1 Energy Exchange Rate Verify (5.52) for the energy exchange rate in the Maxwellian case.

$$\begin{aligned} W^L_{ss'} &= -8\pi \frac{\gamma_{ss'} n_{s'}}{m_{s'}} \frac{v_{\text{th}_s}}{v_{\text{th}_{s'}}} (T_s - T'_s) \int \mathrm{d}\eta\ \eta^4 f_{sM}(\eta) \left[\frac{\text{erf}(\eta')}{\eta'} - \left(1 + \frac{m_{s'} T_s}{m_s T_{s'}}\right) \text{erf}'(\eta') \right] \\ &= -\frac{4}{\sqrt{\pi}} \frac{\gamma_{ss'} n_s n_{s'} (T_s - T_{s'})}{m_{s'} (v^2_{\text{th}_s} + v^2_{\text{th}_{s'}})^{3/2}} \end{aligned} \tag{5.52}$$

(There is an obvious mistake of leaving out the $f_{sM}(\eta)$ in Hazeltine and Meiss in (5.52) and (5.51). They are also very imprecise in that the η's in the square brackets should be $\eta' = v/v_{\text{th}_{s'}}$ rather than $\eta = v/v_{\text{th}_s}$.)

Solution:

We begin with ($\eta = v/v_{\text{th}_s}$, $\xi = v_\parallel/v$, $\eta' = v/v_{\text{th}_{s'}}$, $\frac{m_{s'} T_s}{m_s T_{s'}} = v^2_{\text{th}_s}/v^2_{\text{th}_{s'}}$)

$$W^L_{ss'} \equiv \int \mathrm{d}^3 v\ \frac{1}{2} m_s v^2 C_{ss'} \tag{5.24}$$

$$C_{ss'}(f_{sM}, f_{s'M}) = -2 \frac{\gamma_{ss'} n_{s'}}{m'_s v_{\text{th}_{s'}} v^2_{\text{th}_s}} \left(1 - \frac{T_{s'}}{T_s}\right) f_{sM} \left[\frac{\text{erf}(\eta')}{\eta'} - \left(1 + \frac{m_{s'} T_s}{m_s T_{s'}}\right) \text{erf}'(\eta') \right] \tag{5.51}$$

So we find using $\mathrm{d}v = v_{\text{th}_s} \mathrm{d}\eta$, $f_{sM} = \frac{n_s}{\pi^{3/2} v^3_{\text{th}_s}} e^{-v^2/v^2_{\text{th}_s}} = \frac{n_s}{\pi^{3/2} v^3_{\text{th}_s}} e^{-\eta^2}$

$$W^L_{ss'} = \frac{m_s}{2} 2\pi \int_{-1}^{1} \mathrm{d}\xi \int_0^\infty \mathrm{d}v\ v^2 v^2 \left[-2 \frac{\gamma_{ss'} n_{s'}}{m'_s v_{\text{th}_{s'}} v^2_{\text{th}_s}} \left(1 - \frac{T_{s'}}{T_s}\right) f_{sM} \left[\frac{\text{erf}(\eta')}{\eta'} - \left(1 + \left(\frac{v_{\text{th}_s}}{v_{\text{th}_{s'}}}\right)^2\right) \text{erf}'(\eta') \right]\right] \tag{1}$$

$$= -\frac{4 m_s \pi \gamma_{ss'} n_{s'}}{m'_s v_{\text{th}_{s'}} v^2_{\text{th}_s}} \left(1 - \frac{T_{s'}}{T_s}\right) \int_0^\infty \mathrm{d}v\ v^4 f_{sM} \left[\frac{\text{erf}(\eta')}{\eta'} - \left(1 + \frac{v^2_{\text{th}_s}}{v^2_{\text{th}_{s'}}}\right) \text{erf}'(\eta') \right] \tag{2}$$

$$= -\frac{4\pi \gamma_{ss'} n_{s'}}{m_{s'} v_{\text{th}_{s'}} v^2_{\text{th}_s}} \frac{m_s}{T_s} (T_s - T_{s'}) \int_0^\infty \mathrm{d}\eta\ v^5_{\text{th}_s} \eta^4 f_{sM} \left[\frac{\text{erf}(\eta')}{\eta'} - \left(1 + \frac{v^2_{\text{th}_s}}{v^2_{\text{th}_{s'}}}\right) \text{erf}'(\eta') \right] \tag{3}$$

$$= -\frac{8\pi \gamma_{ss'} n_{s'} v^3_{\text{th}_s}}{m'_s v_{\text{th}_{s'}}} \frac{m_s}{2T_s} (T_s - T_{s'}) \int_0^\infty \mathrm{d}\eta\ \eta^4 f_{sM} \left[\frac{\text{erf}(\eta')}{\eta'} - \left(1 + \frac{v^2_{\text{th}_s}}{v^2_{\text{th}_{s'}}}\right) \text{erf}'(\eta') \right] \tag{4}$$

$$= -\frac{8\pi \gamma_{ss'} n_{s'} v^3_{\text{th}_s}}{m'_s v_{\text{th}_{s'}}} \frac{1}{v^2_{\text{th}_s}} (T_s - T_{s'}) \int_0^\infty \mathrm{d}\eta\ \eta^4 f_{sM} \left[\frac{\text{erf}(\eta')}{\eta'} - \left(1 + \frac{v^2_{\text{th}_s}}{v^2_{\text{th}_{s'}}}\right) \text{erf}'(\eta') \right] \tag{5}$$

$$\boxed{= -\frac{8\pi \gamma_{ss'} n_{s'} v_{\text{th}_s}}{m'_s v_{\text{th}_{s'}}} (T_s - T_{s'}) \int_0^\infty \mathrm{d}\eta\ \eta^4 f_{sM} \left[\frac{\text{erf}(\eta')}{\eta'} - \left(1 + \frac{v^2_{\text{th}_s}}{v^2_{\text{th}_{s'}}}\right) \text{erf}'(\eta') \right]} \tag{6}$$

$$= -\frac{8\pi \gamma_{ss'} n_{s'} v_{\text{th}_s}}{m'_s v_{\text{th}_{s'}}} (T_s - T_{s'}) \int_0^\infty \mathrm{d}\eta\ \eta^4 \frac{n_s e^{-\eta^2}}{\pi^{3/2} v^3_{\text{th}_s}} \left[\frac{\text{erf}(\eta')}{\eta'} - \left(1 + \frac{v^2_{\text{th}_s}}{v^2_{\text{th}_{s'}}}\right) \text{erf}'(\eta') \right] \tag{7}$$

$$= -\frac{8\pi \gamma_{ss'} n_{s'}}{m'_s v^2_{\text{th}_s} v_{\text{th}_{s'}}} (T_s - T_{s'}) \int_0^\infty \mathrm{d}\eta\ \eta^4 \frac{n_s e^{-\eta^2}}{\pi^{3/2}} \left[\frac{\text{erf}(\eta')}{\eta'} - \left(1 + \frac{v^2_{\text{th}_s}}{v^2_{\text{th}_{s'}}}\right) \text{erf}'(\eta') \right] \tag{8}$$

We focus on the integral now. We use that $\eta' = \eta \frac{v_{\text{th}_s}}{v_{\text{th}_{s'}}} \equiv \eta\gamma$ and $\text{erf}'(\eta') = \frac{2}{\sqrt{\pi}} e^{-\eta^2 \gamma^2}$.

Thus,

$$\int_0^\infty \mathrm{d}\eta\ \eta^4 e^{-\eta^2}\frac{\mathrm{erf}(\eta\gamma)}{\eta\gamma} = \frac{1}{\gamma}\int_0^\infty \mathrm{d}\eta\ \eta^3 e^{-\eta^2}\,\mathrm{erf}(\eta\gamma) = \frac{1}{\gamma}\int_0^\infty \mathrm{d}\eta\ \frac{\mathrm{d}}{\mathrm{d}\eta}\left[\frac{-e^{-\eta^2}}{2}(\eta^2+1)\right]\mathrm{erf}(\eta\gamma) \tag{9}$$

$$= \left.\frac{-e^{-\eta^2}}{2\gamma}(\eta^2+1)\,\mathrm{erf}(\eta\gamma)\right|_{\eta=0}^{\infty} - \frac{1}{2\gamma}\int_0^\infty \mathrm{d}\eta\ (-e^{-\eta^2})(\eta^2+1)\frac{\mathrm{d}}{\mathrm{d}\eta}\,\mathrm{erf}(\eta\gamma) \tag{10}$$

$$= \frac{1}{2\gamma}\int_0^\infty \mathrm{d}\eta\ e^{-\eta^2}\left(\eta^2+1\right)\mathrm{erf}'(\eta\gamma)\gamma = \frac{1}{2}\int_0^\infty \mathrm{d}\eta\ e^{-\eta^2}\left(\eta^2+1\right)\frac{2}{\sqrt{\pi}}e^{-\eta^2\gamma^2} \tag{11}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{d}\eta\ e^{-\eta^2(1+\gamma^2)}\left(\eta^2+1\right) = \frac{1}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{4(1+\gamma^2)^{3/2}} + \frac{\sqrt{\pi}}{2\sqrt{1+\gamma^2}}\right] = \frac{1}{4(1+\gamma^2)^{3/2}}\left[1+2+2\gamma^2\right] \tag{12}$$

$$= \frac{3+2\gamma^2}{4(1+\gamma^2)^{3/2}} \tag{13}$$

and

$$\int_0^\infty \mathrm{d}\eta\ \eta^4 e^{-\eta^2}\left(1+\gamma^2\right)\frac{2}{\sqrt{\pi}}e^{-\eta^2\gamma^2} = \frac{2(1+\gamma^2)}{\sqrt{\pi}}\int_0^\infty \mathrm{d}\eta\ e^{-\eta^2(1+\gamma^2)}\eta^4 \tag{14}$$

$$= \frac{2(1+\gamma^2)}{\sqrt{\pi}}\left[\frac{3\sqrt{\pi}}{8(1+\gamma^2)^{5/2}}\right] = \frac{3}{4(1+\gamma^2)^{3/2}} \tag{15}$$

so that

$$\int_0^\infty \mathrm{d}\eta\ \eta^4 e^{-\eta^2}\left[\frac{\mathrm{erf}(\eta')}{\eta'} - \left(1+\gamma^2\right)\mathrm{erf}'(\eta')\right] = \frac{3+2\gamma^2}{4(1+\gamma^2)^{3/2}} - \frac{3}{4(1+\gamma^2)^{3/2}} \tag{16}$$

$$= \frac{\gamma^2}{2(1+\gamma^2)^{3/2}} = \frac{v_{\mathrm{th}_s}^2}{2v_{\mathrm{th}_{s'}}^2\left(1+\frac{v_{\mathrm{th}_s}^2}{v_{\mathrm{th}_{s'}}^2}\right)^{3/2}} = \frac{v_{\mathrm{th}_s}^2 v_{\mathrm{th}_{s'}}}{2\left(v_{\mathrm{th}_{s'}}^2+v_{\mathrm{th}_s}^2\right)^{3/2}} \tag{17}$$

So that we find

$$-\frac{8\gamma_{ss'}n_s n_{s'}}{\sqrt{\pi}m_s' v_{\mathrm{th}_s}^2 v_{\mathrm{th}_{s'}}}\left(T_s - T_{s'}\right)\int_0^\infty \mathrm{d}\eta\ \eta^4 e^{-\eta^2}\left[\frac{\mathrm{erf}(\eta')}{\eta'} - \left(1+\frac{v_{\mathrm{th}_s}^2}{v_{\mathrm{th}_{s'}}^2}\right)\mathrm{erf}'(\eta')\right] \tag{18}$$

$$= -\frac{8\gamma_{ss'}n_s n_{s'}}{\sqrt{\pi}m_{s'} v_{\mathrm{th}_s}^2 v_{\mathrm{th}_{s'}}}(T_s - T_{s'})\frac{v_{\mathrm{th}_s}^2 v_{\mathrm{th}_{s'}}}{2\left(v_{\mathrm{th}_{s'}}^2+v_{\mathrm{th}_s}^2\right)^{3/2}} \tag{19}$$

$$= \frac{4\gamma_{ss'}n_s n_{s'}(T_s - T_{s'})}{\sqrt{\pi}m_{s'}(v_{\mathrm{th}_s}^2+v_{\mathrm{th}_{s'}}^2)^{3/2}} \tag{20}$$

And so we recover the correct result,

$$\boxed{W_{ss'}^L = \frac{4}{\sqrt{\pi}}\frac{\gamma_{ss'}n_s n_{s'}\left(T_s - T_{s'}\right)}{\left(v_{\mathrm{th}_s}^2+v_{\mathrm{th}_{s'}}^2\right)^{3/2}}} \tag{21}$$

2 Momentum Conservation Starting with (5.33), show explicitly that the like-particle collision operator conserves momentum:

$$\int \mathrm{d}^3v \; m_s \mathbf{v} C_{ss'} \equiv 0 \tag{22}$$

$$C_{ss'} = \frac{\gamma_{ss'}}{2}\frac{\partial}{\partial \mathbf{v}} \cdot \int \mathrm{d}^3v' \; \overleftrightarrow{\mathbf{U}} \cdot \left[\frac{f'_{s'}}{m_s}\frac{\partial f_s}{\partial \mathbf{v}} - \frac{f_s}{m_{s'}}\frac{\partial f'_{s'}}{\partial \mathbf{v}'}\right] \tag{5.33}$$

Solution:

We then have

$$\int \mathrm{d}^3v \; m_s \mathbf{v} C_{ss'} = \frac{m_s\gamma_{ss'}}{2}\int \mathrm{d}^3v \; \mathbf{v}\frac{\partial}{\partial \mathbf{v}} \cdot \underbrace{\int \mathrm{d}^3v' \; \overleftrightarrow{\mathbf{U}} \cdot \left[\frac{f'_{s'}}{m_s}\frac{\partial f_s}{\partial \mathbf{v}} - \frac{f_s}{m_{s'}}\frac{\partial f'_{s'}}{\partial \mathbf{v}'}\right]}_{\mathbf{T}(\mathbf{v})} \tag{23}$$

$$= \frac{m_s\gamma_{ss'}}{2}\int \mathrm{d}^3v \; \mathbf{v}\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{T} = \frac{m_s\gamma_{ss'}}{2}\int \mathrm{d}^3v \; \left[\frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{T}\mathbf{v}) - \mathbf{T} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{v}}\right] \tag{24}$$

We see this is correct through

$$v_i \partial_j T_j = \partial_j (v_i T_j) - T_j \partial_j v_i = \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{T}\mathbf{v}) - \mathbf{T} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{v}} \tag{25}$$

and using Gauss's Law with $\partial \mathbf{v}/\partial \mathbf{v} = \mathbb{1}$ we see

$$\int \mathrm{d}^3v \; m_s \mathbf{v} C_{ss'} = \frac{m_s\gamma_{ss'}}{2}\left[\oint \mathrm{d}\mathbf{S}_v \; \cdot \mathbf{T}\mathbf{v} - \int \mathrm{d}^3v \; \mathbf{T}\right] \tag{26}$$

Now, as $\mathbf{T}(\mathbf{v}) = \mathbf{T}(f_s(\mathbf{v}))$ and $f_s(\mathbf{v}) = 0$ at infinitely far away where the surface is, we see the surface integral vanishes. We then are left

$$I_{ss'} = \frac{m_s\gamma_{ss'}}{2}\int \mathrm{d}^3v \; \int \mathrm{d}^3v' \; \overleftrightarrow{\mathbf{U}} \cdot \left[\frac{f'_{s'}}{m_s}\frac{\partial f_s}{\partial \mathbf{v}} - \frac{f_s}{m_{s'}}\frac{\partial f'_{s'}}{\partial \mathbf{v}'}\right] \tag{27}$$

Note that we have (switching s and s' and $\mathbf{v}$ and $\mathbf{v}'$)

$$I_{s's} = \frac{m_{s'}\gamma_{s's}}{2}\int \mathrm{d}^3v \; \int \mathrm{d}^3v' \; \overleftrightarrow{\mathbf{U}} \cdot \left[\frac{f_s}{m_{s'}}\frac{\partial f'_{s'}}{\partial \mathbf{v}'} - \frac{f'_{s'}}{m_s}\frac{\partial f_s}{\partial \mathbf{v}}\right] \tag{28}$$

As

$$m_s\gamma_{ss'} = \frac{4\pi m_s e_s^2 e_{s'}^2}{m_s}\ln\Lambda = m_{s'}\gamma_{s's} \tag{29}$$

We see that

$$I_{ss'} + I_{s's} = 0 \tag{30}$$

Thus, we see that

$$\int \mathrm{d}^3v \; m_s \mathbf{v} C_{ss'} + \int \mathrm{d}^3v \; m_{s'} \mathbf{v} C_{s's} = 0 \tag{31}$$

Now, if we have like-particles, then $s = s'$ and hence

$$2\int \mathrm{d}^3v \; m_s\mathbf{v}C_{ss} = 0 \tag{32}$$

$$\boxed{\int \mathrm{d}^3v \; m_s\mathbf{v}C_{ss} \equiv 0} \tag{33}$$

3 Diffusion Tensor Relate the diffusion tensor $\overleftrightarrow{\mathbf{D}}$ of (5.40) to derivatives of the Rosenbluth potential. Then use (5.49) to compute $\overleftrightarrow{\mathbf{D}}$ explicitly for the case of a Maxwellian distribution.

$$\overleftrightarrow{\mathbf{D}}_{ss'} = \frac{\gamma_{ss'}}{2m_s}\int \mathrm{d}^3v\ \overleftrightarrow{\mathbf{U}} f_s \tag{5.40}$$

$$\begin{aligned} G_{sM} &= \frac{n_s v_{\mathrm{th}_s}}{2\eta}\left[\eta\,\mathrm{erf}'(\eta) + (1+2\eta^2)\,\mathrm{erf}(\eta)\right] \\ H_{sM} &= \frac{n_s}{v_{\mathrm{th}_s}\eta}\,\mathrm{erf}(\eta) \end{aligned} \tag{5.49}$$

Solution:

We remember that

$$\overleftrightarrow{\mathbf{U}} = \frac{u^2\mathbb{1} - \mathbf{u}\mathbf{u}}{u^3} \tag{34}$$

$$U_{\alpha\beta} = \frac{u^2\delta_{\alpha\beta} - u_\alpha u_\beta}{u^3} \tag{35}$$

$$\mathbf{u} = \mathbf{v} - \mathbf{v}' \tag{36}$$

$$u_\alpha = v_\alpha - v'_\alpha \tag{37}$$

Now as

$$\overleftrightarrow{\mathbf{U}} = \frac{\partial^2 u}{\partial\mathbf{u}\partial\mathbf{u}} = \frac{\partial}{\partial\mathbf{u}}\left(\frac{\mathbf{u}}{u}\right) \tag{38}$$

$$U_{\alpha\beta} = \frac{\partial^2 u}{\partial u_\alpha \partial u_\beta} = \frac{\partial}{\partial u_\alpha}\left(\frac{u_\beta}{u}\right) \tag{39}$$

We note that $\frac{\partial}{\partial\mathbf{v}} = \frac{\partial}{\partial\mathbf{u}}$ and similarly $\frac{\partial}{\partial\mathbf{v}'} = -\frac{\partial}{\partial\mathbf{u}}$ because

$$\frac{\partial}{\partial\mathbf{v}} = \frac{\partial\mathbf{u}}{\partial\mathbf{v}}\cdot\frac{\partial}{\partial\mathbf{u}} = \frac{\partial(\mathbf{v}-\mathbf{v}')}{\partial\mathbf{v}}\frac{\partial}{\partial\mathbf{u}} = \left(\mathbb{1} - \overleftrightarrow{\mathbf{0}}\right)\cdot\frac{\partial}{\partial\mathbf{u}} = \frac{\partial}{\partial\mathbf{u}} \tag{40}$$

$$\frac{\partial}{\partial v_\alpha} = \frac{\partial u_\beta}{\partial v_\alpha}\frac{\partial}{\partial u_\beta} = \frac{\partial(v_\beta - v'_\beta)}{\partial v_\alpha}\frac{\partial}{\partial u_\beta} = (\delta_{\alpha\beta} - 0)\frac{\partial}{\partial u_\beta} = \frac{\partial}{\partial u_\alpha} \tag{41}$$

$$\frac{\partial}{\partial\mathbf{v}'} = \frac{\partial\mathbf{u}'}{\partial\mathbf{v}}\cdot\frac{\partial}{\partial\mathbf{u}} = \frac{\partial(\mathbf{v}-\mathbf{v}')}{\partial\mathbf{v}'}\frac{\partial}{\partial\mathbf{u}} = \left(\overleftrightarrow{\mathbf{0}} - \mathbb{1}\right)\cdot\frac{\partial}{\partial\mathbf{u}} = -\frac{\partial}{\partial\mathbf{u}} \tag{42}$$

$$\frac{\partial}{\partial v'_\alpha} = \frac{\partial u_\beta}{\partial v'_\alpha}\frac{\partial}{\partial u_\beta} = \frac{\partial(v_\beta - v'_\beta)}{\partial v'_\alpha}\frac{\partial}{\partial u_\beta} = (0 - \delta_{\alpha\beta})\frac{\partial}{\partial u_\beta} = -\frac{\partial}{\partial u_\alpha} \tag{43}$$

Thus

$$\overleftrightarrow{\mathbf{U}} = \frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{u}}{u}\right) = \frac{\partial}{\partial\mathbf{v}}\frac{\partial}{\partial\mathbf{v}}u = \frac{\partial^2 u}{\partial\mathbf{v}\partial\mathbf{v}} \tag{44}$$

$$U_{\alpha\beta} = \frac{\partial}{\partial v_\alpha}\left(\frac{u_\beta}{u}\right) = \frac{\partial}{\partial v_\alpha}\frac{\partial}{\partial v_\beta}u = \frac{\partial^2 u}{\partial v_\alpha \partial v_\beta} \tag{45}$$

$$\overleftrightarrow{\mathbf{U}} = -\frac{\partial}{\partial\mathbf{v}'}\left(\frac{\mathbf{u}}{u}\right) = -\frac{\partial}{\partial\mathbf{v}}\left(-\frac{\partial}{\partial\mathbf{v}}u\right) = \frac{\partial^2 u}{\partial\mathbf{v}'\partial\mathbf{v}'} \tag{46}$$

$$U_{\alpha\beta} = -\frac{\partial}{\partial v'_\alpha}\left(\frac{u_\beta}{u}\right) = \frac{\partial}{\partial v'_\alpha}\left(-\frac{\partial}{\partial v'_\beta}u\right) = \frac{\partial^2 u}{\partial v'_\alpha \partial v'_\beta} \tag{47}$$

This yields

$$\int \mathrm{d}^3v\ \overleftrightarrow{\mathbf{U}} f_s = \frac{\partial}{\partial \mathbf{v}' \partial \mathbf{v}'} \int \mathrm{d}^3v\ u f_s \tag{48}$$

$$\int \mathrm{d}^3v\ U_{\alpha\beta} f_s = \frac{\partial}{\partial v'_\alpha \partial v'_\beta} \int \mathrm{d}^3v\ u f_s \tag{49}$$

Now we let $\mathbf{v} \leftrightarrow \mathbf{v}'$ (that is, swap $\mathbf{v}$ and $\mathbf{v}'$, which doesn't affect u) so that

$$\frac{\partial}{\partial \mathbf{v} \partial \mathbf{v}} \int \mathrm{d}^3v'\ u f'_s = \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} G_s(\mathbf{v}) \tag{50}$$

$$\frac{\partial}{\partial v_\alpha \partial v_\beta} \int \mathrm{d}^3v'\ u f'_s = \frac{\partial^2}{\partial v_\alpha \partial v_\beta} G_s(\mathbf{v}) \tag{51}$$

So we find

$$\boxed{\overleftrightarrow{\mathbf{D}}_{ss'} = \frac{\gamma_{ss'}}{2m_s} \int \mathrm{d}^3v\ \overleftrightarrow{\mathbf{U}} f_s = \frac{\gamma_{ss'}}{2m_s} \frac{\partial^2 G_s(\mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}} = \frac{\gamma_{ss'}}{2m_s} \frac{\partial^2 G_s(\mathbf{v})}{\partial v_\alpha \partial v_\beta}} \tag{52}$$

For the Maxwellian case we have

$$G_s(\mathbf{v}) = \frac{n_s v_{\mathrm{th}_s}}{2 \frac{v}{v_{\mathrm{th}_s}}} \left[\frac{v}{v_{\mathrm{th}_s}} \operatorname{erf}' \left(\frac{v}{v_{\mathrm{th}_s}} \right) + \left(1 + 2 \frac{v^2}{v_{\mathrm{th}_s}^2} \right) \operatorname{erf} \left(\frac{v}{v_{\mathrm{th}_s}} \right) \right] \tag{53}$$

We use that

$$\frac{\partial v}{\partial \mathbf{v}} = \frac{\partial \sqrt{\mathbf{v} \cdot \mathbf{v}}}{\partial \mathbf{v}} = \frac{1}{2\sqrt{\mathbf{v} \cdot \mathbf{v}}} \frac{\partial \mathbf{v} \cdot \mathbf{v}}{\partial \mathbf{v}} = \frac{\mathbb{1} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbb{1}}{2v} = \frac{\mathbf{v}}{v} \tag{54}$$

$$\frac{\partial v}{\partial v_\alpha} = \frac{\partial \sqrt{v_\alpha v_\alpha}}{\partial v_\alpha} = \frac{1}{2\sqrt{v^2}} \frac{\partial v_\alpha^2}{\partial v_\alpha} = \frac{2v_\alpha}{2v} = \frac{v_\alpha}{v} \tag{55}$$

$$\tag{56}$$

Thus, we have (using $\operatorname{erf}'(\eta) = \frac{2}{\sqrt{\pi}} e^{-\eta^2}$)

$$\frac{\partial G_s}{\partial \mathbf{v}} = \frac{n_s v_{\mathrm{th}_s}^2}{2} \frac{\partial}{\partial \mathbf{v}} \left[\frac{1}{v} \left\{ \frac{v}{v_{\mathrm{th}_s}} \frac{2}{\sqrt{\pi}} e^{-v^2/v_{\mathrm{th}_s}^2} + \left(1 + \frac{2v^2}{v_{\mathrm{th}_s}^2} \right) \operatorname{erf} \left(\frac{v}{v_{\mathrm{th}_s}} \right) \right\} \right] \tag{57}$$

$$= \frac{n_s v_{\mathrm{th}_s}^2}{2} \frac{\partial}{\partial \mathbf{v}} \left[\frac{2}{\sqrt{\pi} v_{\mathrm{th}_s}} e^{-v^2/v_{\mathrm{th}_s}^2} + \left(\frac{1}{v} + \frac{2v}{v_{\mathrm{th}_s}^2} \right) \operatorname{erf} \left(\frac{v}{v_{\mathrm{th}_s}} \right) \right] \tag{58}$$

Now let's use

$$\frac{\partial}{\partial \mathbf{v}} = \frac{\partial v}{\partial \mathbf{v}} \frac{\partial}{\partial v} = \frac{\mathbf{v}}{v} \frac{\partial}{\partial v} \tag{59}$$

and take the derivatives one at a time

$$\frac{\partial}{\partial v} \left(\frac{2}{\sqrt{\pi} v_{\mathrm{th}_s}} e^{-v^2/v_{\mathrm{th}_s}^2} \right) = \frac{2}{\sqrt{\pi} v_{\mathrm{th}_s}} \frac{-2v}{v_{\mathrm{th}_s}^2} e^{-v^2 v_{\mathrm{th}_s}^2} = -\frac{4v}{\sqrt{\pi} v_{\mathrm{th}_s}^3} e^{-v^2/v_{\mathrm{th}_s}^2} \tag{60}$$

$$\frac{\partial}{\partial v} \left(\frac{1}{v} \operatorname{erf} \left(\frac{v}{v_{\mathrm{th}_s}} \right) \right) = \frac{-1}{v^2} \operatorname{erf} \left(\frac{v}{v_{\mathrm{th}_s}} \right) + \frac{1}{v} \frac{1}{v_{\mathrm{th}_s}} \operatorname{erf}' \left(\frac{v}{v_{\mathrm{th}_s}} \right) = \frac{2}{\sqrt{\pi} v v_{\mathrm{th}_s}} e^{-v^2/v_{\mathrm{th}_s}^2} - \frac{\operatorname{erf} \left(\frac{v}{v_{\mathrm{th}_s}} \right)}{v^2} \tag{61}$$

$$\frac{\partial}{\partial v} \left(\frac{2v}{v_{\mathrm{th}_s}^2} \operatorname{erf} \left(\frac{v}{v_{\mathrm{th}_s}} \right) \right) = \frac{2}{v_{\mathrm{th}_s}^2} \operatorname{erf} \left(\frac{v}{v_{\mathrm{th}_s}} \right) + \frac{2v}{v_{\mathrm{th}_s}^2} \frac{1}{v_{\mathrm{th}_s}} \operatorname{erf}' \left(\frac{v}{v_{\mathrm{th}_s}} \right) = \frac{2}{v_{\mathrm{th}_s}^2} \operatorname{erf} \left(\frac{v}{v_{\mathrm{th}_s}} \right) + \frac{4v}{\sqrt{\pi} v_{\mathrm{th}_s}^3} e^{-v^2/v_{\mathrm{th}_s}^2} \tag{62}$$

Thus,

$$\begin{aligned}\frac{\partial G_s}{\partial \mathbf{v}} &= \frac{n_s v_{\text{th}_s}^2}{2}\frac{\mathbf{v}}{v}\left[-\cancel{\frac{4v}{\sqrt{\pi}v_{\text{th}_s}^3}e^{-v^2/v_{\text{th}_s}^2}} + \frac{2}{\sqrt{\pi}vv_{\text{th}_s}}e^{-v^2/v_{\text{th}_s}^2} - \frac{\text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)}{v^2}\right.\\ &\qquad\left. + \frac{2}{v_{\text{th}_s}^2}\text{erf}\left(\frac{v}{v_{\text{th}_s}}\right) + \cancel{\frac{4v}{\sqrt{\pi}v_{\text{th}_s}^3}e^{-v^2/v_{\text{th}_s}^2}}\right] \qquad (63)\\ &= \frac{n_s v_{\text{th}_s}^2}{2}\mathbf{v}\left[\frac{2}{\sqrt{\pi}v^2 v_{\text{th}_s}}e^{-v^2/v_{\text{th}_s}^2} + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{2}{vv_{\text{th}_s}^2} - \frac{1}{v^3}\right)\right] \qquad (64)\end{aligned}$$

Now we take the second derivative

$$\frac{\partial^2 G_s}{\partial \mathbf{v}\partial \mathbf{v}} = \frac{n_s v_{\text{th}_s}^2}{2}\frac{\partial}{\partial \mathbf{v}}\left\{\mathbf{v}\left[\frac{2}{\sqrt{\pi}v^2 v_{\text{th}_s}}e^{-v^2/v_{\text{th}_s}^2} + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{2}{vv_{\text{th}_s}^2} - \frac{1}{v^3}\right)\right]\right\} \qquad (65)$$

$$\begin{aligned}&= \frac{n_s v_{\text{th}_s}^2}{2}\left\{\frac{\partial \mathbf{v}}{\partial \mathbf{v}}\left[\frac{2}{\sqrt{\pi}v^2 v_{\text{th}_s}}e^{-v^2/v_{\text{th}_s}^2} + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{2}{vv_{\text{th}_s}^2} - \frac{1}{v^3}\right)\right]\right.\\ &\qquad\left. + \mathbf{v}\frac{\mathbf{v}}{v}\frac{\partial}{\partial v}\left[\frac{2}{\sqrt{\pi}v^2 v_{\text{th}_s}}e^{-v^2/v_{\text{th}_s}^2} + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{2}{vv_{\text{th}_s}^2} - \frac{1}{v^3}\right)\right]\right\} \qquad (66)\end{aligned}$$

Now as $\frac{\partial \mathbf{v}}{\partial \mathbf{v}} = \mathbb{1}$, we only need to calculate the second portion

$$\frac{\partial}{\partial v}\left[\frac{2}{\sqrt{\pi}v^2 v_{\text{th}_s}}e^{-v^2/v_{\text{th}_s}^2} + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{2}{vv_{\text{th}_s}^2} - \frac{1}{v^3}\right)\right] \qquad (67)$$

So

$$\frac{2}{\sqrt{\pi}v_{\text{th}_s}}\frac{\partial}{\partial v}\left[\frac{1}{v^2}e^{-v^2/v_{\text{th}_s}^2}\right] = \frac{2}{\sqrt{\pi}v_{\text{th}_s}}\left[\frac{-2}{v^3} - \frac{2v}{v^2 v_{\text{th}_s}^2}\right]e^{-v^2/v_{\text{th}_s}^2} = -\frac{4}{\sqrt{\pi}v_{\text{th}_s}v}\left[\frac{1}{v^2} + \frac{1}{v_{\text{th}_s}^2}\right]e^{-v^2/v_{\text{th}_s}^2} \qquad (68)$$

$$\begin{aligned}\frac{\partial}{\partial v}\left[\text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{2}{vv_{\text{th}_s}^2} - \frac{1}{v^3}\right)\right] &= \frac{1}{v_{\text{th}_s}}\text{erf}'\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{2}{vv_{\text{th}_s}^2} - \frac{1}{v^3}\right) + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{-2}{v^2 v_{\text{th}_s}^2} + \frac{3}{v^4}\right) \qquad (69)\\ &= \frac{2e^{-v^2/v_{\text{th}_s}^2}}{\sqrt{\pi}v_{\text{th}_s}}\left(\frac{2}{vv_{\text{th}_s}^2} - \frac{1}{v^3}\right) + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{-2}{v^2 v_{\text{th}_s}^2} + \frac{3}{v^4}\right) \qquad (70)\end{aligned}$$

Thus we find

$$\begin{aligned}\frac{\partial^2 G_s}{\partial \mathbf{v}\partial \mathbf{v}} &= \frac{n_s v_{\text{th}_s}^2}{2}\left\{\mathbb{1}\left[\frac{2}{\sqrt{\pi}v^2 v_{\text{th}_s}}e^{-v^2/v_{\text{th}_s}^2} + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{2}{vv_{\text{th}_s}^2} - \frac{1}{v^3}\right)\right]\right.\\ &\quad\left. + \mathbf{v}\mathbf{v}\left[-\frac{2e^{-v^2/v_{\text{th}_s}^2}}{\sqrt{\pi}v_{\text{th}_s}}\left[\frac{2}{v^4} + \cancel{\frac{2}{v_{\text{th}_s}^2 v^2}}\right] + \frac{2e^{-v^2/v_{\text{th}_s}^2}}{\sqrt{\pi}v_{\text{th}_s}}\left(\cancel{\frac{2}{v^2 v_{\text{th}_s}^2}} - \frac{1}{v^4}\right) + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{-2}{v^3 v_{\text{th}_s}^2} + \frac{3}{v^5}\right)\right]\right\} \qquad (71)\end{aligned}$$

Changing back to pure erf and erf′ notation with $\eta = v/v_{\text{th}_s}$ we find

$$\begin{aligned}\frac{\partial^2 G_s}{\partial \mathbf{v}\partial \mathbf{v}} &= \frac{n_s v_{\text{th}_s}^2}{2}\Bigg\{\mathbb{1}\left[\frac{2}{\sqrt{\pi}v^2 v_{\text{th}_s}}e^{-v^2/v_{\text{th}_s}^2} + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{2}{v_{\text{th}_s}^2 v} - \frac{1}{v^3}\right)\right] \\ &\qquad + \mathbf{v}\mathbf{v}\left[-\frac{6e^{-v^2/v_{\text{th}_s}^2}}{\sqrt{\pi}v_{\text{th}_s}v^4} + \text{erf}\left(\frac{v}{v_{\text{th}_s}}\right)\left(\frac{-2}{v^3 v_{\text{th}_s}^2} + \frac{3}{v^5}\right)\right]\Bigg\}\end{aligned} \tag{72}$$

$$\begin{aligned}&= \frac{n_s v_{\text{th}_s}^2}{2}\Bigg\{\mathbb{1}\left[\frac{1}{\eta^2 v_{\text{th}_s}^3}\text{erf}'(\eta) + \text{erf}(\eta)\left(\frac{2}{v_{\text{th}_s}^3 \eta} - \frac{1}{\eta^3 v_{\text{th}_s}^3}\right)\right] \\ &\qquad + \mathbf{v}\mathbf{v}\left[-\frac{3\,\text{erf}'(\eta)}{v_{\text{th}_s}^5 \eta^4} + \text{erf}(\eta)\left(\frac{-2}{\eta^3 v_{\text{th}_s}^5} + \frac{3}{\eta^5 v_{\text{th}_s}^5}\right)\right]\Bigg\}\end{aligned} \tag{73}$$

Now we let $\boldsymbol{\eta} = \mathbf{v}/v_{\text{th}_s}$ and

$$\begin{aligned}\frac{\partial^2 G_s}{\partial \mathbf{v}\partial \mathbf{v}} &= \frac{n_s v_{\text{th}_s}^2}{2}\Bigg\{\frac{\mathbb{1}}{v_{\text{th}_s}^3}\left[\frac{1}{\eta^2}\text{erf}'(\eta) + \text{erf}(\eta)\left(\frac{2}{\eta} - \frac{1}{\eta^3}\right)\right] \\ &\qquad + \boldsymbol{\eta}\boldsymbol{\eta}\left[-\frac{3\,\text{erf}'(\eta)}{v_{\text{th}_s}^3 \eta^4} + \text{erf}(\eta)\left(\frac{-2}{\eta^3 v_{\text{th}_s}^3} + \frac{3}{\eta^5 v_{\text{th}_s}^3}\right)\right]\Bigg\}\end{aligned} \tag{74}$$

$$\begin{aligned}&= \frac{n_s}{2v_{\text{th}_s}}\Bigg\{\mathbb{1}\frac{1}{\eta^3}\left[\eta\,\text{erf}'(\eta) + \text{erf}(\eta)\left(2\eta^2 - 1\right)\right] \\ &\qquad + \boldsymbol{\eta}\boldsymbol{\eta}\left[-\frac{3\,\text{erf}'(\eta)}{\eta^4} + \text{erf}(\eta)\left(\frac{-2}{\eta^3} + \frac{3}{\eta^5}\right)\right]\Bigg\}\end{aligned} \tag{75}$$

$$\begin{aligned}&= \frac{n_s}{2v_{\text{th}_s}}\Bigg\{\mathbb{1}\frac{1}{\eta^3}\left[\eta\,\text{erf}'(\eta) + \text{erf}(\eta)\left(2\eta^2 - 1\right)\right] \\ &\qquad + \boldsymbol{\eta}\boldsymbol{\eta}\frac{3}{\eta^5}\left[-\text{erf}'(\eta) + \text{erf}(\eta)\left(1 - \frac{2\eta^2}{3}\right)\right]\Bigg\}\end{aligned} \tag{76}$$

If we define

$$M_1(\eta) = \frac{1}{\eta^3}\left[\eta\,\text{erf}'(\eta) + \text{erf}(\eta)\left(2\eta^2 - 1\right)\right] \tag{77}$$

$$M_2(\eta) = \frac{3}{\eta^5}\left[\text{erf}'(\eta) + \text{erf}(\eta)\left(\frac{2\eta^2}{3} - 1\right)\right] \tag{78}$$

noting that

$$M_2(\eta) = \frac{1}{v}\frac{\partial}{\partial v}(-M_1(\eta)) = -\frac{1}{v}\frac{\partial \eta}{\partial v}\frac{\partial}{\partial \eta}M_1(\eta) = -\frac{1}{\eta}\frac{\partial M_1(\eta)}{\partial \eta} \tag{79}$$

we have using $\overleftrightarrow{\mathbf{D}} = \frac{\gamma_{ss'}}{m_s}\frac{\partial^2 G_s}{\partial \mathbf{v}\partial \mathbf{v}}$ that

$$\overleftrightarrow{\mathbf{D}}_{ss'} = \frac{\gamma_{ss'} n_s}{4 m_s v_{\text{th}_s}} \left[\mathbb{1} M_1(\eta) - \boldsymbol{\eta}\boldsymbol{\eta} M_2(\eta)\right] \tag{80}$$

$$D_{ss',\alpha\beta} = \frac{\gamma_{ss'} n_s}{4 m_s v_{\text{th}_s}} \left[\delta_{\alpha\beta} M_1(\eta) - \eta_\alpha \eta_\beta \eta M_2(\eta)\right] \tag{81}$$

with $\boldsymbol{\eta} = \mathbf{v}/v_{\text{th}}$, $\eta = v/v_{\text{th}}$ and

$$M_1(\eta) = \frac{1}{\eta^3} \left[\eta \,\text{erf}'(\eta) + \text{erf}(\eta) \left(2\eta^2 - 1\right)\right] \tag{82}$$

$$M_2(\eta) = \frac{3}{\eta^5} \left[\text{erf}'(\eta) + \text{erf}(\eta) \left(\frac{2\eta^2}{3} - 1\right)\right] = -\frac{1}{\eta} \frac{\partial M_1}{\partial \eta} \tag{83}$$

4 Krook Collision Operator The *Krook collision operator* is a model linearized operator defined by

$$C_K(f_1) \equiv -\nu \left[f_1 - \frac{n_1}{n_M} f_M \right] \tag{84}$$

where ν is a constant representing the collision frequency and

$$n_1 \equiv \int \mathrm{d}^3v \ f_1 \tag{85}$$

Discuss this operator in terms of the general physical constraints imposed in §5.1. Specify in particular which conservation laws C_K obeys.

Solution:

We see that $C_K(f_1)$ is obviously bilinear, that it will vanish when $f_1 = f_M$. We see that it should have Galilean invariance, as it has no direct dependence on $\mathbf{v}$ or $\mathbf{v}'$. It does not have all the spatial symmetry because there is position information in n_1. For the H theorem we see

$$\Theta = -\sum_s \int \mathrm{d}^3v \ \ln f_s C_s = -\sum_s \int \mathrm{d}^3v \ \ln f_s \left\{ -\nu \left[f_s - \frac{n_s}{n_M} f_M \right] \right\} \tag{86}$$

$$= \nu \sum_s \int \mathrm{d}^3v \ \left[f_s - \frac{n_s}{n_M} f_M \right] \ln f_s \tag{87}$$

Hence, it is difficult to tell if the H theorem is satisfied as I see no easy simplifications from here. However, since it is taking particles from f_s and putting them into f_M, it is intuitively clear that it should be increasing entropy.

For conservation, we see that it does conserve particles, as

$$\int \mathrm{d}^3v \ C_1 = \int \mathrm{d}^3v \ -\nu \left[f_1 - \frac{n_1}{n_M} f_M \right] = -\nu \left[\int \mathrm{d}^3v \ f_1 - \frac{n_1}{n_M} \int \mathrm{d}^3v \ f_M \right] \tag{88}$$

$$= -\nu \left[n_1 - \frac{n_1}{\cancel{n_M}} \cancel{n_M} \right] = 0 \tag{89}$$

We use the definitions

$$\int \mathrm{d}^3v \ m_1 \mathbf{v} f_1 = n_1 \delta \mathbf{u} \tag{90}$$

$$\int \mathrm{d}^3v \ m_1 \mathbf{v} f_M = n_M \mathbf{u} \tag{91}$$

$$\int \mathrm{d}^3v \ \frac{m_1 v^2}{2} f_1 = \frac{3n_1}{2} T_1 \tag{92}$$

$$\int \mathrm{d}^3v \ \frac{m_1 v^2}{2} f_M = \frac{3n_M}{2} T \tag{93}$$

As for momentum, we find

$$\int \mathrm{d}^3v\ m_1\mathbf{v}C_1 = \int \mathrm{d}^3v\ \ -\nu m_1\mathbf{v}\left[f_1 - \frac{n_1}{n_M}f_M\right] = -\nu m_1\left[\int \mathrm{d}^3v\ \mathbf{v}f_1 - \frac{n_1}{n_M}\int \mathrm{d}^3v\ \mathbf{v}f_M\right] \tag{94}$$

$$= -\nu\left[n_1\delta\mathbf{u} - \frac{n_1}{\cancel{n_M}}\cancel{n_M}\mathbf{u}\right] = -\nu n_1\left[\delta\mathbf{u} - \mathbf{u}\right] \neq 0 \tag{95}$$

and for energy

$$\int \mathrm{d}^3v\ \frac{1}{2}m_1v^2C_1 = \int \mathrm{d}^3v\ \ -\nu\frac{m_1}{2}v^2\left[f_1 - \frac{n_1}{n_M}f_M\right] = -\frac{\nu m_1}{2}\left[\int \mathrm{d}^3v\ v^2f_1 - \frac{n_1}{n_M}\int \mathrm{d}^3v\ v^2f_M\right] \tag{96}$$

$$= -\nu\left[\frac{3n_1}{2}T_1 - \frac{n_1}{\cancel{n_M}}\frac{3\cancel{n_M}}{2}T\right] = -\frac{3\nu}{2}\left[T_1 - T\right] \neq 0 \tag{97}$$

Thus, only particles are conserved, and momentum and energy are not conserved.

5 Collisionality A precise measure of the collisionality parameter ν/ω_t is given by the ratio of the times τ_c/τ_t, where τ_c is the Coulomb collision time, given for ions and electrons by (5.78) and (5.79), and τ_t is the transit time defined by (4.136).

$$\tau_e = \frac{3}{16\sqrt{\pi}} \frac{m_e^2 v_{\mathrm{th}_e}^3}{e^4 Z_{\mathrm{eff}} n_e \ln \Lambda} \tag{5.78}$$

$$\tau_i = \frac{3\sqrt{2}}{16\sqrt{\pi}} \frac{m_i^2 v_{\mathrm{th}_i}^3}{(Ze^4) n_i \ln \Lambda} \tag{5.79}$$

$$\tau_t \equiv \oint \frac{\mathrm{d}\theta}{|u^\theta|} \tag{4.136}$$

5.a Scaling How does this ratio scale with species mass? with temperature?

Solution:

(As an aside, it is fairly obvious that ν/ω_t should scale the same as τ_t/τ_c, *not* τ_c/τ_t. We know that plasmas become less collisional as they increase in temperature.)

We expect that the transit time, τ_t should scale as $1/\omega_t$, with ω_t the transit frequency. Thus

$$\tau_t \sim \frac{L}{v_{\mathrm{th}}} \sim \frac{L m^{1/2}}{T^{1/2}} \tag{98}$$

Now we use that

$$m^2 v_{\mathrm{th}}^3 \sim m^2 \frac{T^{3/2}}{m^{3/2}} \sim m^{1/2} T^{3/2} \tag{99}$$

Thus

$$\frac{\tau_c}{\tau_t} \sim \frac{\cancel{m^{1/2}} T^{3/2}}{\frac{\cancel{m^{1/2}}}{T^{1/2}}} \sim T^{3/2+1/2} \sim T^2 \tag{100}$$

$$\frac{\nu}{\omega_t} \sim \frac{\tau_t}{\tau_c} \sim T^{-2} \tag{101}$$

So, there is no mass in the scale (as predicted in the text, when stating that electrons and ions have the same collisionality parameter at comparable temperatures), while the collisionality scales with inverse temperature quadratically.

5.b Standard Tokamak Collisionalities Consider the Standard Tokamak with $Z_{\text{eff}} = 1$. At what temperatures are the ion and electron collisionalities approximately unity?

toroidal field (B_T)	50 kG
major radius (R_0)	300 cm
minor radius (a)	80 cm
safety factor (q)	$q \simeq 1$ (on axis) $q \simeq 3$ (at edge)
central density (n)	10^{14} cm^{-3}
central temperature ($T_i = T_e = T$)	10 keV

Table 1: The Standard Tokamak parameters.

Solution:

Let's do it for electrons, as it should be approximately the same for both. Thus, (switching to SI, so putting $e^4 \to e^4/(4\pi\epsilon_0)^2$)

$$\frac{\tau_e}{\tau_t} = \frac{3(4\pi\epsilon_0)^2}{16\sqrt{\pi}} \frac{2^{3/2} T_e^{3/2} \sqrt{m_e}}{e^4 n_e \ln\Lambda} \frac{1}{\frac{a\sqrt{m_e}}{\sqrt{2T_e}}} = \frac{3(4\pi\epsilon_0)^2}{4\sqrt{\pi}} \frac{T_e^2}{e^4 n_e a \ln\Lambda} \tag{102}$$

for $\tau_e/\tau_t \approx 1$ we require ((103) is in J^2 while (104) is in eV^2)

$$T_e^2 \approx \frac{4\sqrt{\pi} e^4 n_e a \ln\Lambda}{3(4\pi\epsilon_0)^2} \approx \frac{4\sqrt{\pi}}{3(4\pi)^2 (8.85\times 10^{-12}\ \text{F/m})^2} (1.6\times 10^{-19}\ \text{C})^4 (10^{20}\ \text{m}^{-3})(0.8\ \text{m})(17) \tag{103}$$

$$T_e^2 \approx \frac{4\sqrt{\pi} e^4 n_e \ln\Lambda}{3(4\pi\epsilon_0)^2} \approx \frac{4\sqrt{\pi}}{3(4\pi)^2 (8.85\times 10^{-12}\ \text{F/m})^2} (1.6\times 10^{-19}\ \text{C})^2 (10^{20}\ \text{m}^{-3})(0.8\ \text{m})(17) \tag{104}$$

$$\approx 6664.649\ \text{eV}^2 \tag{105}$$

$$\boxed{T_e \approx 81\ \text{eV}} \tag{106}$$

As a test, let's try proton ions,

$$\frac{\tau_i}{\tau_t} = \frac{3\sqrt{2}(4\pi\epsilon_0)^2}{16\sqrt{\pi}} \frac{2^{3/2} T_i^{3/2} \sqrt{m_i}}{e^4 n_i \ln\Lambda} \frac{1}{\frac{a\sqrt{m_i}}{\sqrt{2T_i}}} = \frac{3\sqrt{2}(4\pi\epsilon_0)^2}{4\sqrt{\pi}} \frac{T_i^2}{e^4 n_e a \ln\Lambda} \tag{107}$$

Thus, we see that $T_i^2 = T_e^2/\sqrt{2}$ and so $T_i \approx 68$ eV which is quite comparable.

In summary, for collisionality of unity we require

$$\boxed{T_e \sim 81\ \text{eV} \qquad T_i \sim 68\ \text{eV}} \tag{108}$$

Based on scaling, this would put the Standard Tokamak at very low collisionality, as is expected for a tokamak.

6 Charge Exchange An important atomic interaction occurring near the plasma boundary is charge exchange: a neutral H-atom from the wall delivers its electron to an ion from the plasma interior. If the ion distribution is denoted by f_i and the neutral distribution by f_n, then the charge-exchange operator is represented approximately by

$$X(f_i, f_n) = \int \mathrm{d}^3v' \ \sigma_x |\mathbf{v} - \mathbf{v}'| \left[f_i(\mathbf{v}) f_n(\mathbf{v}') - f_n(\mathbf{v}) f_i(\mathbf{v}') \right] , \tag{109}$$

where σ_x is the charge exchange cross-section. The operator appears in the ion kinetic equation alongside the Coulomb collision operator, $C(f_i)$. The product $\sigma_x|\mathbf{v} - \mathbf{v}'|$ is a weak function of velocity, often taken to be constant.

6.a Ion-Neutral Particle Conservation It is physically obvious that charge exchange conserves both ions and neutrals. Verify that the operator X has this conservation property, analogous to (5.20).

$$\int \mathrm{d}^3v \ C_{ss'} = 0 \tag{5.20}$$

Solution:
First we write

$$\int \mathrm{d}^3v \ X(f_i, f_n) = \int \mathrm{d}^3v \ \int \mathrm{d}^3v' \ \sigma_x |\mathbf{v} - \mathbf{v}'| \left[f_i(\mathbf{v}) f_n(\mathbf{v}') - f_n(\mathbf{v}) f_i(\mathbf{v}') \right] \tag{110}$$

We now note that

$$X(f_i, f_n) = -X(f_n, f_i) \tag{111}$$

We now realize that interchanging $\mathbf{v}$ and $\mathbf{v}'$ should have no effect on the answer to (110), as $\mathbf{v}$ and $\mathbf{v}'$ are both dummy variables. That is defining

$$X'(f_i, f_n) = \int \mathrm{d}^3v' \ \sigma_x |\mathbf{v} - \mathbf{v}'| \left[f_i(\mathbf{v}') f_n(\mathbf{v}) - f_n(\mathbf{v}') f_i(\mathbf{v}) \right] \tag{112}$$

where the use of $|\mathbf{v}' - \mathbf{v}| = |\mathbf{v} - \mathbf{v}'|$ and that $\sigma_x = \sigma_x(|\mathbf{v} - \mathbf{v}'|)$. We then should have

$$\int \mathrm{d}^3v \ X(f_i, f_n) = \int \mathrm{d}^3v' \ X'(f_i, f_n) \tag{113}$$

But we notice (using that changing the order of integration does not change the answer)

$$\int \mathrm{d}^3v' \ X'(f_i, f_n) = \int \mathrm{d}^3v \ \int \mathrm{d}^3v' \ \sigma_x |\mathbf{v}' - \mathbf{v}| \left[f_i(\mathbf{v}') f_n(\mathbf{v}) - f_n(\mathbf{v}') f_i(\mathbf{v}) \right] \tag{114}$$

$$= -\int \mathrm{d}^3v \ \int \mathrm{d}^3v' \ \sigma_x |\mathbf{v} - \mathbf{v}'| \left[f_i(\mathbf{v}) f_n(\mathbf{v}') - f_n(\mathbf{v}) f_i(\mathbf{v}') \right] \tag{115}$$

$$= -\int \mathrm{d}^3v \ X(f_i, f_n) \tag{116}$$

and so, in fact, (113) reads

$$\int \mathrm{d}^3v\ X(f_i, f_n) = \int \mathrm{d}^3v'\ X'(f_i, f_n) = -\int \mathrm{d}^3v\ X(f_i, f_n) \tag{117}$$

$$\int \mathrm{d}^3v\ X(f_i, f_n) = -\int \mathrm{d}^3v\ X(f_i, f_n) \tag{118}$$

$$\Rightarrow \tag{119}$$

$$\int \mathrm{d}^3v\ X(f_i, f_n) = 0 \tag{120}$$

as desired.

Let's show that this works even in the case $\sigma_x|\mathbf{v}-\mathbf{v}'|$ independent of velocity even more simply.

We note that for $\sigma_x|\mathbf{v}-\mathbf{v}'|$ independent of velocity we then can approximate this as $\sigma_x u$ for some velocity u. Thus, using $\int \mathrm{d}^3v\ f_i(\mathbf{v}) = n_i$ and $\int \mathrm{d}^3v\ f_n(\mathbf{v}) = n_n$ we find

$$\int \mathrm{d}^3v\ X(f_i, f_n) = \int \mathrm{d}^3v \int \mathrm{d}^3v'\ \sigma_x|\mathbf{v}-\mathbf{v}'|\left[f_i(\mathbf{v})f_n(\mathbf{v}') - f_n(\mathbf{v})f_i(\mathbf{v}')\right] \tag{121}$$

$$= \sigma_x u \int \mathrm{d}^3v\ \left[f_i(\mathbf{v})n_n - f_n(\mathbf{v})n_i\right] = \sigma_x u\left[n_i n_n - n_n n_i\right] = 0 \tag{122}$$

We also realize that

$$\int \mathrm{d}^3v\ \left[X(f_i, f_n) + X(f_n, f_i)\right] = 0 \tag{123}$$

which shows conservation of particles for any combination of a specific neutral and specific ion species, just as one would expect.

6.b Charge-Exchange Mean-Free Path The charge-exchange mean-free path, λ_x, is defined as the average distance traveled by a neutral between charge exchange events. Assuming that both distributions are roughly Maxwellian, with comparable temperatures, estimate λ_x in terms of σ_x and the ion density.

Solution:

I argue that λ_x should depend on $\sigma_x n_i$, as that is the likelihood of a particle hitting an ion in a unit volume. This can be more rigorously shown in a slab geometry, each particle taking an area σ and there being $n_i A$ particles in the slab (A being the area of the slab). Thus,

$$\boxed{\lambda_x \sim \frac{1}{n_i \sigma_x}} \tag{124}$$

6.c Only Solution Short Mean-Free Path When λ_x is sufficiently short, the lowest order neutral distribution satisfies $X(f_i, f_n) = 0$. An obvious solution is $f_n = N(\mathbf{x}) f_i$, where N is an arbitrary function of position. Modify the H-theorem argument of (5.98) to show that $f_n = N(\mathbf{x}) f_i$ is the *only* solution.

$$\Theta = \frac{1}{4} \sum_{s,s'} m_s \gamma_{ss'} \int \mathrm{d}^3 v \mathrm{d}^3 v' \; f_s f'_{s'} A_\alpha U_{\alpha\beta} A_\beta \tag{5.98}$$

Solution:

It makes more sense to use the form

$$\Theta_2[f, g] = -\sum_{ss'} \int \mathrm{d}^3 v \; \widehat{f}_s C_{ss'1}(\widehat{g}) \tag{5.118}$$

but instead defining

$$\Theta[g_1, g_2] = \int \mathrm{d}^3 v \; \frac{g_1 n_i}{n_n f} X(g_2, f) \tag{125}$$

with g_1 and g_2 any two neutral distributions, n_n the neutral density and f_i the ion distribution.

Writing this out completely yields

$$\Theta[g_1, g_2] = -\int \mathrm{d}^3 v \int \mathrm{d}^3 v' \; \frac{\sigma_x}{n_n} |\mathbf{v} - \mathbf{v}'| \frac{g_1(\mathbf{v}) n_i}{f(\mathbf{v})} \left[f(\mathbf{v}) g_2(\mathbf{v}') - g_2(\mathbf{v}) f(\mathbf{v}') \right] \tag{126}$$

If we exchange indices $\mathbf{v} \leftrightarrow \mathbf{v}'$ then

$$\Theta[g_1, g_2] = -\int \mathrm{d}^3 v \int \mathrm{d}^3 v' \; \frac{\sigma_x}{n_n} |\mathbf{v} - \mathbf{v}'| \frac{g_1(\mathbf{v}') n_i}{f(\mathbf{v}')} \left[f(\mathbf{v}') g_2(\mathbf{v}) - g_2(\mathbf{v}') f(\mathbf{v}) \right] \tag{127}$$

Thus, we find

$$2\Theta[g_1, g_2] \tag{128}$$

$$= -\int \mathrm{d}^3v \int \mathrm{d}^3v' \ \frac{\sigma_x}{n_n}|\mathbf{v}-\mathbf{v}'|n_i \left[g_1(\mathbf{v})g_2(\mathbf{v}') - g_1(\mathbf{v})g_2(\mathbf{v})\frac{f(\mathbf{v}')}{f(\mathbf{v})} + g_1(\mathbf{v}')g_2(\mathbf{v}') - g_1(\mathbf{v}')g_2(\mathbf{v}')\frac{f(\mathbf{v})}{f(\mathbf{v}')}\right] \tag{129}$$

$$= -\int \mathrm{d}^3v \int \mathrm{d}^3v' \ \frac{n_i\sigma_x}{n_n}|\mathbf{v}-\mathbf{v}'|f(\mathbf{v})f(\mathbf{v}') \left[\frac{g_1(\mathbf{v})g_2(\mathbf{v}')}{f(\mathbf{v})f(\mathbf{v}')} - \frac{g_1(\mathbf{v})g_2(\mathbf{v})}{f(\mathbf{v})^2} + \frac{g_1(\mathbf{v}')g_2(\mathbf{v}')}{f(\mathbf{v})f(\mathbf{v}')} - \frac{g_1(\mathbf{v}')g_2(\mathbf{v}')}{f(\mathbf{v}')^2}\right] \tag{130}$$

$$= \int \mathrm{d}^3v \int \mathrm{d}^3v' \ \frac{n_i\sigma_x}{n_n}|\mathbf{v}-\mathbf{v}'|f(\mathbf{v})f(\mathbf{v}') \left[\frac{g_1(\mathbf{v})}{f(\mathbf{v})}\frac{g_2(\mathbf{v})}{f(\mathbf{v})} + \frac{g_1(\mathbf{v}')}{f(\mathbf{v}')}\frac{g_2(\mathbf{v}')}{f(\mathbf{v}')} - \frac{g_1(\mathbf{v})}{f(\mathbf{v})}\frac{g_2(\mathbf{v}')}{f(\mathbf{v}')} - \frac{g_1(\mathbf{v}')}{f(\mathbf{v}')}\frac{g_2(\mathbf{v})}{f(\mathbf{v})}\right] \tag{131}$$

$$= \int \mathrm{d}^3v \int \mathrm{d}^3v' \ \frac{n_i\sigma_x}{n_n}|\mathbf{v}-\mathbf{v}'|f(\mathbf{v})f(\mathbf{v}') \left[\frac{g_1(\mathbf{v})}{f(\mathbf{v})}\left(\frac{g_2(\mathbf{v})}{f(\mathbf{v})} - \frac{g_2(\mathbf{v}')}{f(\mathbf{v}')}\right) + \frac{g_1(\mathbf{v}')}{f(\mathbf{v}')}\left(\frac{g_2(\mathbf{v}')}{f(\mathbf{v}')} - \frac{g_2(\mathbf{v})}{f(\mathbf{v})}\right)\right] \tag{132}$$

$$= \int \mathrm{d}^3v \int \mathrm{d}^3v' \ \frac{n_i\sigma_x}{n_n}|\mathbf{v}-\mathbf{v}'|f(\mathbf{v})f(\mathbf{v}') \left[\frac{g_1(\mathbf{v})}{f(\mathbf{v})}\left(\frac{g_2(\mathbf{v})}{f(\mathbf{v})} - \frac{g_2(\mathbf{v}')}{f(\mathbf{v}')}\right) - \frac{g_1(\mathbf{v}')}{f(\mathbf{v}')}\left(\frac{g_2(\mathbf{v})}{f(\mathbf{v})} - \frac{g_2(\mathbf{v}')}{f(\mathbf{v}')}\right)\right] \tag{133}$$

$$= \int \mathrm{d}^3v \int \mathrm{d}^3v' \ \frac{n_i\sigma_x}{n_n}|\mathbf{v}-\mathbf{v}'|f(\mathbf{v})f(\mathbf{v}') \left[\left(\frac{g_1(\mathbf{v})}{f(\mathbf{v})} - \frac{g_1(\mathbf{v}')}{f(\mathbf{v}')}\right)\left(\frac{g_2(\mathbf{v})}{f(\mathbf{v})} - \frac{g_2(\mathbf{v}')}{f(\mathbf{v}')}\right)\right] \tag{134}$$

Thus, if we take $g_1 = g_2 = g$ we find

$$\Theta[g, g] = \int \mathrm{d}^3v \int \mathrm{d}^3v' \ \frac{n_i\sigma_x}{2n_n}|\mathbf{v}-\mathbf{v}'|f(\mathbf{v})f(\mathbf{v}') \left(\frac{g(\mathbf{v})}{f(\mathbf{v})} - \frac{g(\mathbf{v}')}{f(\mathbf{v}')}\right)^2 \tag{135}$$

We note that $\Theta[g, g] \geq 0$ and that $\Theta[g, g] = 0$ only when $g/f = N(\mathbf{x})$ where N is independent of velocity.

As $\Theta[g, g]$ is the rate of neutral entropy production, we then see that $g = N(\mathbf{x})f$ is the only way for $\Theta[g, g] = 0$ non-trivially.

7 Slowing Down Alpha particles, the ash of fusion reactions, are born in a D-T plasma at 3.5 MeV. Sufficiently well-confined alphas ultimately equilibrate to the ion temperature T_i, but one expects a "slowing-down" tail of hot alphas, with distribution $f_{\alpha h}$ and thermal speed v_{th_h}. The slowing-down distribution is determined primarily by energy scattering with Maxwellian scatterers.

7.a Collision Operator Assuming the $f_{\alpha h}$ depends only on kinetic energy w, show that the relevant collision operator is

$$C_{\text{slow}} = \sum_s C(f_{\alpha h}, f_{sM}) = \sum_s \left(\frac{\gamma_{\alpha s} n_s v_{\text{th}_s}^3}{2v}\frac{\partial}{\partial w}\right)\left\{\phi\left(\frac{v}{v_{\text{th}_s}}\right)\left[\frac{\partial f_{\alpha h}}{\partial w} + \frac{m_\alpha}{T_s} f_{\alpha h}\right]\right\} \tag{136}$$

where

$$\phi(x) = \operatorname{erf}(x) - x\operatorname{erf}'(x)\,. \tag{137}$$

Solution:

There is very clearly an error in the above formula. $\partial f_\alpha/\partial w$ and $(m_\alpha/T_s)f_\alpha$ clearly do not have the same units if w is in energy, as T_s/m_α does not have units of energy. Thus, the formula simply is incorrect and it is almost impossible to know what the correct form would be.

Let's begin with

$$\begin{aligned} C_{ss'} &= \frac{\gamma_{ss'}}{2m_s}\left\{\frac{\partial^2}{\partial v_\alpha \partial v_\beta}\left(f_s \frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta}\right) - 2\left(1+\frac{m_s}{m_{s'}}\right)\frac{\partial}{\partial v_\alpha}\left(f_s\frac{\partial H_{s'}}{\partial v_\alpha}\right)\right\} \\ &= \frac{\gamma_{ss'}}{2m_s}\left\{\frac{\partial^2}{\partial \mathbf{v}\partial \mathbf{v}} : f_s \frac{\partial^2 G_{s'}}{\partial \mathbf{v}\partial \mathbf{v}} - 2\left(1+\frac{m_s}{m_{s'}}\right)\frac{\partial}{\partial \mathbf{v}}\cdot\left(f_s\frac{\partial H_{s'}}{\partial \mathbf{v}}\right)\right\} \end{aligned} \tag{138}$$

We also know that (with $\eta = \frac{v}{v_{\text{th}_{s'}}}$)

$$G_{s'M} = \frac{n_{s'} v_{\text{th}_{s'}}}{2\eta}\left(\eta\operatorname{erf}'(\eta) + \left(1+2\eta^2\right)\operatorname{erf}(\eta)\right) \tag{139}$$

$$H_{s'M} = \frac{n_{s'}}{v_{\text{th}_{s'}}\eta}\operatorname{erf}(\eta) \tag{140}$$

$$\frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\alpha} = 2H_{s'} \tag{141}$$

$$\frac{\partial^2 H_{s'}}{\partial v_\alpha \partial v_\alpha} = -4\pi f_{s'} \tag{142}$$

We then find

$$\frac{\partial^2}{\partial v_\alpha \partial v_\beta}\left(f_s \frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta}\right) = \frac{\partial}{\partial v_\alpha}\left(\frac{\partial f_s}{\partial v_\beta}\frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta} + f_s \frac{\partial^3 G_{s'}}{\partial v_\alpha \partial v_\beta \partial v_\beta}\right) \tag{143}$$

$$= \frac{\partial^2 f_s}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta} + \frac{\partial f_s}{\partial v_\beta}\frac{\partial^3 G_{s'}}{\partial v_\alpha \partial v_\alpha \partial v_\beta} + \frac{\partial f_s}{\partial v_\alpha}\frac{\partial^3 G_{s'}}{\partial v_\alpha \partial v_\beta \partial v_\beta} + f_s \frac{\partial^4 G_{s'}}{\partial v_\alpha \partial v_\alpha \partial v_\beta \partial v_\beta} \tag{144}$$

$$= \frac{\partial^2 f_s}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta} + \frac{\partial f_s}{\partial v_\beta}\frac{\partial (2H_{s'})}{\partial v_\beta} + \frac{\partial f_s}{\partial v_\alpha}\frac{\partial (2H_{s'})}{\partial v_\alpha} + f_s \frac{\partial^2 (2H_{s'})}{\partial v_\beta \partial v_\beta} \tag{145}$$

$$= \frac{\partial^2 f_s}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta} + \frac{\partial f_s}{\partial v_\beta}\frac{\partial (2H_{s'})}{\partial v_\beta} + \frac{\partial f_s}{\partial v_\alpha}\frac{\partial (2H_{s'})}{\partial v_\alpha} + f_s 2(-4\pi f_{s'}) \tag{146}$$

$$= \frac{\partial^2 f_s}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta} + 4\frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} - 8\pi f_s f_{s'} \tag{147}$$

$$= \frac{\partial^2 f_s}{\partial \mathbf{v}\partial \mathbf{v}} : \frac{\partial^2 G_{s'}}{\partial \mathbf{v}\partial \mathbf{v}} + 4\frac{\partial f_s}{\partial \mathbf{v}} \cdot \frac{\partial H_{s'}}{\partial \mathbf{v}} - 8\pi f_s f_{s'} \tag{148}$$

and

$$\frac{\partial}{\partial v_\alpha}\left(f_s \frac{\partial H_{s'}}{\partial v_\alpha}\right) = \frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} + f_s \frac{\partial^2 H_{s'}}{\partial v_\alpha \partial v_\alpha} \tag{149}$$

$$= \frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} + f_s\left(-4\pi f_{s'}\right) \tag{150}$$

$$= \frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} - 4\pi f_s f_{s'} \tag{151}$$

$$= \frac{\partial f_s}{\partial \mathbf{v}} \cdot \frac{\partial H_{s'}}{\partial \mathbf{v}} - 4\pi f_s f_{s'} \tag{152}$$

and so

$$\frac{\partial^2}{\partial v_\alpha \partial v_\beta}\left(f_s \frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta}\right) - 2\left(1 + \frac{m_s}{m_{s'}}\right)\frac{\partial}{\partial v_\alpha}\left(f_s \frac{\partial H_{s'}}{\partial v_\alpha}\right) \tag{153}$$

$$= \frac{\partial^2 f_s}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta} + 4\frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} - 8\pi f_s f_{s'} - 2\left(1 + \frac{m_s}{m_{s'}}\right)\left(\frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} - 4\pi f_s f_{s'}\right) \tag{154}$$

$$= \frac{\partial^2 f_s}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta} + 4\frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} - \cancel{8\pi f_s f_{s'}} - 2\frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} + \cancel{8\pi f_s f_{s'}} - 2\frac{m_s}{m_{s'}}\frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} + 8\pi\frac{m_s}{m_{s'}} f_s f_{s'} \tag{155}$$

$$= \frac{\partial^2 f_s}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta} + 2\frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} - 2\frac{m_s}{m_{s'}}\frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} + 8\pi\frac{m_s}{m_{s'}} f_s f_{s'} \tag{156}$$

$$= \frac{\partial^2 f_s}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 G_{s'}}{\partial v_\alpha \partial v_\beta} + 2\left(1 - \frac{m_s}{m_{s'}}\right)\frac{\partial f_s}{\partial v_\alpha}\frac{\partial H_{s'}}{\partial v_\alpha} + 8\pi\frac{m_s}{m_{s'}} f_s f_{s'} \tag{157}$$

$$= \frac{\partial^2 f_s}{\partial \mathbf{v}\partial \mathbf{v}} : \frac{\partial^2 G_{s'}}{\partial \mathbf{v}\partial \mathbf{v}} + 2\left(1 - \frac{m_s}{m_{s'}}\right)\frac{\partial f_s}{\partial \mathbf{v}} \cdot \frac{\partial H_{s'}}{\partial \mathbf{v}} + 8\pi\frac{m_s}{m_{s'}} f_s f_{s'} \tag{158}$$

Now we can use

$$\frac{\partial}{\partial v_\alpha} = \frac{\partial v}{\partial v_\alpha}\frac{\partial}{\partial v} = \frac{\partial\sqrt{\sum_\alpha v_\alpha v_\alpha}}{\partial v_\alpha}\frac{\partial}{\partial v} = \frac{1}{2v}2v_\alpha\frac{\partial}{\partial v} = \frac{v_\alpha}{v}\frac{\partial}{\partial v} \tag{159}$$

$$\frac{\partial}{\partial \mathbf{v}} = \frac{\mathbf{v}}{v}\frac{\partial}{\partial v} \tag{160}$$

as $f_s = f_s(w) = f_s(v)$, $G_{s'} = G_{s'}(v)$, and $H_{s'} = H_{s'}(v)$. Thus,

$$\frac{\partial^2 f_s}{\partial\mathbf{v}\partial\mathbf{v}} = \frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{v}}{v}\frac{\partial f_s}{\partial v}\right) = \frac{\partial\mathbf{v}}{\partial\mathbf{v}}\frac{1}{v}\frac{\partial f_s}{\partial v} + \mathbf{v}\frac{\partial}{\partial\mathbf{v}}\left(\frac{1}{v}\frac{\partial f_s}{\partial\mathbf{v}}\right) = \mathbb{1}\frac{1}{v}\frac{\partial f_s}{\partial v} + \frac{\mathbf{v}\mathbf{v}}{v}\frac{\partial}{\partial v}\left(\frac{1}{v}\frac{\partial f_s}{\partial v}\right) \tag{161}$$

$$= \mathbb{1}\left(\frac{1}{v}\frac{\partial f_s}{\partial v}\right) + \frac{\mathbf{v}\mathbf{v}}{v^2}\left(\frac{\partial^2 f_s}{\partial v^2} - \frac{1}{v}\frac{\partial f_s}{\partial v}\right) \tag{162}$$

and so obviously by analogy

$$\frac{\partial^2 G_{s'}}{\partial\mathbf{v}\partial\mathbf{v}} = \mathbb{1}\left(\frac{1}{v}\frac{\partial G_{s'}}{\partial v}\right) + \frac{\mathbf{v}\mathbf{v}}{v^2}\left(\frac{\partial^2 G_{s'}}{\partial v^2} - \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\right) \tag{163}$$

and so

$$\frac{\partial^2 f_s}{\partial\mathbf{v}\partial\mathbf{v}} : \frac{\partial^2 G_{s'}}{\partial\mathbf{v}\partial\mathbf{v}} = \left[\mathbb{1}\left(\frac{1}{v}\frac{\partial f_s}{\partial v}\right) + \frac{\mathbf{v}\mathbf{v}}{v^2}\left(\frac{\partial^2 f_s}{\partial v^2} - \frac{1}{v}\frac{\partial f_s}{\partial v}\right)\right] : \left[\mathbb{1}\left(\frac{1}{v}\frac{\partial G_{s'}}{\partial v}\right) + \frac{\mathbf{v}\mathbf{v}}{v^2}\left(\frac{\partial^2 G_{s'}}{\partial v^2} - \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\right)\right] \tag{164}$$

$$\begin{aligned} &= \frac{1}{v^2}\frac{\partial f_s}{\partial v}\frac{\partial G_{s'}}{\partial v} + \frac{1}{v}\frac{\partial f_s}{\partial v}\frac{v^2}{v^2}\left(\frac{\partial^2 G_{s'}}{\partial v^2} - \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\right) + \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\frac{v^2}{v^2}\left(\frac{\partial^2 f_s}{\partial v^2} - \frac{1}{v}\frac{\partial f_s}{\partial v}\right) \\ &\quad + \frac{v^4}{v^4}\left(\frac{\partial^2 f_s}{\partial v^2} - \frac{1}{v}\frac{\partial f_s}{\partial v}\right)\left(\frac{\partial^2 G_{s'}}{\partial v^2} - \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\right) \end{aligned} \tag{165}$$

$$\begin{aligned} &= \frac{1}{v^2}\frac{\partial f_s}{\partial v}\frac{\partial G_{s'}}{\partial v} + \frac{1}{v}\frac{\partial f_s}{\partial v}\left(\frac{\partial^2 G_{s'}}{\partial v^2} - \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\right) + \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\left(\frac{\partial^2 f_s}{\partial v^2} - \frac{1}{v}\frac{\partial f_s}{\partial v}\right) \\ &\quad + \left(\frac{\partial^2 f_s}{\partial v^2} - \frac{1}{v}\frac{\partial f_s}{\partial v}\right)\left(\frac{\partial^2 G_{s'}}{\partial v^2} - \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\right) \end{aligned} \tag{166}$$

We then have

$$\frac{\partial f_s}{\partial\mathbf{v}} \cdot \frac{\partial H_{s'}}{\partial\mathbf{v}} = \frac{\mathbf{v}}{v}\frac{\partial f_s}{\partial v} \cdot \frac{\mathbf{v}}{v}\frac{\partial H_{s'}}{\partial v} = \frac{v^2}{v^2}\frac{\partial f_s}{\partial v}\frac{\partial H_{s'}}{\partial v} = \frac{\partial f_s}{\partial v}\frac{\partial H_{s'}}{\partial v} \tag{167}$$

and thus altogether

$$\begin{aligned} \frac{C_{ss'}}{\frac{\gamma_{ss'}}{2m_s}} &= \frac{1}{v^2}\frac{\partial f_s}{\partial v}\frac{\partial G_{s'}}{\partial v} + \frac{1}{v}\frac{\partial f_s}{\partial v}\left(\frac{\partial^2 G_{s'}}{\partial v^2} - \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\right) + \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\left(\frac{\partial^2 f_s}{\partial v^2} - \frac{1}{v}\frac{\partial f_s}{\partial v}\right) \\ &\quad + \left(\frac{\partial^2 f_s}{\partial v^2} - \frac{1}{v}\frac{\partial f_s}{\partial v}\right)\left(\frac{\partial^2 G_{s'}}{\partial v^2} - \frac{1}{v}\frac{\partial G_{s'}}{\partial v}\right) + 2\left(1 - \frac{m_s}{m_{s'}}\right)\frac{\partial f_s}{\partial v}\frac{\partial H_{s'}}{\partial v} + 8\pi\frac{m_s}{m_{s'}}f_s f_{s'} \end{aligned} \tag{168}$$

Of course, we are now forced to use representations for $G_{s'}$ and $H_{s'}$. We begin with

$$\frac{\partial G_{s'}}{\partial v} = \frac{\partial\eta}{\partial v}\frac{\partial G_{s'}}{\partial\eta} = \frac{1}{v_{\text{th}_{s'}}}\frac{\partial G_{s'}}{\partial\eta} = \frac{n_{s'}}{2}\frac{\partial}{\partial\eta}\left(\text{erf}'(\eta) + \frac{1+2\eta^2}{\eta}\text{erf}(\eta)\right) \tag{169}$$

Using that $\text{erf}'(\eta) = \frac{2}{\sqrt{\pi}}e^{-\eta^2}$ so $\text{erf}''(\eta) = -2\eta\,\text{erf}'(\eta)$ we find

$$\frac{2}{n_{s'}}\frac{\partial G_{s'}}{\partial v} = -2\eta\,\text{erf}'(\eta) + \frac{4\eta^2 - (1+2\eta^2)}{\eta^2}\text{erf}(\eta) + \frac{1+2\eta^2}{\eta}\text{erf}'(\eta) \tag{170}$$

$$= \left(\cancel{-2\eta} + \frac{1}{\eta} + \cancel{2\eta}\right)\text{erf}'(\eta) + \frac{(2\eta^2-1)\,\text{erf}(\eta)}{\eta^2} = \frac{\text{erf}'(\eta)}{\eta} + \frac{(2\eta^2-1)\,\text{erf}(\eta)}{\eta^2} \tag{171}$$

Thus,

$$\frac{2}{n_{s'}}\frac{\partial^2 G_{s'}}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\operatorname{erf}'(\eta)}{\eta} + \frac{(2\eta^2-1)\operatorname{erf}(\eta)}{\eta^2}\right) = \frac{1}{v_{\mathrm{th}_{s'}}}\frac{\partial}{\partial \eta}\left(\frac{\operatorname{erf}'(\eta)}{\eta} + \frac{(2\eta^2-1)\operatorname{erf}(\eta)}{\eta^2}\right) \tag{172}$$

$$= \frac{1}{v_{\mathrm{th}_{s'}}}\left(\frac{\eta[-2\eta\operatorname{erf}'(\eta)] - \operatorname{erf}'(\eta)}{\eta^2} + \frac{\eta^2[4\eta\operatorname{erf}(\eta) + (2\eta^2-1)\operatorname{erf}'(\eta)] - 2\eta(2\eta^2-1)\operatorname{erf}(\eta)}{\eta^4}\right) \tag{173}$$

$$= \frac{1}{v_{\mathrm{th}_{s'}}}\left[\left(-2 - \frac{1}{\eta^2} + 2 - \frac{1}{\eta^2}\right)\operatorname{erf}'(\eta) + \left(\frac{4}{\eta} - \frac{4}{\eta} + \frac{2}{\eta^3}\right)\right] = \frac{2}{v_{\mathrm{th}_{s'}}}\left[\frac{-\operatorname{erf}'(\eta)}{\eta^2} + \frac{\operatorname{erf}(\eta)}{\eta^3}\right] \tag{174}$$

All that is now left is the rather simple

$$\frac{\partial H_{s'}}{\partial v} = \frac{\partial \eta}{\partial v}\frac{\partial H_{s'}}{\partial \eta} = \frac{1}{v_{\mathrm{th}_{s'}}}\frac{n_{s'}}{v_{\mathrm{th}_{s'}}}\frac{\partial}{\partial \eta}\left(\frac{\operatorname{erf}(\eta)}{\eta}\right) = \frac{n_{s'}}{v_{\mathrm{th}_{s'}}^2}\frac{\eta\operatorname{erf}'(\eta) - \operatorname{erf}(\eta)}{\eta^2} \tag{175}$$

Thus, we now find (using $2\pi f_{s'} = \frac{2n_{s'}}{\sqrt{\pi}v_{\mathrm{th}_{s'}}^3}e^{-\eta^2} = \frac{n_{s'}}{v_{\mathrm{th}_{s'}}^3}\operatorname{erf}'(\eta)$)

$$\begin{aligned}\frac{2m_s C_{ss'}}{\gamma_{ss'}} &= \frac{n_{s'}}{2v^2}\frac{\partial f_s}{\partial v}\left(\frac{\operatorname{erf}'(\eta)}{\eta} - \frac{\operatorname{erf}(\eta)}{\eta^2}\right) + \frac{n_{s'}}{2v}\frac{\partial f_s}{\partial v}\left[\frac{-2\operatorname{erf}'(\eta)}{\eta^2 v_{\mathrm{th}_{s'}}} + \frac{2\operatorname{erf}(\eta)}{v_{\mathrm{th}_{s'}}\eta^3} - \frac{1}{v}\left(\frac{\operatorname{erf}'(\eta)}{\eta} - \frac{\operatorname{erf}(\eta)}{\eta^2}\right)\right] \\ &+ \frac{n_{s'}}{2v}\left(\frac{\operatorname{erf}'(\eta)}{\eta} - \frac{\operatorname{erf}(\eta)}{\eta^2}\right)\left(\frac{\partial^2 f_s}{\partial v^2} - \frac{1}{v}\frac{\partial f_s}{\partial v}\right) \\ &+ \left(\frac{\partial^2 f_s}{\partial v^2} - \frac{1}{v}\frac{\partial f_s}{\partial v}\right)\frac{n_{s'}}{2}\left(\frac{-2\operatorname{erf}'(\eta)}{\eta^2 v'_{\mathrm{th}_s}} + \frac{2\operatorname{erf}(\eta)}{v_{\mathrm{th}_{s'}}\eta^3} - \frac{1}{v}\left(\frac{\operatorname{erf}'(\eta)}{\eta} - \frac{\operatorname{erf}(\eta)}{\eta^2}\right)\right) \\ &+ 2\left(1 - \frac{m_s}{m_{s'}}\right)\frac{\partial f_s}{\partial v}\frac{n_{s'}}{v_{\mathrm{th}_{s'}}^2}\frac{\eta\operatorname{erf}'(\eta) - \operatorname{erf}(\eta)}{\eta^2} + \frac{4n_{s'}m_s}{m_{s'}v_{\mathrm{th}_{s'}}^3}f_s\operatorname{erf}'(\eta)\end{aligned} \tag{176}$$

we see that the coefficient of $\frac{n_{s'}}{2}\frac{\partial^2 f_s}{\partial v^2}$ will be

$$\frac{1}{v}\left(\frac{\operatorname{erf}'(\eta)}{\eta} - \frac{\operatorname{erf}(\eta)}{\eta^2}\right) + \frac{-2\operatorname{erf}'(\eta)}{\eta^2 v_{\mathrm{th}_{s'}}} + \frac{2\operatorname{erf}(\eta)}{v_{\mathrm{th}_{s'}}\eta^3} - \frac{1}{v}\left(\frac{\operatorname{erf}'(\eta)}{\eta} - \frac{\operatorname{erf}(\eta)}{\eta}\right) \tag{177}$$

and the coefficient of $\frac{\partial f_s}{\partial v}$ will be

$$\frac{n_{s'}}{2v^2}\beta + \frac{n_{s'}}{2v}\delta - \frac{n_{s'}}{2v^2}\beta - \frac{n_{s'}}{2v}\delta + \frac{2n_{s'}}{v_{\mathrm{th}_{s'}}^2}\left(1 - \frac{m_s}{m_{s'}}\right)\frac{\eta\operatorname{erf}'(\eta) - \operatorname{erf}(\eta)}{\eta^2} \tag{178}$$

with

$$\beta = \left(\frac{\operatorname{erf}'(\eta)}{\eta} - \frac{\operatorname{erf}(\eta)}{\eta^2}\right) \tag{179}$$

$$\delta = \left[\frac{-2\operatorname{erf}'(\eta)}{\eta^2 v_{\mathrm{th}_{s'}}} + \frac{2\operatorname{erf}(\eta)}{v_{\mathrm{th}_{s'}}\eta^3} - \frac{1}{v}\left(\frac{\operatorname{erf}'(\eta)}{\eta} - \frac{\operatorname{erf}(\eta)}{\eta^2}\right)\right] \tag{180}$$

and so we finally reach the conclusion

$$\frac{2m_s C_{ss'}}{\gamma_{ss'}} = \frac{n_{s'}}{2}\frac{\partial^2 f_s}{\partial v^2}\left(\frac{-2\,\mathrm{erf}'(\eta)}{\eta^2 v_{\mathrm{th}_{s'}}} + \frac{2\,\mathrm{erf}(\eta)}{v_{\mathrm{th}_{s'}}\eta^3}\right) + \frac{\partial f_s}{\partial v}\frac{2n_{s'}}{v^2_{\mathrm{th}_{s'}}}\left(1 - \frac{m_s}{m_{s'}}\right)\frac{\eta\,\mathrm{erf}'(\eta) - \mathrm{erf}(\eta)}{\eta^2} + f_s\frac{4n_{s'}m_s}{m_{s'}v^3_{\mathrm{th}_{s'}}}\,\mathrm{erf}'(\eta) \tag{181}$$

Introducing $\phi(\eta) = \mathrm{erf}(\eta) - \eta\,\mathrm{erf}'(\eta)$ we find

$$\frac{2m_s C_{ss'}}{\gamma_{ss'}} = \frac{n_{s'}}{v_{\mathrm{th}_{s'}}\eta^3}\frac{\partial^2 f_s}{\partial v^2}\phi(\eta) + \frac{\partial f_s}{\partial v}\frac{2n_{s'}}{v^2_{\mathrm{th}_{s'}}}\left(1 - \frac{m_s}{m_{s'}}\right)\frac{-\phi(\eta)}{\eta^2} + f_s\frac{4n_{s'}m_s}{m_{s'}v^3_{\mathrm{th}_{s'}}}\,\mathrm{erf}'(\eta) \tag{182}$$

$$= \frac{n_{s'}}{v_{\mathrm{th}_{s'}}\eta^3}\frac{\partial^2 f_s}{\partial v^2}\phi(\eta) + \frac{\partial f_s}{\partial v}\frac{2n_{s'}\phi(\eta)}{v^2_{\mathrm{th}_{s'}}\eta^2}\left(\frac{m_s}{m_{s'}} - 1\right) + f_s\frac{4n_{s'}m_s\,\mathrm{erf}'(\eta)}{m_{s'}v^3_{\mathrm{th}_{s'}}} \tag{183}$$

We can now switch to derivatives with respect to $\frac{1}{2}m_s v^2 = w$ so that

$$\frac{\partial f_s}{\partial v} = \frac{\partial w}{\partial v}\frac{\partial f_s}{\partial w} = m_s v\frac{\partial f_s}{\partial w} \tag{184}$$

$$\frac{\partial^2 f_s}{\partial v^2} = m_s v\frac{\partial}{\partial w}\left[m_s v\frac{\partial f_s}{\partial w}\right] = m_s^2 v\left[\frac{\partial v}{\partial w}\frac{\partial f_s}{\partial w} + v\frac{\partial^2 f_s}{\partial w^2}\right] = m_s\frac{\partial f_s}{\partial w} + m_s^2 v^2\frac{\partial^2 f_s}{\partial w^2} \tag{185}$$

This leads to

$$\frac{2m_s C_{ss'}}{\gamma_{ss'}} = \frac{n_{s'}m_s v}{v_{\mathrm{th}_{s'}}}\left[\frac{1}{v}\frac{\partial f_s}{\partial w} + m_s v\frac{\partial^2 f_s}{\partial w^2}\right]\frac{\phi(\eta)}{\eta^3} + \frac{\partial f_s}{\partial w}\frac{2n_{s'}\phi(\eta)m_s v}{v^2_{\mathrm{th}_{s'}}\eta^2}\left(\frac{m_s}{m_{s'}} - 1\right) + f_s\frac{4n_{s'}m_s\,\mathrm{erf}'(\eta)}{m_{s'}v^3_{\mathrm{th}_{s'}}} \tag{186}$$

$$= n_{s'}m_s^2 v\frac{\phi(\eta)}{\eta^2}\frac{\partial^2 f_s}{\partial w^2} + \frac{\partial f_s}{\partial w}\left[\frac{n_{s'}m_s}{v}\frac{\phi(\eta)}{\eta^2} + \frac{2n_{s'}m_s}{v_{\mathrm{th}_{s'}}}\frac{\phi(\eta)}{\eta}\left(\frac{m_s}{m_{s'}} - 1\right)\right] + f_s\frac{4n_{s'}m_s\,\mathrm{erf}'(\eta)}{m_{s'}v^3_{\mathrm{th}_{s'}}} \tag{187}$$

$$= n_{s'}m_s\left(m_s v\frac{\partial^2 f_s}{\partial w^2}\frac{\phi(\eta)}{\eta^2} + \frac{\partial f_s}{\partial w}\left[\frac{\phi(\eta)}{v\eta^2} + \frac{2\phi(\eta)}{v_{\mathrm{th}_{s'}}\eta}\left(\frac{m_s}{m_{s'}} - 1\right)\right] + f_s\frac{4\,\mathrm{erf}'(\eta)}{m_{s'}v^3_{\mathrm{th}_{s'}}}\right) \tag{188}$$

Simplifying a bit we find, $(\phi'(\eta) = 2\eta^2\,\mathrm{erf}'(\eta))$ we find

$$C_{ss'} = \frac{n_{s'}\gamma_{ss'}}{2}\left(m_s v\frac{\partial^2 f_s}{\partial w^2}\frac{\phi(\eta)}{\eta^2} + \frac{\partial f_s}{\partial w}\left[\frac{\phi(\eta)}{v\eta^2} + \frac{2\phi(\eta)}{v_{\mathrm{th}_{s'}}\eta}\left(\frac{m_s}{m_{s'}} - 1\right)\right] + f_s\frac{2\phi'(\eta)}{m_{s'}v^3_{\mathrm{th}_{s'}}\eta^2}\right) \tag{189}$$

$$= \frac{n_{s'}\gamma_{ss'}v^3_{\mathrm{th}_{s'}}}{2v}\left(\frac{m_s v^2}{v^3_{\mathrm{th}_{s'}}}\frac{\partial^2 f_s}{\partial w^2}\frac{\phi(\eta)}{\eta^2} + \frac{\partial f_s}{\partial w}\left[\frac{\phi(\eta)}{v^3_{\mathrm{th}_{s'}}\eta^2} + \frac{2\phi(\eta)}{v^3_{\mathrm{th}_{s'}}}\left(\frac{m_s}{m_{s'}} - 1\right)\right] + f_s\frac{2\phi'(\eta)}{m_{s'}v^5_{\mathrm{th}_{s'}}\eta}\right) \tag{190}$$

$$= \frac{n_{s'}\gamma_{ss'}v^3_{\mathrm{th}_{s'}}}{2v}\left(\frac{m_s}{v_{\mathrm{th}_{s'}}}\phi(\eta)\frac{\partial^2 f_s}{\partial w^2} + \frac{\partial f_s}{\partial w}\left[\frac{\phi(\eta)}{v_{\mathrm{th}_{s'}}v^2} + \frac{2\phi(\eta)}{v^3_{\mathrm{th}_{s'}}}\left(\frac{m_s}{m_{s'}} - 1\right)\right] + f_s\frac{2\phi'(\eta)}{m_{s'}v v^4_{\mathrm{th}_{s'}}}\right) \tag{191}$$

$$\tag{192}$$

We can then switch $s \to \alpha$ and $s' \to s$ to find

$$C_{\alpha s} = \frac{n_s \gamma_{\alpha s}}{2} \left(m_\alpha v \frac{\partial^2 f_{\alpha h}}{\partial w^2} \frac{\phi(\eta)}{\eta^2} + \frac{\partial f_{\alpha h}}{\partial w} \left[\frac{\phi(\eta)}{v\eta^2} + \frac{2\phi(\eta)}{v_{\text{th}_s}\eta} \left(\frac{m_\alpha}{m_s} - 1 \right) \right] + f_{\alpha h} \frac{2\phi'(\eta)}{m_s v_{\text{th}_s}^3 \eta^2} \right) \tag{193}$$

$$= \frac{n_s \gamma_{\alpha s} v_{\text{th}_s}^3}{2v} \left(\frac{m_\alpha v^2}{v_{\text{th}_s}^3} \frac{\partial^2 f_{\alpha h}}{\partial w^2} \frac{\phi(\eta)}{\eta^2} + \frac{\partial f_{\alpha h}}{\partial w} \left[\frac{\phi(\eta)}{v_{\text{th}_s}^3 \eta^2} + \frac{2\phi(\eta)}{v_{\text{th}_s}^3} \left(\frac{m_\alpha}{m_s} - 1 \right) \right] + f_{\alpha h} \frac{2\phi'(\eta)}{m_s v_{\text{th}_s}^5 \eta} \right) \tag{194}$$

$$= \frac{n_s \gamma_{\alpha s} v_{\text{th}_s}^3}{2v} \left(\frac{m_\alpha}{v_{\text{th}_s}} \phi(\eta) \frac{\partial^2 f_{\alpha h}}{\partial w^2} + \frac{\partial f_{\alpha h}}{\partial w} \left[\frac{\phi(\eta)}{v^2 v_{\text{th}_s}} + \frac{2\phi(\eta)}{v_{\text{th}_s}^3} \left(\frac{m_\alpha}{m_s} - 1 \right) \right] + f_{\alpha h} \frac{2\phi'(\eta)}{m_s v v_{\text{th}_s}^4} \right) \tag{195}$$

Expanding the original expression yields

$$C_{\alpha s} = \frac{\gamma_{\alpha s} n_s v_{\text{th}_s}^3}{2v} \left(\frac{\partial \phi(\eta)}{\partial w} \left[\frac{\partial f_{\alpha h}}{\partial w} + \frac{m_\alpha}{T_s} f_{\alpha h} \right] + \phi(\eta) \left[\frac{\partial^2 f_{\alpha h}}{\partial w^2} + \frac{m_\alpha}{T_s} \frac{\partial f_{\alpha h}}{\partial w} \right] \right) \tag{196}$$

$$= \frac{\gamma_{\alpha s} n_s v_{\text{th}_s}^3}{2v} \left(\frac{1}{m_\alpha v v_{\text{th}_s}} \phi'(\eta) \left[\frac{\partial f_{\alpha h}}{\partial w} + \frac{m_\alpha}{T_s} f_{\alpha h} \right] + \phi(\eta) \frac{\partial^2 f_{\alpha h}}{\partial w^2} + \frac{m_\alpha \phi(\eta)}{T_s} \frac{\partial f_{\alpha h}}{\partial w} \right) \tag{197}$$

$$= \frac{\gamma_{\alpha s} n_s v_{\text{th}_s}^3}{2v} \left(\phi(\eta) \frac{\partial^2 f_{\alpha h}}{\partial w^2} + \frac{\partial f_{\alpha h}}{\partial w} \left[\frac{\phi'(\eta)}{m_\alpha v v_{\text{th}_s}} + \frac{m_\alpha \phi(\eta)}{T_s} \right] + f_{\alpha h} \frac{\phi'(\eta)}{v v_{\text{th}_s} T_s} \right) \tag{198}$$

Note that these cannot match.

7.b Approximation of Collision Operator Under typical conditions ($T_i = T_e \simeq 10$ keV) $v_{\text{th}_i} \ll v_{\text{th}_h} \ll v_{\text{th}_e}$, show that the slowing-down operator is approximated by

$$C_{\text{slow}} \simeq \frac{1}{\tau_s}\frac{m_i}{m_\alpha}\left(\frac{v_c}{v}\right)^3 w\frac{\partial}{\partial w}\left[\phi_s(v)\left(f_{\alpha h} + \frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w}\right)\right] \tag{5.137}$$

where the slowing-down time is $\tau_s \equiv [m_i n_i/(m_e n_e)]\tau_e$, the critical speed satisfies $v_c^3 \equiv (3\pi^{1/2}/4)(m_e/m_i)v_{\text{th}_e}^3$ and $\phi_s(v) \equiv (2m_\alpha/m_i)[(n_i/n_e) + (v/v_c)^3]$.

Solution:

Here we use that

$$\text{erf}(x) \xrightarrow{x\to 0} \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3}\right) \tag{199}$$
$$\text{erf}'(x) \xrightarrow{x\to 0} \frac{2}{\sqrt{\pi}}\left(1 - x^2\right) \tag{200}$$
$$\text{erf}(x) \xrightarrow{x\to\infty} 1 \tag{201}$$
$$\text{erf}'(x) \xrightarrow{x\to\infty} 0 \tag{202}$$

and hence

$$\phi(x) \xrightarrow{x\to 0} \frac{2}{\sqrt{\pi}}\left(\cancel{x} - \frac{x^3}{3}\right) - x\frac{2}{\sqrt{\pi}}\left(\cancel{1} - x^2\right) = \frac{2x^3}{\sqrt{\pi}}\left(-\frac{1}{3} - (-1)\right) = \frac{4}{3\sqrt{\pi}}x^3 \tag{203}$$
$$\phi(x) \xrightarrow{x\to\infty} 1 - 0 = 1 \tag{204}$$

We then use that $\frac{v}{v_{\text{th}_i}} \to \infty$ and $\frac{v}{v_{\text{th}_e}} \to 0$ based off of the relations on v_{th_h}.

Using the form provided (which is, as noted incorrect unit wise)

$$\begin{aligned} C_{\text{slow}} &= \frac{\gamma_{\alpha e} n_e v_{\text{th}_e}^3}{2v}\frac{\partial}{\partial w}\left\{\frac{4}{3\sqrt{\pi}}\frac{v^3}{v_{\text{th}_e}^3}\left[\frac{\partial f_{\alpha h}}{\partial w} + \frac{m_\alpha}{T} f_{\alpha h}\right]\right\} \\ &\quad + \frac{\gamma_{\alpha i} n_i v_{\text{th}_i}^3}{2v}\frac{\partial}{\partial w}\left\{\frac{\partial f_{\alpha h}}{\partial w} + \frac{m_\alpha}{T} f_{\alpha h}\right\} \end{aligned} \tag{205}$$

$$= \frac{m_\alpha}{2vT}\frac{\partial}{\partial w}\left\{\left[\frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w} + f_{\alpha h}\right]\left[\frac{4\gamma_{\alpha e} n_e v^3}{3\sqrt{\pi}} + \gamma_{\alpha i} n_i v_{\text{th}_i}^3\right]\right\} \tag{206}$$

$$= \frac{m_\alpha}{2vT}\frac{m_e v_{\text{th}_e}^3}{m_i}\frac{\partial}{\partial w}\left\{\left[\frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w} + f_{\alpha h}\right]\left[\gamma_{\alpha e} n_e v^3 \frac{4 m_i}{3\sqrt{\pi} m_e v_{\text{th}_e}^3} + \frac{m_i}{m_e v_{\text{th}_e}^3}\gamma_{\alpha i} n_i v_{\text{th}_i}^3\right]\right\} \tag{207}$$

$$= \frac{m_\alpha}{2vT}\frac{m_e v_{\text{th}_e}^3}{m_i}\frac{\partial}{\partial w}\left\{\left[\frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w} + f_{\alpha h}\right]\left[\gamma_{\alpha e} n_e \frac{v^3}{v_c^3} + \frac{m_i}{m_e v_{\text{th}_e}^3}\gamma_{\alpha i} n_i v_{\text{th}_i}^3\right]\right\} \tag{208}$$

$$= \frac{m_\alpha \gamma_{\alpha e} n_e}{2vT}\frac{m_e v_{\text{th}_e}^3}{m_i}\frac{\partial}{\partial w}\left\{\left[\frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w} + f_{\alpha h}\right]\left[\frac{v^3}{v_c^3} + \frac{m_i v_{\text{th}_i}^3}{m_e v_{\text{th}_e}^3}\frac{\gamma_{\alpha i} n_i}{\gamma_{\alpha e} n_e}\right]\right\} \tag{209}$$

$$= \frac{m_\alpha \gamma_{\alpha e} n_e}{2vT}\frac{m_e v_{\text{th}_e}^3}{m_i}\frac{\partial}{\partial w}\left\{\left[\frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w} + f_{\alpha h}\right]\left[\frac{v^3}{v_c^3} + \frac{m_i m_e^{3/2}}{m_e m_i^{3/2}}\frac{Z_i^2 n_i}{n_e}\right]\right\} \tag{210}$$

$$= \frac{m_\alpha \gamma_{\alpha e} n_e}{2vT}\frac{m_e v_{\text{th}_e}^3}{m_i}\frac{\partial}{\partial w}\left\{\left[\frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w} + f_{\alpha h}\right]\left[\frac{v^3}{v_c^3} + \sqrt{\frac{m_e}{m_i}}\frac{Z_i^2 n_i}{n_e}\right]\right\} \tag{211}$$

(212)

which will yield

$$= \frac{\gamma_{\alpha e} n_e}{T} \frac{4}{3\sqrt{\pi}} \frac{v_c^3}{v^3} \frac{m_\alpha v^2}{2} \frac{\partial}{\partial w} \left\{ \left[\frac{T}{m_\alpha} \frac{\partial f_{\alpha h}}{\partial w} + f_{\alpha h} \right] \left[\frac{v^3}{v_c^3} + \sqrt{\frac{m_e}{m_i}} \frac{Z_i^2 n_i}{n_e} \right] \right\} \tag{213}$$

$$= \frac{\gamma_{\alpha e} n_e}{T} \frac{4}{3\sqrt{\pi}} \frac{v_c^3}{v^3} w \frac{\partial}{\partial w} \left\{ \left[\frac{T}{m_\alpha} \frac{\partial f_{\alpha h}}{\partial w} + f_{\alpha h} \right] \left[\frac{v^3}{v_c^3} + \sqrt{\frac{m_e}{m_i}} \frac{Z_i^2 n_i}{n_e} \right] \right\} \tag{214}$$

We use that

$$\gamma_{\alpha e} = \frac{4\pi e_\alpha^2 e_e^2 \ln \Lambda}{m_\alpha} \tag{215}$$

$$\tau_e = \frac{3 m_e^2 v_{\mathrm{th}_e}^3}{16\sqrt{\pi} e^4 Z_{\mathrm{eff}} n_e \ln \Lambda} \tag{216}$$

$$\gamma_{\alpha e} = \frac{1}{\tau_e} \frac{3\sqrt{\pi} m_e^2 v_{\mathrm{th}_e}^3}{4 m_\alpha n_e} \tag{217}$$

so that

$$C_{\mathrm{slow}} = \frac{1}{\tau_e} \frac{3\sqrt{\pi} m_e^2 v_{\mathrm{th}_e}^3}{4 m_\alpha T} \frac{4}{3\sqrt{\pi}} \frac{v_c^3}{v^3} w \frac{\partial}{\partial w} \left\{ \left[\frac{T}{m_\alpha} \frac{\partial f_{\alpha h}}{\partial w} + f_{\alpha h} \right] \left[\frac{v^3}{v_c^3} + \sqrt{\frac{m_e}{m_i}} \frac{Z_i^2 n_i}{n_e} \right] \right\} \tag{218}$$

$$= \frac{1}{\tau_e} \frac{m_e^2 v_{\mathrm{th}_e}^3}{m_\alpha T} \frac{v_c^3}{v^3} w \frac{\partial}{\partial w} \left\{ \left[\frac{T}{m_\alpha} \frac{\partial f_{\alpha h}}{\partial w} + f_{\alpha h} \right] \left[\frac{v^3}{v_c^3} + \sqrt{\frac{m_e}{m_i}} \frac{Z_i^2 n_i}{n_e} \right] \right\} \tag{219}$$

which is about as close as one can get since the result is clearly wrong (by units) anyway.

8 Properties of Slowing-Down Collision Operator The slowing down distribution $f_{\alpha h}$ is determined by the obvious balance

$$C_{\text{slow}}(f_{\alpha h}) = S_\alpha \delta(w - w_0) \tag{220}$$

where S_α measures the fusion source and w_0 is the birth energy, 3.5 MeV. Using the results of 7, and neglecting contributions of order $\exp\left(-v_{\text{th}_h}^2/v_{\text{th}_i}^2\right)$, show that

$$f_{\alpha h} = \begin{cases} \frac{s}{\phi_s} & \text{for } w < w_0 \\ \frac{s}{\phi_s} \exp\left[\frac{m_\alpha}{T}(w - w_0)\right] & \text{for } w > w_0 \end{cases} \tag{221}$$

where $s \equiv (m_\alpha/m_i)(\tau_s/[2\pi v_c^3])S_\alpha$.

Solution:

We then use that

$$\frac{1}{\tau_s}\frac{m_i}{m_\alpha}\frac{v_c^3}{v^3} w \frac{\partial}{\partial w}\left[\phi_s(v)\left(f_{\alpha h} + \frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w}\right)\right] = S_\alpha \delta(w - w_0) \tag{222}$$

So for $w \neq w_0$ this equation states (for C some constant)

$$\left[\phi_s(v)\left(f_{\alpha h} + \frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w}\right)\right] = C \tag{223}$$

$$\frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w} = \frac{C}{\phi_s} - f_{\alpha h} \tag{224}$$

Thus,

$$f_{\alpha h} = Ae^{-m_\alpha w/T} + \int^w \frac{C_1}{\phi_s}\,\mathrm{d}w' \tag{225}$$

where $C_1 = \frac{m_\alpha}{T}C$.

Now, we need $f_{\alpha h} \to 0$ as $w \to \infty$ and hence $\int^w \frac{C}{\phi_s}\,\mathrm{d}w'\Big|_{w=\infty} = 0$ for $f_{\alpha h}^+$. We must also require that for $w \to 0$ that $f_{\alpha h} \to 0$ (because these are fast alphas there can't be any that aren't moving) and so here $A = 0$. Now, as the function $\int^w \frac{C}{\phi_s}\,\mathrm{d}w'$ doesn't satisfy these properties unless we choose $C = 0$ for $w \to \infty$ and $A = 0$ for $w \to 0$. At $w = w_0$ we want continuity and so

$$Ae^{-m_\alpha w_0/T} = \int^w \frac{C}{\phi_s}\,\mathrm{d}w'\Big|_{w=w_0} \tag{226}$$

which will give $\frac{A}{C}$.

When crossing from $w < w_0$ to $w > w_0$ we require a continuous solution and so

$$\lim_{\epsilon\to 0}\int_{w_0-\epsilon}^{w_0+\epsilon} \mathrm{d}w\ \frac{\partial}{\partial w}\left[\phi_s(v)\left(f_{\alpha h} + \frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w}\right)\right] = \lim_{\epsilon\to 0}\int_{w_0-\epsilon}^{w_0+\epsilon} \mathrm{d}w\ S_\alpha\, \delta(w - w_0)\frac{\tau_s v^3 m_\alpha}{m_i v_c^3 w} \tag{227}$$

$$\phi_s(v^+)f_{\alpha h}^+ + \frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w}^+ - \phi_s(v^+)f_{\alpha h}^- - \frac{T}{m_\alpha}\frac{\partial f_{\alpha h}}{\partial w}^- = S_\alpha\frac{\tau_s v_0^3 m_\alpha}{m_i v_c^3 w_0} \tag{228}$$

Of course continuity of $\phi_s(v) f_{\alpha h}$ implies that we need only consider the deriviatives and hence

$$\frac{\partial f_{\alpha h}}{\partial w}^{+} - \frac{\partial f_{\alpha h}}{\partial w}^{-} = S_\alpha \frac{\tau_s v_0^3 m_\alpha^2}{T m_i v_c^3 w_0} \tag{229}$$

$$\frac{-A m_\alpha}{T} e^{-m_\alpha w_0/T} - \frac{C}{\phi_s} = S_\alpha \frac{\tau_s v_0^3 m_\alpha^2}{T m_i v_c^3 w_0} \tag{230}$$

9 Comparison of Guiding-Center/Perturbation Distributions Equation (5.136) for the perturbed guiding-center distribution in the isothermal case, can be compared to (4.66), for the perturbation in $\widetilde{f}$. The comparison is made in terms of a toroidally confined plasma with thermal speed v_{th}, gradient scale-length L, poloidal magnetic field B_P, and toroidal field B_T.

$$f = -\frac{Iu}{\Omega}\frac{\mathrm{d}f_M}{\mathrm{d}\chi} \tag{5.136}$$

$$\widetilde{f} \simeq -\boldsymbol{\rho} \cdot \nabla\overline{f} \tag{4.66}$$

9.a Perturbation Sizes in Tokamak and RFP Compare the sizes of the two perturbations in a typical tokamak and in a reversed-field pinch.

Solution:

We see that (I will distinguish L as a or R for minor and major radius, respectively)

$$f \sim \frac{(RB_0)(v_{\text{th}})}{\Omega}\frac{f_M}{B_P(\pi\left((R+a)^2-(R-a)^2\right))} \approx f_M\frac{\cancel{R}B_0 v_{\text{th}}}{\Omega B_P \cancel{R}a} \tag{231}$$

$$\widetilde{f} \sim \frac{v_{\text{th}}}{\Omega}\frac{\overline{f}}{a} = \overline{f}\frac{v_{\text{th}}}{a\Omega} \tag{232}$$

We can write this in terms of

$$f \sim f_M\frac{B_0 v_{\text{th}}}{a\Omega B_P} \equiv f_M\widetilde{s}_1 \tag{233}$$

$$f = \overline{f}\frac{v_{\text{th}}}{\Omega a} \equiv \overline{f}\widetilde{s}_2 \tag{234}$$

And so (remembering $B_0 \sim B_T$)

$$s = \frac{\widetilde{s}_1}{\widetilde{s}_2} = \frac{\frac{B_0\cancel{v_{\text{th}}}}{\cancel{a\Omega}B_P}}{\frac{\cancel{v_{\text{th}}}}{\cancel{a\Omega}}} = \frac{B_0}{B_P} \sim \frac{B_T}{B_P} \tag{235}$$

For a typical tokamak $s \gg 1$ while for a typical RFP $s \ll 1$ as $B_T = 0$ isnide the RFP.

9.b Differences Between Distributions While (4.66) is generally valid, (5.136) requires constant temperature. What is the origin of this difference?

Solution:

In (4.66) the derivation did not assume a near Maxwellian initially, and so it is less restrictive, while in (5.136) we started our perturbation from a near Maxwellian, and so it is unsurprising that by starting with a more stringent intial distribution it is less valid generally.

9.c Physical Analogy Discuss the physical analogy between the two distributions, using (4.33).

$$\mathbf{v}_{\mathrm{gc}} = \hat{\mathbf{b}} v_{\parallel} + \mathcal{O}(\delta) \tag{4.33}$$

Solution:

We see that we would expect a contribution of the form

$$\mathbf{v}_{\mathrm{gc}} \cdot \nabla f \approx \hat{\mathbf{b}} v_{\parallel} \cdot \nabla f \tag{236}$$

To non-dimensionalize we see that we should divide by Ω, and thus the correction should be of the form $\frac{v}{\Omega}$. Thus the analogy is that of the normalized convective derivative, or flow of particles causing changes in the distribution.

10 Angular Momentum Conservation Consequences In a general axisymmetric system, we expect the *collisionless* distribution to have the form

$$f(\mathbf{x}, \mathbf{v}) = F(p_\zeta, \mu, U) \tag{237}$$

where p_ζ is the (covariant) angular momentum, $p_\zeta = mv_\zeta + (e/c)A_\zeta$, and the function F is unspecified. After recalling (3.124), use an appropriate gyroradius expansion to relate F to perturbations given by (5.128) and (5.133). Thus (5.128) is derived from angular momentum conservation.

$$A_\zeta = -\chi(r, t) \tag{3.124}$$

$$f_1 = k + g \tag{5.128}$$

$$k = -\frac{Iu}{\Omega}\frac{\mathrm{d}f_M}{\mathrm{d}\chi} \tag{5.133}$$

Solution:

This is collisionless and hence we see that the form will be

$$u\,\nabla_\| F_1 = -\mathbf{v}_D \cdot \nabla F_0 \tag{238}$$

with $F_0 = F_0(p_\zeta, \mu_0, U_0)$ with μ_0 and U_0 being the magnetic moment and energy to lowest order. Thus, we find

$$u\,\nabla_\| F_1 = -\mathbf{v}_D \left(\frac{\partial F_0}{\partial p_\zeta} \cdot \nabla p_\zeta + \cancel{\frac{\partial F_0}{\partial \mu_0}} \cdot \nabla \mu_0 + \cancel{\frac{\partial F_0}{\partial U_0}} \cdot \nabla U_0\right) = -\mathbf{v}_D \cdot \left(\frac{\partial F_0}{\partial A_\zeta}\nabla A_\zeta\right) = -\mathbf{v}_D \cdot \frac{\partial F_0}{\partial \chi}\nabla\chi \tag{239}$$

and so everything follows exactly as derived from before with the replacement $F_1 = k$ and $F_0 = f_M$. which is similar to the k term.