1 General Stability of Simple Differential Equation

Find conditions on $f(x)$ such that the first order differential equation $\dot{x} = f(x)$ has a fixed point at $x = 0$ and

1.a Spectral Stability

has the property of spectral stability.

Solution:

The requirement of being a fixed point at $x = 0$ implies that $f(0) = 0$. Thus, we see that $x_e = 0$ is an equilibrium point. We can then linearize around this point and find for Δx that

$$\frac{\mathrm{d}\Delta x}{\mathrm{d}t} = f(\Delta x) \approx f'(0)\Delta x + \mathcal{O}(\Delta x^2) \tag{1}$$

$$\Delta x \sim e^{f'(0)t} \tag{2}$$

Thus, we see we have spectral stability if $f'(0) < 0$.

1.b Asymptotic Linear Stability

has the property of asymptotic linear stability.

Solution:

We follow the same procedure and see that we require $f'(0) < 0$ again.

1.c Lyapunov Stability

has the property of Lyapunov stability.

Solution:

This requires that we solve

$$\frac{\mathrm{d}\Delta x}{\mathrm{d}t} = f(\Delta x) \tag{3}$$

This means we'd need to solve

$$\int_{x_0}^{x} \frac{\mathrm{d}\Delta x}{f(\Delta x)} = t - t_0 \tag{4}$$

for Δx.

1.d Global Stability

has the property of global stability (all initial conditions approach $x = 0$ as $t \to \infty$).

Solution:

This is true if $f'(0) < 0$ as then we will approach equilibrium which will go to zero as $t \to \infty$.

2 Lyapunov Stability Two dimensional, incompressible fluid motion is governed by the equation

$$\frac{\partial U}{\partial t} + [\phi\,,\, U] = 0$$

where $[f\,,\, g] = \hat{\mathbf{z}} \cdot \nabla f \times \nabla g$, and $U = \nabla^2 \phi$ is the vorticity, and $\mathbf{V} = \hat{\mathbf{z}} \times \nabla \phi$ is the velocity. Consider a fluid in a bounded disk, with the condition $\phi =$ constant on the boundary.

2.a Conserved Quantities Show that $H = \frac{1}{2}\int \mathrm{d}^2x\ |\nabla\phi|^2$ and $G = \int \mathrm{d}^2x\ g(U)$ for any function $g(U)$ are conserved quantities.

Solution:

First let's note that

$$\frac{\partial U}{\partial t} + [\phi\,,\, U] = \frac{\partial U}{\partial t} + \hat{\mathbf{z}} \cdot \nabla\phi \times \nabla U = \frac{\partial U}{\partial t} + \nabla U \cdot (\hat{\mathbf{z}} \times \nabla\phi) = \frac{\partial U}{\partial t} + \mathbf{V} \cdot \nabla U = \frac{\mathrm{d}U}{\mathrm{d}t} \tag{5}$$

Take $\int \mathrm{d}^2x\ \phi$ of the equation and we see

$$\int \mathrm{d}^2x\ \phi \frac{\partial\, \nabla^2\phi}{\partial t} + \int \mathrm{d}^2x\ \phi \hat{\mathbf{z}} \cdot \nabla\phi \times \nabla\nabla^2\phi = 0 \tag{6}$$

We use

$$\begin{aligned} \phi\hat{\mathbf{z}} \cdot \nabla\phi \times \nabla\nabla^2\phi = \phi z_i \epsilon_{ijk} (\partial_j\phi)\partial_k\partial_m\partial_m\phi &= \phi z_i \epsilon_{ijk} \{\partial_k\left([\partial_j\phi]\partial_m\partial_m\phi\right) - \phi(\partial_m\partial_m\phi)\cancel{\partial_k\partial_j\phi}\} \\ &= \phi z_i \epsilon_{ijk} \partial_k\left([\partial_j\phi]\partial_m\partial_m\phi\right) = -\phi\hat{\mathbf{z}} \cdot \boldsymbol{\nabla} \times (\nabla\phi\, \nabla^2\phi) \end{aligned} \tag{7}$$

Note also, that because of the constancy of $\hat{\mathbf{z}}$ this is

$$\begin{aligned} &= \phi\partial_k\left(\epsilon_{ijk} z_i [\partial_j\phi]\partial_m\partial_m\phi\right) = \partial_k\left(\phi\epsilon_{kij} z_i [\partial_j\phi]\partial_m\partial_m\phi\right) - (\partial_k\phi)\epsilon_{kij} z_i \partial_m\partial_m\phi[\partial_j\phi]\phi \\ &= \boldsymbol{\nabla} \cdot \left(\phi\, \nabla^2\phi\hat{\mathbf{z}} \times \nabla\phi\right) - \nabla\phi \cdot \nabla^2\phi\hat{\mathbf{z}} \times \nabla\phi \end{aligned} \tag{8}$$

Thus

$$\begin{aligned} \int \mathrm{d}^2x\ \hat{\mathbf{z}} \cdot \nabla\phi \times \nabla\nabla^2\phi &= \int \mathrm{d}^2x\ \phi\boldsymbol{\nabla} \cdot (\nabla^2\phi\hat{\mathbf{z}} \times \nabla\phi) \\ &= \cancel{\int \mathrm{d}^2x\ \boldsymbol{\nabla} \cdot (\phi\, \nabla^2\phi\hat{\mathbf{z}} \times \nabla\phi)} - \int \mathrm{d}^2x\ \nabla\phi \cdot \nabla^2\phi\hat{\mathbf{z}} \times \nabla\phi = 0 \end{aligned} \tag{9}$$

because of our boundary conditions (i.e., that $\nabla\phi = 0$ along the boundary because ϕ is constant on the boundary).

We also have

$$\phi\frac{\partial}{\partial t}\nabla^2\phi = \phi\frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \nabla\phi = \phi\boldsymbol{\nabla} \cdot \frac{\partial}{\partial t}\nabla\phi = \cancel{\boldsymbol{\nabla} \cdot (\phi\frac{\partial}{\partial t}\nabla\phi)} - \nabla\phi \cdot \frac{\partial}{\partial t}\nabla\phi = -\frac{1}{2}\frac{\partial}{\partial t}|\nabla\phi|^2 \tag{10}$$

where the cancellation is again because of the boundary condition. Thus we find

$$-\int \mathrm{d}^2x\ \frac{1}{2}\frac{\partial}{\partial t}|\nabla\phi|^2 = -\frac{\partial}{\partial t}\int \mathrm{d}^2x\ \frac{1}{2}|\nabla\phi|^2 = -\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \tag{11}$$

and so H is conserved.

For G we use [define $F = \frac{\partial g}{\partial U}$ and so $F = F(U(\mathbf{x}, t))$]

$$\frac{\partial g(U)}{\partial t} = \frac{\partial g}{\partial U}\frac{\partial U}{\partial t} = -\frac{\partial g}{\partial U}\hat{\mathbf{z}} \cdot \nabla\phi \times \nabla U \tag{12}$$

$$\nabla F = \frac{\partial F}{\partial \mathbf{x}} = \frac{\partial F}{\partial U}\frac{\partial U}{\partial \mathbf{x}} = \frac{\partial F}{\partial U}\nabla U \tag{13}$$

so that

$$\begin{aligned} F\hat{\mathbf{z}} \cdot \nabla\phi \times \nabla U &= \hat{\mathbf{z}} \cdot \nabla(\phi F) \times \nabla U - \phi\hat{\mathbf{z}} \cdot \nabla F \times \nabla U = \hat{\mathbf{z}} \cdot \nabla(\phi F) \times \nabla U - \cancel{\phi\frac{\partial F}{\partial U}\hat{\mathbf{z}} \cdot \nabla U \times \nabla U} \\ &= \hat{\mathbf{z}} \cdot \boldsymbol{\nabla} \times (\nabla(\phi F)U) = \boldsymbol{\nabla} \cdot (U\hat{\mathbf{z}} \times \nabla(\phi F)) \end{aligned} \tag{14}$$

Thus, we find

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^2x\ g(U) = \int \mathrm{d}^2x\ \frac{\partial g(U)}{\partial t} = -\int \mathrm{d}^2x\ F\hat{\mathbf{z}} \cdot \nabla\phi \times \nabla U = -\int \mathrm{d}^2x\ \boldsymbol{\nabla} \cdot (U\hat{\mathbf{z}} \times \nabla(\phi F)) = 0 \tag{15}$$

via boundary conditions on ϕ.

2.b Functional Determines Equilibrium Solution Consider the functional $F[\phi] = H + G$. Show that $\delta F = 0$ determines an equilibrium solution.

Solution:

We take

$$\delta F = \delta H + \delta G \tag{16}$$

Then clearly $\delta F = 0$ implies $\delta H = -\delta G$. We have

$$\delta H = \delta \int \mathrm{d}^2 x \; \frac{1}{2} |\nabla \phi|^2 = \int \mathrm{d}^2 x \; [\nabla \delta\phi \cdot \nabla \phi] \tag{17}$$

$$\delta G = \delta \int \mathrm{d}^2 x \; g(U) = \int \mathrm{d}^2 x \; \left[\frac{\partial g}{\partial U} \delta U\right] = \int \mathrm{d}^2 x \; \underbrace{J}_{\frac{\partial g}{\partial U}} \nabla^2 \delta\phi = \int \mathrm{d}^2 x \; [\boldsymbol{\nabla} \cdot (J \delta\phi) - \nabla J \cdot \nabla \delta\phi] \tag{18}$$

We note that $\boldsymbol{\nabla} \cdot (J\delta\phi) = 0$ when integrated along the boundary because $\delta\phi = 0$ at the boundary because of our imposed boundary conditions.

Thus

$$\delta H + \delta G = \int \mathrm{d}^2 x \; \nabla \delta\phi \cdot [\nabla \phi - \nabla J] \tag{19}$$

Thus, we need to check if $\nabla\phi = \nabla J$ is a possible solution. Take $\phi = J + C$ for some constant C. Then

$$U = \nabla^2 \phi = \nabla^2 J \tag{20}$$

$$[\phi \, , U] = [J + C \, , U] = \hat{\mathbf{z}} \cdot (\nabla J \times \nabla U) = \hat{\mathbf{z}} \cdot \frac{\partial J}{\partial U} \nabla U \times \nabla U = 0 \tag{21}$$

Therefore, we must have an equilibrium solution as $\frac{\partial \nabla^2 J}{\partial t} = 0$.

2.c Functional and Lyapunov Stability Show that $\delta^2 F$ is positive definite if $g'' > 0$, and therefore that the flow is (formally) Lyapunov stable when the velocity profile has no inflection points. Complete Lyapunov stability follows upon demonstrating (7.9), see (Arnol'd,1965)

$$\delta^2 F\,[\xi] \geq C \|\xi\|^2 \text{ for any } \xi \tag{7.9}$$

Solution:

We need to calculate $\delta^2 F$. This is given by

$$\delta^2 F = \int \mathrm{d}^2 x \; \frac{1}{2} |\nabla \delta\phi|^2 + \int \mathrm{d}^2 x \; \frac{\partial^2 g}{\partial U^2} (\delta U)^2 = \int \mathrm{d}^2 x \; \left[\frac{|\nabla \delta\phi|^2}{2} + (\nabla^2 \delta\phi)^2 \frac{\partial^2 g}{\partial U^2}\right] \tag{22}$$

It is then obvious that if $g'' \equiv \frac{\partial^2 g}{\partial U^2} > 0$ that this is necessarily a positive number.

3 Derivation of Cylindrical Reduced MHD Eigenvalue Equation Derive (7.73), verifying (7.71) and (7.72).

$$k_\parallel \nabla_\perp^2(k_\parallel\varphi) + \epsilon k_\parallel k_\perp J_0'\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[k_\parallel^2 r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right] + \frac{k_\parallel^2}{r^2}(1-m^2)\varphi \tag{7.71}$$

$$U = \nabla_\perp^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right] + \frac{1}{r^2}(1-m^2)\varphi \tag{7.72}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{[\omega^2-(k_\parallel v_A)^2]r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right\} + [\omega^2-(k_\parallel v_A)^2]\frac{(1-m^2)}{r^2}\varphi - (k_\perp v_A)^2\beta'\kappa_r\varphi = 0 \tag{7.73}$$

Solution:

We begin with the equation

$$\frac{\omega^2}{v_A^2}\nabla_\perp^2\varphi = k_\parallel \nabla_\perp^2 k_\parallel\varphi + \epsilon k_\parallel k_\perp J_0'\varphi + k_\perp^2\kappa_r\beta'\varphi \tag{23}$$

First, let's show (7.71).

$$k_\parallel \nabla_\perp^2(k_\parallel\varphi) + \epsilon k_\parallel k_\perp J_0'\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[k_\parallel^2 r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right] + \frac{k_\parallel^2}{r^2}(1-m^2)\varphi \tag{24}$$

Let's expand the left hand side first

$$\nabla_\perp^2(k_\parallel\varphi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(k_\parallel\varphi)}{\partial r}\right) - k_\perp^2 k_\parallel\varphi \tag{25}$$

We also use $-r/(aq) = \psi_0'$ and

$$J_0' = \frac{\mathrm{d}}{\mathrm{d}r}\nabla_\perp^2\psi_0 = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\psi_0'\right]\right] = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{-r}{aq}\right]\right] = -\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{aq}\right]\right] \tag{26}$$

Note there is no $-k_\perp^2\psi_0$ because $\nabla_\perp f = (\hat{\mathbf{r}}\frac{\partial}{\partial r} + i\hat{\boldsymbol{\theta}}k_\perp)\widetilde{f}$, but in fact, we see that there would also be a term $r\frac{\partial f_0}{\partial r}$ included for the equilibrium current.

We use ($n = 0$ for equilibrium current)

$$\frac{R_0 k_\parallel}{m} = \frac{1}{q} \tag{27}$$

so

$$\frac{r^2}{aq} = \frac{r^2 R_0 k_\parallel}{am} = \frac{1}{m\epsilon}r^2 k_\parallel \tag{28}$$

Thus,

$$J_0' = -\frac{1}{m\epsilon}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 k_\parallel\right)\right] \tag{29}$$

Thus,

$$k_\parallel \nabla_\perp^2(k_\parallel\varphi) + \epsilon k_\parallel k_\perp J_0'\varphi = \frac{k_\parallel}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(k_\parallel\varphi)}{\partial r}\right) - k_\perp^2 k_\parallel^2\varphi + \epsilon k_\parallel k_\perp \varphi \frac{-1}{m\epsilon}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\partial(r^2 k_\parallel)}{\partial r}\right] \tag{30}$$

$$= k_\parallel \frac{\partial^2(k_\parallel\varphi)}{\partial r^2} + \frac{k_\parallel}{r}\frac{\partial(k_\parallel\varphi)}{\partial r} - k_\perp^2 k_\parallel^2\varphi - \frac{k_\parallel k_\perp\varphi}{m}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\partial k_\parallel}{\partial r} + 2k_\parallel\right] \tag{31}$$

$$= k_\parallel^2\frac{\partial^2\varphi}{\partial r^2} + 2k_\parallel\frac{\partial k_\parallel}{\partial r}\frac{\partial\varphi}{\partial r} + k_\parallel\varphi\frac{\partial^2 k_\parallel}{\partial r^2} + \frac{k_\parallel^2}{r}\frac{\partial\varphi}{\partial r} + \frac{k_\parallel\varphi}{r}\frac{\partial k_\parallel}{\partial r} - k_\perp^2k_\parallel^2\varphi - \frac{k_\parallel k_\perp\varphi}{m}\left[\frac{\partial k_\parallel}{\partial r} + r\frac{\partial^2 k_\parallel}{\partial r^2} + 2\frac{\partial k_\parallel}{\partial r}\right] \tag{32}$$

$$= k_\parallel^2\frac{\partial^2\varphi}{\partial r^2} + 2k_\parallel\frac{\partial k_\parallel}{\partial r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2 k_\parallel}{\partial r^2}\left[k_\parallel\varphi - \frac{rk_\parallel k_\perp\varphi}{m}\right] + \frac{k_\parallel^2}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial k_\parallel}{\partial r}\left[\frac{k_\parallel\varphi}{r} - \frac{3k_\parallel k_\perp\varphi}{m}\right] - k_\perp^2k_\parallel^2\varphi \tag{33}$$

$$= k_\parallel^2\frac{\partial^2\varphi}{\partial r^2} + 2k_\parallel\frac{\partial k_\parallel}{\partial r}\frac{\partial\varphi}{\partial r} + k_\parallel\varphi\frac{\partial^2 k_\parallel}{\partial r^2}\left[1 - \frac{rk_\perp}{m}\right] + \frac{k_\parallel^2}{r}\frac{\partial\varphi}{\partial r} + k_\parallel\varphi\frac{\partial k_\parallel}{\partial r}\left[\frac{1}{r} - \frac{3k_\perp}{m}\right] - k_\perp^2k_\parallel^2\varphi \tag{34}$$

We can use $k_\perp = m/r$ so that

$$= k_\parallel^2\frac{\partial^2\varphi}{\partial r^2} + 2k_\parallel\frac{\partial k_\parallel}{\partial r}\frac{\partial\varphi}{\partial r} + \cancel{k_\parallel\varphi\frac{\partial^2 k_\parallel}{\partial r^2}\left[1 - \frac{rm}{rm}\right]} + \frac{k_\parallel^2}{r}\frac{\partial\varphi}{\partial r} + k_\parallel\varphi\frac{\partial k_\parallel}{\partial r}\left[\frac{1}{r} - \frac{3m}{mr}\right] - \frac{m^2k_\parallel^2}{r^2}\varphi \tag{35}$$

$$= k_\parallel^2\frac{\partial^2\varphi}{\partial r^2} + 2k_\parallel\frac{\partial k_\parallel}{\partial r}\frac{\partial\varphi}{\partial r} + \frac{k_\parallel^2}{r}\frac{\partial\varphi}{\partial r} - \frac{2k_\parallel\varphi}{r}\frac{\partial k_\parallel}{\partial r} - \frac{m^2k_\parallel^2}{r^2}\varphi \tag{36}$$

Now for the right hand side,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[k_\parallel^2 r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right] + \frac{k_\parallel^2}{r^2}(1-m^2)\varphi \tag{37}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[k_\parallel^2 r^3\left[\frac{1}{r}\frac{\partial\varphi}{\partial r} - \frac{\varphi}{r^2}\right]\right] + \frac{k_\parallel^2}{r^2}(1-m^2)\varphi \tag{38}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[k_\parallel^2 r^2\frac{\partial\varphi}{\partial r} - rk_\parallel\varphi\right] + \frac{k_\parallel^2}{r^2}(1-m^2)\varphi \tag{39}$$

$$= k_\parallel^2\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial\varphi}{\partial r}\left[2k_\parallel r^2\frac{\partial k_\parallel}{\partial r} + 2rk_\parallel^2\right] - k_\parallel^2\varphi - 2k_\parallel r\varphi\frac{\partial k_\parallel}{\partial r} - rk_\parallel^2\frac{\partial\varphi}{\partial r}\right) + \frac{k_\parallel^2}{r^2}(1-m^2)\varphi \tag{40}$$

$$= k_\parallel^2\frac{\partial^2\varphi}{\partial r^2} + 2k_\parallel\frac{\partial\varphi}{\partial r}\frac{\partial k_\parallel}{\partial r} + \frac{\cancel{2}k_\parallel^2}{r}\frac{\partial\varphi}{\partial r} - \cancel{\frac{k_\parallel^2\varphi}{r^2}} - \frac{2k_\parallel\varphi}{r}\frac{\partial k_\parallel}{\partial r} - \cancel{\frac{k_\parallel^2}{r}\frac{\partial\varphi}{\partial r}} + \frac{k_\parallel^2}{r^2}(\cancel{1} - m^2)\varphi \tag{41}$$

$$= k_\parallel^2\frac{\partial^2\varphi}{\partial r^2} + 2k_\parallel\frac{\partial\varphi}{\partial r}\frac{\partial k_\parallel}{\partial r} + \frac{k_\parallel^2}{r}\frac{\partial\varphi}{\partial r} - \frac{2k_\parallel\varphi}{r}\frac{\partial k_\parallel}{\partial r} - \frac{m^2k_\parallel^2}{r^2}\varphi \tag{42}$$

Thus we have proven (7.71). We then take this and put it into our original equation and so find on the left hand side

$$\nabla_\perp^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\varphi}{\partial r}\right] - k_\perp^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} - \frac{m^2}{r^2}\varphi \tag{43}$$

We then note

$$\begin{aligned}
&\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right] + \frac{1}{r^2}\left(1-m^2\right)\varphi && (44)\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\varphi}{\partial r} - r\varphi\right] + \frac{1}{r^2}\left(1-m^2\right)\varphi && (45)\\
&= \frac{1}{r^2}\left(2r\frac{\partial\varphi}{\partial r} + r^2\frac{\partial^2\varphi}{\partial r^2} - \varphi - r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\left(1-m^2\right)\varphi && (46)\\
&= \frac{\cancel{2}}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial r^2} - \cancel{\frac{\varphi}{r^2}} - \cancel{\frac{1}{r}\frac{\partial\varphi}{\partial r}} + \frac{1}{r^2}\left(\cancel{1}-m^2\right)\varphi && (47)\\
&= \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} - \frac{m^2}{r^2}\varphi && (48)
\end{aligned}$$

So that we can write $\nabla_\perp^2\varphi$ in that manner if we so wish. Then we have

$$\frac{\omega^2}{v_A^2}\nabla_\perp^2\varphi = k_\parallel\nabla_\perp^2(k_\parallel\varphi) + \epsilon k_\parallel k_\perp J_0'\varphi + k_\perp^2\kappa_r\beta'\varphi \tag{49}$$

$$\frac{\omega^2}{v_A^2}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right] + \frac{1}{r^2}\left(1-m^2\right)\varphi\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[k_\parallel^2 r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right] + \frac{k_\parallel^2}{r^2}(1-m^2)\varphi + \frac{m^2}{r^2}\kappa_r\beta'\varphi \tag{50}$$

$$\omega^2\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right] + \frac{1}{r^2}\left(1-m^2\right)\varphi\right) - \frac{v_A^2}{r^2}\frac{\partial}{\partial r}\left[k_\parallel^2 r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right] - \frac{k_\parallel^2 v_A^2}{r^2}(1-m^2)\varphi - (k_\perp v_A)^2\kappa_r\beta'\varphi = 0 \tag{51}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[(\omega^2 - v_A^2k_\parallel^2)r^3\frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\right] + \frac{1}{r^2}(\omega^2 - v_A^2k_\parallel^2)\left(1-m^2\right)\varphi - (k_\perp v_A)^2\kappa_r\beta'\varphi = 0 \tag{52}$$

Which is a full derivation of the required equation.

4 Differential Equation Consider the equation

$$x^2\xi'' + 2x\xi' + (\omega^2 x^2 - 2)\xi = 0$$

with the boundary conditions $\lim_{x\to 0} x\xi(x) = \xi(1) = 0$.

4.a Sturmian Equation Show this is a Sturmian equation.

Solution:

A Sturmian equation is of the form

$$[f(x;\lambda)\xi']' - g(x;\lambda)\xi = 0 \tag{53}$$

We require

$$f(x;\lambda)\xi''(x) + f'(x;\lambda)\xi'(x) - g(x;\lambda)\xi = 0 \tag{54}$$

Thus choosing $f(x;\lambda) = x^2$ and $g(x;\lambda) = 2 - \omega^2 x^2$ we see that the above equation yields

$$x^2\xi'' + 2x\xi' + (\omega^2 x^2 - 2)\xi = 0 \tag{55}$$

and so is clearly Sturmian.

4.b Variational Principle Obtain the variational principle and show that there are no solutions for $\omega^2 < 0$.

Solution:

It is known for a Sturmian problem that the functional to minimize through variation is

$$F[\xi] = \int \mathrm{d}x \ \left[f(\xi')^2 + g\xi^2\right] \tag{56}$$

as

$$\delta F[\xi] = \int \mathrm{d}x \ [2f\xi'\delta\xi' + 2g\xi\delta\xi] \tag{57}$$

$$= \int \mathrm{d}x \ [\cancel{(2f\xi'\delta\xi)'} - (2f\xi')'\delta\xi + 2g\xi\delta\xi] \tag{58}$$

$$= \int \mathrm{d}x \ \delta\xi \ [(2f\xi')' + 2g\xi] \tag{59}$$

Which if we minimize is equivalent to $(f\xi')' + g\xi = 0$, as desired.

We can use

$$\Lambda = \frac{\int \mathrm{d}x \ [x^2(\xi')^2 + (2 - \omega^2 x^2)\xi^2]}{\int \mathrm{d}x \ \xi^2} \tag{60}$$

to find a possible eigenvalue.

I think we need more information before we can prove there are no solutions for $\omega^2 < 0$. That is, Hazeltine and Meiss should actually have some example of what they're even looking for in the book.

4.c Spherical Bessel Equation and Eigenvalues Convert the equation to the spherical Bessel equation and obtain the eigenvalues.

Solution:

The spherical bessel function differential equation is

$$x^2\xi'' + 2x\xi' + (x^2 - n[n+1])\xi = 0 \tag{61}$$

If we define $\omega x \equiv y$ then

$$x^2\omega^2\xi''(y) + 2x\omega\xi'(y) + (y^2 - 2)\xi(y) = 0 \tag{62}$$
$$y^2\xi''(y) + 2y\xi'(y) + (y^2 - 2)\xi(y) = 0 \tag{63}$$

Then we see that our equation is the spherical Bessel functions with $n = 1$.

The eigenvalues will be obtained by noting that the boundary conditions require us to choose the spherical Bessel function of the first kind and $\xi(1) = 0$ requires us to $\xi(1) = j_1(\omega) = 0$ so the ω are the roots of j_1.

5 Curvature of Helical Field Line in a Cylindrical Plasma Show that the curvature of a helical field line in a cylindrical plasma is given by $\kappa_r = -r/(Rq)^2$. Combining this with (7.98) gives a familiar form of the Suydam criterion.

$$q^2\beta'\left(\frac{Rq}{r_s}\right)^2 < \frac{q'^2}{4} \text{ for stability} \tag{7.98}$$

Solution:

We need to find $\kappa_r = \hat{\mathbf{r}} \cdot \boldsymbol{\kappa} = \hat{\mathbf{r}} \cdot \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}$.

We use in cylindrical coordinates that

$$(\mathbf{A} \cdot \nabla\mathbf{B})_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta} + A_z\frac{\partial B_r}{\partial z} - \frac{A_\theta B_\theta}{r} \tag{64}$$

In our case

$$\mathbf{B} = B_z(r)\hat{\mathbf{z}} + B_\theta(r)\hat{\boldsymbol{\theta}} \tag{65}$$

$$\hat{\mathbf{b}} = \frac{B_z}{\sqrt{B_z^2 + B_\theta^2}}\hat{\mathbf{z}} + \frac{B_\theta}{\sqrt{B_z^2 + B_\theta^2}}\hat{\boldsymbol{\theta}} = b_z\hat{\mathbf{z}} + b_\theta\hat{\boldsymbol{\theta}} \tag{66}$$

So that

$$(\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}})_r = -\frac{b_\theta b_\theta}{r} = -\frac{1}{r}\frac{B_\theta^2}{B_\theta^2 + B_z^2} = \frac{-1}{r}\frac{1}{1 + \frac{B_z^2}{B_\theta^2}} \tag{67}$$

We have

$$q = \frac{r}{R}\frac{B_z}{B_\theta} \tag{68}$$

$$\frac{Rq}{r} = \frac{B_z}{B_\theta} \tag{69}$$

Thus,

$$\kappa_r = \frac{-1}{r}\frac{1}{1 + \frac{R^2q^2}{r^2}} = \frac{-r}{r^2 + R^2q^2} \tag{70}$$

Note if we use a tokamak ordering so $q \sim \mathcal{O}(1)$ and $\frac{r}{R} \sim \mathcal{O}(\epsilon)$ we find

$$\kappa_r = \frac{-r}{R^2q^2}\frac{1}{\frac{r^2}{R^2q^2} + 1} = \frac{-r}{R^2q^2}\left(1 + \mathcal{O}(\epsilon^2)\right) \approx \frac{-r}{R^2q^2} \tag{71}$$

which is the desired answer.

6 Verify Energy Principle Verify (7.124), keeping track of the partial integrations so as to obtain the boundary terms.

$$\delta^2 W[\xi] = \frac{1}{2}\int \mathrm{d}^3x \, \left[\frac{Q^2}{4\pi} - \frac{1}{c}\boldsymbol{\xi}\cdot\mathbf{J}\times\mathbf{Q} + \frac{5}{3}P(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 + (\boldsymbol{\xi}\cdot\nabla P)(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})\right] + \text{B.T.} \tag{7.124}$$

Solution:

We begin with the form

$$\delta^2 W[\xi] = \frac{1}{2}\int \mathrm{d}^3x \, \left[\left(P + \frac{B^2}{8\pi}\right)\left[\mathrm{Tr}\,[\nabla\boldsymbol{\xi}]^2 - (\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2\right] + \frac{5}{3}P\,(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 + \frac{1}{4\pi}\left\{\mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right\}^2\right] \tag{72}$$

Remember that $\mathbf{Q} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B})$. Let's first deal with $\mathrm{Tr}[\nabla\boldsymbol{\xi}]^2 = \nabla\boldsymbol{\xi} : \nabla\boldsymbol{\xi}$ (a strange notation...). Define $P_T = P + B^2/(4\pi)$ for convenience for now.

$$\int \mathrm{d}^3x \; P_T\partial_i\xi^j\partial_j\xi^i = \int \mathrm{d}^3x \; \partial_i(P_T\xi^j)\partial_j\xi^i - \xi^j\partial_i(P_T)\partial_j\xi^i \tag{73}$$

The first term

$$\partial_i(P_T\xi^j)\partial_j\xi^i = \partial_i(P_T\xi^j\partial_j\xi^i) - P_T\xi^j\partial_i\partial_j\xi^i = \text{B.T.} - P_T\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) \tag{74}$$

and the second term, using $\nabla P_T = \mathbf{B}\cdot\nabla\mathbf{B}$ so that

$$\xi^j\partial_i P_T\partial_j\xi^i = (\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi})\cdot(\mathbf{B}\cdot\nabla\mathbf{B}) \tag{75}$$

and altogether

$$P_T\,\nabla\boldsymbol{\xi} : \nabla\boldsymbol{\xi} \to \text{B.T.} - P_T\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\nabla\xi)\cdot(\mathbf{B}\cdot\nabla B) \tag{76}$$

Now for the other non-trace term,

$$\int \mathrm{d}^3x \; P_T\partial_i\xi^i\partial_j\xi^j = \int \mathrm{d}^3x \; \partial_i(P_T\xi^i)\partial_j\xi^j - \xi^i\partial_i(P_T)\partial_j\xi^j \tag{77}$$

$$\partial_i(P_T\xi^i)\partial_j\xi^j = \partial_i(P_T\xi^i\partial_j\xi^j) - P_T\xi^i\partial_i\partial_j\xi^j = \text{B.T.} - P_T\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) \tag{78}$$

and the second term, using $\nabla P_T = \mathbf{B}\cdot\nabla\mathbf{B}$ so that

$$\xi^i\partial_i P_T\partial_j\xi^j = (\boldsymbol{\xi}\cdot\mathbf{B}\cdot\nabla\mathbf{B})\boldsymbol{\nabla}\cdot\boldsymbol{\xi} \tag{79}$$

and altogether

$$P_T(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 \to \text{B.T.} - P_T\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\mathbf{B}\cdot\nabla\mathbf{B})\boldsymbol{\nabla}\cdot\boldsymbol{\xi} \tag{80}$$

We can also note that

$$
\begin{aligned}
Q^2 &= [(\mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})) - \boldsymbol{\xi}\cdot\nabla\mathbf{B}]\cdot[(\mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})) - \boldsymbol{\xi}\cdot\nabla\mathbf{B}] && (81)\\
&= (\mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}))^2 - 2(\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})) + (\boldsymbol{\xi}\cdot\nabla\mathbf{B})^2 && (82)
\end{aligned}
$$

We can also use

$$
\begin{aligned}
\frac{4\pi\mathbf{J}}{c}\times\mathbf{Q} &= (\boldsymbol{\nabla}\times\mathbf{B})\times(\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B})) = \epsilon_{ijk}\epsilon_{jlm}\partial_l(B_m)\epsilon_{knp}\partial_n(\epsilon_{prs}\xi_r B_s) && (83)\\
&= [\epsilon_{jki}\epsilon_{jlm}]\,[\epsilon_{knp}\epsilon_{rsp}]\,\partial_l(B_m)\partial_n(\xi_r B_s) = [\delta_{kl}\delta_{im} - \delta_{km}\delta_{li}]\,[\delta_{kr}\delta_{ns} - \delta_{ks}\delta_{rn}]\,\partial_l(B_m)\partial_n(\xi_r B_s) && (84)\\
&= [\delta_{kl}\delta_{im}\delta_{kr}\delta_{ns} - \delta_{kl}\delta_{im}\delta_{ks}\delta_{rn} - \delta_{km}\delta_{li}\delta_{kr}\delta_{ns} + \delta_{km}\delta_{li}\delta_{ks}\delta_{rn}]\,\partial_l(B_m)\partial_n(\xi_r B_s) && (85)\\
&= [\delta_{lr}\delta_{im}\delta_{ns} - \delta_{ls}\delta_{im}\delta_{rn} - \delta_{rm}\delta_{li}\delta_{ns} + \delta_{ms}\delta_{li}\delta_{rn}]\,\partial_l(B_m)\partial_n(\xi_r B_s) && (86)\\
&= \partial_r(B_i)\partial_s(\xi_r B_s) - \partial_s(B_i)\partial_r(\xi_r B_s) - \partial_i(B_r)\partial_s(\xi_r B_s) + \partial_i(B_s)\partial_r(\xi_r B_s) && (87)\\
&= (\mathbf{B}\cdot\nabla\boldsymbol{\xi})\cdot\nabla\mathbf{B} - [\boldsymbol{\nabla}\cdot(\boldsymbol{\xi}\mathbf{B})]\cdot\nabla\mathbf{B} - \nabla\mathbf{B}\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\xi}) + \nabla\mathbf{B}\cdot[\boldsymbol{\nabla}\cdot(\boldsymbol{\xi}\mathbf{B})] && (88)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\boldsymbol{\xi}\cdot\frac{4\pi\mathbf{J}}{c}\times\mathbf{Q} && (89)\\
&= (\mathbf{B}\cdot\nabla\boldsymbol{\xi})\cdot(\nabla\mathbf{B}\cdot\boldsymbol{\xi}) - [\boldsymbol{\nabla}\cdot(\boldsymbol{\xi}\mathbf{B})]\cdot(\nabla\mathbf{B}\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\xi}) + (\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot[\boldsymbol{\nabla}\cdot(\boldsymbol{\xi}\mathbf{B})] && (90)
\end{aligned}
$$

Note that

$$
\begin{aligned}
(\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot[\boldsymbol{\nabla}\cdot(\boldsymbol{\xi}\mathbf{B})] &= (\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot[\mathbf{B}\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla\mathbf{B}] && (91)\\
&= (\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot[\mathbf{B}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}] + (\boldsymbol{\xi}\cdot\nabla\mathbf{B})^2 && (92)
\end{aligned}
$$

We can then note that $Q^2 - \frac{4\pi}{c}\boldsymbol{\xi}\cdot\mathbf{J}\times\mathbf{Q}$ is the same as

$$
\begin{aligned}
&= (\mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}))^2 - \cancel{2}(\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})) + \cancel{(\boldsymbol{\xi}\cdot\nabla\mathbf{B})^2}\\
&\quad - (\mathbf{B}\cdot\nabla\boldsymbol{\xi})\cdot(\nabla\mathbf{B}\cdot\boldsymbol{\xi}) + [\boldsymbol{\nabla}\cdot(\boldsymbol{\xi}\mathbf{B})]\cdot(\nabla\mathbf{B}\cdot\boldsymbol{\xi}) && (93)\\
&\quad + \cancel{(\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\xi})} - \cancel{(\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot[\mathbf{B}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}]} - \cancel{(\boldsymbol{\xi}\cdot\nabla\mathbf{B})^2}\\
&= (\mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}))^2 - (\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}))\\
&\quad - (\mathbf{B}\cdot\nabla\boldsymbol{\xi})\cdot(\nabla\mathbf{B}\cdot\boldsymbol{\xi}) + (\boldsymbol{\nabla}\cdot\boldsymbol{\xi})\mathbf{B}\cdot(\nabla\mathbf{B}\cdot\boldsymbol{\xi}) + (\boldsymbol{\xi}\cdot\nabla\mathbf{B})\cdot(\nabla\mathbf{B}\cdot\boldsymbol{\xi}) && (94)
\end{aligned}
$$

We know that the P_T terms are of the form

$$
\text{B.T.} - (\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi})\cdot(\mathbf{B}\cdot\nabla\mathbf{B}) + (\mathbf{B}\cdot\nabla\mathbf{B})\cdot\boldsymbol{\xi}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) \qquad (95)
$$

7 Linearized Energy Derive the linearized energy $\delta^2 W\left[\phi, \psi\right]$, for low-beta, large-aspect ratio reduced MHD, (7.62). Compare the result with (7.126).

$$\begin{aligned} &\frac{1}{v_A^2}\frac{\mathrm{d}U}{\mathrm{d}t} = -\nabla_{\parallel} J - \frac{1}{R_0}\left[x\,,\,p\right] \\ &\frac{1}{c}\frac{\partial \psi}{\partial t} + \nabla_{\parallel}\varphi = \frac{\eta c}{4\pi} J \\ &\frac{\mathrm{d}p}{\mathrm{d}t} = c\left[\beta\,,\,\varphi\right] \end{aligned} \tag{7.62}$$

$$\begin{aligned} \delta^2 W\left[\xi\right] = \frac{1}{2}\int \mathrm{d}^3x \, \Bigg[&\frac{Q_\perp^2}{4\pi} + \frac{B^2}{4\pi}\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right|^2 + \frac{5}{3}P\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right)^2 + (\boldsymbol{\xi}_\perp\cdot\nabla P)(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}) \\ &- \frac{1}{c}J_\parallel \hat{\mathbf{b}}\cdot(\mathbf{Q}_\perp\times\boldsymbol{\xi}_\perp)\Bigg] + \text{B.T.} \end{aligned} \tag{7.126}$$

Solution:

8 θ-Pinch Consider a cylindrical plasma, described by $\mathbf{B} = B(r)\hat{\mathbf{z}}$ and $P = P(r)$. Use the Fourier representation (7.64) for $\boldsymbol{\xi} = \xi_r \hat{\mathbf{r}} + \xi_\theta \nabla\theta = \xi_\zeta \nabla\zeta$ (note that the Fourier amplitudes are complex). The argument at the end of §7.6 implies that one can consider incompressible displacements $\boldsymbol{\xi}$. Use this to eliminate $\xi_\parallel = \xi_\zeta$.

$$\psi = \psi_0 + \sum_{m,n} \widetilde{\psi}_{m,n} e^{im\theta - n\zeta} \tag{7.64}$$

$$\delta^2 W[\xi] = \frac{1}{2}\int \mathrm{d}^3x \left[\frac{Q_\perp^2}{4\pi} + \frac{B^2}{4\pi}\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right|^2 + \frac{5}{3}P\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right)^2 + (\boldsymbol{\xi}_\perp\cdot\nabla P)(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}) - \frac{1}{c}J_\parallel \hat{\mathbf{b}}\cdot(\mathbf{Q}_\perp\times\boldsymbol{\xi}_\perp)\right] + \text{B.T.} \tag{7.126}$$

8.a Reduced Energy Principle Show that (7.126) can be reduced to the form

$$\delta^2 W = \frac{\pi}{2}R\int_0^a \frac{\mathrm{d}r}{r}B^2\left\{\left|k\xi_\theta - \frac{im}{k}(r\xi_r)'\right|^2 + \left(\frac{n}{kR}\right)^2\left[\left|(r\xi_r)'\right|^2 + k^2r^2|\xi_r|^2\right]\right\}$$

where $k^2 = (m/r)^2 + (n/R)^2$.

Solution:

8.b Find ξ_θ Determine ξ_θ by minimizing $\delta^2 W$.

Solution:

8.c Unstable Modes Substitute this into the above expression showing that $\delta^2 W$ is positive for $n \neq 0$. Thus the only possible unstable mode is that for $n = 0$ for which $\delta^2 W$ is zero.

Solution:

9 Screw Pinch Consider the cylindrical field (7.62), with $P = P(r)$.

$$\begin{aligned}
&\frac{1}{v_A^2}\frac{\mathrm{d}U}{\mathrm{d}t} = -\nabla_\| J - \frac{1}{R_0}\left[x\,,\,p\right] \\
&\frac{1}{c}\frac{\partial\psi}{\partial t} + \nabla_\|\varphi = \frac{\eta c}{4\pi}J \\
&\frac{\mathrm{d}p}{\mathrm{d}t} = c\left[\beta\,,\,\varphi\right]
\end{aligned} \tag{7.62}$$

9.a Incompressibility Removing Components of $\boldsymbol{\xi}$ Eliminate $\xi_\|$ using the incompressibility constraint, and show that $\boldsymbol{\xi}_\perp$ can be eliminated from $\delta^2 W$ algebraically, just as ξ_θ was in exercise 8.

Solution:

9.b Energy Principle for Screw Pinch The resulting expression is a quadratic form in ξ_r and ξ_r'. After judicious integration by parts to eliminate the $\xi_r\xi_r'$ terms, show that $\delta^2 W$ becomes

$$\begin{aligned}
\delta^2 W = \frac{\pi}{2}R\int_0^a \frac{\mathrm{d}r}{r}\Bigg\{ &\left(\frac{rk_\|}{k}^2\right)|\xi_r'|^2 \\
&+\left[\left(\frac{n}{Rk}\right)^2 r\beta' + [(kr)^2-1]\left(\frac{k_\|}{k}\right)^2 - \frac{n^2Rk_\|}{(kR)^4}(n+mq)\right]|\xi_r|^2\Bigg\}
\end{aligned}$$

Solution:

9.c Large Aspect Ratio Limit Taking the large aspect ratio limit, compare this result to the $\omega^2 = 0$ limit of (7.87), noting that $\varphi/r = \xi_r$.

$$\begin{aligned}
&-\int_0^a \mathrm{d}r\; r^3[\omega^2 - (k_\| v_A)^2]\left[\left(\frac{\varphi}{r}\right)'\right]^2 \\
&= \int_0^a \mathrm{d}r\;[\omega^2 - (k_\| v_A)^2](m^2-1)\left(\frac{\varphi}{r}\right)^2 + m^2\int_0^a \mathrm{d}r\; r\beta'\kappa_r\left(\frac{\varphi}{r}\right)^2
\end{aligned} \tag{7.87}$$

Solution:

9.d Slab Model The sheared slab has the equilibrium field $\mathbf{B} = B_0[\hat{\mathbf{z}} + \hat{\mathbf{y}}F(x/L_s)]$, where (x, y, z) are rectangular coordinates. Linearize ideal, zero-beta, reduced MHD (7.55) using the model $F = \tanh(x/L_s)$. Consider perturbations with $k_\parallel(x = 0) = 0$, and show that

$$\Delta' L_s = 2\left(\frac{1}{kL_s} - kL_s\right)$$

where

$$\begin{aligned}
&\frac{1}{v_A^2}\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{B_0^2}{\bar{B}^2}\nabla_\parallel J_\parallel + \hat{\mathbf{b}}_0 \cdot \boldsymbol{\kappa}_0 \times \nabla_\perp p \\
&\frac{1}{c}\frac{\partial\psi}{\partial t} + \frac{1}{B_0}\nabla_\parallel(B_0\varphi) = \frac{\eta c}{4\pi}J \\
&\frac{\mathrm{d}p}{\mathrm{d}t} = c\hat{\mathbf{b}}_0 \cdot \nabla\beta \times \nabla_\perp\varphi
\end{aligned} \tag{7.55}$$

Solution:

10 Standard Tokamak Estimates Use the approximate dispersion relation (7.158) to estimate the growth rate (γ), layer width (w), shear-Alfvén width (x_A) and resistive skin depth (x_R) of an $m \geq 2$ tearing mode in the Standard Tokamak. Use the estimates $\sigma_s \simeq 10^{18}$ s^{-1}, $L_s \simeq qR$ and $\Delta' \simeq 1/a$. Discuss the physical significance of the ratio w/ρ_i.

$$\gamma \sim (\Delta')^{4/5} \left(\frac{\eta c^2}{4\pi}\right)^{3/5} (k_\parallel' v_A)^{2/5} \tag{7.158}$$

toroidal field (B_T)	50 kG
major radius (R_0)	300 cm
minor radius (a)	80 cm
safety factor (q)	$q \simeq 1$ (on axis) $q \simeq 3$ (at edge)
central density (n)	10^{14} cm^{-3}
central temperature ($T_i = T_e = T$)	10 keV

Table 1: The Standard Tokamak parameters.

Solution:

11 Drift Model Electron Response Consider a cylindrical system with uniform temperature and negligible magnetic curvature (but nonvanishing shear). Use the drift model of §6.5 to derive the linear electron response to $E_\parallel$; compare your answer to the kinetic result, (7.190). See Hazeltine and Meiss (1985).

$$
\begin{aligned}
J_\parallel &= \sigma_* E_\parallel \\
\sigma_* &= \frac{i}{4\pi}\omega_{pe}^2 \frac{\omega - \omega_{*e}}{k_\parallel^2 v_{\mathrm{th}_e}^2} Z'(z)
\end{aligned}
\tag{7.190}
$$

Solution:

12 Ideal $m = 1$ mode Electrostatic Potential Show that the electrostatic potential for the ideal $m = 1$ mode, at low beta, has the approximate form

$$\Phi = C/2 - (C/\pi)\arctan\left(\frac{x}{x_A}\right)$$

where C is a constant, $x = r - r_s$ and x_A is the shear-Alfvén width. See Rosenbluth *et. al.* (1973).

Solution:

13 Mercier to Suydam Criterion Show that the Mercier criterion (7.235) reduces to the Suydam criterion (7.98) in the appropriate limit. Note that q' means $\mathrm{d}q/\mathrm{d}r$ in the former and $\mathrm{d}q/\mathrm{d}v$ in the latter.

$$q^2\beta'\left(\frac{Rq}{r_s}\right)^2\kappa_r < \frac{q'^2}{4} \tag{7.98}$$

$$\frac{q'^2}{4} - \widetilde{\beta}'\left[\left\langle\frac{B^2}{|\nabla\chi|^2}\right\rangle\langle\kappa_v\rangle + \left\langle\left(\widetilde{\beta}'\frac{hB^2}{|\nabla\chi|^2} - q'\right)\left(h\left\langle\frac{B^2}{|\nabla\chi|^2}\right\rangle - \left\langle\frac{hB^2}{|\nabla\chi|^2}\right\rangle\right)\right\rangle\right] > 0 \tag{7.235}$$

Solution:

14 Normal Curvature Form Reducing Mercier Criterion Show, starting with (3.120) or (3.121), that the flux surface average of the normal curvature, at low beta, is proportional to $q^2 - 1$. Thus the Mercier criterion reduces to

$$\frac{q'^2}{4} + \frac{\beta'}{r} q^2 (1 - q^2) > 0$$

showing that modes $q > 1$ are stable. See Ware and Haas (1966).

$$\kappa_r \approx -\frac{\partial}{\partial r} (\ln R) - \left(\frac{B_p}{B} \right)^2 \frac{1}{r} \tag{3.120}$$

$$\kappa_\theta = (1 - b_\theta b^\theta) \frac{\partial}{\partial \theta} (\ln B) = \left(\frac{B_T}{B} \right)^2 \frac{\partial}{\partial \theta} (\ln B) \tag{3.121}$$

Solution:

15 Averaged Ballooning Equation Compute φ_3 and derive the averaged ballooning equation (7.244) for the envelope φ_0.

$$-\omega^2(1+z^2)\varphi_0 = s^2\frac{\partial}{\partial z}\left(1+z^2\right)\frac{\partial}{\partial z}\varphi_0 + \left[\rho\bar{\kappa} + \frac{\rho^2(2s-\frac{3}{8}\rho^2)}{(1+z^2)^2}\right]\varphi_0 \tag{7.244}$$

Solution:

16 Ballooning Equation for Resistive, Flute-Reduced MHD Obtain the ballooning equation for resistive, flute-reduced MHD, deriving (7.250). Show that when $\eta \gg x_R^{-2}$ the line bending term is small, and in this limit the mode has the form $\Phi \sim e^{-(\eta/w)^2}$ and obtain the mode width, w.

$$\frac{\partial}{\partial \eta} \frac{F^2}{1 + x_R^2 F^2} \frac{\partial}{\partial \eta} \phi \tag{7.250}$$

Solution: