1 Verify Expressions

Verify (6.10) for the pressure tensor, and (6.11), for the heat flux.

$$\overleftrightarrow{\mathbf{P}}_s = \overleftrightarrow{\mathbf{p}}_s + m_s n_s \mathbf{V}_s \mathbf{V}_s \tag{6.10}$$

$$\mathbf{Q}_s = \overleftrightarrow{\mathbf{q}}_s + \mathbf{V}_s \cdot \overleftrightarrow{\mathbf{p}}_s + \frac{3}{2} p_s \mathbf{V}_s + \frac{1}{2} m_s n_s V_s^2 \mathbf{V}_s \tag{6.11}$$

Solution:

We use that

$$\overleftrightarrow{\mathbf{P}}_s = \int \mathrm{d}^3 v \; m_s f_s \mathbf{v}\mathbf{v} \tag{1}$$

$$\overleftrightarrow{\mathbf{p}}_s = \int \mathrm{d}^3 v \; m_s f_s (\mathbf{v} - \mathbf{V}_s)(\mathbf{v} - \mathbf{V}_s) \tag{2}$$

$$p_s = \frac{1}{3} \operatorname{Tr}(\overleftrightarrow{\mathbf{p}}_s) = \frac{1}{3} \int \mathrm{d}^3 v \; f_s m_s |\mathbf{v} - \mathbf{V}_s|^2 \tag{3}$$

$$n_s = \int \mathrm{d}^3 v \; f_s \tag{4}$$

$$n_s \mathbf{V}_s = \int \mathrm{d}^3 v \; f_s \mathbf{v} \tag{5}$$

$$\mathbf{Q}_s = \int \mathrm{d}^3 v \; \frac{m_s f_s}{2} v^2 \mathbf{v} \tag{6}$$

$$\mathbf{q}_s = \int \mathrm{d}^3 v \; \frac{m_s f_s}{2} |\mathbf{v} - \mathbf{V}_s|^2 (\mathbf{v} - \mathbf{V}_s) \tag{7}$$

We also use that $\mathbf{V}_s$, m_s are independent of $\mathbf{v}$ and thus can be taken out of the $\int \mathrm{d}^3 v$ integrals. Thus for (6.10),

$$\overleftrightarrow{\mathbf{p}}_s = \int \mathrm{d}^3 v \; m_s f_s \left[\mathbf{v}\mathbf{v} - \mathbf{v}\mathbf{V}_s - \mathbf{V}_s\mathbf{v} + \mathbf{V}_s\mathbf{V}_s\right] \tag{8}$$

$$= \overleftrightarrow{\mathbf{P}}_s - \underbrace{\left(\int \mathrm{d}^3 v \; m_s f_s \mathbf{v}\right)}_{=m_s n_s \mathbf{V}_s} \mathbf{V}_s - \mathbf{V}_s \underbrace{\left(\int \mathrm{d}^3 v \; m_s f_s \mathbf{v}\right)}_{=m_s n_s \mathbf{V}_s} + m_s \mathbf{V}_s \mathbf{V}_s \underbrace{\int \mathrm{d}^3 v \; f_s}_{n_s} \tag{9}$$

$$\overleftrightarrow{\mathbf{p}}_s = \overleftrightarrow{\mathbf{P}}_s + (1-2) m_s n_s \mathbf{V}_s \mathbf{V}_s \tag{10}$$

$$\overleftrightarrow{\mathbf{P}}_s = \overleftrightarrow{\mathbf{p}}_s + m_s n_s \mathbf{V}_s \mathbf{V}_s \tag{11}$$

as desired.

Now for (6.11)

$$\mathbf{q}_s = \int \mathrm{d}^3 v \; \frac{m_s f_s}{2} |\mathbf{v} - \mathbf{V}_s|^2 (\mathbf{v} - \mathbf{V}_s) \tag{12}$$

$$\mathbf{q}_s = \int \mathrm{d}^3 v \; \frac{m_s f_s}{2} (\mathbf{v} - \mathbf{V}_s) \cdot (\mathbf{v} - \mathbf{V}_s)(\mathbf{v} - \mathbf{V}_s) \tag{13}$$

$$\mathbf{q}_s = \int \mathrm{d}^3 v \; \frac{m_s f_s}{2} (v^2 - 2\mathbf{v} \cdot \mathbf{V}_s + V_s^2) \mathbf{v} - \underbrace{\int \mathrm{d}^3 v \; \frac{m_s f_s}{2} |\mathbf{v} - \mathbf{V}_s|^2 \mathbf{V}_s}_{\frac{3}{2} p_s \mathbf{V}_s} \tag{14}$$

$$\mathbf{q}_s = \underbrace{\int \mathrm{d}^3 v \ \frac{m_s f_s}{2} v^2 \mathbf{v}}_{\mathbf{Q}_s} - \int \mathrm{d}^3 v \ \frac{m_s f_s}{2} 2\mathbf{v} \cdot \mathbf{V}_s \mathbf{v} + \underbrace{\int \mathrm{d}^3 v \ \frac{m_s f_s}{2} V_s^2 \mathbf{v}}_{\frac{m_s}{2} V_s^2 n_s \mathbf{V}_s} - \frac{3}{2} p_s \mathbf{V}_s \tag{15}$$

$$\mathbf{q}_s = \mathbf{Q}_s - \mathbf{V}_s \cdot \underbrace{\int \mathrm{d}^3 v \ m_s f_s \mathbf{v}\mathbf{v}}_{\overleftrightarrow{\mathbf{P}}_s = \overleftrightarrow{\mathbf{p}}_s + m_s n_s \mathbf{V}_s \mathbf{V}_s} + \frac{1}{2} m_s n_s V_s^2 \mathbf{V}_s - \frac{3}{2} p_s \mathbf{V}_s \tag{16}$$

$$\mathbf{q}_s = \mathbf{Q}_s - \mathbf{V}_s \cdot \left(\overleftrightarrow{\mathbf{p}}_s + m_s n_s \mathbf{V}_s \mathbf{V}_s \right) + \frac{1}{2} m_s n_s V_s^2 \mathbf{V}_s - \frac{3}{2} p_s \mathbf{V}_s \tag{17}$$

$$\mathbf{q}_s = \mathbf{Q}_s - \mathbf{V}_s \cdot \overleftrightarrow{\mathbf{p}}_s - \frac{1}{2} m_s n_s \mathbf{V}_s \mathbf{V}_s - \frac{3}{2} p_s \mathbf{V}_s \tag{18}$$

$$\mathbf{Q}_s = \mathbf{q}_s + \mathbf{V}_s \cdot \overleftrightarrow{\mathbf{p}}_s + \frac{1}{2} m_s n_s \mathbf{V}_s \mathbf{V}_s + \frac{3}{2} p_s \mathbf{V}_s \tag{19}$$

as desired.

2 Cold Plasma In some beam plasmas the flow velocity far exceeds the thermal velocity. The cold plasma model idealizes this situation by taking each species temperate to vanish; the distributions are delta functions centered at the species flow speed $\mathbf{V}_s$.

Derive an exact set of closed fluid equations for a cold plasma with a single ion species. Discuss the relation between energy conservation, (6.23), and momentum conservation, (6.24), in the cold plasma case.

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p + \frac{1}{2}mnV^2\right) + \boldsymbol{\nabla}\cdot\mathbf{Q} = W + \mathbf{V}\cdot(\mathbf{F} + en\mathbf{E}) \tag{6.23}$$

$$\frac{\partial}{\partial t}(mn\mathbf{V}) + \boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{P}} - en\left(\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B}\right) = \mathbf{F} \tag{6.24}$$

Solution:

In this case we have $f_s = n_s\delta^3(\mathbf{v}-\mathbf{V}_s)$ which automatically satisfies the definitions for n_s and $\mathbf{V}_s$. Then we have

$$\mathbf{Q}_s = \int \mathrm{d}^3v\ f_s\frac{1}{2}m_sv^2\mathbf{v} = \frac{1}{2}n_sm_sV_s^2\mathbf{V}_s \tag{20}$$

$$\overleftrightarrow{\mathbf{P}}_s = \int \mathrm{d}^3v\ f_sm_s\mathbf{v}\mathbf{v} = n_sm_s\mathbf{V}_s\mathbf{V}_s \tag{21}$$

We note that we can solve

$$M^N_{\alpha\beta\ldots\tau} = \int \mathrm{d}^3v\ fv_\alpha v_\beta\ldots v_\tau = n_s\mathbf{V}_s^N \tag{22}$$

where $\alpha,\beta,\ldots,\tau$ are N factors. We then see that

$$\int \mathrm{d}^3v\ \mathbf{E}\cdot\frac{\partial f}{\partial\mathbf{v}}\mathbf{v}^N = \mathbf{E}\cdot\cancel{\int \mathrm{d}^3v\ \frac{\partial}{\partial\mathbf{v}}\left[f\mathbf{v}^N\right]} - \mathbf{E}\cdot\int \mathrm{d}^3v\ f\frac{\partial\mathbf{v}^N}{\partial\mathbf{v}} \tag{23}$$

$$= -\mathbf{E}\cdot\int \mathrm{d}^3v\ n_s\delta^3(\mathbf{v}-\mathbf{V}_s)N\mathbb{1}\mathbf{v}^{N-1} = -Nn_s\mathbf{E}\mathbf{v}^{N-1} \tag{24}$$

$$\int \mathrm{d}^3v\ \mathbf{v}\times\mathbf{B}\cdot\frac{\partial f}{\partial\mathbf{v}}\mathbf{v}^N = \mathbf{B}\cdot\int \mathrm{d}^3v\ \frac{\partial f}{\partial\mathbf{v}}\times\mathbf{v}^N = -\mathbf{B}\times\left[\int \mathrm{d}^3v\ \cancel{\frac{\partial}{\partial\mathbf{v}}\times\left(f\mathbf{v}^{N+1}\right)} - \int \mathrm{d}^3v\ \mathbf{v}Nf_s\mathbf{v}^{N-1}\right] \tag{25}$$

$$= \mathbf{B}\times Nn_s\mathbf{V}_s^N = Nn_s\mathbf{B}\times\mathbf{V}_s^N \tag{26}$$

The last identity is easier to see in index form,

$$\int \mathrm{d}^3v\ \epsilon_{ijk}v_iB_j\frac{\partial f_s}{\partial v_k}\overbrace{v_l\cdots v_m}^{N} = \epsilon_{ijk}B_j\int \mathrm{d}^3v\ v_iv_l\cdots v_m\frac{\partial f_s}{\partial v_k} \tag{27}$$

$$= \epsilon_{ijk}B_j\int \mathrm{d}^3v\ \left[\cancel{\frac{\partial}{\partial v_k}\left(f_sv_iv_l\cdots v_m\right)} - f_s\frac{\partial}{\partial v_k}\left(v_iv_l\cdots v_m\right)\right] \tag{28}$$

$$= -\int \mathrm{d}^3v\ \epsilon_{ijk}B_jf_s\big(\cancel{\delta_{ik}v_l\cdots v_m} + \overbrace{\delta_{lk}v_i\cdots v_m + \cdots + \delta_{mk}v_iv_l\cdots}^{N}\big) \tag{29}$$

$$= -\int \mathrm{d}^3v\ Nf_s\epsilon_{ijk}v_iB_j\overbrace{v_l\cdots v_n}^{N-1} = \int \mathrm{d}^3v\ Nf_s\mathbf{B}\times\mathbf{v}^N \tag{30}$$

Thus we find

$$\boxed{\frac{\partial}{\partial t}\left(n_s \mathbf{V}_s^N\right) + \boldsymbol{\nabla}\cdot\left(n_s \mathbf{V}_s^{N+1}\right) - \frac{e}{m} N n_s \mathbf{E}\mathbf{V}_s^{N-1} + N n_s \Omega \hat{\mathbf{b}} \times \mathbf{V}_s^N = \mathbf{C}^{(N)}} \tag{31}$$

with the definition

$$\mathbf{C}^{(N)} = \int \mathrm{d}^3 v \; C \mathbf{v}^N \tag{32}$$

with C the collision operator. Or, in index notation,

$$\frac{\partial}{\partial t}\left(n_s (V_{s,\alpha} \cdots V_{s,\sigma} V_{s,\tau})\right) + \frac{\partial}{\partial x_\alpha}\left(n_s V_{s,\alpha} \cdots V_{s,\sigma} V_{s,\tau} V_{s,\upsilon}\right) - \frac{e}{m} N n_s E_i V_{s,\alpha} \cdots V_{s,\sigma} \tag{33}$$

$$+ \epsilon_{ijk} N n_s \Omega b_i V_{s,j} V_{s,\alpha} \cdots V_{s,\sigma} = C^{(N)}_{\alpha \ldots \tau} \tag{34}$$

where $\alpha, \ldots, \sigma, \tau, \upsilon$ are $N+1$ quantities with the definition

$$C^{(N)}_{\alpha \ldots \tau} = \int \mathrm{d}^3 v \; C v_\alpha \cdots v_\sigma v_\tau \tag{35}$$

with C the collision operator.

We begin with (6.24) and see that

$$\frac{\partial}{\partial t}\left(m_s n_s \mathbf{V}_s\right) + \boldsymbol{\nabla}\cdot\left(n_s m_s \mathbf{V}_s \mathbf{V}_s\right) - en\left(\mathbf{E} + \frac{1}{c}\mathbf{V}_s + \mathbf{B}\right) = \mathbf{F}_s \tag{36}$$

We see that $W = p = 0$ in this cold limit and so (6.23) yields

$$\frac{\partial}{\partial t}\left(\frac{1}{2} m_s n_s V_s^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2} m_s n_s V_s^2 \mathbf{V}_s\right) = \mathbf{V}\cdot\left(\mathbf{F}_s + en\mathbf{E}\right) \tag{37}$$

Thus, if we take $\mathbf{V}_s \cdot$ (36) we find

$$\mathbf{V}_s \cdot \frac{\partial}{\partial t}\left(m_s n_s \mathbf{V}_s\right) + \mathbf{V}_s \cdot \boldsymbol{\nabla}\cdot\left(n_s m_s \mathbf{V}_s \mathbf{V}_s\right) - \mathbf{V}_s \cdot en\left(\mathbf{E} + \cancel{\frac{1}{c}\mathbf{V}_s + \mathbf{B}}\right) = \mathbf{V}_s \cdot \mathbf{F}_s \tag{38}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2} m_s n_s V_s^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2} n_s m_s V_s^2 \mathbf{V}_s\right) = \mathbf{V}\cdot\left(\mathbf{F}_s + en\mathbf{E}\right) \tag{39}$$

which means that energy conservation is subsumed by momentum conservation and adds no new information.

3 Contravariant Components of $\overleftrightarrow{\mathbf{P}}_{\mathbf{CGL}}$

Compute all contravariant components of the stress tensor $\overleftrightarrow{\mathbf{P}}_{\text{CGL}}$ with respect to flux coordinates (r, θ, ζ), in terms of $P_\perp$, $P_\parallel$ and components of $\mathbf{B}$.

Solution:

We have

$$\overleftrightarrow{\mathbf{P}}_{\text{CGL}} = P_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} + \left(\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right) P_\perp \tag{40}$$

with $\hat{\mathbf{b}} = \frac{\mathbf{B}}{|\mathbf{B}|}$. Now we can rewrite this as

$$\overleftrightarrow{\mathbf{P}}_{\text{CGL}} = \left(\frac{P_\parallel - P_\perp}{B^2}\right) \mathbf{BB} + \mathbb{1} P_\perp \tag{41}$$

We can then use that in flux coordinates that

$$B^r = 0 \tag{42}$$

$$B^\theta = \frac{\chi'}{\sqrt{g}} \tag{43}$$

$$B^\zeta = \frac{q\chi'}{\sqrt{g}} \tag{44}$$

$$\sqrt{g} = 1/(\nabla r \cdot \nabla\theta \times \nabla\zeta) \tag{45}$$

The identity tensor should be invariant under a transformation of coordinates in the sense that it will still be diagonal. Thus,

$$\overleftrightarrow{\mathbf{P}}^{rr}_{\text{CGL}} = P_\perp \tag{46}$$

$$\overleftrightarrow{\mathbf{P}}^{r\theta}_{\text{CGL}} = \overleftrightarrow{\mathbf{P}}^{\theta r}_{\text{CGL}} = 0 \tag{47}$$

$$\overleftrightarrow{\mathbf{P}}^{r\zeta}_{\text{CGL}} = \overleftrightarrow{\mathbf{P}}^{\zeta r}_{\text{CGL}} = 0 \tag{48}$$

$$\overleftrightarrow{\mathbf{P}}^{\theta\zeta}_{\text{CGL}} = \overleftrightarrow{\mathbf{P}}^{\zeta\theta}_{\text{CGL}} = \frac{P_\parallel B^\theta B^\theta}{B^2} \tag{49}$$

$$\overleftrightarrow{\mathbf{P}}^{\theta\theta}_{\text{CGL}} = P_\parallel \frac{B^\theta B^\theta}{B^2} - P_\perp \tag{50}$$

$$\overleftrightarrow{\mathbf{P}}^{\zeta\zeta}_{\text{CGL}} = P_\parallel \frac{B^\zeta B^\zeta}{B^2} - P_\perp \tag{51}$$

4 Law for Evolution of Parallel Stress Tensor Express (6.60) explicitly as a law for evolution of the parallel stress tensor. Assume for simplicity that the magnetic field is uniform.

$$\hat{\mathbf{b}} \cdot \overleftrightarrow{\mathbf{S}} \cdot \hat{\mathbf{b}} = 0 \tag{6.60}$$

Solution:

We remember that

$$\overleftrightarrow{\mathbf{S}} = \Omega\left(\hat{\mathbf{b}} \times \overleftrightarrow{\mathbf{S}} + \left[\hat{\mathbf{b}} \times \overleftrightarrow{\mathbf{S}}\right]^{\intercal}\right) = -\left\{\frac{\partial \overleftrightarrow{\mathbf{P}}}{\partial t} + m\boldsymbol{\nabla} \cdot \overset{\overset{\rightarrow}{\leftrightarrow}}{\mathbf{M}}^{(3)} - en(\mathbf{EV} + \mathbf{VE}) - m\overleftrightarrow{\mathbf{C}}^{(2)}\right\} \tag{52}$$

Thus,

$$\frac{\partial P_\parallel}{\partial t} + \hat{\mathbf{b}} \cdot \left(m\boldsymbol{\nabla} \cdot \overset{\overset{\rightarrow}{\leftrightarrow}}{\mathbf{M}}^{(3)}\right) \cdot \hat{\mathbf{b}} - 2enE_\parallel V_\parallel - mC_\parallel^{(2)} = 0 \tag{53}$$

$$\boxed{\frac{\partial P_\parallel}{\partial t} = -m\boldsymbol{\nabla} \cdot \mathbf{M}_\parallel^{(3)} + 2enE_\parallel V_\parallel + mC_\parallel^{(2)}} \tag{54}$$

where

$$\hat{\mathbf{b}} \cdot \left(m\boldsymbol{\nabla} \cdot \overset{\overset{\rightarrow}{\leftrightarrow}}{\mathbf{M}}^{(3)}\right) \cdot \hat{\mathbf{b}} = mb_i b_k \partial_j M_{jik} = m\partial_j(b_i b_k M_{jik}) \equiv m\boldsymbol{\nabla} \cdot \mathbf{M}_\parallel^{(3)} \tag{55}$$

$$b_i M_{jik} b_k \equiv \mathbf{M}_\parallel^{(3)} \tag{56}$$

5 Collisional Moment with No Unlike-Species Collisions When unlike-species collisions are omitted (a common approximation to ion kinetics), the first nonvanishing collisional moment is $C^{(2)}$. Show that

$$C^{(2)}_{\alpha\beta} = -\frac{m\gamma}{2}\left(K_{\alpha\beta} + K_{\beta\alpha}\right) \tag{57}$$

$$K_{\alpha\beta} = \int \mathrm{d}^3v\mathrm{d}^3v'\ f_M f'_M (v_\alpha - v'_\alpha) U_{\beta\lambda}\left(\frac{\partial \widehat{f}}{\partial v_\lambda} - \frac{\partial \widehat{f}'}{\partial v'_\lambda}\right) \tag{58}$$

Here the tensor $U_{\alpha\beta} = \frac{1}{u^3}\left(u^2\delta_{\alpha\beta} - u_\alpha u_\beta\right)$ with $u_\alpha = v_\alpha - v'_\alpha$.

Solution:

I'm assuming we're using the Coulomb collision operator. Thus,

$$C_{st} = \frac{\gamma_{st}}{2}\frac{\partial}{\partial v_\lambda}\cdot\int \mathrm{d}^3v'\ U_{\lambda\mu}\left[\frac{f'_t}{m_s}\frac{\partial f_s}{\partial v_\mu} - \frac{f_s}{m_t}\frac{\partial f'_t}{\partial v'_\mu}\right] \tag{59}$$

Assuming

$$f'_s = f'_M\left(1 + \widehat{f}'_s\right) \tag{60}$$

$$f_s = f_M\left(1 + \widehat{f}_s\right) \tag{61}$$

with

$$f_M = \frac{n_s}{\pi^{3/2}v^3_{\mathrm{th}_s}}e^{-v^2/v^2_{\mathrm{th}_s}} \tag{62}$$

$$\frac{\partial f_M}{\partial v_\lambda} = \frac{\partial v}{\partial v_\lambda}\frac{\partial f_M}{\partial v} = \frac{v_\lambda}{v}\frac{-2v}{v^2_{\mathrm{th}_s}}f_M = -\frac{2v_\lambda}{v^2_{\mathrm{th}_s}}f_M \tag{63}$$

We see that (using $s = t$ for only like particle collisions)

$$C^{(2)}_{\alpha\beta} = \int \mathrm{d}^3v\ v_\alpha v_\beta \frac{\gamma}{2}\frac{\partial}{\partial v_\lambda}\int \mathrm{d}^3v'\ U_{\lambda\mu}\left[\frac{f'_s}{m_s}\frac{\partial f_s}{\partial v_\mu} - \frac{f_s}{m_s}\frac{\partial f'_s}{\partial v'_\mu}\right] \tag{64}$$

$$= \frac{\gamma}{2m_s}\int \mathrm{d}^3v\int \mathrm{d}^3v'\ v_\alpha v_\beta \frac{\partial}{\partial v_\lambda}U_{\lambda\mu}\left[f'_s\frac{\partial f_s}{\partial v_\mu} - f_s\frac{\partial f'_s}{\partial v'_\mu}\right] \tag{65}$$

$$= \frac{\gamma}{2m_s}\int \mathrm{d}^3v\int \mathrm{d}^3v'\ v_\alpha v_\beta \frac{\partial}{\partial v_\lambda}U_{\lambda\mu}\left[f'_M\left(1 + \widehat{f}'_s\right)\frac{\partial}{\partial v_\mu}f_M\left(1 + \widehat{f}_s\right) - f_M\left(1 + \widehat{f}_s\right)\frac{\partial}{\partial v'_\mu}f'_M\left(1 + \widehat{f}'_s\right)\right] \tag{66}$$

$$\begin{aligned} = \frac{\gamma}{2m_s}\int \mathrm{d}^3v\int \mathrm{d}^3v'\ v_\alpha v_\beta \frac{\partial}{\partial v_\lambda}U_{\lambda\mu}\Bigg[&f'_M\left(1 + \widehat{f}'_s\right)\left\{-\frac{2v_\mu}{v^2_{\mathrm{th}_s}}f_M(1 + \widehat{f}_s) + f_M\frac{\partial \widehat{f}_s}{\partial v_\mu}\right\} \\ &- f_M\left(1 + \widehat{f}_s\right)\left\{-\frac{2v'_\mu}{v^2_{\mathrm{th}_s}}f'_M(1 + \widehat{f}'_s) + f'_M\frac{\partial \widehat{f}'_s}{\partial v'_\mu}\right\}\Bigg] \end{aligned} \tag{67}$$

If we then call the bracketed terms to the right of $U_{\lambda\mu}$ the variable T_μ we can integrate by parts

$$C^{(2)}_{\alpha\beta} = \frac{\gamma}{2m_s}\int \mathrm{d}^3v\int \mathrm{d}^3v'\ \left(\cancel{\frac{\partial}{\partial v_\lambda}\left[v_\alpha v_\beta U_{\lambda\mu}T_\mu\right]} - U_{\lambda\mu}T_\mu\frac{\partial}{\partial v_\lambda}\left[v_\alpha v_\beta\right]\right) \tag{68}$$

$$= -\frac{\gamma}{2m_s}\int \mathrm{d}^3v\int \mathrm{d}^3v'\ U_{\lambda\mu}T_\mu\left[\delta_{\alpha\lambda}v_\beta + \delta_{\lambda\beta}v_\alpha\right] \tag{69}$$

Now let's simplify T_μ by ridding ourselves of terms with two perturbed terms (or more),

$$\begin{aligned} T_\mu &= -\frac{2v_\mu}{v_{\mathrm{th}_s}^2} f_M' f_M \left(1 + \widehat{f}_s' + \widehat{f}_s\right) + f_M f_M' \frac{\partial \widehat{f}_s}{\partial v_\mu} - \left[-\frac{2v_\mu'}{v_{\mathrm{th}_s}^2} f_M' f_M \left(1 + \widehat{f}_s' + \widehat{f}_s\right) + f_M f_M' \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] && (70) \\ &= \frac{2 f_M f_M'}{v_{\mathrm{th}_s}^2} (1 + \widehat{f}_s' + \widehat{f}_s)(v_\mu' - v_\mu) + f_M f_M' \left[\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] && (71) \end{aligned}$$

Thus, the integrand reads

$$[v_\beta U_{\alpha\mu} + v_\alpha U_{\beta\mu}] \left[2 f_M f_M' (1 + \widehat{f}_s' + \widehat{f}_s)(v_\mu' - v_\mu) + f_M f_M' \left[\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] \right] \tag{72}$$

Now we use that $v_\mu - v_\mu' = u_\mu$ and $U_{\zeta\mu} u_\mu = 0$. Thus we have

$$C_{\alpha\beta}^{(2)} = -\frac{\gamma}{2m_s} \int \mathrm{d}^3 v \mathrm{d}^3 v' \ \ [v_\beta U_{\alpha\mu} + v_\alpha U_{\beta\mu}] \, f_M f_M' \left[\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] \tag{73}$$

We then note that if we switch v_τ and v_τ' that $U_{\tau\sigma}$ remains the same, f_M and f_M' simply swap places, and $\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \to \frac{\partial \widehat{f}_s'}{\partial v_\mu'} - \frac{\partial \widehat{f}_s}{\partial v_\mu} = -\left(\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right)$. Thus,

$$\begin{aligned} C_{\alpha\beta}^{(2)} &= -\frac{\gamma}{2m_s} \int \mathrm{d}^3 v' \mathrm{d}^3 v \ \ - \left[v_\beta' U_{\alpha\mu} + v_\alpha' U_{\beta\mu} \right] f_M' f_M \left[\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] && (74) \\ &= \frac{\gamma}{2m_s} \int \mathrm{d}^3 v' \mathrm{d}^3 v \ \ \left[v_\beta' U_{\alpha\mu} + v_\alpha' U_{\beta\mu} \right] f_M' f_M \left[\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] && (75) \\ &= \frac{\gamma}{2m_s} \int \mathrm{d}^3 v \mathrm{d}^3 v' \ \ \left[v_\beta' U_{\alpha\mu} + v_\alpha' U_{\beta\mu} \right] f_M f_M' \left[\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] && (76) \end{aligned}$$

Symmetrizing by adding (73) and (76) yields

$$\begin{aligned} 2C_{\alpha\beta}^{(2)} &= -\frac{\gamma}{2m_s} \int \mathrm{d}^3 v \mathrm{d}^3 v' \ \ [v_\beta U_{\alpha\mu} + v_\alpha U_{\beta\mu}] \, f_M f_M' \left[\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] \\ &\quad + \frac{\gamma}{2m_s} \int \mathrm{d}^3 v \mathrm{d}^3 v' \ \ \left[v_\beta' U_{\alpha\mu} + v_\alpha' U_{\beta\mu} \right] f_M f_M' \left[\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] && (77) \\ &= \frac{\gamma}{2m_s} \int \mathrm{d}^3 v \mathrm{d}^3 v' \ \ \left[(v_\beta' - v_\beta) U_{\alpha\mu} + (v_\alpha' - v_\alpha) U_{\beta\mu} \right] f_M f_M' \left[\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] && (78) \\ &= -\frac{\gamma}{2m_s} \int \mathrm{d}^3 v \mathrm{d}^3 v' \ \ \left[(v_\beta - v_\beta') U_{\alpha\mu} + (v_\alpha - v_\alpha') U_{\beta\mu} \right] f_M f_M' \left[\frac{\partial \widehat{f}_s}{\partial v_\mu} - \frac{\partial \widehat{f}_s'}{\partial v_\mu'} \right] && (79) \\ &= -\frac{\gamma}{2m_s} \left(K_{\beta\alpha} + K_{\alpha\beta} \right) && (80) \end{aligned}$$

Thus,

$$\boxed{C_{\alpha\beta}^{(2)} = -\frac{\gamma}{4m_s} \left(K_{\alpha\beta} + K_{\beta\alpha} \right)} \tag{81}$$

It is easy to see that Hazeltine and Meiss have inconsistent units (again), and so they are probably incorrect on the factor of 2 as well.

6 Collisional Viscosity We have noted that collisional viscosity is relatively small. It is also surprisingly easy to calculate, at least whenever the gyroaveraged distribution can be approximated by a displaced Maxwellian,

$$\bar{f} = f_M \left(1 + 2\frac{v_\parallel V_\parallel}{v_{\rm th}^2}\right).$$

6.a Collisional Viscosity Expression Use this $\bar{f}$ and the method of (6.62) *et seq.* to derive an expression for $\overleftrightarrow{\mathbf{\Pi}}_c$ as a collisional moment of the distribution $-\frac{1}{\Omega}\hat{\mathbf{b}} \times \mathbf{v} \cdot \nabla\left[(2V_\parallel v_\parallel / v_{\rm th}^2) f_M\right]$.

$$P^0_{\alpha\beta} = \frac{1}{4\Omega}\left[\epsilon_{\beta\kappa\lambda} b_\kappa \left(S_{\alpha\lambda} + 3b_\alpha b_\gamma S_{\lambda\gamma}\right) + \epsilon_{\alpha\kappa\lambda} b_\kappa (S_{\beta\lambda} + 3b_\beta b_\gamma S_{\lambda\gamma})\right] \tag{6.62}$$

Solution:

By symmetry we see in the discussion following (6.62) that

$$\Pi_{c00} = C_{00}^{(2)} = 0 \tag{82}$$

$$\Pi_{c12} = \frac{mC_{13}^{(2)}}{\Omega} \tag{83}$$

$$\Pi_{c13} = \frac{mC_{12}^{(2)}}{\Omega} \tag{84}$$

$$\Pi_{c23} = \frac{m(C_{33}^{(2)} - C_{22}^{(2)})}{4\Omega} \tag{85}$$

$$\Pi_{c22} = -\Pi_{c33} = \frac{mC_{23}^{(2)}}{2\Omega} \tag{86}$$

Now, as $\hat{\mathbf{b}} \times \hat{\mathbf{v}}/\Omega = \boldsymbol{\rho}$, we see that the distribution is

$$-\frac{1}{\Omega}\hat{\mathbf{b}} \times \mathbf{v} \cdot \nabla[(2V_\parallel v_\parallel / v_{\rm th}^2) f_M] = -\boldsymbol{\rho} \cdot \nabla(\bar{f} - f_M) \tag{87}$$

Let's look at Π_{c12}. We see

$$\Pi_{c12} = \frac{m}{\Omega}\int \mathrm{d}^3 v \; v_1 v_3 C(f) \tag{88}$$

As argued in the text, the gyroaveraged part vanishes, and so we need the gyrodependent part $\widetilde{f}$, which by (4.66) in Hazeltine and Meiss yields

$$\widetilde{f} = -\boldsymbol{\rho} \cdot \nabla\bar{f} \tag{89}$$

We would then have

$$C(\widetilde{f}) = C(-\boldsymbol{\rho} \cdot \nabla\bar{f}) = C(-\boldsymbol{\rho} \cdot \nabla[f_M + \frac{2v_\parallel V_\parallel}{v_{\rm th}^2} f_M]) \tag{90}$$

Now as $\boldsymbol{\rho} \cdot \nabla f_M$ is a part of the gyroaveraged part that vanishes (in other words, it is not of the correct order), we can say it vanishes and are just left with

$$C(-\boldsymbol{\rho} \cdot \nabla\left[\frac{2v_\parallel V_\parallel}{v_{\rm th}^2} f_M\right]) \tag{91}$$

as required.

6.b Krook Model Collisional Viscosity Expression Use the Krook model collision operator to compute the collisional viscosity component Π_{c12}. Assume for simplicity that $\mathbf{B}$, n, and T are spatially constant, and that $\hat{\mathbf{b}} \cdot \nabla V_\parallel = 0$.

[For comparison, the exact result is $\Pi_{ci12} = -(6/5)\frac{p_i}{\Omega_i^2 \tau_i}\hat{\mathbf{e}}_2 \cdot \nabla V_\parallel$. Even this version is straightforward, if rather lengthy, to calculate; if you try it, begin with the formula given in Problem 5.]

Solution:

We first recognize that

$$\boldsymbol{\rho} = \frac{v_\perp}{\Omega}(\hat{\mathbf{e}}_2 \sin\gamma + \hat{\mathbf{e}}_3 \cos\gamma) = v_3\hat{\mathbf{e}}_2 + v_2\hat{\mathbf{e}}_3 \tag{92}$$

We use

$$\Pi_{c12} = \frac{m}{\Omega}C_{13}^{(2)} = \frac{m}{\Omega}\int \mathrm{d}^3v\ v_1 v_3 C(-\boldsymbol{\rho} \cdot \nabla(\bar{f} - f_M)) \tag{93}$$

With the Krook model, we find

$$C(f_1) = -\nu\left[f_1 - \frac{n_1}{n_M}f_M\right] \tag{94}$$

$$n_1 \equiv \int \mathrm{d}^3v\ f_1 \tag{95}$$

We can simply ignore the Maxwellian part and use $C(f_1) = -\nu f_1$ as the Maxwellian part will simply cancel out (it doesn't matter what n_1 is).

Then we find

$$\Pi_{c12} = \frac{m\nu}{\Omega^2}\int_{-\infty}^{\infty} \mathrm{d}v_1\, \mathrm{d}v_2 \mathrm{d}v_3\ \ -v_1 v_3(v_3\hat{\mathbf{e}}_2 + \cancel{v_2\hat{\mathbf{e}}_3}) \cdot \nabla\left(\frac{2v_1 V_\parallel}{v_{\mathrm{th}}^2}f_M\right) \tag{96}$$

$$= -\frac{m\nu}{\Omega^2}\hat{\mathbf{e}}_2 \cdot \nabla\frac{V_\parallel}{v_{\mathrm{th}}^2}\int_{-\infty}^{\infty} \mathrm{d}v_1 \mathrm{d}v_2 \mathrm{d}v_3\ v_1^2 v_3^2 \frac{n}{\pi^{3/2}v_{\mathrm{th}}^3}e^{-(v_1^2+v_2^2+v_3^2)/v_{\mathrm{th}}^2} \tag{97}$$

$$= -\frac{m\nu}{\Omega^2 v_{\mathrm{th}}^2}\hat{\mathbf{e}}_2 \cdot \nabla V_\parallel \int_{-\infty}^{\infty} \mathrm{d}v_1 \mathrm{d}v_3\ v_1^2 v_3^2 \frac{n}{\pi v_{\mathrm{th}}^2}e^{-(v_1^2+v_3^2)/v_{\mathrm{th}}^2} \tag{98}$$

$$= \frac{-mn\nu}{\Omega^2 v_{\mathrm{th}}^2}\hat{\mathbf{e}}_2 \cdot \nabla V_\parallel \left(\frac{1}{\pi v_{\mathrm{th}}^2}\frac{\pi v_{\mathrm{th}}^6}{4}\right) \tag{99}$$

$$= \frac{-m v_{\mathrm{th}}^2 n\nu}{4\Omega^2}\hat{\mathbf{e}}_2 \cdot \nabla V_\parallel \tag{100}$$

$$= \frac{-nT\nu}{2\Omega^2}\hat{\mathbf{e}}_2 \cdot \nabla V_\parallel \tag{101}$$

$$= \frac{-p\nu}{2\Omega^2}\hat{\mathbf{e}}_2 \cdot \nabla V_\parallel = -\frac{p}{2\Omega^2\tau}\hat{\mathbf{e}}_2 \cdot \nabla V_\parallel \tag{102}$$

which is very close to the correct answer aside from a numerical pre-factor.

7 Symmetric Second Rank Tensor Identity Suppose that a symmetric second rank tensor $\overleftrightarrow{\mathbf{T}}$ satisfies

$$\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{e}} = 0 = \mathrm{Tr}(\overleftrightarrow{\mathbf{T}})\,, \tag{103}$$

$$\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{T}} + \left(\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{T}}\right)^{\intercal} = \overleftrightarrow{\mathbf{S}}\,, \tag{104}$$

where $\hat{\mathbf{e}}$ is a specified unit vector and $\overleftrightarrow{\mathbf{S}}$ a known tensor. Show that a particular solution for $\overleftrightarrow{\mathbf{T}}$ is

$$\overleftrightarrow{\mathbf{T}} = -\frac{1}{4}\left\{\left[\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} + 3\hat{\mathbf{e}} \times [(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}]\right] + \left[\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} + 3\hat{\mathbf{e}} \times [(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}]\right]^{\intercal}\right\}. \tag{105}$$

Solution:

Let's show it is a particular solution by showing that the given $\overleftrightarrow{\mathbf{T}}$ satisfies all the given properties. We have in component form that

$$\overleftrightarrow{\mathbf{T}} = T_{il} = -\frac{1}{4}\left\{\epsilon_{ijk}e_j S_{kl} + 3\epsilon_{ijk}e_j e_m S_{mk} e_l + \epsilon_{ljk}e_j S_{ki} + 3\epsilon_{ljk}e_j e_m S_{mk} e_i\right\} \tag{106}$$

First

$$\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{e}} = -\frac{1}{4}\left\{\cancel{\hat{\mathbf{e}} \cdot \left[\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} + 3\hat{\mathbf{e}} \times [(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}]\right]} \cdot \hat{\mathbf{e}} + \hat{\mathbf{e}} \cdot \left[\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} + 3\hat{\mathbf{e}} \times [(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}]\right]^{\intercal} \cdot \hat{\mathbf{e}}\right\} \tag{107}$$

Here we have used for arbitrary tensor $\overleftrightarrow{\mathbf{L}}$ and arbitrary vector $\mathbf{a}$ that

$$\mathbf{a} \cdot \left(\mathbf{a} \times \overleftrightarrow{\mathbf{L}}\right) = \mathbf{0} \tag{108}$$

$$\left(\overleftrightarrow{\mathbf{L}} \times \mathbf{a}\right) \cdot \mathbf{a} = \mathbf{0} \tag{109}$$

which are proved by (a $i \leftrightarrow j$ indicates that the dummy indices i and j have been swapped.)

$$\mathbf{a} \cdot \left(\mathbf{a} \times \overleftrightarrow{\mathbf{L}}\right) = a_i\left(\epsilon_{ijk}a_j L_{kl}\right) = \epsilon_{ijk}a_i a_j L_{kl} \overset{i\leftrightarrow j}{=} \epsilon_{jik}a_j a_i L_{kl} = -\epsilon_{ijk}a_j a_i L_{kl} = \epsilon_{ijk}a_i a_j L_{kl} \tag{110}$$

$$\Rightarrow \epsilon_{ijk}a_i a_j L_{kl} = -\epsilon_{ijk}a_i a_j L_{kl} = 0 \tag{111}$$

$$\left(\overleftrightarrow{\mathbf{L}} \times \mathbf{a}\right) \cdot \mathbf{a} = \left(\epsilon_{ijk}L_{lj}a_k\right)a_i = \epsilon_{ijk}a_i a_k L_{lj} \overset{i\leftrightarrow k}{=} \epsilon_{kji}a_k a_i L_{lj} = -\epsilon_{ijk}a_i a_k L_{lj} \tag{112}$$

$$\Rightarrow \epsilon_{ijk}a_i a_k L_{lj} = -\epsilon_{ijk}a_i a_k L_{lj} = 0 \tag{113}$$

Note, though, that in general

$$\left(\mathbf{a} \times \overleftrightarrow{\mathbf{L}}\right) \cdot \mathbf{a} \neq \mathbf{0} \tag{114}$$

$$\mathbf{a} \cdot \left(\overleftrightarrow{\mathbf{L}} \times \mathbf{a}\right) \neq \mathbf{0} \tag{115}$$

We can now use the further identities

$$\left[\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{L}}\right]^{\intercal} = -\overleftrightarrow{\mathbf{L}}^{\intercal} \times \hat{\mathbf{e}} \tag{116}$$

$$[\epsilon_{ijk} e_j L_{kl}]^{\intercal} = \epsilon_{ljk} e_j L_{ki} = -L^{\intercal}_{ik} \epsilon_{lkj} e_j = -\overleftrightarrow{\mathbf{L}}^{\intercal} \times \hat{\mathbf{e}} \tag{117}$$

Thus,

$$\hat{\mathbf{e}} \cdot \left(\left[\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}}\right]^{\intercal} \cdot \hat{\mathbf{e}}\right) = \hat{\mathbf{e}} \cdot \left(-\overleftrightarrow{\mathbf{S}}^{\intercal} \times \hat{\mathbf{e}} \cdot \hat{\mathbf{e}}\right) = \hat{\mathbf{e}} \cdot \mathbf{0} = 0 \tag{118}$$

$$\hat{\mathbf{e}} \cdot \left(\left[\hat{\mathbf{e}} \times [(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}]\right]^{\intercal} \cdot \hat{\mathbf{e}}\right) = \hat{\mathbf{e}} \cdot \left(-[(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}]^{\intercal} \times \hat{\mathbf{e}} \cdot \hat{\mathbf{e}}\right) = \hat{\mathbf{e}} \cdot \mathbf{0} = 0 \tag{119}$$

and so

$$\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{e}} = 0 \tag{120}$$

as required. Now let's test the trace,

$$\begin{aligned}
\mathrm{Tr}[\overleftrightarrow{\mathbf{T}}] = T_{ii} &= -\frac{1}{4}\left\{\epsilon_{ijk} e_j S_{ki} + 3\epsilon_{ijk} e_j e_m S_{mk} e_i + \epsilon_{ijk} e_j S_{ki} + 3\epsilon_{ijk} e_j e_m S_{mk} e_i\right\} &(121)\\
&= -\frac{1}{4}\left\{\epsilon_{ijk} e_j S_{ki} + 3\epsilon_{ijk} e_j e_m S_{mk} e_i + \epsilon_{kji} e_j S_{ik} + 3\epsilon_{kji} e_j e_m S_{mi} e_k\right\} &(122)\\
&= -\frac{1}{4}\left\{\cancel{\epsilon_{ijk} e_j S_{ki}} + \cancel{3\epsilon_{ijk} e_j e_m S_{mk} e_i} - \cancel{\epsilon_{ijk} e_j S_{ik}} - \cancel{3\epsilon_{ijk} e_j e_m S_{mi} e_k}\right\} &(123)\\
&= 0 &(124)
\end{aligned}$$

where in (122) I switched the dummy indices k and i in the last two terms (which will not change the value).

Now let's show

$$\overleftrightarrow{\mathbf{T}} = -\frac{1}{4}\left\{\left[\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} + 3\hat{\mathbf{e}} \times [(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}]\right] + \left[\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} + 3\hat{\mathbf{e}} \times [(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}]\right]^{\intercal}\right\} \tag{125}$$

with the substitution $\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{T}} + (\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{T}})^{\intercal} = \overleftrightarrow{\mathbf{S}}$ inserted. We can see that $\overleftrightarrow{\mathbf{S}}$ must be symmetric thus we may use

$$\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} + 3\hat{\mathbf{e}} \times [(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}] = \hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} + 3\hat{\mathbf{e}} \times [(\overleftrightarrow{\mathbf{S}} \cdot \hat{\mathbf{e}})\hat{\mathbf{e}}] = \hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} \cdot (\mathbb{1} + 3\hat{\mathbf{e}}\hat{\mathbf{e}}) \tag{126}$$

So,

$$\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} \cdot (\mathbb{1} + 3\hat{\mathbf{e}}\hat{\mathbf{e}}) = \epsilon_{ijk} e_j S_{kl} (\delta_{lm} + 3 e_l e_m) \tag{127}$$

and

$$S_{kl} = \epsilon_{knp} e_n T_{pl} + \epsilon_{lnp} e_n T_{pk} \tag{128}$$

yielding

$$\begin{aligned}
\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} \cdot (\mathbb{1} + 3\hat{\mathbf{e}}\hat{\mathbf{e}}) &= \epsilon_{ijk} e_j (\epsilon_{knp} e_n T_{pl} + \epsilon_{lnp} e_n T_{pk})(\delta_{lm} + 3 e_l e_m) &(129)\\
&= [(\delta_{in}\delta_{jp} - \delta_{ip}\delta_{nj})\, e_j e_n T_{pl} + \epsilon_{ijk}\epsilon_{lnp} e_j e_n T_{pk}](\delta_{lm} + 3 e_l e_m) &(130)\\
&= [(e_p e_i T_{pl} - e_j e_j T_{il}) + \epsilon_{ijk}\epsilon_{lnp} e_j e_n T_{pk}](\delta_{lm} + 3 e_l e_m) &(131)
\end{aligned}$$

Now let's calculate $\epsilon_{ijk}\epsilon_{lnp}e_j e_n T_{pk}$ with

$$\epsilon_{ijk}\epsilon_{lnp} = \begin{vmatrix} \delta_{il} & \delta_{in} & \delta_{ip} \\ \delta_{jl} & \delta_{jn} & \delta_{jp} \\ \delta_{kl} & \delta_{kn} & \delta_{kp} \end{vmatrix} = \delta_{il}\left(\delta_{jn}\delta_{kp} - \delta_{jp}\delta_{kn}\right) + \delta_{in}\left(\delta_{jp}\delta_{kl} - \delta_{jl}\delta_{kp}\right) + \delta_{ip}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right) \tag{132}$$

and so finding

$$\epsilon_{ijk}\epsilon_{lnp}e_j e_n T_{pk} = \delta_{il}\left(e_j e_j T_{kk} - e_p T_{pn} e_n\right) + e_p e_i T_{pl} - e_i e_l T_{pp} + e_l e_n T_{in} - e_n e_n T_{il} \tag{133}$$

$$= \delta_{il}\left(T_{kk} - e_p T_{pn} e_n\right) + e_i e_p T_{pl} - e_i e_l T_{pp} + T_{in} e_n e_l - T_{il} \tag{134}$$

$$\tag{135}$$

altogether then

$$\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} \cdot (\mathbb{1} + 3\hat{\mathbf{e}}\hat{\mathbf{e}}) = [(e_p e_i T_{pl} - T_{il}) + \delta_{il}\left(T_{kk} - e_p T_{pn} e_n\right) + e_i e_p T_{pl} + T_{in} e_n e_l - e_i e_l T_{pp} - T_{il}](\delta_{lm} + 3e_l e_m) \tag{136}$$

$$= [2e_p e_i T_{pl} - 2T_{il} + \delta_{il}\left(T_{kk} - e_p T_{pn} e_n\right) + T_{in} e_n e_l - e_i e_l T_{pp}](\delta_{lm} + 3e_l e_m) \tag{137}$$

Let's do this term by term.

$$e_p e_i T_{pl}(\delta_{lm} + 3e_l e_m) = e_p e_i T_{pm} + 3e_i e_m e_p T_{pl} e_l \tag{138}$$

$$T_{il}(\delta_{lm} + 3e_l e_m) = T_{im} + 3T_{il} e_l e_m \tag{139}$$

$$\delta_{il}(\delta_{lm} + 3e_l e_m) = \delta_{im} + 3e_i e_m \tag{140}$$

$$T_{in} e_n e_l(\delta_{lm} + 3e_l e_m) = T_{in} e_n e_m + 3T_{in} e_n e_l e_l e_m = 4T_{in} e_n e_m \tag{141}$$

$$e_i e_l T_{pp}(\delta_{lm} + 3e_l e_m) = e_i e_m T_{pp} + 3e_i e_m T_{pp} = 4e_i e_m T_{pp} \tag{142}$$

and so altogether this yields

$$\begin{aligned}\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} \cdot (\mathbb{1} + 3\hat{\mathbf{e}}\hat{\mathbf{e}}) = 2e_i e_p T_{pm} + 6e_i e_m(e_p T_{pl} e_l) - 2T_{im} - 6T_{ip} e_p e_m \\ + (\delta_{im} + 3e_i e_m)(T_{kk} - e_p T_{pl} e_l) + 4T_{in} e_n e_m - 4e_i e_m T_{kk}\end{aligned} \tag{143}$$

We may note that

$$3e_i e_m(T_{kk} - e_p T_{pl} e_l) + 6e_i e_m(e_p T_{pl} e_l) - 4e_i e_m(T_{kk}) = e_i e_m\left(3T_{kk} - 4T_{kk} - 3e_p T_{pl} e_l + 6e_p T_{pl} e_l\right) \tag{144}$$

$$= e_i e_m\left(-T_{kk} + 3e_p T_{pl} e_l\right) = -e_i e_m(T_{kk} - e_p T_{pl} e_l) + 2e_i e_m(e_p T_{pl} e_l) \tag{145}$$

so that

$$\begin{aligned}\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} \cdot (\mathbb{1} + 3\hat{\mathbf{e}}\hat{\mathbf{e}}) = 2e_i e_p T_{pm} - 2T_{im} - 2T_{ip} e_p e_m \\ + (\delta_{im} - e_i e_m)(T_{kk} - e_p T_{pl} e_l) + 2e_i e_m(e_p T_{pl} e_l)\end{aligned} \tag{146}$$

So now we may add the transpose which will yield (utilizing $T_{ip} = T_{pi}$)

$$\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} \cdot (\mathbb{1} + 3\hat{\mathbf{e}}\hat{\mathbf{e}}) + \left(\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} \cdot (\mathbb{1} + 3\hat{\mathbf{e}}\hat{\mathbf{e}})\right)^{\intercal} \tag{147}$$

$$\begin{aligned} &= 2e_i e_p T_{pm} - 2T_{im} - 2T_{ip} e_p e_m + 2e_p T_{pi} e_m - 2T_{mi} - 2e_i T_{mp} e_p \\ &\quad + 2(\delta_{im} - e_i e_m)(T_{kk} - e_p T_{pl} e_l) + 4e_i e_m (e_p T_{pl} e_l) \end{aligned} \tag{148}$$

$$\begin{aligned} &= \cancel{2e_i e_p T_{pm}} - 2T_{im} - \cancel{2T_{ip} e_p e_m} + \cancel{2T_{ip} e_p e_m} - 2T_{im} - \cancel{2e_i e_p T_{pm}} \\ &\quad + 2(\delta_{im} - e_i e_m)(T_{kk} - e_p T_{pl} e_l) + 4e_i e_m (e_p T_{pl} e_l) \end{aligned} \tag{149}$$

$$= -4T_{im} + 2(\delta_{im} - e_i e_m)\left(T_{kk} - e_p T_{pl} e_l\right) + 4e_i e_m \left(e_p T_{pl} e_l\right) \tag{150}$$

$$= -4\overleftrightarrow{\mathbf{T}} + 2(\mathbb{1} - \hat{\mathbf{e}}\hat{\mathbf{e}})(\cancel{\mathrm{Tr}(\overleftrightarrow{\mathbf{T}})} - \cancel{\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{e}}}) + 4\hat{\mathbf{e}}\hat{\mathbf{e}}\cancel{(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{e}})} \tag{151}$$

$$= -4\overleftrightarrow{\mathbf{T}} \tag{152}$$

Thus, as required, we find

$$\overleftrightarrow{\mathbf{T}} = -\frac{1}{4}\left\{\left[\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} + 3\hat{\mathbf{e}} \times [(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}]\right] + \left[\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} + 3\hat{\mathbf{e}} \times [(\hat{\mathbf{e}} \cdot \overleftrightarrow{\mathbf{S}})\hat{\mathbf{e}}]\right]^{\intercal}\right\} \tag{153}$$

$$= -\frac{1}{4}\left\{\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} \cdot (\mathbb{1} + 3\hat{\mathbf{e}}\hat{\mathbf{e}}) + \left(\hat{\mathbf{e}} \times \overleftrightarrow{\mathbf{S}} \cdot (\mathbb{1} + 3\hat{\mathbf{e}}\hat{\mathbf{e}})\right)^{\intercal}\right\} = -\frac{1}{4}(-4\overleftrightarrow{\mathbf{T}}) = \overleftrightarrow{\mathbf{T}} \tag{154}$$

8 Ideal MHD Plasma Has No Electric Field Show that ideal MHD requires the electric field in the plasma rest frame to vanish.

Solution:

Let's consider the electric field in a laboratory frame. In ideal MHD, we use the ideal Ohm's law

$$\mathbf{E} = -\frac{1}{c}\mathbf{V} \times \mathbf{B} \tag{155}$$

where $\mathbf{V}$ is the flow velocity of the plasma. Now if we are going with the plasma, $\mathbf{V} = \mathbf{0}$ and so in ideal MHD we must have $\mathbf{E} = \mathbf{0}$.

9 Radial Electric Field to Satisfy MHD Ordering In the Standard Tokamak, what radial electric field, in Volts/cm, is necessary for the poloidal $\mathbf{E} \times \mathbf{B}$ drift to satisfy the MHD ordering?

toroidal field (B_T)	50 kG
major radius (R_0)	300 cm
minor radius (a)	80 cm
safety factor (q)	$q \simeq 1$ (on axis) $q \simeq 3$ (at edge)
central density (n)	10^{14} cm^{-3}
central temperature ($T_i = T_e = T$)	10 keV

Table 1: The Standard Tokamak parameters.

Solution:

We require

$$\frac{E_r}{B_T} = v_{\text{th}} \tag{156}$$

$$E_r = v_{\text{th}} B_T \simeq (299\,792\,458 \text{ m/s})\sqrt{\frac{(10 \text{ keV})}{511 \text{ keV}}}(5 \text{ T}) = 2.1 \times 10^8 \text{ V/m} = 210 \text{ MV/m} \tag{157}$$

$$= 2.1 \times 10^6 \text{ V/cm} = 2.1 \text{ MV/cm} \tag{158}$$

For comparison, for the drift ordering we have

$$\delta \simeq \frac{v_{\text{th}}}{L\Omega} = \frac{v_{\text{th}} m_i}{eLB} = \frac{\sqrt{T_i m_i}}{eLB} = \frac{\sqrt{(10 \text{ keV})(1.67 \times 10^{-27} \text{ kg})(1.6 \times 10^{-16} \text{ J/keV})}}{(.8 \text{ m})(1.6 \times 10^{-19} \text{ C})(5 \text{ T})} \tag{159}$$

$$= 2 \times 10^{-3} \tag{160}$$

bringing down the voltage to $E_r \simeq 4 \times 10^5$ V/m = 400 kV/m = 4 kV/cm. While this is large, it seems more likely to be achieved.

10 MHD Law for Magnetic Field Evolution Derive from (6.106) the MHD law for evolution of the field magnitude **B**. Compare the result to the pressure law, (6.102).

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} - \mathbf{B}\cdot\nabla\mathbf{V} + \mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{V} = \mathbf{0} \tag{6.106}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} + \frac{5}{3}P\boldsymbol{\nabla}\cdot\mathbf{V} = 0 \tag{6.102}$$

Solution:

Let's begin with

$$\frac{\partial\mathbf{B}}{\partial t} + c\boldsymbol{\nabla}\times\mathbf{E} = 0 \tag{161}$$

$$\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B} = 0 \tag{162}$$

We then find that

$$-c\boldsymbol{\nabla}\times\mathbf{E} = \boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B}) = \epsilon_{ijk}\partial_j\left(\epsilon_{klm}V_lB_m\right) = \epsilon_{ijk}\epsilon_{lmk}\partial_j\left(V_lB_m\right) \tag{163}$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}\right)\partial_j\left(V_lB_m\right) = \partial_m\left(V_iB_m\right) - \partial_l(V_lB_i) \tag{164}$$

$$= B_m\partial_mV_i + V_i\partial_mB_m - B_i\partial_lV_l - v_l\partial_lB_i \tag{165}$$

$$= \mathbf{B}\cdot\nabla\mathbf{V} + \mathbf{V}\cancel{\boldsymbol{\nabla}\cdot\mathbf{B}} - \mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{V} - \mathbf{V}\cdot\nabla\mathbf{B} \tag{166}$$

Thus,

$$\frac{\partial\mathbf{B}}{\partial t} = -c\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{B}\cdot\nabla\mathbf{V} - \mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{V} - \mathbf{V}\cdot\nabla\mathbf{B} \tag{167}$$

$$\underbrace{\frac{\partial\mathbf{B}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{B}}_{\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}} - \mathbf{B}\cdot\nabla\mathbf{V} + \mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{V} = \mathbf{0} \tag{168}$$

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} - \mathbf{B}\cdot\nabla\mathbf{V} + \mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{V} = \mathbf{0} \tag{169}$$

Usually I'd call the field magnitude $\sqrt{\mathbf{B}\cdot\mathbf{B}} = B$, which given Hazeltine and Meiss's multiple typos in the book, could certainly be the case of their meaning, and so I will also do that. First we take $\mathbf{B}\cdot$(6.106) and find

$$\mathbf{B}\cdot\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} - \mathbf{B}\cdot(\mathbf{B}\cdot\nabla\mathbf{V}) + \mathbf{B}\cdot\mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{V} = \mathbf{B}\cdot\mathbf{0} \tag{170}$$

$$\frac{1}{2}\frac{\mathrm{d}B^2}{\mathrm{d}t} - \mathbf{B}\cdot(\mathbf{B}\cdot\nabla\mathbf{V}) + B^2\boldsymbol{\nabla}\cdot\mathbf{V} = 0 \tag{171}$$

$$B\frac{\mathrm{d}B}{\mathrm{d}t} - (\mathbf{B}\cdot\nabla\mathbf{V})\cdot\mathbf{B} + B^2\boldsymbol{\nabla}\cdot\mathbf{V} = 0 \tag{172}$$

$$\boxed{\frac{\mathrm{d}B}{\mathrm{d}t} + B\boldsymbol{\nabla}\cdot\mathbf{V} = (\mathbf{B}\cdot\nabla\mathbf{V})\cdot\frac{\mathbf{B}}{B} = B\hat{\mathbf{b}}\hat{\mathbf{b}}:\nabla\mathbf{V}} \tag{173}$$

Compared to the pressure evolution, we see that there is an extra term due to shear in the velocity field along the magnetic field.

11 Drift-Wave Consistency Show that drift-wave consistency requires the wavevector $\mathbf{k}$ to satisfy

$$\sqrt{\frac{m_e}{m_i}}\frac{\rho_i}{L} \ll \frac{k_\parallel}{k_\perp} \ll \frac{\rho_i}{L}\,.$$

Thus drift waves propagate in a direction nearly, but not exactly, perpendicular to the magnetic field.

Solution:

This comes from the necessity that

$$k_\parallel v_{\mathrm{th}_i} \ll \omega_* \ll k_\parallel v_{\mathrm{th}_e} \tag{174}$$

with $\omega_* \equiv |\omega_*| = k_\perp \frac{cT}{eB}\frac{P_0'}{P_0}$. We see that (suppressing factors of 2 and $\sqrt{2}$ from thermal velocities from now on as they will make little difference in these limits)

$$\omega_* = k_\perp \overbrace{\frac{cm_i}{eB}}^{1/\Omega_i}\overbrace{\frac{T}{m_i}}^{\sim v_{\mathrm{th}_i}^2}\overbrace{\frac{P_0'}{P_0}}^{\sim 1/L} \sim \frac{k_\perp v_{\mathrm{th}_i}^2}{L\Omega_i} = \frac{k_\perp \rho_i v_{\mathrm{th}_i}}{L} \tag{175}$$

using $v_{\mathrm{th}_i}/L = \rho_i$ and thus

$$k_\parallel v_{\mathrm{th}_i} \ll \omega_* \Rightarrow k_\parallel \cancel{v_{\mathrm{th}_i}} \ll \frac{k_\perp \rho_i \cancel{v_{\mathrm{th}_i}}}{L} \tag{176}$$

$$\frac{k_\parallel}{k_\perp} \ll \frac{\rho_i}{L} \tag{177}$$

and with $\dfrac{v_{\mathrm{th}_i}}{v_{\mathrm{th}_e}} = \sqrt{\dfrac{2T}{m_i}}\sqrt{\dfrac{m_e}{2T}} = \sqrt{\dfrac{m_e}{m_i}}$ we find

$$\omega_* \ll k_\parallel v_{\mathrm{th}_e} \Rightarrow \frac{k_\perp \rho_i v_{\mathrm{th}_i}}{L} \ll k_\parallel v_{\mathrm{th}_e} \tag{178}$$

$$\frac{k_\parallel}{k_\perp} \gg \frac{\rho_i}{L}\frac{v_{\mathrm{th}_i}}{v_{\mathrm{th}_e}} = \sqrt{\frac{m_e}{m_i}}\frac{\rho_i}{L} \tag{179}$$

which combined yields

$$\sqrt{\frac{m_e}{m_i}}\frac{\rho_i}{L} \ll \frac{k_\parallel}{k_\perp} \ll \frac{\rho_i}{L} \tag{180}$$

as a consistency condition.

12 Drift Wave Dispersion Relation for Cold Ions Derive the drift-wave dispersion relation for the cold-ion case — that is, the $v_{\text{th}_i} \to 0$ limit of (6.181) — by linearizing (6.163) and (6.164).

$$1 - \frac{5}{3}\frac{k_\parallel^2 v_{\text{th}_i}^2}{(\omega-\omega_E)^2} = \frac{\omega_*}{\omega-\omega_E} \tag{6.181}$$

$$n_e \exp\left(-\frac{e\Phi}{T_e}\right) \simeq \text{constant} \tag{6.163}$$

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} \simeq 0 \tag{6.164}$$

Solution:

We have that

$$n_e \exp\left(-\frac{e\Phi}{T_e}\right) = (n_{e0}+\widetilde{n}_e)\left(1-\frac{e\widetilde{\Phi}}{T_e}\right) \approx n_{e0} \tag{181}$$

$$\widetilde{n}_e - n_{e0}\frac{e\widetilde{\Phi}}{T_e} = 0 \tag{182}$$

$$\frac{\widetilde{n}_e}{n_{e0}} = \frac{e\widetilde{\Phi}}{T_e} \tag{183}$$

and

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = \frac{\partial n_i}{\partial t} + \mathbf{V}_i \cdot \nabla n_i = 0 \tag{184}$$

$$\tag{185}$$

Now, by momentum balance we find

$$m_i n_i \frac{\partial \mathbf{V}_i}{\partial t} = n_i q_i\left(\mathbf{E}+\mathbf{V}_i\times\mathbf{B}\right) - \cancel{\nabla p_i} \tag{186}$$

$$m_i n_{i0}\frac{\partial \widetilde{\mathbf{V}}_i}{\partial t} = n_{i0}q_i\left(\widetilde{\mathbf{E}}+\widetilde{\mathbf{V}}_i\times\mathbf{B}_0\right) \tag{187}$$

so along the field line we find

$$-i\omega m_i n_{i0}\widetilde{V}_{i\parallel} = n_{i0}q_i E_\parallel = -iq_i n_{i0}k_\parallel\widetilde{\Phi} \tag{188}$$

$$\widetilde{V}_{i\parallel} = \frac{q_i k_\parallel}{m_i\omega}\widetilde{\Phi} \tag{189}$$

and taking $\mathbf{B}_0\times$ the momentum balance equation yields

$$\overbrace{m_i n_{i0}\frac{\partial\widetilde{\mathbf{V}}_i\times\mathbf{B}_0}{\partial t}}^{\mathcal{O}(\omega/\Omega)} = n_{i0}q_i\left(\mathbf{B}_0\times\widetilde{\mathbf{E}}+\widetilde{\mathbf{V}}_{i\perp}B_0^2\right) \tag{190}$$

$$\widetilde{\mathbf{V}}_{i\perp} = \frac{\widetilde{\mathbf{E}}\times\mathbf{B}_0}{B_0^2} + \underbrace{\frac{m_i}{q_{i0}}\frac{\partial\widetilde{\mathbf{V}}_i\times\mathbf{B}_0}{\partial t}}_{\mathcal{O}(\omega/\Omega)} \tag{191}$$

which is just an $\mathbf{E} \times \mathbf{B}$ drift. Thus,

$$\frac{\partial \widetilde{n}_i}{\partial t} + \widetilde{\mathbf{V}}_i \cdot \nabla n_{i0} = 0 \tag{192}$$

$$-i\omega \widetilde{n}_i + \widetilde{\mathbf{V}}_{i\parallel} \cdot \nabla n_{i0} + \widetilde{\mathbf{V}}_{i\perp} \cdot \nabla n_{i0} = 0 \tag{193}$$

We find (letting $\nabla n_{i0} = \frac{n_{i0}}{L_n}\hat{\mathbf{x}}$ and $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$)

$$\frac{\widetilde{\mathbf{E}} \times \mathbf{B}_0}{B_0^2} \cdot \nabla n_{i0} = \frac{n_{i0}}{L_n}\hat{\mathbf{x}} \cdot \frac{-\nabla\widetilde{\Phi} \times \hat{\mathbf{z}}}{B_0} = -i\frac{n_{i0} k_y \widetilde{\Phi}}{L_n B_0} \tag{194}$$

Then

$$\widetilde{\mathbf{V}}_{i\parallel} \cdot \nabla n_{i0} = 0 \tag{195}$$

quite obviously for ∇n_{i0} in the $\hat{\mathbf{x}}$ direction and $\widetilde{\mathbf{V}}_{i\parallel}$ in the $\hat{\mathbf{z}}$ direction.

Thus we have

$$-i\omega \widetilde{n}_i + -i \overbrace{\frac{k_y T_{e0}}{eB_0 L_n}}^{-\omega_*} \frac{n_{i0} e\widetilde{\Phi}}{T_{e0}} = 0 \tag{196}$$

$$\omega \widetilde{n}_i - \omega_* \frac{n_{i0} e\widetilde{\Phi}}{T_{e0}} = 0 \tag{197}$$

$$\frac{\widetilde{n}_i}{n_{i0}} = \frac{\omega_{*e}}{\omega}\frac{e\widetilde{\Phi}}{T_{e0}} \tag{198}$$

and so by quasineutrality

$$\widetilde{n}_e = \widetilde{n}_i \tag{199}$$

$$n_{i0}\frac{\omega_*}{\omega}\frac{e\widetilde{\Phi}}{T_{e0}} = n_{e0}\frac{e\widetilde{\Phi}}{T_{e0}} \tag{200}$$

$$1 - \frac{\omega_*}{\omega} = 0 \tag{201}$$

$$\boxed{\omega = \omega_*} \tag{202}$$

It's clear that if there were a Doppler shift, we would simply have $\omega = \omega_* + \omega_E$ where $\omega_E = \frac{k_y c \Phi_0'}{B_0}$ with $\Phi_0'(x) = E_0$.

To prove this, let's we include an equilibrium electric field in the $\hat{\mathbf{x}}$ direction. We then have

$$\mathbf{E}_0 = -\mathbf{V}_{0i} \times \mathbf{B}_0 \tag{203}$$

$$\mathbf{E}_0 \times \mathbf{B}_0 = \mathbf{B}_0 \times (\mathbf{V}_{0i} \times \mathbf{B}_0) = \mathbf{B}_0^2 V_{0i\perp} \tag{204}$$

$$\mathbf{V}_{0i\perp} = \frac{\mathbf{E}_0 \times \mathbf{B}_0}{B_0^2} = \frac{E_0}{B_0}\hat{\mathbf{x}} \times \hat{\mathbf{z}} = \frac{-E_0}{B_0}\hat{\mathbf{y}} = \frac{\Phi_0'}{B_0}\hat{\mathbf{y}} \tag{205}$$

where $\Phi_0' = E_0$, and so we see there is an equilibrium $\mathbf{V}_0$. We see this won't change the perturbed equation if we ignore parallel velocity. We can ignore the $\hat{\mathbf{z}}$ or parallel velocity due to the electric field for the same reasons we could ignore the parallel electric field above: it doesn't contribute to the dispersion relation.

Thus, we find

$$\frac{\partial \widetilde{n}_i}{\partial t} + \widetilde{\mathbf{V}}_i \cdot \nabla n_{i0} + \mathbf{V}_{0i\perp} \cdot \nabla \widetilde{n}_i = 0 \tag{206}$$

$$-i\omega\widetilde{n}_i + \cancel{\widetilde{\mathbf{V}}_{i\parallel} \cdot \nabla n_{i0}} + \widetilde{\mathbf{V}}_{i\perp} \cdot \nabla n_{i0} + \frac{\Phi_0'}{B_0}\hat{\mathbf{y}} \cdot i\mathbf{k}\widetilde{n}_i = 0 \tag{207}$$

$$-i\omega\widetilde{n}_i + \widetilde{\mathbf{V}}_{i\perp} \cdot \nabla n_{i0} + \frac{\overbrace{ik_y\Phi_0'}^{i\omega_E}}{B_0}\widetilde{n}_i = 0 \tag{208}$$

$$-i(\omega - \omega_e)\widetilde{n}_i + \widetilde{\mathbf{V}}_{i\perp} \cdot \nabla n_{i0} + \frac{ik_y\Phi_0'}{B_0}\widetilde{n}_i = 0 \tag{209}$$

and so we now proceed as before replacing ω with $\omega - \omega_E$.

$$-i(\omega - \omega_E)\widetilde{n}_i + -i\,\frac{\overbrace{k_yT_{e0}}^{-\omega_*}}{eB_0L_n}\,\frac{n_{i0}e\widetilde{\Phi}}{T_{e0}} = 0 \tag{210}$$

$$(\omega - \omega_E)\widetilde{n}_i - \omega_*\frac{n_{i0}e\widetilde{\Phi}}{T_{e0}} = 0 \tag{211}$$

$$\frac{\widetilde{n}_i}{n_{i0}} = \frac{\omega_{*e}}{\omega - \omega_E}\frac{e\widetilde{\Phi}}{T_{e0}} \tag{212}$$

and so by quasineutrality

$$\widetilde{n}_e = \widetilde{n}_i \tag{213}$$

$$n_{i0}\frac{\omega_*}{\omega - \omega_E}\frac{e\widetilde{\Phi}}{T_{e0}} = n_{e0}\frac{e\widetilde{\Phi}}{T_{e0}} \tag{214}$$

$$1 - \frac{\omega_*}{\omega - \omega_E} = 0 \tag{215}$$

$$\boxed{\omega - \omega_E = \omega_*} \tag{216}$$

as promised.