\sin^2 x}$$

Solution:

We have

$$\sigma \left\langle \sqrt{1-\lambda(1-\epsilon\cos\theta)} \right\rangle = \frac{\oint \mathrm{d}\theta\ \dfrac{\sigma\sqrt{1-\lambda(1-\epsilon\cos\theta)}}{\mathbf{B}\cdot\nabla\theta}}{\oint \frac{\mathrm{d}\theta}{\mathbf{B}\cdot\nabla\theta}} \tag{11.1}$$

We can use $\mathbf{B}\cdot\nabla\theta = B^\theta$ which is not a function of θ for $\epsilon \ll 1$ as it is basically a constant. Thus

$$\sigma \left\langle \sqrt{1-\lambda(1-\epsilon\cos\theta)} \right\rangle = \frac{\frac{\sigma}{B^\theta}\oint \mathrm{d}\theta\ \sqrt{1-\lambda(1-\epsilon\cos\theta)}}{\frac{1}{B^\theta}\oint \mathrm{d}\theta} = \frac{\sigma}{2\pi}\oint \mathrm{d}\theta\ \sqrt{1-\lambda(1-\epsilon\cos\theta)} \tag{11.2}$$

Now use $\cos\theta = 1 - 2\sin^2(\theta/2)$ with $k^2 = \frac{1-\lambda(1-\epsilon)}{2\epsilon\lambda}$ so $\lambda(1-\epsilon\cos\theta) = \lambda[1-\epsilon+2\epsilon\sin^2(\theta/2)]$ and we find

$$\sigma\left\langle\sqrt{1-\lambda(1-\epsilon\cos\theta)}\right\rangle = \frac{\sigma}{2\pi}\oint \mathrm{d}\theta\ \sqrt{1-\lambda(1-\epsilon)-2\epsilon\lambda\sin^2(\theta/2)} \tag{11.3}$$

$$= \frac{\sigma}{2\pi}\oint \mathrm{d}\theta\ \sqrt{2\epsilon\lambda}\sqrt{k^2-\sin^2(\theta/2)} = \frac{\sigma\sqrt{2\epsilon\lambda}}{2\pi}\oint \mathrm{d}\theta\ \sqrt{k^2-\sin^2(\theta/2)} \tag{11.4}$$

For circulating particles, we have $\oint \mathrm{d}\theta \to \int_0^{2\pi} \mathrm{d}\theta$ so that choosing $u = \theta/2$ and factoring out k^2 we find

$$\sigma\left\langle\sqrt{1-\lambda(1-\epsilon\cos\theta)}\right\rangle = \frac{\sigma k\sqrt{2\epsilon\lambda}}{\pi}\int_0^{\pi} \mathrm{d}u\ \sqrt{1-\frac{\sin^2 u}{k^2}} \tag{11.5}$$

We can break the integral into parts, $I_2 = \int_0^{\pi} \mathrm{d}u\ \sqrt{1+k^{-2}\sin^2 u}$, and for the second integral take $t = -u+\pi$ and use $\sin(\pi-t) = \sin(\pi)\cos(t) - \cos(\pi)\sin(t) = -\sin t$, so

$$I_2 = \int_0^{\pi/2} \mathrm{d}u\ \sqrt{1-k^{-2}\sin^2 u} + \int_{-\pi/2+\pi}^{-\pi+\pi} \mathrm{d}t\ -\sqrt{1-k^{-2}\sin^2 t} = E(k^{-1}) + \int_0^{\pi/2} \mathrm{d}t\ \sqrt{1-k^{-2}\sin^2 t} \tag{11.6}$$

$$= 2E(k^{-1}) \tag{11.7}$$

Thus, we find

$$\sigma\left\langle\sqrt{1-\lambda(1-\epsilon\cos\theta)}\right\rangle = \frac{2\sigma k\sqrt{2\epsilon\lambda}}{\pi}E(k^{-1}) \tag{11.8}$$

We can use that

$$2\epsilon\lambda k^2 = 1-\lambda(1-\epsilon) \tag{11.9}$$

$$\lambda(2\epsilon k^2+1-\epsilon) = 1 \tag{11.10}$$

$$\lambda = \frac{1}{1+2k^2\epsilon-\epsilon} \tag{11.11}$$

so that

$$k\sqrt{\lambda} = \sqrt{\frac{1}{k^{-2}(1-\epsilon)+2\epsilon}} \tag{11.12}$$

So

$$\sigma\left\langle\sqrt{1-\lambda(1-\epsilon\cos\theta)}\right\rangle = \sqrt{\frac{2\epsilon}{k^{-2}(1-\epsilon)+2\epsilon}}\frac{2E(k^{-1})}{\pi} \tag{11.13}$$

where we can "ignore" the σ for circulating particles (because it was taken care of when we made the limits $[0, 2\pi]$ in θ).

11.6.1.2 Change Variables

Perform the remaining integral (HS-11.24) by changing the integration variable to k^{-2} and using the numerical result

$$I = \int_0^1 \left[1 - \frac{\pi}{2E(k^{-1})}\right] \frac{\mathrm{d}k^{-2}}{k^{-3}} = -0.621$$

Solution:

We have

$$\frac{3}{4}\int_0^{\lambda_c} \frac{\lambda\,\mathrm{d}\lambda}{\langle 1-\lambda/h\rangle} = \frac{3\pi}{4}\int_0^{\lambda_c} \mathrm{d}\lambda\, \frac{\lambda\sqrt{k^{-2}(1-\epsilon)+2\epsilon}}{2\sqrt{2\epsilon}E(-k)} \tag{11.14}$$

We can then use

$$\lambda = \frac{k^{-2}}{k^{-2}(1-\epsilon)+2\epsilon} \tag{11.15}$$

$$\mathrm{d}\lambda = \frac{[k^{-2}(1-\epsilon)+2\epsilon]\,\mathrm{d}k^{-2} - k^{-2}(1-\epsilon)\,\mathrm{d}k^{-2}}{(k^{-2}(1-\epsilon)+2\epsilon)^2} = \frac{2\epsilon\,\mathrm{d}k^{-2}}{(k^{-2}(1-\epsilon)+2\epsilon)^2} \tag{11.16}$$

Hence, we find

$$\frac{3\pi}{4}\int_0^{\lambda_c} \mathrm{d}\lambda\, \frac{\lambda\sqrt{k^{-2}(1-\epsilon)+2\epsilon}}{2\sqrt{2\epsilon}E(-k)} = \frac{3\pi}{8}\int_0^{\frac{2\epsilon\lambda_c}{1-(1-\epsilon)\lambda_c}} \mathrm{d}k^{-2}\, \frac{2\epsilon}{(k^{-2}(1-\epsilon)+2\epsilon)^2}\frac{\sqrt{k^{-2}(1-\epsilon)+2\epsilon}}{\sqrt{2\epsilon}E(-k)}\frac{k^{-2}}{k^{-2}(1-\epsilon)+2\epsilon} \tag{11.17}$$

$$= \frac{3\pi}{8}\int_0^{\frac{2\epsilon\lambda_c}{1-(1-\epsilon)\lambda_c}} \mathrm{d}k^{-2}\, \frac{2\epsilon}{(k^{-2}(1-\epsilon)+2\epsilon)^{5/2}}\frac{k^{-2}}{\sqrt{2\epsilon}E(-k)} \tag{11.18}$$

Now we can use that $\epsilon \ll 1$ so that $k^{-2}/(k^{-2})^{5/2} = 1/k^{-3}$ with $\lambda_c = 1-\epsilon$,

$$\frac{2\epsilon\lambda_c}{1-(1-\epsilon)\lambda_c} = \frac{2\epsilon(1-\epsilon)}{1-(1-\epsilon)^2} = \frac{2\epsilon(1-\epsilon)}{\epsilon(2-\epsilon)} = \frac{2-2\epsilon}{2-\epsilon} \tag{11.19}$$

so that with $\lim_{\epsilon\to 0}$, we get this equal to 1 and so we find

$$-\frac{3\sqrt{2\epsilon}}{4}\int_0^1 \mathrm{d}k^{-2}\, \frac{\pi}{2E(k^{-1})k^{-3}} = \frac{3\sqrt{2\epsilon}}{4}\int_0^1 \mathrm{d}k^{-2}\, \left[1 - \frac{\pi}{2k^{-3}E(k^{-1})}\right] - \frac{3\sqrt{2\epsilon}}{4}\int_0^1 \mathrm{d}k^{-2}\, k^3 \tag{11.20}$$

We find

$$\int_0^1 \mathrm{d}k^{-2}\, k^3 = \int_0^1 \mathrm{d}k\, k^3(-2k^{-3}) = -2 \tag{11.21}$$

Thus,

$$f_t = 1 - \frac{3}{4}\int_0^{\lambda_c} \frac{\lambda\,\mathrm{d}\lambda}{\langle 1-\lambda/h\rangle} \approx 1 - \frac{3\sqrt{2\epsilon}}{4}(-0.621+2) \approx 1 - 1.46\sqrt{\epsilon} \tag{11.22}$$

We note that this is incompatible with our Ansatz and so we must have missed an order $\epsilon^{-1/2}$ term in the integral by taking $k^{-2}/(k^{-2}+2\epsilon)^{5/2} \to k^3$. Thus, for

$$\frac{3\sqrt{2\epsilon}}{4}\left(\int_0^1 \frac{\pi}{2E(k^{-1})}\frac{k^{-2}\,\mathrm{d}k^{-2}}{(k^{-2}+2\epsilon)^{5/2}}\right) = \frac{3\sqrt{\epsilon}}{4}\left(\int_0^1 \frac{k^{-2}\,\mathrm{d}k^{-2}}{(k^{-2}+2\epsilon)^{5/2}} - I\right) \tag{11.23}$$

we would get

$$\int_0^1 \frac{k^{-2}\,\mathrm{d}k^{-2}}{(k^{-2}+2\epsilon)^{5/2}} = \int_0^1 \frac{x\,\mathrm{d}x}{(x+2\epsilon)^{5/2}} \tag{11.24}$$

use $u = (\alpha x+\beta)(x+2\epsilon)^{-3/2}$ so

$$\mathrm{d}u = \frac{-3(\alpha x+\beta)\,\mathrm{d}x}{2(x+2\epsilon)^{5/2}} + \alpha\,\mathrm{d}x(x+2\epsilon)^{-3/2} = \frac{-3(\alpha x+\beta)\,\mathrm{d}x + 2\alpha(x+2\epsilon)\,\mathrm{d}x}{2(x+2\epsilon)^{5/2}} = \frac{(-\alpha x+4\alpha\epsilon-3\beta)\,\mathrm{d}x}{2(x+2\epsilon)^{5/2}} \tag{11.25}$$

If we choose $4\alpha\epsilon - 3\beta = 0$ we find $\beta = \frac{4}{3}\alpha\epsilon$. For convenience we can choose $\alpha = -2$ and we have

$$\mathrm{d}u = \frac{x\,\mathrm{d}x}{(x+2\epsilon)^{5/2}} \tag{11.26}$$

So we get

$$\int_0^1 \frac{x\,\mathrm{d}x}{(x+2\epsilon)^{5/2}} = \int_{u_0}^{u_1} \mathrm{d}u = [u]_{u_0}^{u_1} = \left[\frac{-2x-\frac{8\epsilon}{3}}{(x+2\epsilon)^{3/2}}\right]_0^1 \tag{11.27}$$

$$= \frac{-2-\frac{8\epsilon}{3}}{(1+2\epsilon)^{3/2}} - \frac{-\frac{8\epsilon}{3}}{(2\epsilon)^{3/2}} \approx \frac{2\sqrt{2}}{3\sqrt{\epsilon}} - 2 + \mathcal{O}(\epsilon) \tag{11.28}$$

And so we get

$$f_t = 1 - \frac{3}{4}\int_0^{\lambda_c} \frac{\lambda\,\mathrm{d}\lambda}{\langle 1-\lambda/h\rangle} \approx 1 - \frac{3\sqrt{2\epsilon}}{4}\left(\frac{2\sqrt{2}}{3\sqrt{\epsilon}} - 2 + 0.621\right) \approx -\frac{3\sqrt{2}}{4}(1.379)\sqrt{\epsilon} + \mathcal{O}(\epsilon) \approx 1.46\sqrt{\epsilon} \tag{11.29}$$

11.6.2 Approximate Formula

Since the definition of f_t involves two nested integrals, this quantity is difficult to calculate for realistic equilibria. An approximate formula is therefore useful. An excellent approximation to f_t is given by

$$f_t \simeq 0.25 f_{tl} + 0.75 f_{tu}$$

where

$$f_{tl} = 1 - \frac{3}{4}\int_0^{\lambda_c} \mathrm{d}\lambda\;\lambda\left\langle \frac{1}{\sqrt{1-\lambda/h}}\right\rangle$$

$$f_{tu} = 1 - \frac{3}{4}\int_0^{\lambda_c} \mathrm{d}\lambda\;\frac{\lambda}{\sqrt{1-\lambda\langle h^{-1}\rangle}}$$

are rigorous upper and lower bounds on f_t (Lin-Liu and Miller, 1995).

11.6.2.1 Identity

Start by proving that for any function $A(\theta)$

$$\langle A \rangle \leq \langle A^2 \rangle^{1/2}$$
$$\frac{1}{\langle A \rangle} \leq \left\langle \frac{1}{A} \right\rangle$$

Solution:

First consider

$$\left\langle (A - \langle A \rangle)^2 \right\rangle \geq 0 \quad (11.30)$$
$$\left\langle A^2 \right\rangle - 2\langle A \rangle^2 + \langle A \rangle^2 \geq 0 \quad (11.31)$$
$$\left\langle A^2 \right\rangle - \langle A \rangle^2 \geq 0 \quad (11.32)$$
$$\left\langle A^2 \right\rangle \geq \langle A \rangle^2 \quad (11.33)$$
$$\langle A \rangle^2 \leq \left\langle A^2 \right\rangle \quad (11.34)$$
$$\langle A \rangle \leq \sqrt{\langle A^2 \rangle} \quad (11.35)$$

Then consider

$$\left\langle \left(\frac{1}{\sqrt{A}} - \frac{\sqrt{A}}{\langle A \rangle} \right)^2 \right\rangle \geq 0 \quad (11.36)$$
$$\left\langle \left(\frac{1}{A} - \frac{2}{\langle A \rangle} + \frac{A}{\langle A \rangle^2} \right) \right\rangle \geq 0 \quad (11.37)$$
$$\left\langle \frac{1}{A} \right\rangle - \frac{2}{\langle A \rangle} + \frac{\langle A \rangle}{\langle A \rangle^2} \geq 0 \quad (11.38)$$
$$\left\langle \frac{1}{A} \right\rangle - \frac{2}{\langle A \rangle} + \frac{1}{\langle A \rangle} \geq 0 \quad (11.39)$$
$$\left\langle \frac{1}{A} \right\rangle - \frac{1}{\langle A \rangle} \geq 0 \quad (11.40)$$
$$\frac{1}{\langle A \rangle} \leq \left\langle \frac{1}{A} \right\rangle \quad (11.41)$$

Alternatively, we could consider the Cauchy-Schwartz inequality. The Cauchy-Schwartz inequality states

$$\langle fg \rangle^2 \leq \left\langle f^2 \right\rangle \left\langle g^2 \right\rangle \quad (11.42)$$

Take $f = A$ and $g = 1$ to get (because $\langle 1 \rangle = 1$)

$$\langle A \rangle^2 \leq \left\langle A^2 \right\rangle \left\langle 1^2 \right\rangle = \left\langle A^2 \right\rangle \quad (11.43)$$
$$\langle A \rangle \leq \sqrt{\langle A^2 \rangle} \quad (11.44)$$

Then take $f = \sqrt{A}$ and $g = 1/\sqrt{A}$ and this states

$$1 \leq \langle A \rangle \left\langle \frac{1}{A} \right\rangle \tag{11.45}$$

$$\frac{1}{\langle A \rangle} \leq \left\langle \frac{1}{A} \right\rangle \tag{11.46}$$

as desired.

11.6.2.2 Inequality

Show that $f_{tl} \leq f_t \leq f_{tu}$ for any magnetic equilibrium.

Solution:

Take $f_{tu} - f_{tl}$ and we find

$$f_{tu} - f_{tl} = \frac{3}{4} \int_0^{\lambda_c} \mathrm{d}\lambda \; \lambda \left[\left\langle \frac{1}{\sqrt{1 - \lambda/h}} \right\rangle - \frac{1}{\sqrt{1 - \lambda/\langle h \rangle}} \right] \tag{11.47}$$

If we show that the integrand is greater than or equal to zero, then we have proved that $f_{tu} \geq f_t \geq f_{tl}$ because by definition f_t is between f_{tu} and f_{tl}. We begin by noting that λ is independent of θ in this case as it is the variable of integration, so that

$$\langle \lambda/h \rangle = \lambda/\langle h \rangle \tag{11.48}$$

so that we have

$$\left\langle \frac{1}{\sqrt{1 - \lambda/h}} \right\rangle - \frac{1}{\sqrt{\langle 1 - \lambda/h \rangle}} \tag{11.49}$$

We have from $\left\langle \sqrt{A} \right\rangle \leq \sqrt{\langle A \rangle}$ that

$$\sqrt{\langle 1 - \lambda/h \rangle} \geq \left\langle \sqrt{1 - \lambda/h} \right\rangle \tag{11.50}$$

$$\frac{1}{\sqrt{\langle 1 - \lambda/h \rangle}} \leq \frac{1}{\left\langle \sqrt{1 - \lambda/h} \right\rangle} \tag{11.51}$$

So

$$-\frac{1}{\sqrt{\langle 1 - \lambda/h \rangle}} \geq -\frac{1}{\left\langle \sqrt{1 - \lambda/h} \right\rangle} \tag{11.52}$$

$$\left\langle \frac{1}{\sqrt{1 - \lambda/h}} \right\rangle - \frac{1}{\sqrt{\langle 1 - \lambda/h \rangle}} \geq \left\langle \frac{1}{\sqrt{1 - \lambda/h}} \right\rangle - \frac{1}{\left\langle \sqrt{1 - \lambda/h} \right\rangle} \tag{11.53}$$

So let's focus on the right hand side of the inequality. From $\left\langle \frac{1}{A} \right\rangle - \frac{1}{\langle A \rangle} \geq 0$ with $A = \sqrt{1 - \lambda/h}$ we have

$$\left\langle \frac{1}{\sqrt{1 - \lambda/h}} \right\rangle - \frac{1}{\sqrt{\langle 1 - \lambda/h \rangle}} \geq \left\langle \frac{1}{\sqrt{1 - \lambda/h}} \right\rangle - \frac{1}{\left\langle \sqrt{1 - \lambda/h} \right\rangle} \geq 0 \tag{11.54}$$

Thus,

$$\left\langle \frac{1}{\sqrt{1-\lambda/h}} \right\rangle - \frac{1}{\sqrt{1-\lambda/\langle h\rangle}} = \left\langle \frac{1}{\sqrt{1-\lambda/h}} \right\rangle - \frac{1}{\sqrt{\langle 1-\lambda/h\rangle}} \geq 0 \tag{11.55}$$

and we have established what we wished to prove as this implies for $\lambda_c > \lambda > 0$ that

$$\int_0^{\lambda_c} \mathrm{d}\lambda\ \lambda \left[\left\langle \frac{1}{\sqrt{1-\lambda/h}} \right\rangle - \frac{1}{\sqrt{1-\lambda/\langle h\rangle}} \right] \geq 0 \tag{11.56}$$

and so

$$f_{tu} \geq f_{tl} \tag{11.57}$$

and so clearly

$$f_{tu} \geq f_t \geq f_{tl} \tag{11.58}$$

11.6.2.3 Another Identity

Show that

$$f_{tl} = 1 - \langle y^2\rangle \left\langle y^{-2}\left[1-\left(1+\frac{y}{2}\right)\sqrt{1-y}\right]\right\rangle$$

where $y = B/B_{\max} = \lambda_c/h$, by interchanging the flux-surface average and the integral in the definition of f_{tl}.

Solution:

We look at the integral

$$\int_0^{\lambda_c} \mathrm{d}\lambda\, \frac{\lambda}{\sqrt{1-\lambda/h}} \tag{11.59}$$

Try the substitution $u = (\alpha\lambda+\beta)\sqrt{1-\lambda/h}$. Then

$$\mathrm{d}u = \frac{-(\alpha\lambda+\beta)}{2h\sqrt{1-\lambda/h}}\,\mathrm{d}\lambda + \alpha\sqrt{1-\lambda/h}\,\mathrm{d}\lambda = \frac{\frac{-\alpha\lambda}{2h} - \frac{\beta}{2h} + \alpha - \frac{\alpha\lambda}{h}}{\sqrt{1-\lambda/h}}\,\mathrm{d}\lambda \tag{11.60}$$

So if we choose $\alpha = \frac{\beta}{2h}$ and $\beta = -4h^2/(3)$ we find

$$\mathrm{d}u = \frac{\frac{-3\alpha\lambda}{2h}}{\sqrt{1-\lambda/h}}\,\mathrm{d}\lambda = \frac{\frac{-3\lambda}{2h}\frac{\beta}{2h}}{\sqrt{1-\lambda/h}}\,\mathrm{d}\lambda = \frac{\lambda}{\sqrt{1-\lambda/h}}\,\mathrm{d}\lambda \tag{11.61}$$

$$u = \left(\frac{-4h^2}{3}\frac{1}{2h}\lambda + \frac{-4h^2}{3}\right)\sqrt{1-\lambda/h} = \left(\frac{-2h}{3}\lambda - \frac{4h^2}{3}\right)\sqrt{1-\lambda/h} \tag{11.62}$$

So we get

$$\int_0^{\lambda_c} \mathrm{d}\lambda \, \frac{\lambda}{\sqrt{1-\lambda/h}} = \int_{u_0}^{u_1} \mathrm{d}u \;\; = [u]_{u_0}^{u_1} = \left[\left(\frac{-2h}{3}\lambda - \frac{4h^2}{3}\right)\sqrt{1-\lambda/h}\right]_0^{\lambda_c} \tag{11.63}$$

$$= \left(\frac{-2h\lambda_c}{3} - \frac{4h^2}{3}\right)\sqrt{1-\lambda_c/h} - \left(\frac{-4h^2}{3}\sqrt{1}\right) \tag{11.64}$$

$$= \frac{4h}{3}\left(h - (h + \frac{\lambda_c}{2})\sqrt{1-\lambda_c/h}\right) \tag{11.65}$$

$$= \frac{4h^2}{3}\left(1 - (1+\frac{y}{2})\sqrt{1-y}\right) \tag{11.66}$$

$$\tag{11.67}$$

Thus,

$$f_{tl} = 1 - \frac{3}{4}\int_0^{\lambda_c} \mathrm{d}\lambda \; \lambda \left\langle \frac{1}{\sqrt{1-\lambda/h}} \right\rangle = 1 - \left\langle \frac{3}{4}\int_0^{\lambda_c} \mathrm{d}\lambda \; \lambda \frac{1}{\sqrt{1-\lambda/h}} \right\rangle \tag{11.68}$$

$$= 1 - \left\langle \frac{3}{4}\frac{4h^2}{3}\left(1 - \left(1+\frac{y}{2}\right)\sqrt{1-y}\right)\right\rangle = 1 - \left\langle h^2\left(1 - \left(1+\frac{y}{2}\right)\sqrt{1-y}\right)\right\rangle \tag{11.69}$$

$$= 1 - \lambda_c^2 \left\langle y^{-2}\left(1 - \left(1+\frac{y}{2}\right)\sqrt{1-y}\right)\right\rangle \tag{11.70}$$

We can use that λ_c is independent of averaging, $\langle\lambda_c\rangle = \lambda_c$ and $\langle h^2\rangle = 1$ from $h = \frac{B_0}{B}$ and $B_0^2 = \langle B^2\rangle$ so that on the outside we can take $\lambda_c^2 = \frac{\lambda_c^2}{1} = \frac{\lambda_c^2}{\langle h^2\rangle} = \left\langle \frac{\lambda_c^2}{h^2}\right\rangle = \langle y^2\rangle$ and so

$$f_{tl} = 1 - \langle y^2\rangle \left\langle y^{-2}\left(1 - \left(1+\frac{y}{2}\right)\sqrt{1-y}\right)\right\rangle \tag{11.71}$$

as desired.

11.6.2.4 Large-Aspect-Ratio

Apply these formulas to the standard, large-aspect-ratio equilibrium, $h = B_0/B = 1 + \epsilon\cos\theta$, and calculate f_{tl} and f_{tu} to lowest order in $\sqrt{\epsilon}$.

Solution:

First let's calculate $\langle h\rangle$. We find that it must be

$$\langle h\rangle = \langle 1 + \epsilon\cos\theta\rangle = 1 + \epsilon\langle\cos\theta\rangle = 1 \tag{11.72}$$

$$\left\langle \frac{1}{h}\right\rangle = \left\langle \frac{1}{1+\epsilon\cos\theta}\right\rangle \approx \left\langle 1 - \epsilon\cos\theta + \epsilon^2\cos^2\theta + \mathcal{O}(\epsilon^3)\right\rangle = 1 + \frac{\epsilon^2}{2} + \mathcal{O}(\epsilon^3) \tag{11.73}$$

Where $\oint \mathrm{d}\theta \;\; \cos^2\theta = \pi$ and $\oint \mathrm{d}\theta \;\; = 2\pi$ has been used.

Thus, we find (identify $1+\frac{\epsilon^2}{2}$ with h^{-1}) from before

$$\int_0^{\lambda_c} \mathrm{d}\lambda \, \frac{\lambda}{\sqrt{1-\lambda\left(1+\frac{\epsilon^2}{2}\right)}} = \frac{4}{3(1+\frac{\epsilon^2}{2})^2}\left(1-\left[1+\frac{(1+\frac{\epsilon^2}{2})\lambda_c}{2}\right]\sqrt{1-\lambda_c(1+\epsilon^2/2)}\right) \tag{11.74}$$

$$\frac{4}{3}\left(1-\epsilon^2+\mathcal{O}(\epsilon^3)\right)\left(1-\left[1+\frac{(1+\frac{\epsilon^2}{2})\lambda_c}{2}\right]\sqrt{1-\lambda_c}\left(1-\frac{\lambda_c\epsilon^2}{4}+\mathcal{O}(\epsilon^3)\right)\right) \tag{11.75}$$

Thus, order by order we see

$$=\frac{4}{3}\left(1-\left[1+\frac{\lambda_c}{2}\right]\sqrt{1-\lambda_c}\right)+\frac{4\epsilon^2}{3}\left(-1-\left[1+\frac{\lambda_c}{2}\right]\sqrt{1-\lambda_c}+\frac{\lambda_c^2}{8}\sqrt{1-\lambda_c}\right)\epsilon^2+\mathcal{O}(\epsilon^3) \tag{11.76}$$

Hence to $\mathcal{O}(\sqrt{\epsilon})$

$$f_{tu} = 1-\left(1-\left[1+\frac{\lambda_c}{2}\right]\sqrt{1-\lambda_c}\right)=\left[1+\frac{\lambda_c}{2}\right]\sqrt{1-\lambda_c} \tag{11.77}$$

We can now use that $\lambda_c = 1-\epsilon+\mathcal{O}(\epsilon^2)$ so that

$$\left[1+\frac{\lambda_c}{2}\right]\sqrt{1-\lambda_c}=\left[1+\frac{1}{2}-\frac{\epsilon}{2}+\mathcal{O}(\epsilon)^2\right]\sqrt{1-1+\epsilon+\mathcal{O}(\epsilon^2)}=\frac{3}{2}\sqrt{\epsilon}+\mathcal{O}(\epsilon) \tag{11.78}$$

Thus,

$$f_{tu} = \frac{3\sqrt{\epsilon}}{2}+\mathcal{O}(\epsilon) \tag{11.79}$$

Now for the trickier f_{tl}. Here we can use that

$$y=\frac{\lambda_c}{h}=\frac{1-\epsilon+\mathcal{O}(\epsilon^2)}{1+\epsilon\cos\theta+\mathcal{O}(\epsilon^2)}=(1-\epsilon)\left(1-\epsilon\cos\theta\right)+\mathcal{O}(\epsilon^2)=1-\epsilon(1-\cos\theta)+\mathcal{O}(\epsilon^2) \tag{11.80}$$

$$y^2 = 1-2\epsilon(1-\cos\theta)+\mathcal{O}(\epsilon^2) \tag{11.81}$$

$$1/y=\frac{h}{\lambda_c}=\frac{1+\epsilon\cos\theta+\mathcal{O}(\epsilon^2)}{1-\epsilon+\mathcal{O}(\epsilon^2)}=(1+\epsilon)\left(1+\epsilon\cos\theta\right)+\mathcal{O}(\epsilon^2)=1+\epsilon(1-\cos\theta)+\mathcal{O}(\epsilon^2) \tag{11.82}$$

$$1/y^2 = 1+2\epsilon(1-\cos\theta)+\mathcal{O}(\epsilon^2) \tag{11.83}$$

$$\sqrt{1-y}=\sqrt{1-1-\epsilon(1-\cos\theta)+\mathcal{O}(\epsilon)^2}=\sqrt{\epsilon}\sqrt{1-\cos\theta}+\mathcal{O}(\epsilon) \tag{11.84}$$

Thus,

$$\left\langle y^{-2}\left(1-\frac{2+y}{2}\sqrt{1-y}\right)\right\rangle=\left\langle\left(1+2\epsilon(1-\cos\theta)\right)\left(1-\frac{2+1-\epsilon(1-\cos\theta)}{2}\sqrt{\epsilon}\sqrt{1-\cos\theta}+\mathcal{O}(\epsilon^2)\right)\right\rangle \tag{11.85}$$

$$=\left\langle 1-\frac{3}{2}\sqrt{\epsilon}\sqrt{1-\cos\theta}\right\rangle+\mathcal{O}(\epsilon) \tag{11.86}$$

We also have

$$\langle y^2 \rangle = \langle 1 - 2\epsilon(1 - \cos\theta) \rangle = 1 + \mathcal{O}(\epsilon) \tag{11.87}$$

Thus, find

$$f_{tl} = 1 - \left\langle 1 - \frac{3}{2}\sqrt{\epsilon}\sqrt{1 - \cos\theta} \right\rangle + \mathcal{O}(\epsilon) = \frac{3}{2}\sqrt{\epsilon}\left\langle \sqrt{1 - \cos\theta} \right\rangle \tag{11.88}$$

We use $1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$ and so

$$\int_0^{2\pi} \mathrm{d}\theta\ \sqrt{2}\sin\left(\frac{\theta}{2}\right) = 2\sqrt{2}\int_0^{\pi} \mathrm{d}\zeta\ \ \sin\zeta = -2\sqrt{2}\left[\cos(\pi) - \cos(0)\right] = 4\sqrt{2} \tag{11.89}$$

So $\left\langle \sqrt{1 - \cos\theta} \right\rangle = 4\sqrt{2}/(2\pi) = \frac{2\sqrt{2}}{\pi}$ and

$$f_{tl} = \frac{3}{2}\frac{2\sqrt{2}}{\pi}\epsilon + \mathcal{O}(\epsilon) = \frac{3\sqrt{2}}{\pi}\sqrt{\epsilon} + \mathcal{O}(\epsilon) \approx 1.35\sqrt{\epsilon} + \mathcal{O}(\epsilon) \tag{11.90}$$

Thus,

$$\begin{aligned} f_t &\simeq 0.25 f_{tl} + 0.75 f_{tu} = \left[0.25(1.35) + 0.75(1.5)\right]\sqrt{\epsilon} + \mathcal{O}(\epsilon) = \left[0.3375 + 1.125\right]\sqrt{\epsilon} + \mathcal{O}(\epsilon) \\ &\approx 1.46\sqrt{\epsilon} + \mathcal{O}(\epsilon) \end{aligned} \tag{11.91}$$

11.6.3 Onsager Symmetry

Verify Onsager symmetry of the electron transport coefficients in the banana regime.

Solution:

We first note that Onsager symmetry implies that $L^{ee}_{jk} = L^{ee}_{kj}$, i.e., that L_{jk} is a symmetric tensor. We use that

$$\dot{S}_e = -\sum_{jk} L^{ee}_{jk} A^e_j A^e_k \tag{11.92}$$

$$\Gamma^e_j = \sum_k L^{ee}_{jk} A^e_k \tag{11.93}$$

with

$$A^e_1 = \frac{\mathrm{d}\ln p_e}{\mathrm{d}\psi} \tag{11.94}$$

$$A^e_2 = \frac{\mathrm{d}\ln T_e}{\mathrm{d}\psi} \tag{11.95}$$

$$A^e_3 = \frac{E^{(A)}_{\parallel}}{T_e} \tag{11.96}$$

and that

$$\Gamma^e_1 = \langle \mathbf{\Gamma}_e \cdot \nabla\psi \rangle \tag{11.97}$$

$$\Gamma^e_2 = \langle \mathbf{q}_e \cdot \nabla\psi \rangle / T_e \tag{11.98}$$

$$\Gamma^e_3 = \langle j_{e\parallel} - j_{es} \rangle \tag{11.99}$$

We have

$$\Gamma_1^e = -f_t \frac{n_e I^2 T_e}{m_e \Omega_e^2 \tau_e}\left[1.53\left(1+\frac{T_i}{T_e}\right)\frac{\mathrm{d}\ln n_e}{\mathrm{d}\psi} - 0.59\frac{\mathrm{d}\ln T_e}{\mathrm{d}\psi} - \frac{0.26}{T_e}\frac{\mathrm{d}T_i}{\mathrm{d}\psi}\right] - 1.66 f_t \frac{n_e I E_\parallel^{(A)}}{B_0} \quad (11.100)$$

$$\Gamma_2^e = -f_t \frac{n_e I^2 T_e}{m_e \Omega_e^2 \tau_e}\left[-2.12\left(1+\frac{T_i}{T_e}\right)\frac{\mathrm{d}\ln n_e}{\mathrm{d}\psi} + 2.51\frac{\mathrm{d}\ln T_e}{\mathrm{d}\psi} - \frac{0.37}{T_e}\frac{\mathrm{d}T_i}{\mathrm{d}\psi}\right] + 1.19 f_t \frac{n_e I E_\parallel^{(A)}}{B_0} \quad (11.101)$$

$$\Gamma_3^e = -f_t n_e T_e R\left[1.66\left(1+\frac{T_i}{T_e}\right)\frac{\mathrm{d}\ln n_e}{\mathrm{d}\psi} + 0.47\frac{\mathrm{d}\ln T_e}{\mathrm{d}\psi} - \frac{0.29}{T_e}\frac{\mathrm{d}T_i}{\mathrm{d}\psi}\right] - 1.31 f_t \sigma E_\parallel^a \quad (11.102)$$

for $\epsilon \ll 1$ and $Z = 1$. We then use $p_e = n_e T_e$ so $\ln n_e = \ln p_e - \ln T_e$ and we write

$$\Gamma_1^e = -f_t \frac{n_e I^2 T_e}{m_e \Omega_e^2 \tau_e}\left[1.53\left(1+\frac{T_i}{T_e}\right)\frac{\mathrm{d}\ln p_e}{\mathrm{d}\psi} - 2.12\frac{\mathrm{d}\ln T_e}{\mathrm{d}\psi} - \frac{1.53 T_i}{T_e}\frac{\mathrm{d}\ln T_e}{\mathrm{d}\psi} - \frac{0.26}{T_e}\frac{\mathrm{d}T_i}{\mathrm{d}\psi}\right] - 1.66 f_t \frac{n_e I E_\parallel^{(A)}}{B_0} \quad (11.103)$$

$$\Gamma_2^e = -f_t \frac{n_e I^2 T_e}{m_e \Omega_e^2 \tau_e}\left[-2.12\left(1+\frac{T_i}{T_e}\right)\frac{\mathrm{d}\ln p_e}{\mathrm{d}\psi} + 4.63\frac{\mathrm{d}\ln T_e}{\mathrm{d}\psi} + \frac{2.12 T_i}{T_e}\frac{\mathrm{d}\ln T_e}{\mathrm{d}\psi} - \frac{0.37}{T_e}\frac{\mathrm{d}T_i}{\mathrm{d}\psi}\right] + 1.19 f_t \frac{n_e I E_\parallel^{(A)}}{B_0} \quad (11.104)$$

$$\Gamma_3^e = -f_t n_e T_e R\left[1.66\left(1+\frac{T_i}{T_e}\right)\frac{\mathrm{d}\ln p_e}{\mathrm{d}\psi} - 1.19\frac{\mathrm{d}\ln T_e}{\mathrm{d}\psi} - \frac{1.66 T_i}{T_e}\frac{\mathrm{d}\ln T_e}{\mathrm{d}\psi} - \frac{0.29}{T_e}\frac{\mathrm{d}T_i}{\mathrm{d}\psi}\right] - 1.31 f_t \sigma E_\parallel^a \quad (11.105)$$

For

$$\Gamma_1^e = L_{11}^{ee} A_1^e + L_{12}^{ee} A_2^e + L_{13}^{ee} A_3^e \quad (11.106)$$

$$\Gamma_2^e = L_{21}^{ee} A_1^e + L_{22}^{ee} A_2^e + L_{23}^{ee} A_3^e \quad (11.107)$$

$$\Gamma_3^e = L_{31}^{ee} A_1^e + L_{32}^{ee} A_2^e + L_{33}^{ee} A_3^e \quad (11.108)$$

We can ignore the factors in front of the numbers and the non electron quantities (set $T_i = 0$) and we then see that

$$L_{jk}^{ee} \propto \begin{bmatrix} 1.53 & -2.12 & 1.66 \\ -2.12 & 4.63 & -1.19 \\ 1.66 & -1.19 & 1.31 \end{bmatrix} \quad (11.109)$$

which is clearly symmetric. Thus we have Onsager symmetry.

Chapter 12

The Moment Approach to Neoclassical Theory

12.3 Collisional Regime

If we have

$$\overleftrightarrow{\mathbf{\Pi}} = -\frac{3}{2}\eta W_{zz}\left(\mathbf{bb} - \frac{1}{3}\mathbb{1}\right) \tag{12.1}$$

then

$$\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{\Pi}} = B_i\partial_j\Pi_{ij} = \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\left(-\frac{3}{2}\eta W_{zz}\left(\mathbf{bb} - \frac{1}{3}\mathbb{1}\right)\right) = B_i\partial_j\left(-\frac{3}{2}\eta W_{zz}\left(b_ib_j - \frac{1}{3}\delta_{ij}\right)\right) \tag{12.2}$$

$$= -B_i\frac{3}{2}\eta W_{zz}\partial_j\left(b_ib_j - \frac{\delta_{i\cancel{j}}}{\cancel{3}}\right) - B_i\frac{3}{2}\eta[b_ib_j - \frac{\delta_{ij}}{3}]\partial_jW_{zz} \tag{12.3}$$

$$= -B_i\frac{3}{2}\eta W_{zz}[(b_i\partial_jb_j) + b_j\partial_jb_i)] - \frac{3}{2}\eta[Bb_j - \frac{Bb_j}{3}]\partial_jW_{zz} \tag{12.4}$$

We then use

$$B_ib_i\partial_jb_j = B\boldsymbol{\nabla}\cdot\mathbf{b} = B\boldsymbol{\nabla}\cdot(\mathbf{B}/B) = \cancel{\boldsymbol{\nabla}\cdot\mathbf{B}} + B\,\nabla(1/B)\cdot\mathbf{B} = -\frac{B}{B^2}\mathbf{B}\cdot\,\nabla B = -\nabla_\parallel B \tag{12.5}$$

$$B_ib_j\partial_jb_i = Bb_ib_j\partial_jb_i = Bb_j\partial(b_ib_i) = Bb_j\partial(1) = 0 \tag{12.6}$$

Thus, we find

$$\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{\Pi}} = -\frac{3}{2}\eta W_{zz}\left(-\nabla_\parallel B\right) - \eta B\,\nabla_\parallel W_{zz} = \frac{3}{2}\eta W_{zz}\,\nabla_\parallel B - \eta B\,\nabla_\parallel W_{zz} \tag{12.7}$$

If we take the flux surface average, we can use that $\langle\mathbf{B}\cdot\,\nabla f\rangle = \left\langle B\,\nabla_\parallel f\right\rangle = 0$ for any f single-valued. Thus,

$$\left\langle\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{\Pi}}\right\rangle = \left\langle-\frac{3}{2}\eta W_{zz}\left(-\nabla_\parallel B\right) - \cancel{\eta B\,\nabla_\parallel W_{zz}}\right\rangle = \frac{3}{2}\eta\left\langle W_{zz}\,\nabla_\parallel B\right\rangle \tag{12.8}$$

where we have used η is a constant with respect to the average (and is actually a constant).

Next, we note that $\mathbf{b} \cdot \nabla \mathbf{b} = \nabla_{\|} \mathbf{b} = -\mathbf{b} \times \boldsymbol{\nabla} \times \mathbf{b} = \boldsymbol{\kappa}$ and so for low β

$$\kappa = \frac{\nabla_{\perp} B}{B} \tag{12.9}$$

We then use that

$$\hat{\boldsymbol{\varphi}} \cdot \nabla B = 0 \tag{12.10}$$

because

$$\nabla B = \cancel{\frac{\partial B}{\partial \varphi}} \nabla \varphi + \frac{\partial B}{\partial \theta} \nabla \theta + \frac{\partial B}{\partial \psi} \nabla \psi \tag{12.11}$$

due to axisymmetry. Thus,

$$\hat{\boldsymbol{\varphi}} \nabla B = \hat{\boldsymbol{\varphi}} \cdot \hat{\mathbf{b}} \nabla_{\|} B + \hat{\boldsymbol{\varphi}} \cdot \nabla_{\perp} B = 0 \tag{12.12}$$

$$-\hat{\boldsymbol{\varphi}} \cdot \mathbf{b} \nabla_{\|} B = \hat{\boldsymbol{\varphi}} \cdot \nabla_{\perp} B \tag{12.13}$$

So we may state (with $\hat{\boldsymbol{\varphi}} \cdot \mathbf{B} = \frac{I}{R}$ in our coordinate system)

$$\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\kappa} = \frac{\hat{\boldsymbol{\varphi}} \cdot \nabla_{\perp} B}{B} = -\frac{\hat{\boldsymbol{\varphi}} \cdot \hat{\mathbf{b}} \nabla_{\|} B}{B} = -\frac{I}{RB^2} \nabla_{\|} B \tag{12.14}$$

12.8 Chapter 12 Exercises

12.8.1 Banana Regime Ion Heat Flux

Derive the banana-regime ion heat flux (HS-11.31) in a pure plasma with large aspect ratio by using the moment approach.

Solution:

For this we remind ourselves that we have

$$I_j^a = \sum_{b,k} L_{jk}^{ab} A_k^b \tag{12.15}$$

$$L_{jk}^{ab} = 3 \left\langle \left(\nabla_{\|} B \right)^2 \right\rangle \frac{I^2 T_b}{e_a e_b B_0^4} \left(\frac{\mu_{aj} \mu_{bk}}{\mu_1} - \mu_{a,j+k-1} \delta_{ab} \right) \tag{12.16}$$

with

$$A_1^a = \frac{\mathrm{d} \ln p_a}{\mathrm{d}\psi} + \frac{e_a}{T_a} \frac{\mathrm{d}\varphi}{\mathrm{d}\psi} \tag{12.17}$$

$$A_2^a = \frac{\mathrm{d} \ln T_a}{\mathrm{d}\psi} \tag{12.18}$$

$$I_1^a = \langle \boldsymbol{\Gamma}_a \cdot \nabla \psi \rangle^{BP} \tag{12.19}$$

$$I_2^a = \langle \mathbf{q}_a \cdot \nabla \psi \rangle^{BP} / T_a \tag{12.20}$$

We are only interested in I_2^i. We use that the ion-electron contribution should be quite small due to the weak interaction. Thus we need only I_2^{ii} for a good calculation. We are given for the banana regime that

$$\mu_{ak} = \frac{m_a n_a \langle B^2 \rangle}{3 \left\langle \left(\nabla_\| B \right)^2 \right\rangle} \frac{f_t}{f_c} \left\{ \nu_D^a \left(x^2 - \frac{5}{2} \right)^{k-1} \right\} \tag{12.21}$$

$$\mu_1 = \sum_a \mu_{a1} \tag{12.22}$$

This gives us all the required information.

We have

$$I_2^i = L_{21}^{ii} A_1^i + L_{22}^{ii} A_1^i \tag{12.23}$$

$$L_{21}^{ii} = 3 \left\langle \left(\nabla_\| B \right)^2 \right\rangle \frac{I^2 T_b}{Z^2 e^2 B_0^4} \left(\frac{\mu_{i2}\mu_{i1}}{\mu_1} - \mu_{i,2+1-1} \right) = 3 \left\langle \left(\nabla_\| B \right)^2 \right\rangle \frac{I^2 T_b}{Z^2 e^2 B_0^4} \left(\frac{\mu_{i2}\mu_{i1}}{\mu_1} - \mu_{i,2} \right) \tag{12.24}$$

$$L_{22}^{ii} = 3 \left\langle \left(\nabla_\| B \right)^2 \right\rangle \frac{I^2 T_b}{Z^2 e^2 B_0^4} \left(\frac{\mu_{i2}\mu_{i2}}{\mu_1} - \mu_{i,2+2-1} \right) = 3 \left\langle \left(\nabla_\| B \right)^2 \right\rangle \frac{I^2 T_b}{Z^2 e^2 B_0^4} \left(\frac{\mu_{i2}\mu_{i2}}{\mu_1} - \mu_{i,3} \right) \tag{12.25}$$

Note

$$\mu_{i1} = \frac{m_i n_i \langle B^2 \rangle}{3 \left\langle \left(\nabla_\| B \right)^2 \right\rangle} \frac{f_t}{f_c} \left\{ \nu_D^i \right\} \tag{12.26}$$

$$\mu_{i2} = \frac{m_i n_i \langle B^2 \rangle}{3 \left\langle \left(\nabla_\| B \right)^2 \right\rangle} \frac{f_t}{f_c} \left\{ \nu_D^i \left(x^2 - \frac{5}{2} \right)^{1} \right\} \tag{12.27}$$

$$\mu_{i3} = \frac{m_i n_i \langle B^2 \rangle}{3 \left\langle \left(\nabla_\| B \right)^2 \right\rangle} \frac{f_t}{f_c} \left\{ \nu_D^i \left(x^2 - \frac{5}{2} \right)^{2} \right\} \tag{12.28}$$

Note that $\mu_{i1} = \mu_1$ in this case so that $L_{21} = 0$. We are left with

$$L_{22}^{ii} = \frac{m_i n_i T_b \langle B^2 \rangle}{Z^2 e^2 B_0^4} \frac{f_t}{f_c} \left(\frac{\left\{ \nu_D^i \left(x^2 - \frac{5}{2} \right) \right\}^2}{\left\{ \nu_D^i \right\}} - \left\{ \nu_D^i \left(x^2 - \frac{5}{2} \right)^2 \right\} \right) \tag{12.29}$$

Since $\langle B^2 \rangle = B_0^2$ this simplifies. We then use that

$$\langle F(v) \rangle = \frac{8}{3\sqrt{\pi}} \int_0^\infty \mathrm{d}x \; F(x) e^{-x^2} x^4 \tag{12.30}$$

$$\tau_i = \frac{12\pi^{3/2} \sqrt{m_i T_i^3} \epsilon_0^2}{n_i Z^4 e^4 \ln \Lambda} = \frac{12\pi^{3/2} v_{\mathrm{th}_i}^3 \sqrt{m_i \frac{m_i^3}{2^3}} \epsilon_0^2}{n_i Z^4 e^4 \ln \Lambda} = \frac{3\sqrt{2}\pi^{3/2} v_{\mathrm{th}_i}^3 m_i^2 \epsilon_0^2}{n_i Z^4 e^4 \ln \Lambda} = \sqrt{2}\tau_{ii} \tag{12.31}$$

$$\nu_D^{ii} = \widehat{\nu}_{ii} \frac{\phi(x) - G(x)}{x^3} \tag{12.32}$$

$$\widehat{\nu}_{ii} = \frac{n_i Z^4 e^4 \ln \Lambda}{4\pi \epsilon_0^2 m_i^2 v_{\mathrm{th}_i}^3} = \frac{3\sqrt{2\pi}}{4\tau_i} \tag{12.33}$$

However, we can use

$$\{\nu_D^{ii}\}\,\tau_{ii} = \sqrt{1+1^2} + \ln\frac{1}{1+\sqrt{1+1^2}} \approx 0.533 \tag{12.34}$$

$$\{\nu_D^{ii}x^2\}\,\tau_{ii} = \frac{1}{\sqrt{1+1^2}} \approx 0.707 \tag{12.35}$$

$$\{\nu_D^{ii}x^4\}\,\tau_{ii} = 2\frac{1+5/4}{(1+1^2)^{3/2}} \approx 1.59 \tag{12.36}$$

$$\{\nu_D^{ii}\}\,\tau_i \approx 0.754 \tag{12.37}$$

$$\{\nu_D^{ii}x^2\}\,\tau_i \approx 1. \tag{12.38}$$

$$\{\nu_D^{ii}x^4\}\,\tau_i \approx 2.25 \tag{12.39}$$

$$\tag{12.40}$$

Thus,

$$\frac{\{\nu_D^{ii}(x^2-5/2)\}^2}{\{\nu_D^{ii}\}} - \left\{\nu_D^{ii}\left(x^4 - 5x^2 + \frac{25}{4}\right)\right\} = \frac{\left(\frac{1}{\tau_i} - \frac{5}{2}(0.754)\frac{1}{\tau_i}\right)^2}{0.754/\tau_i} - \left[\frac{2.25}{\tau_i} - \frac{5}{\tau_i} + \frac{25}{4\tau_i}(0.754)\right] \tag{12.41}$$

$$= \frac{1}{\tau_i}\left(\frac{0.783}{0.754} - 1.774\right) \approx \frac{1.038 - 1.962}{\tau_i} \approx -\frac{0.92}{\tau_i} \tag{12.42}$$

Thus,

$$L_{22}^{ii} = -0.92\frac{f_t}{f_c}\frac{m_i n_i I^2 T_i}{\tau_i Z^2 e^2 B_0^2} \tag{12.43}$$

which matches for $Z=1$ the form given in the book.

12.8.2 Electron Particle and Heat Flux

In the same way, calculate the electron particle and heat fluxes at low collisionality and large aspect ratio.

Solution:

This time we cannot ignore the electron-ion contribution as it is important. Thus we need to get L_{jk}^{ei} and L_{jk}^{ee}. We also use $\nu_D^e = \nu_D^{ee} + \nu_D^{ei}$ for this calculation.

We remind ourselves that for $x_{ab} = v_{\text{th}_b}/v_{\text{th}_a}$ that

$$\{\nu_D^{ab}\}\,\tau_{ab} = \sqrt{1+x_{ab}^2} + x_{ab}^2\ln\left(\frac{x_{ab}}{1+\sqrt{1+x_{ab}^2}}\right) \tag{12.44}$$

$$\{\nu_D^{ab}x_a^2\}\,\tau_{ab} = \frac{1}{\sqrt{1+x_{ab}^2}} \tag{12.45}$$

$$\{\nu_D^{ab}x_a^4\}\,\tau_{ab} = 2\frac{1+\frac{5}{4}x_{ab}^2}{(1+x_{ab}^2)^{3/2}} \tag{12.46}$$

We use that for electrons we have $x_{ee} = 1$

$$\left\{\nu_D^{ee}\right\}\tau_{ee} = 0.533 \tag{12.47}$$
$$\left\{\nu_D^{ee} x_e^2\right\}\tau_{ee} = 0.707 \tag{12.48}$$
$$\left\{\nu_D^{ee} x_e^4\right\}\tau_{ee} = 1.59 \tag{12.49}$$

while for ions we have $x_{ei} = v_{\mathrm{th}_i}/v_{\mathrm{th}_e} \ll 1 \approx 0$ and so

$$\left\{\nu_D^{ei}\right\}\tau_{ei} = 1 \tag{12.50}$$
$$\left\{\nu_D^{ei} x_e^2\right\}\tau_{ei} = 1 \tag{12.51}$$
$$\left\{\nu_D^{ei} x_e^4\right\}\tau_{ei} = 2 \tag{12.52}$$

We thus find

$$\mu_{e1} = \frac{m_e n_e \left\langle B^2\right\rangle}{3\left\langle \left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left\{\nu_D^{ee} + \nu_D^{ei}\right\} \tag{12.53}$$

$$\mu_{e2} = \frac{m_e n_e \left\langle B^2\right\rangle}{3\left\langle \left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left\{\left(\nu_D^{ee} + \nu_D^{ei}\right)\left(x^2 - \frac{5}{2}\right)^1\right\} \tag{12.54}$$

$$\mu_{e3} = \frac{m_e n_e \left\langle B^2\right\rangle}{3\left\langle \left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left\{\left(\nu_D^{ee} + \nu_D^{ei}\right)\left(x^2 - \frac{5}{2}\right)^2\right\} \tag{12.55}$$

Thus,

$$\mu_{e1} = \frac{m_e n_e \left\langle B^2\right\rangle}{3\left\langle \left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left([0.533]\tau_{ee}^{-1} + \tau_{ei}^{-1}\right) \tag{12.56}$$

$$\mu_{e2} = \frac{m_e n_e \left\langle B^2\right\rangle}{3\left\langle \left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left(\left[0.707 - \frac{5}{2}(0.533)\right]\tau_{ee}^{-1} + \left[1 - \frac{5}{2}\right]\tau_{ei}^{-1}\right) \tag{12.57}$$

$$\mu_{e3} = \frac{m_e n_e \left\langle B^2\right\rangle}{3\left\langle \left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left(\left[1.59 - 5(0.707) + \frac{25}{4}(0.533)\right]\tau_{ee}^{-1} + \left[2 - 5(1) + \frac{25}{4}\right]\tau_{ei}^{-1}\right) \tag{12.58}$$

or, more simply

$$\mu_{e1} = \frac{m_e n_e \left\langle B^2\right\rangle}{3\left\langle \left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left([0.533]\tau_{ee}^{-1} + \tau_{ei}^{-1}\right) \tag{12.59}$$

$$\mu_{e2} = \frac{m_e n_e \left\langle B^2\right\rangle}{3\left\langle \left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left([-0.626]\,\tau_{ee}^{-1} + [-1.5]\tau_{ei}^{-1}\right) \tag{12.60}$$

$$\mu_{e3} = \frac{m_e n_e \left\langle B^2\right\rangle}{3\left\langle \left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left([1.386]\,\tau_{ee}^{-1} + [3.25]\tau_{ei}^{-1}\right) \tag{12.61}$$

With $\tau_{ee} \approx \tau_{ei}$ for a pure plasma and we can then write

$$\mu_{e1} = \frac{m_e n_e \langle B^2 \rangle}{3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle} \frac{f_t}{f_c} (1.533) \tau_{ei}^{-1} \tag{12.62}$$

$$\mu_{e2} = \frac{m_e n_e \langle B^2 \rangle}{3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle} \frac{f_t}{f_c} (-2.125) \tau_{ei}^{-1} \tag{12.63}$$

$$\mu_{e3} = \frac{m_e n_e \langle B^2 \rangle}{3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle} \frac{f_t}{f_c} (4.637) \tau_{ei}^{-1} \tag{12.64}$$

Let's construct our matrix piece by piece. We also use $\mu_1 = \mu_{e1} + \mu_{i1} \sim \mu_{i1}$. Because $\mu_{i1} \sim m_i \tau_{ii}^{-1}$ and $m_i \tau_{ii}^{-1} \gg m_e \tau_{ee}^{-1}$ we can say that $\mu_{i1} \gg \mu_{i1}$ and ignore the μ_{e1} contribution to μ_1. Then we can use $\mu_{ei}/\mu_{i1} \ll 1 \approx 0$ and so we get the easy simplifications

$$L_{11}^{ee} = 3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle \frac{I^2 T_e}{e^2 B_0^4} \left(\cancel{\frac{\mu_{e1}\mu_{e1}}{\mu_{i1}}} - \mu_{e1} \right) = -1.533 \frac{f_t}{f_c} \frac{m_e n_e}{\tau_{ei}} \frac{I^2 T_e}{e^2 B^2} \tag{12.65}$$

$$L_{12}^{ee} = 3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle \frac{I^2 T_e}{e^2 B_0^4} \left(\cancel{\frac{\mu_{e1}\mu_{e2}}{\mu_{i1}}} - \mu_{e2} \right) = 2.125 \frac{f_t}{f_c} \frac{m_e n_e}{\tau_{ei}} \frac{I^2 T_e}{e^2 B^2} \tag{12.66}$$

$$L_{21}^{ee} = 3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle \frac{I^2 T_e}{e^2 B_0^4} \left(\cancel{\frac{\mu_{e2}\mu_{e1}}{\mu_{i1}}} - \mu_{e2} \right) = 2.125 \frac{f_t}{f_c} \frac{m_e n_e}{\tau_{ei}} \frac{I^2 T_e}{e^2 B^2} \tag{12.67}$$

$$L_{22}^{ee} = 3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle \frac{I^2 T_e}{e^2 B_0^4} \left(\cancel{\frac{\mu_{e2}\mu_{e2}}{\mu_{e1}}} - \mu_{e3} \right) = -(4.637) \frac{f_t}{f_c} \frac{m_e n_e}{\tau_{ei}} \frac{I^2 T_e}{e^2 B^2} \tag{12.68}$$

And for the other matrix $\mu_{i2}/\mu_{i1} \approx -0.885/0.754 \approx -1.174$

$$L_{11}^{ei} = -3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle \frac{I^2 T_i}{e^2 B_0^4} \left(\frac{\mu_{e1}\mu_{i1}}{\mu_{i1}} \right) = -1.533 \frac{f_t}{f_c} \frac{m_e n_e}{\tau_{ei}} \frac{I^2 T_i}{e^2 B^2} \tag{12.69}$$

$$L_{12}^{ei} = -3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle \frac{I^2 T_i}{e^2 B_0^4} \left(\frac{\mu_{e1}\mu_{i2}}{\mu_{i1}} \right) = -(1.533)(-1.174) \frac{f_t}{f_c} \frac{m_e n_e}{\tau_{ei}} \frac{I^2 T_i}{e^2 B^2} \tag{12.70}$$

$$L_{21}^{ei} = -3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle \frac{I^2 T_i}{e^2 B_0^4} \left(\frac{\mu_{e2}\mu_{i1}}{\mu_{i1}} \right) = (2.125) \frac{f_t}{f_c} \frac{m_e n_e}{\tau_{ei}} \frac{I^2 T_i}{e^2 B^2} \tag{12.71}$$

$$L_{22}^{ei} = -3\left\langle \left(\nabla_\parallel B \right)^2 \right\rangle \frac{I^2 T_i}{e^2 B_0^4} \left(\frac{\mu_{e2}\mu_{i2}}{\mu_{i1}} \right) = (2.125)(-1.174) \frac{f_t}{f_c} \frac{m_e n_e}{\tau_{ei}} \frac{I^2 T_i}{e^2 B^2} \tag{12.72}$$

Helander gets something different, but his result does not make sense to me. I would think he has a typo as his matrix would seem to make less sense than the one I just constructed.

Summarizing

$$L^{ee} = \frac{f_t}{f_c} \frac{m_e n_e}{\tau_{ei}} \frac{I^2 T_e}{e^2 B^2} \begin{bmatrix} -1.533 & 2.125 \\ 2.125 & -4.637 \end{bmatrix} \tag{12.73}$$

$$L^{ei} = \frac{f_t}{f_c} \frac{m_e n_e}{\tau_{ei}} \frac{I^2 T_i}{e^2 B^2} \begin{bmatrix} -1.533 & 1.800 \\ 2.125 & -2.495 \end{bmatrix} \tag{12.74}$$

12.8.3 Banana Regime Impurity Transport

Calculate the neoclassical transport of highly charged impurities in the banana regime. Express the flux in terms of the impurity strength parameter $\alpha = Z_{\text{eff}} - 1 = n_Z \frac{Z^2}{n_i}$.

Solution:

This is completely in analogy with electron-ion collisions, we have the ion-impurity collisions. Thus we write $\nu_D^i = \nu_D^{ii} + \nu_D^{iZ}$ and we get the same results, switching the charges and masses appropriately We use that for ions we have $x_{ii} = 1$

$$\left\{\nu_D^{ii}\right\}\tau_{ii} = 0.533 \tag{12.75}$$
$$\left\{\nu_D^{ii}x_i^2\right\}\tau_{ii} = 0.707 \tag{12.76}$$
$$\left\{\nu_D^{ii}x_i^4\right\}\tau_{ii} = 1.59 \tag{12.77}$$

while for impurities we have $x_{iZ} = v_{\text{th}_i}/v_{\text{th}_z} \ll 1 \approx 0$ and so

$$\left\{\nu_D^{iZ}\right\}\tau_{iZ} = 1 \tag{12.78}$$
$$\left\{\nu_D^{iZ}x_i^2\right\}\tau_{iZ} = 1 \tag{12.79}$$
$$\left\{\nu_D^{iZ}x_i^4\right\}\tau_{iZ} = 2 \tag{12.80}$$

We thus find

$$\mu_{i1} = \frac{m_i n_i \left\langle B^2\right\rangle}{3\left\langle\left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left\{\nu_D^{ii} + \nu_D^{iZ}\right\} \tag{12.81}$$

$$\mu_{i2} = \frac{m_i n_i \left\langle B^2\right\rangle}{3\left\langle\left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left\{\left(\nu_D^{ii} + \nu_D^{iZ}\right)\left(x^2 - \frac{5}{2}\right)^1\right\} \tag{12.82}$$

$$\mu_{i3} = \frac{m_i n_i \left\langle B^2\right\rangle}{3\left\langle\left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left\{\left(\nu_D^{ii} + \nu_D^{iZ}\right)\left(x^2 - \frac{5}{2}\right)^2\right\} \tag{12.83}$$

So that

$$\mu_{e1} = \frac{m_i n_i \left\langle B^2\right\rangle}{3\left\langle\left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left([0.533]\tau_{ii}^{-1} + \tau_{iZ}^{-1}\right) \tag{12.84}$$

$$\mu_{e2} = \frac{m_i n_i \left\langle B^2\right\rangle}{3\left\langle\left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left([-0.626]\,\tau_{ii}^{-1} + [-1.5]\tau_{iZ}^{-1}\right) \tag{12.85}$$

$$\mu_{e3} = \frac{m_i n_i \left\langle B^2\right\rangle}{3\left\langle\left(\nabla_\| B\right)^2\right\rangle}\frac{f_t}{f_c}\left([1.386]\,\tau_{ii}^{-1} + [3.25]\tau_{iZ}^{-1}\right) \tag{12.86}$$

With $\tau_{ii}/\tau_{iZ} = \alpha$ and using that $\mu_{Zk} \gg \mu_{ik}$ (because $\nu_D^{iZ} \gg \nu_D^{ii}$ [basically for the same reasons

$\mu_{ik} \gg \mu_{ek}$]).

$$\mu_{e1} = \frac{m_i n_i \left\langle B^2 \right\rangle}{3\left\langle \left(\nabla_\| B\right)^2 \right\rangle} \frac{f_t}{f_c}(0.533 + \alpha)\tau_{ii}^{-1} \tag{12.87}$$

$$\mu_{e2} = \frac{m_i n_i \left\langle B^2 \right\rangle}{3\left\langle \left(\nabla_\| B\right)^2 \right\rangle} \frac{f_t}{f_c}(-0.626 - 1.5\alpha)\tau_{ii}^{-1} \tag{12.88}$$

$$\mu_{e3} = \frac{m_i n_i \left\langle B^2 \right\rangle}{3\left\langle \left(\nabla_\| B\right)^2 \right\rangle} \frac{f_t}{f_c}(1.386 + 3.25\alpha)\tau_{ii}^{-1} \tag{12.89}$$

And so, we find

$$L_{11}^{ii} = 3\left\langle \left(\nabla_\| B\right)^2 \right\rangle \frac{I^2 T_i}{e^2 B_0^4}\left(\cancel{\frac{\mu_{i1}\mu_{i1}}{\mu_{Z1}}} - \mu_{i1}\right) = -(0.533 + \alpha)\frac{f_t}{f_c}\frac{m_i n_i}{\tau_{ii}}\frac{I^2 T_i}{e^2 B^2} \tag{12.90}$$

$$L_{12}^{ii} = 3\left\langle \left(\nabla_\| B\right)^2 \right\rangle \frac{I^2 T_i}{e^2 B_0^4}\left(\cancel{\frac{\mu_{i1}\mu_{i2}}{\mu_{Z1}}} - \mu_{i2}\right) = -(-0.626 - 1.5\alpha)\frac{f_t}{f_c}\frac{m_i n_i}{\tau_{ii}}\frac{I^2 T_i}{e^2 B^2} \tag{12.91}$$

$$L_{21}^{ii} = 3\left\langle \left(\nabla_\| B\right)^2 \right\rangle \frac{I^2 T_i}{e^2 B_0^4}\left(\cancel{\frac{\mu_{i2}\mu_{i1}}{\mu_{Z1}}} - \mu_{i2}\right) = -(-0.626 - 1.5\alpha)\frac{f_t}{f_c}\frac{m_i n_i}{\tau_{ii}}\frac{I^2 T_i}{e^2 B^2} \tag{12.92}$$

$$L_{22}^{ii} = 3\left\langle \left(\nabla_\| B\right)^2 \right\rangle \frac{I^2 T_i}{e^2 B_0^4}\left(\cancel{\frac{\mu_{i2}\mu_{i2}}{\mu_{Z1}}} - \mu_{i3}\right) = -(1.386 + 3.25\alpha)\frac{f_t}{f_c}\frac{m_i n_i}{\tau_{ii}}\frac{I^2 T_i}{e^2 B^2} \tag{12.93}$$

We could calculate the other coefficients for L^{iZ} but they are smaller because they have a Z in the denominator and so they can be ignored as a first approximation.

Thus, we can use that $Z\left\langle \mathbf{\Gamma}_Z \cdot \nabla\psi \right\rangle = -\left\langle \mathbf{\Gamma}_i \cdot \nabla\psi \right\rangle$ to find the impurity transport and find

$$\left\langle \mathbf{\Gamma}_Z \cdot \nabla\psi \right\rangle = -\frac{1}{Z}(L_{11}^{ii} A_1^i + L_{12} A_2^i) \tag{12.94}$$

$$= -\frac{1}{Z}\left(-(0.533 + \alpha)\frac{f_t}{f_c}\frac{m_i n_i}{\tau_{ii}}\frac{I^2 T_i}{e^2 B^2}\left\{\frac{\mathrm{d}\ln p_i}{\mathrm{d}\psi} + \frac{e}{T_i}\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}\right\} + (0.626 + 1.5\alpha)\frac{f_t}{f_c}\frac{m_i n_i}{\tau_{ii}}\frac{I^2 T_i}{e^2 B^2}\frac{\mathrm{d}\ln T_i}{\mathrm{d}\psi}\right) \tag{12.95}$$

$$= -\frac{1}{Z}\frac{f_t}{f_c}\frac{m_i n_i}{\tau_{ii}}\frac{I^2 T_i}{e^2 B^2}\left(-(0.533 + \alpha)\left\{\left[\frac{\mathrm{d}\ln n_i}{\mathrm{d}\psi} + \frac{\mathrm{d}\ln T_i}{\mathrm{d}\psi}\right] + \frac{e}{T_i}\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}\right\} + (0.626 - 1.5\alpha)\frac{\mathrm{d}\ln T_i}{\mathrm{d}\psi}\right) \tag{12.96}$$

$$= -\frac{f_t}{f_c}\frac{I^2 n_i T_i}{Z m_i \Omega_i^2 \tau_{ii}}\left((0.093 + 0.5\alpha)\frac{\mathrm{d}\ln T_i}{\mathrm{d}\psi} + (-0.533 - \alpha)\left\{\frac{\mathrm{d}\ln n_i}{\mathrm{d}\psi} + \frac{e}{T_i}\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}\right\}\right) \tag{12.97}$$

$$= \frac{f_t}{f_c}\frac{I^2 n_i T_i}{Z m_i \Omega_i^2 \tau_{ii}}\left(-(0.093 + 0.5\alpha)\frac{\mathrm{d}\ln T_i}{\mathrm{d}\psi} + (0.533 + \alpha)\left\{\frac{\mathrm{d}\ln n_i}{\mathrm{d}\psi} + \frac{e}{T_i}\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}\right\}\right) \tag{12.98}$$

The thing to note is that in this banana-regime that if the temperature gradient is large enough, then the impurity transport will be outward, overtaking the usual inward transport of impurities due to the density gradient.

Chapter 13

Advanced Topics

13.6 Chapter 13 Exercises

13.6.1 Poloidal Rotation of Ions from Constant Force

Calculate the poloidal rotation in a low collisionality plasma caused by a constant force acting on ions.

Solution:

We use (HS-12.45) and insert an appropriate force on the right hand side to find

$$\widehat{\mu}_{i1} u_{i\theta} + \widehat{\mu}_{i2} \frac{2q_{i\theta}}{p_i} = \frac{\left\langle BR_{i\|} \right\rangle}{\langle B^2 \rangle} \tag{13.1}$$

$$\widehat{\mu}_{i2} u_{i\theta} + \widehat{\mu}_{i3} \frac{2q_{i\theta}}{p_i} = l_{22}^{ii} \left(\frac{V_{2i} B}{\langle B^2 \rangle} + \frac{2q_{i\theta}}{5p_i} \right) \tag{13.2}$$

We solve for $2q_{i\theta}/(5p_i)$ in the second equation and sub this into the first

$$\frac{2q_{i\theta}}{p_i} \left(\widehat{\mu}_{i3} - l_{22}^{ii} \right) = l_{22}^{ii} \frac{V_{2i} B}{\langle B^2 \rangle} - \widehat{\mu}_{i2} u_{i\theta} \tag{13.3}$$

$$\widehat{\mu}_{i1} u_{i\theta} + \widehat{\mu}_{i2} \left[\frac{l_{22}^{ii} \frac{V_{2i} B}{\langle B^2 \rangle} - \widehat{\mu}_{i2} u_{i\theta}}{\widehat{\mu}_{i3} - l_{22}^{ii}} \right] = \frac{\left\langle BR_{i\|} \right\rangle}{\langle B^2 \rangle} \tag{13.4}$$

$$u_{i\theta} \left(\widehat{\mu}_{i1} - \frac{\widehat{\mu}_{i2}^2}{\widehat{\mu}_{i3} - l_{22}^{ii}} \right) = \frac{1}{\langle B^2 \rangle} \left[\left\langle BR_{i\|} \right\rangle - \frac{\widehat{\mu}_{i2} l_{22}^{ii} V_{2i} B}{\widehat{\mu}_{i3} - l_{22}^{ii}} \right] \tag{13.5}$$

$$u_{i\theta} = \frac{1}{\langle B^2 \rangle} \left[\left\langle BR_{i\|} \right\rangle - \frac{\widehat{\mu}_{i2} l_{22}^{ii} V_{2i} B}{\widehat{\mu}_{i3} - l_{22}^{ii}} \right] \Bigg/ \left(\widehat{\mu}_{i1} - \frac{\widehat{\mu}_{i2}^2}{\widehat{\mu}_{i3} - l_{22}^{ii}} \right) \tag{13.6}$$

We can simplify using

$$\widehat{\mu}_{i1} - \frac{\widehat{\mu}_{i2}^2}{\widehat{\mu}_{i3} - l_{22}^{ii}} = \frac{\widehat{\mu}_{i1} (\widehat{\mu}_{i3} - l_{22}^{ii}) - \widehat{\mu}_{i2}^2}{\widehat{\mu}_{i3} - l_{22}^{ii}} \tag{13.7}$$

and so

$$u_{i\theta} = \frac{1}{\langle B^2 \rangle} \left[\frac{(\widehat{\mu}_{i3} - l_{22}^{ii}) \left\langle BR_{i\|} \right\rangle}{\widehat{\mu}_{i1} (\widehat{\mu}_{i3} - l_{22}^{ii}) - \widehat{\mu}_{i2}^2} - \frac{\widehat{\mu}_{i3} - l_{22}^{ii}}{\widehat{\mu}_{i1}(\widehat{\mu}_{i3} - l_{22}^{ii}) - \widehat{\mu}_{i2}^2} \frac{\widehat{\mu}_{i2} l_{22}^{ii} V_{2i} B}{\widehat{\mu}_{i3} - l_{22}^{ii}} \right] \tag{13.8}$$

$$u_{i\theta} = \frac{1}{\langle B^2 \rangle} \left[\frac{(\widehat{\mu}_{i3} - l_{22}^{ii}) \left\langle BR_{i\|} \right\rangle - \widehat{\mu}_{i2} l_{22}^{ii} V_{2i} B}{\widehat{\mu}_{i1} (\widehat{\mu}_{i3} - l_{22}^{ii}) - \widehat{\mu}_{i2}^2} \right] \tag{13.9}$$

with $y = f_t/f_c$ and

$$\widehat{\mu}_{i1} = 0.533 \frac{y m_i n_i}{\tau_{ii}} \tag{13.10}$$

$$\widehat{\mu}_{i2} = -0.625 \frac{y m_i n_i}{\tau_{ii}} \tag{13.11}$$

$$\widehat{\mu}_{i3} = 1.387 \frac{y m_i n_i}{\tau_{ii}} \tag{13.12}$$

$$l_{22}^{ii} = -\sqrt{2} \frac{m_i n_i}{\tau_{ii}} \approx -1.414 \frac{m_i n_i}{\tau_{ii}} \tag{13.13}$$

we find

$$\widehat{\mu}_{i1} \left(\widehat{\mu}_{i3} - l_{22}^{ii}\right) - \widehat{\mu}_{i2}^2 \approx 0.533 \frac{y m_i^2 n_i^2}{\tau_{ii}^2} (1.387y + 1.414) - 0.391 \frac{y^2 m_i^2 n_i^2}{\tau_{ii}^2} \approx \frac{y m_i^2 n_i^2}{\tau_{ii}^2} (0.349y + 0.754) \tag{13.14}$$

$$\widehat{\mu}_{i3} - l_{22}^{ii} \approx \frac{m_i n_i}{\tau_{ii}} (1.387y + 1.414) \tag{13.15}$$

$$\mu_{i2} l_{22}^{ii} \approx \frac{y m_i^2 n_i^2}{\tau_{ii}^2} (-0.625)(-1.414) \approx 0.885 \frac{y m_i^2 n_i^2}{\tau_{ii}^2} \tag{13.16}$$

Thus,

$$u_{i\theta} = \frac{1}{\langle B^2 \rangle} \left[\frac{\frac{\tau_{ii}}{m_i n_i} (1.387y + 1.414) \left\langle BR_{i\|} \right\rangle - 0.885 y V_{2i} B}{y(0.349y + 0.754)} \right] \tag{13.17}$$

$$u_{i\theta} = \frac{1}{\langle B^2 \rangle} \left[\frac{1.88 \frac{\tau_{ii}}{m_i n_i} (0.98y + 1) \left\langle BR_{i\|} \right\rangle - 1.17 y V_{2i} B}{y(0.462y + 1)} \right] \tag{13.18}$$