1 N **is Constant** Let N be the number of particles inside some fixed volume V. Suppose that the distribution function f vanishes on the boundary of V. Use (4.4) to prove that in this case N must be constant: $\mathrm{d}N/\mathrm{d}t = 0$.

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{4.4}$$

Solution:

Let's assume that V is phase-space volume, as otherwise this isn't quite possible (we need particles with small enough velocities that they cannot escape the region basically). We integrate over $\int \mathrm{d}^3x\mathrm{d}^3v$ and find

$$\int \mathrm{d}^3x\mathrm{d}^3v \left[\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}}\right] = 0 \tag{1}$$

$$\int \mathrm{d}^3x \,\frac{\partial n}{\partial t} + \int \mathrm{d}^3x\mathrm{d}^3v \left[\boldsymbol{\nabla} \cdot (\mathbf{v}f) + \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a}f) - f\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{a}\right] = 0 \tag{2}$$

whereas $\mathbf{v}$ and $\mathbf{x}$ are independent we have $\mathbf{v} \cdot \nabla f = \boldsymbol{\nabla} \cdot (\mathbf{v}f)$. We then convert the two divergences into surface integrals and note that f is zero on the boundary and hence

$$\frac{\mathrm{d}N}{\mathrm{d}t} + \cancel{\int \mathrm{d}^3v\mathrm{d}^3x\, \boldsymbol{\nabla} \cdot (\mathbf{v}f)} + \cancel{\int \mathrm{d}^3x\mathrm{d}^3v\, \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a}f)} - \int \mathrm{d}^3v\mathrm{d}^3x\; f\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{a} = 0 \tag{3}$$

Now we note that

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{a} = \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{e}{m}\left(\mathbf{E}(\mathbf{x},t) + \frac{1}{c}\mathbf{v} \times \mathbf{B}(\mathbf{x},t)\right)\right] \tag{4}$$

which can be written as (with $\frac{\partial}{\partial \mathbf{v}} = \partial_i^v$)

$$\partial_i^v\left(\frac{e}{m}\left[E_i + \epsilon_{ijk}\frac{1}{c}v_jB_k\right]\right) = \frac{e}{mc}\partial_i^v\epsilon_{ijk}v_jB_k = \frac{e}{mc}\left(\epsilon_{ijk}\left[B_k\partial_i^v v_j + \cancel{v_j\partial_i^v B_k}\right]\right) \tag{5}$$

$$= \frac{e}{mc}\epsilon_{ijk}B_k\delta_{ij} = \frac{e}{mc}\epsilon_{iik}B_k = 0 \tag{6}$$

and thus $\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{a} = 0$ and we simply have

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0 \tag{7}$$

as desired.

2 Kinetic Equation in Cylindrical Express the general kinetic equation in terms of cylindrical coordinates (r, θ, z), using the velocity variables ($v^r = \frac{\mathrm{d}r}{\mathrm{d}t}$, $v^\theta = r\frac{\mathrm{d}\theta}{\mathrm{d}t}$, $v^z = \frac{\mathrm{d}z}{\mathrm{d}t}$).

Solution:

We use that the formula is given by

$$\frac{\partial f}{\partial t} + \frac{\mathrm{d}z^i}{\mathrm{d}t}\frac{\partial f}{\partial z^i} = 0 \tag{8}$$

Thus, we see in cylindrical coordinates that we get

$$\frac{\partial f}{\partial t} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\partial f}{\partial r} + \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\partial f}{\partial \theta} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial f}{\partial z} + \frac{\mathrm{d}v^r}{\mathrm{d}t}\frac{\partial f}{\partial v^r} + \frac{\mathrm{d}v^\theta}{\mathrm{d}t}\frac{\partial f}{\partial v^\theta} + \frac{\mathrm{d}v^z}{\mathrm{d}t}\frac{\partial f}{\partial v^z} = 0 \tag{9}$$

$$\boxed{\frac{\partial f}{\partial t} + v^r\frac{\partial f}{\partial r} + \frac{v^\theta}{r}\frac{\partial f}{\partial \theta} + v^z\frac{\partial f}{\partial z} + \frac{\mathrm{d}v^r}{\mathrm{d}t}\frac{\partial f}{\partial v^r} + \frac{\mathrm{d}v^\theta}{\mathrm{d}t}\frac{\partial f}{\partial v^\theta} + \frac{\mathrm{d}v^z}{\mathrm{d}t}\frac{\partial f}{\partial v^z} = 0} \tag{10}$$

3 Disruption in MHD Ordering A disruption in the Standard Tokamak (Table 1) is observed to carry plasma over an appreciable fraction of the minor radius in a time Δt. How small must Δt be in order for the disruption to be described by the MHD ordering?

toroidal field (B_T)	50 kG
major radius (R_0)	300 cm
minor radius (a)	80 cm
safety factor (q)	$q \simeq 1$ (on axis) $q \simeq 3$ (at edge)
central density (n)	10^{14} cm^{-3}
central temperature ($T_i = T_e = T$)	10 keV

Table 1: The Standard Tokamak parameters.

Solution:

For MHD ordering, we have

$$V_E \sim v_{\text{th}} \tag{11}$$

In this case we estimate

$$v_{\text{th}} \sim \sqrt{T_e/m_e} \approx \sqrt{10 \text{ keV}/(511 \text{ keV}/c^2)} \approx 6 \times 10^6 \text{ m/s} \tag{12}$$

Thus, we have

$$\frac{a}{\Delta t} \sim v_{\text{th}} \approx 6 \times 10^6 \text{ m/s} \tag{13}$$

$$\Delta t \sim \frac{.8 \text{ m}}{6 \times 10^6 \text{ m/s}} \approx 1 \times 10^{-7} \text{ s} \tag{14}$$

$$\boxed{\Delta t \sim 0.1\ \mu\text{s} = 100 \text{ ns}} \tag{15}$$

4 Drift Velocity Estimate Using the estimate $\nabla B/B \sim 1/R$, compute the order of magnitude of the drift velocity in the Standard Tokamak (Table 1), in centimeters per second. Similarly discuss the relative size of the two terms in (4.158), for poloidal drift motion in the Standard Tokamak.

$$\mathbf{v}_D = \mathbf{v}_d + \frac{u^2}{\Omega}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)_\perp \tag{4.158}$$

Solution:

We use that

$$\mathbf{v}_D = \frac{1}{\Omega}\hat{\mathbf{b}}\times\left[\left(u^2+\frac{\mu B}{m}\right)\frac{\nabla B}{B}+\frac{e}{m}\nabla\Phi\right]+\frac{u^2}{\Omega}\frac{\left(\boldsymbol{\nabla}\times\mathbf{B}\right)_\perp}{B} \tag{16}$$

$$\mathbf{v}_D = \frac{m}{eB}\hat{\mathbf{b}}\times\left[\left(u^2+\frac{mv_\perp^2 B}{2Bm}\right)\frac{\nabla B}{B}+\frac{e}{m}\nabla\Phi\right]+\frac{mu^2}{eB}\frac{\left(\boldsymbol{\nabla}\times\mathbf{B}\right)_\perp}{B} \tag{17}$$

$$\mathbf{v}_D = \frac{m}{eB}\hat{\mathbf{b}}\times\left[\left(u^2+\frac{v_\perp^2}{2}\right)\frac{\nabla B}{B}+\frac{e}{m}\nabla\Phi\right]+\frac{mu^2}{e}\frac{\left(\boldsymbol{\nabla}\times\mathbf{B}\right)_\perp}{B^2} \tag{18}$$

We use that this will be a combination of grad B and curvature drift. This can be written as

$$\frac{mu^2+mv_\perp^2/2}{eB}\frac{1}{R}\sim\frac{T}{eRB} \tag{19}$$

Thus, the grad B/curvature drift is on the order

$$v_{D1}\sim\frac{(10\text{ keV})(1.6\times10^{-16}\text{ J/keV})}{(1.6\times10^{-19}\text{ C})(3\text{ m})(5\text{ T})}\approx 670\text{ m/s} \tag{20}$$

Assuming drift ordering, we'd find

$$V_E\approx\frac{mv_{\text{th}}^2}{eaB}\approx\frac{T}{eaB}\approx\frac{(10\text{ keV})(1.6\times10^{-16}\text{ J/keV})}{(1.6\times10^{-19}\text{ C})(.8\text{ m})(5\text{ T})} \tag{21}$$

$$V_E\approx 2.5\times10^3\text{ m/s} \tag{22}$$

Now for the last term.

We use $J_\perp B\approx\nabla p\sim nT/a$ so that $\left(\boldsymbol{\nabla}\times\mathbf{B}\right)_\perp=\mu_0\mathbf{J}_\perp$.

And so finally, the last term is given by

$$v_{D2}\frac{mu^2}{e}\frac{\left(\boldsymbol{\nabla}\times\mathbf{B}\right)_\perp}{B^2}\sim\frac{T}{eB^2}\frac{\mu_0\nabla p}{B}\sim\frac{\mu_0 nT^2}{eaB^3} \tag{23}$$

$$\sim\frac{(4\pi\times10^{-7}\text{ H/m})(1\times10^{-20}\text{ m}^{3-})(10\text{ keV})^2\left(1.6\times10^{-16}\text{ J/keV}\right)^2}{(1.6\times10^{-19}\text{ C})(.8\text{ m})(5\text{ T})^3}\sim 20.1\text{ m/s} \tag{24}$$

Hence, we see that the last term is small in comparison to the others, just as expected.

Thus, combining we see our estimates are

$$\boxed{\begin{aligned} v_E &\sim 2.5\times10^3\text{ m/s}=2.5\times10^5\text{ cm/s} && (25)\\ v_{D1}&\sim 6.7\times10^2\text{ m/s}=6.7\times10^4\text{ cm/s} && (26)\\ v_{D2}&\sim 2.0\times10^1\text{ m/s}=2.0\times10^3\text{ cm/s} && (27)\end{aligned}}$$

5 Verify Identities Verify the identities in (4.62).

$$\begin{aligned}\int \mathrm{d}\gamma\ \hat{\mathbf{s}}\hat{\mathbf{s}} &= \frac{1}{2}\left[\hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \gamma\left(\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\right] \\ \int \mathrm{d}\gamma\ \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} &= \frac{1}{2}\left[-\hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \gamma\left(\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\right]\end{aligned} \tag{4.62}$$

Solution:

We can use that

$$\hat{\mathbf{s}} = \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \gamma} \tag{28}$$

$$\hat{\boldsymbol{\rho}} = -\frac{\partial \hat{\mathbf{s}}}{\partial \gamma} \tag{29}$$

$$\mathbb{1} = \hat{\mathbf{b}}\hat{\mathbf{b}} + \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} + \hat{\mathbf{s}}\hat{\mathbf{s}} \tag{30}$$

Thus, we find

$$\int \mathrm{d}\gamma\ \hat{\mathbf{s}}\hat{\mathbf{s}} = \int \mathrm{d}\gamma\ \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \gamma}\hat{\mathbf{s}} = \hat{\boldsymbol{\rho}}\hat{\mathbf{s}} - \int \mathrm{d}\gamma\ \hat{\boldsymbol{\rho}}\frac{\partial \hat{\mathbf{s}}}{\partial \gamma} \tag{31}$$

$$\int \mathrm{d}\gamma\ \hat{\mathbf{s}}\hat{\mathbf{s}} = \hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \int \mathrm{d}\gamma\ \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \int \mathrm{d}\gamma\ \left[\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}} - \hat{\mathbf{s}}\hat{\mathbf{s}}\right] \tag{32}$$

$$2\int \mathrm{d}\gamma\ \hat{\mathbf{s}}\hat{\mathbf{s}} = \hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \int \mathrm{d}\gamma\ \left[\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right] \tag{33}$$

Now as $\mathbb{1}$ and $\hat{\mathbf{b}}\hat{\mathbf{b}}$ are independent of γ we may take them out of the integral and use $\int \mathrm{d}\gamma\ = \gamma$ to find

$$2\int \mathrm{d}\gamma\ \hat{\mathbf{s}}\hat{\mathbf{s}} = \hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \gamma\left[\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right] \tag{34}$$

$$\boxed{\int \mathrm{d}\gamma\ \hat{\mathbf{s}}\hat{\mathbf{s}} = \frac{1}{2}\left[\hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \gamma\left(\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\right]} \tag{35}$$

Similarly,

$$\int \mathrm{d}\gamma\ \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} = \int \mathrm{d}\gamma\ \left(-\frac{\partial \hat{\mathbf{s}}}{\partial \gamma}\hat{\boldsymbol{\rho}}\right) = -\hat{\boldsymbol{\rho}}\hat{\mathbf{s}} - \int \mathrm{d}\gamma\ \left(-\hat{\mathbf{s}}\frac{\partial \hat{\boldsymbol{\rho}}}{\partial \gamma}\right) \tag{36}$$

$$\int \mathrm{d}\gamma\ \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} = -\hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \int \mathrm{d}\gamma\ \hat{\mathbf{s}}\hat{\mathbf{s}} = -\hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \int \mathrm{d}\gamma\ \left[\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}} - \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}\right] \tag{37}$$

$$2\int \mathrm{d}\gamma\ \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} = -\hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \int \mathrm{d}\gamma\ \left[\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right] \tag{38}$$

Now as $\mathbb{1}$ and $\hat{\mathbf{b}}\hat{\mathbf{b}}$ are independent of γ we may take them out of the integral and use $\int \mathrm{d}\gamma\ = \gamma$ to find

$$2\int \mathrm{d}\gamma\ \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} = -\hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \gamma\left[\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right] \tag{39}$$

$$\boxed{\int \mathrm{d}\gamma\ \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} = \frac{1}{2}\left[-\hat{\boldsymbol{\rho}}\hat{\mathbf{s}} + \gamma\left(\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\right]} \tag{40}$$

6 Criticize $J_\parallel/J_\perp$ Ordering Criticize the ordering given by (4.71), by using Ampère's law to estimate $J_\parallel$ in terms of B_P. Thus show that $J_\parallel/J_\perp \sim B/B_P$.

$$J_\parallel \sim J_\perp \sim \frac{cP}{LB} \tag{4.71}$$

Solution:

To keep there from being a charge imbalance, we know that there is a return current (which gives most of the parallel current). The vertical perpendicular current is of the form

$$J_{\perp,v} = \frac{\epsilon \nabla P}{B} \tag{41}$$

The return current must balance this vertically and so must have

$$J_{\parallel,v} = \sin\alpha J_\parallel = \frac{B_P}{B} J_\parallel \tag{42}$$

(α is the angle between toroidal magnetic field $\mathbf{B}_T$ and the full magnetic field $\mathbf{B}$). Thus,

$$J_\parallel = \frac{\epsilon \nabla P}{B_P} \tag{43}$$

This yields the estimate

$$\frac{J_\parallel}{J_\perp} \sim \frac{B}{B_P} \tag{44}$$

as desired.

7 Drift Kinetic Equation Properties The derivation of the drift kinetic equation implicitly assumes $\partial f/\partial\mu \leq f/\mu$. Identify the step in which this assumption enters. At low collision frequency, boundary layers in velocity space can occur, giving the distribution relatively steep dependence on μ: $\partial f/\partial\mu \sim f/\Delta\mu \gg f/\mu$. Use the steady-state version of (4.69) to derive an expression for the smallest $\Delta\mu$ that is consistent with drift-kinetic theory.

$$\left.\frac{\mathrm{d}\mu}{\mathrm{d}t}\right|_{\mathrm{gc}} = -\frac{u\mu}{\Omega}\boldsymbol{\nabla}\cdot\left(\hat{\mathbf{b}}\times\frac{\partial\hat{\mathbf{b}}}{\partial t}\right) - \frac{\mu}{B}\left(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}\right)\hat{\mathbf{b}}\cdot\frac{\partial\mathbf{A}}{\partial t} + \frac{u\mu B}{\Omega}\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot\left(\frac{u\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}}{B}\right) \tag{4.69}$$

Solution:

This assumption occurs at

$$\mathcal{L} \sim C \sim \delta\Omega \tag{4.54}$$

with kinetic equation

$$\frac{\partial\widetilde{f}}{\partial\gamma} = -\frac{(\mathcal{L}-C)}{\Omega}f \tag{4.53}$$

$$\mathcal{L} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{\mathrm{d}\mu}{\mathrm{d}t}\frac{\partial}{\partial\mu} + \frac{\mathrm{d}U}{\mathrm{d}t}\frac{\partial}{\partial U} + \left(\frac{\mathrm{d}\gamma}{\mathrm{d}t} - \Omega\right)\frac{\partial}{\partial\gamma} \tag{45}$$

As is noted in the text, the ordering is meaningful only if f varies moderately, so that $\mu\frac{\partial f}{\partial\mu} \sim 1$.

In steady state, we have $\frac{\partial}{\partial t} \to 0$ and thus

$$\mathbf{v}_{\mathrm{gc}}\cdot\nabla\mu = \frac{u\mu B}{\Omega}\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot\left(\frac{u\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}}{B}\right) \tag{46}$$

Thus (taking $v_{\mathrm{gc}} \sim v_{\parallel} = u$)

$$\frac{\Delta\mu}{\cancel{L}} \sim \frac{uB\mu}{\Omega\cancel{u}}\frac{\cancel{u}J_{\parallel}}{\cancel{L}B} \tag{47}$$

$$\Delta\mu \sim \frac{u\mu(\mu_0 J_{\parallel}/B)}{\Omega} \tag{48}$$

Thus,

$$\boxed{\Delta\mu \sim \frac{B\Omega}{u\mu_0 J_{\parallel}}} \tag{49}$$

to be consistent with the drift kinetic ordering.

8 Annihilation of Orbital Average Explicitly demonstrate the annihilation property of the orbital average, (4.133), in the axisymmetric case.

$$\left\langle \sigma |u| \, \nabla_{\parallel} f \right\rangle_O = 0 \tag{4.133}$$

Solution:

In the axisymmetric case we have in the passing particle region

$$\frac{\left\langle \sigma |u| B \, \nabla_{\parallel} f / u \right\rangle_s}{\left\langle B/u \right\rangle_s} = \frac{\left\langle \sigma^2 B \, \nabla_{\parallel} f \right\rangle_s}{\left\langle B/u \right\rangle_s} = 0 \tag{50}$$

and as $\sigma^2 = 1$ we have by the definition of $\left\langle B \, \nabla_{\parallel} F \right\rangle_s = 0$ and thus the above averaging clearly vanishes.

The trapped case is more interesting in that it says

$$\frac{1}{2} \frac{\left\langle (+1) B |u| \, \nabla_{\parallel} f_+ / |u| \right\rangle}{\left\langle B/|u| \right\rangle} + \frac{1}{2} \frac{\left\langle (-1) B |u| \, \nabla_{\parallel} f_- / |u| \right\rangle}{\left\langle B/|u| \right\rangle} = \frac{\left\langle B \, \nabla_{\parallel} (f_+ - f_-) \right\rangle}{2 \left\langle B/|u| \right\rangle} \tag{51}$$

From the bounce condition we see that $f_+ - f_- = 0$ as required.

Thus both cases vanish and so

$$\left\langle \sigma |u| \, \nabla_{\parallel} f \right\rangle_O = 0 \tag{52}$$

as required.

9 Jacobian and Integrals Verify the velocity space Jacobian of (4.144). Then, using the Maxwellian distribution of (4.181), explicitly perform each integral in (4.146) to verify that

$$\int \mathrm{d}^3v\ f_M = n \quad . \tag{53}$$

(the needed equations are)

$$\int \mathrm{d}^3v\ = \sum_\sigma \int \frac{wB}{m^2|u|}\,\mathrm{d}\lambda\,\mathrm{d}w\,\mathrm{d}\gamma \tag{4.144}$$

$$f_M(\mathbf{x},\mathbf{v}) = \frac{n}{(\sqrt{\pi}v_{\mathrm{th}})^3}\exp\left(-\frac{U-e\Phi}{T}\right) \tag{4.181}$$

$$\int \mathrm{d}^3v\ = \frac{B}{m^2}\sum_\sigma \oint \mathrm{d}\gamma\ \int_{e\Phi}^{\infty}\mathrm{d}U\ \int_0^{(U-e\Phi)/B}\mathrm{d}\mu\ \frac{1}{|u|} \tag{4.146}$$

Solution:

To verify the Jacobian, we simply need to calculate

$$\frac{1}{\mathcal{J}} = \nabla_v\lambda \cdot \nabla_v w \times \nabla_v\gamma \tag{54}$$

Now we have (choosing $\mathcal{J}_\sigma$ such that $u = |u|$ for later)

$$\frac{\partial\lambda}{\partial\mathbf{v}} = \frac{\partial}{\partial\mathbf{v}}\left(\frac{\mu}{w}\right) = \frac{1}{w}\frac{\partial\mu}{\partial\mathbf{v}} - \frac{\mu}{w^2}\frac{\partial w}{\partial\mathbf{v}} \tag{55}$$

$$\frac{\partial w}{\partial\mathbf{v}} = \frac{\partial U}{\partial\mathbf{v}} - e\cancel{\frac{\partial\Phi}{\partial\mathbf{v}}} \tag{56}$$

$$\frac{\partial\gamma}{\partial\mathbf{v}} = -\frac{\hat{\mathbf{b}}\times\mathbf{v}}{v_\perp^2} \tag{57}$$

$$\frac{\partial\mu}{\partial\mathbf{v}} = \frac{m\mathbf{v}_\perp}{B} \tag{58}$$

$$\frac{\partial U}{\partial\mathbf{v}} = m\mathbf{v} \tag{59}$$

Thus,

$$\nabla_v w \times \nabla_v\gamma = m\mathbf{v}\times\left(-\frac{\hat{\mathbf{b}}\times\mathbf{v}}{v_\perp^2}\right) = \frac{-m}{v_\perp^2}\left(\hat{\mathbf{b}}(\mathbf{v}\cdot\mathbf{v}) - \mathbf{v}(\hat{\mathbf{b}}\cdot\mathbf{v})\right) \tag{60}$$

$$= \frac{m}{v_\perp^2}\left(u\mathbf{v} - v^2\hat{\mathbf{b}}\right) \tag{61}$$

$$\nabla_v\lambda \cdot \nabla_v w \times \nabla_v\gamma = \left(\frac{1}{w}\frac{m\mathbf{v}_\perp}{B} - \frac{\mu}{w^2}m\mathbf{v}\right)\cdot\left(\frac{m}{v_\perp^2}\left(u\mathbf{v} - v^2\hat{\mathbf{b}}\right)\right) \tag{62}$$

$$= \frac{m^2v_\perp^2u}{wv_\perp^2B} - 0 - \cancel{\frac{\mu m^2uv^2}{v_\perp^2w^2} + \frac{m^2\mu v^2u}{w^2v_\perp^2}} \tag{63}$$

$$\nabla_v\lambda \cdot \nabla_v w \times \nabla_v\gamma = \frac{m^2u}{wB} = \frac{1}{\mathcal{J}} \tag{64}$$

Remembering that we created two branches of u so that given λ, w, γ we require the sign $\sigma|u|$ for u we then regain the desired Jacobian of (4.144)

$$\int \mathrm{d}^3v \; = \sum_\sigma \int \mathcal{J}_\sigma \,\mathrm{d}\lambda\,\mathrm{d}w\,\mathrm{d}\gamma = \sum_\sigma \int \frac{wB}{m^2|u|}\,\mathrm{d}\lambda\,\mathrm{d}w\,\mathrm{d}\gamma \tag{65}$$

The way to see this is that the $\sigma = +1$ branch will yield an integral exactly the same as the $\sigma = -1$ branch because the integral is in the opposite direction. That is

$$\int \mathrm{d}^3v \; = \sum_\sigma \int \mathcal{J}_\sigma \,\mathrm{d}\lambda\,\mathrm{d}w\,\mathrm{d}\gamma = +\int \frac{wB}{m^2|u|}\,\mathrm{d}\lambda\,\mathrm{d}w\,\mathrm{d}\gamma - \int \frac{wB}{-m^2|u|}\,\mathrm{d}\lambda\,\mathrm{d}w\,\mathrm{d}\gamma \tag{66}$$

$$= 2\int \frac{wB}{m^2|u|}\,\mathrm{d}\lambda\,\mathrm{d}w\,\mathrm{d}\gamma = \sum_\sigma \int \frac{wB}{m^2|u|}\,\mathrm{d}\lambda\,\mathrm{d}w\,\mathrm{d}\gamma \tag{67}$$

Now let's do the integral. We remind ourselves that

$$\frac{1}{2}mu^2 = U - e\Phi - \frac{1}{2}mv_\perp^2 = U - e\Phi - \mu B \tag{68}$$

$$u = \pm\sqrt{\frac{2U}{m} - \frac{2e\Phi}{m} - \mu\frac{2B}{m}} \tag{69}$$

Thus,

$$\int \mathrm{d}^3v\; f_M = \frac{B}{m^2}\sum_\sigma \oint \mathrm{d}\gamma \;\int_{e\Phi}^\infty \mathrm{d}U \;\int_0^{(U-e\Phi)/B} \mathrm{d}\mu\; \frac{n}{|u|(\sqrt{\pi}v_{\mathrm{th}})^3}\exp\left(-\frac{U-e\Phi}{T}\right) \tag{70}$$

$$\int \mathrm{d}^3v\; f_M = \frac{B}{m^2}\sum_\sigma \oint \mathrm{d}\gamma \;\int_{e\Phi}^\infty \mathrm{d}U \;\frac{n}{(\sqrt{\pi}v_{\mathrm{th}})^3}\exp\left(-\frac{U-e\Phi}{T}\right)\int_0^{(U-e\Phi)/B} \mathrm{d}\mu\; \sqrt{\frac{2m}{B}}\frac{1}{\sqrt{\frac{U-e\Phi}{B} - \mu}} \tag{71}$$

And using $S = \frac{U-e\Phi}{B} - \mu$ so $\mathrm{d}S = -\,\mathrm{d}\mu$ we find

$$\int_0^{(U-e\Phi)/B} \mathrm{d}\mu\; \sqrt{\frac{2m}{B}}\frac{1}{\sqrt{\frac{U-e\Phi}{B} - \mu}} = \int_{(U-e\Phi)/B}^0 \mathrm{d}S\; -\sqrt{\frac{2m}{B}}\frac{1}{\sqrt{S}} = \int_0^{(U-e\Phi)/B}\frac{\mathrm{d}S}{\sqrt{S}} \tag{72}$$

$$= \sqrt{\frac{2m}{B}}2\sqrt{S}\Big|_0^{(U-e\Phi)/B} = \frac{\sqrt{2m(U-e\Phi)}}{B} \tag{73}$$

And so $Q = (U - e\Phi)/T$ with $\mathrm{d}Q = \mathrm{d}U/T$

$$\int \mathrm{d}^3v\; f_M = \frac{B}{m^2}\sum_\sigma \oint \mathrm{d}\gamma \;\int_{e\Phi}^\infty \mathrm{d}U \;\frac{n}{(\sqrt{\pi}v_{\mathrm{th}})^3}\exp\left(-\frac{U-e\Phi}{T}\right)\frac{\sqrt{2m(U-e\Phi)}}{B} \tag{74}$$

$$= \frac{2\pi n\sqrt{2m}\cancel{B}}{\cancel{B}m^2\pi^{3/2}\left(\frac{2T}{m}\right)^{3/2}}\sum_\sigma \int_{e\Phi}^\infty \mathrm{d}U\;\sqrt{U-e\Phi}e^{-\frac{U-e\Phi}{T}} \tag{75}$$

$$= \frac{\cancel{2^{3/2}}n\cancel{\sqrt{m}}}{\cancel{\sqrt{m}}\pi^{3/2-1}\cancel{2^{3/2}}T^{3/2}}\sum_\sigma \int_0^\infty \mathrm{d}Q\;T\sqrt{TQ}e^{-Q} \tag{76}$$

$$= \frac{n}{\sqrt{\pi}}\sum_\sigma \int_0^\infty \mathrm{d}Q\;\sqrt{Q}e^{-Q} \tag{77}$$

Now using $q^2 = Q$ so that $2q\,\mathrm{d}q = \mathrm{d}Q$ we find

$$\frac{n}{\sqrt{\pi}} \sum_{\sigma} \int_0^{\infty} \mathrm{d}q\ (2q) q e^{-q^2} = \frac{2n}{\sqrt{\pi}} \sum_{\sigma} \int_0^{\infty} \mathrm{d}q\ q^2 e^{-q^2} \tag{78}$$

$$= \frac{2n}{\cancel{\sqrt{\pi}}} \sum_{\sigma} \frac{\cancel{\sqrt{\pi}}}{4} = \left(\frac{1}{2} + \frac{1}{2}\right) = n \tag{79}$$

as $\sum_{\sigma} \frac{1}{2} = 1$ for our case.

Therefore, we find explicitly that

$$\int \mathrm{d}^3 v\ f_M = n \tag{80}$$

as desired.

10 A^*_ζ is Conserved Often the last term in (4.162), involving $(\boldsymbol{\nabla}\times\mathbf{B})_\parallel$, is omitted. Show that in that case one may write

$$\mathbf{v}_{gc} = \frac{u}{B}\boldsymbol{\nabla}\times\mathbf{A}^* \quad , \quad \mathbf{A}^* = \mathbf{A} + \mathbf{B}\frac{u}{\Omega} \quad . \tag{81}$$

Show that $A^*_\zeta = R^2\,\nabla\zeta\cdot\mathbf{A}^*$ is conserved by guiding-center motion in an axisymmetric system.

$$\mathbf{v}_{\rm gc} = \frac{u}{B}\left[\mathbf{B} + \boldsymbol{\nabla}\times\left(\frac{\mathbf{B}u}{\Omega}\right)\right] - \frac{1}{\Omega}\left(u^2 - \frac{\mu B}{m}\right)\frac{(\boldsymbol{\nabla}\times\mathbf{B})_\parallel}{B} \tag{4.162}$$

Solution:

We then have

$$\boldsymbol{\nabla}\times\mathbf{A}^* = \boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}\times\left(\mathbf{B}\frac{u}{\Omega}\right) \tag{82}$$

$$= \mathbf{B} + \boldsymbol{\nabla}\times\left(\frac{\mathbf{B}}{u}\Omega\right) \tag{83}$$

Thus,

$$\mathbf{v}_{\rm gc} = \frac{u}{B}\boldsymbol{\nabla}\times\mathbf{A}^* = \frac{u}{B}\left[\mathbf{B} + \boldsymbol{\nabla}\times\left(\frac{\mathbf{B}u}{\Omega}\right)\right] \tag{84}$$

We now need to show that A^*_ζ is conserved in guiding center motion. This is equivalent to showing that $\mathrm{d}A^*_\zeta/\mathrm{d}t = 0$. We use that A^*_ζ is related to the conserved ζ canonical angular momentum. The simplest way is to use the Lagrangian (given by Hazeltine & Meiss (2.44))

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{q}{c}\mathbf{A}\cdot\mathbf{v} - q\Phi \tag{2.44}$$

From guiding center theory we have $\mathbf{v} = v_\parallel\hat{\mathbf{b}} + \frac{c\mathbf{E}\times\mathbf{B}}{B^2}$. That is, the perpendicular component is simply $\mathbf{E}\times\mathbf{B}$ drift.

Thus, I write this in the slightly bizarre way

$$\mathcal{L} = mu\hat{\mathbf{b}}\cdot\mathbf{v} + \frac{q}{c}\mathbf{A}\cdot\mathbf{v} - \frac{1}{2}mu^2 + \frac{1}{2}m\frac{c|\mathbf{E}\times\mathbf{B}|^2}{B^4} - q\Phi \tag{85}$$

Then we have (all quantities are independent of ζ by axisymmetry) (we remember that $\frac{\partial\dot{\mathbf{x}}}{\partial\dot{\xi}} = \frac{\partial\mathbf{x}}{\partial\xi}$, the so-called cancelling the dots from classical mechanics)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\zeta}}\right) = \frac{\partial\mathcal{L}}{\partial\zeta} = 0 \tag{86}$$

$$\dot{p}_\zeta = 0 \tag{87}$$

$$p_\zeta = \frac{\partial\mathcal{L}}{\partial\dot{\zeta}} = mu\hat{\mathbf{b}}\cdot\frac{\partial\dot{\mathbf{x}}}{\partial\dot{\zeta}} + \frac{q}{c}\mathbf{A}\cdot\frac{\partial\dot{\mathbf{x}}}{\partial\dot{\zeta}} = mu\hat{\mathbf{b}}\cdot\frac{\partial\mathbf{x}}{\partial\zeta} + \frac{q}{c}\mathbf{A}\cdot\frac{\partial\mathbf{x}}{\partial\zeta} \tag{88}$$

$$= mub_\zeta + \frac{q}{c}A_\zeta = \frac{q}{c}\left(A^*_\zeta\right) \tag{89}$$

as

$$A_\zeta^* = A_\zeta + B_\zeta \frac{u}{\frac{qB}{mc}} = A_\zeta + \frac{umc}{q} b_\zeta \tag{90}$$

We thus find A_ζ^* is conserved.

This is because $\hat{\mathbf{b}} \cdot \frac{\partial \mathbf{x}}{\partial \zeta} = b_\zeta$. This can be seen as

$$\hat{\mathbf{b}} = b_i \frac{\partial \xi^i}{\partial \mathbf{x}} \tag{91}$$

$$\hat{\mathbf{b}} \cdot \frac{\partial \mathbf{x}}{\partial \zeta} = \left[b_i \frac{\partial \xi^i}{\partial \mathbf{x}} \right] \cdot \frac{\partial \mathbf{x}}{\partial \zeta} = b_i \frac{\partial \xi^i}{\partial \zeta} = b_1 \cancelto{0}{\frac{\partial \xi^1}{\partial \zeta}} + b_2 \cancelto{0}{\frac{\partial \xi^2}{\partial \zeta}} + b_\zeta \cancelto{1}{\frac{\partial \zeta}{\partial \zeta}} = b_\zeta \tag{92}$$

11 Tensor Identity Prove that

$$\oint \frac{\mathrm{d}\gamma}{2\pi}\hat{\mathbf{s}}\hat{\boldsymbol{\rho}}\cdot\mathbf{A} = -\frac{1}{2}\hat{\mathbf{b}}\times\mathbf{A} \tag{93}$$

for any vector $\mathbf{A}$.

Solution:

We note that the identity implies $\mathbf{A} \neq \mathbf{A}(\gamma)$ as the left hand side integrates out any γ dependence, and hence it would be contradictory to assume $\mathbf{A} = \mathbf{A}(\gamma)$ as the left hand side would have no γ dependence while the right hand side would.

We can use that

$$\hat{\mathbf{s}} = \hat{\mathbf{e}}_2\cos\gamma - \hat{\mathbf{e}}_3\sin\gamma \tag{94}$$
$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{e}}_2\sin\gamma + \hat{\mathbf{e}}_3\cos\gamma \tag{95}$$

We recognize that $\hat{\mathbf{b}}, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are independent of γ. Then

$$\oint \frac{\mathrm{d}\gamma}{2\pi}\hat{\mathbf{s}}\hat{\boldsymbol{\rho}}\cdot\mathbf{A} = \oint \frac{\mathrm{d}\gamma}{2\pi}\left(\hat{\mathbf{e}}_2\cos\gamma - \hat{\mathbf{e}}_3\sin\gamma\right)\left(\hat{\mathbf{e}}_3\sin\gamma + \hat{\mathbf{e}}_3\cos\gamma\right)\cdot\left(A_b\hat{\mathbf{b}} + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3\right) \tag{96}$$
$$= \oint \frac{\mathrm{d}\gamma}{2\pi}\left[\cancel{\hat{\mathbf{e}}_2 A_2\cos\gamma\sin\gamma} + \hat{\mathbf{e}}_2 A_3\cos^2\gamma - \cancel{\hat{\mathbf{e}}_3 A_3\sin\gamma\cos\gamma} - \hat{\mathbf{e}}_3 A_2\sin^2\gamma\right] \tag{97}$$
$$= \hat{\mathbf{e}}_2 A_3 \oint \frac{\mathrm{d}\gamma}{2\pi}\cos^2\gamma - \hat{\mathbf{e}}_3 A_2 \oint \frac{\mathrm{d}\gamma}{2\pi}\sin^2\gamma \tag{98}$$
$$= \frac{\hat{\mathbf{e}}_2 A_3 - \hat{\mathbf{e}}_3 A_2}{2} \tag{99}$$

Noting that we used $\mathbf{A}$ is not a function of γ to take it's components outside of the integral. We then see that

$$\hat{\mathbf{b}}\times\mathbf{A} = \hat{\mathbf{e}}_3 A_2 - \hat{\mathbf{e}}_2 A_3 \tag{100}$$

and so find

$$\oint \frac{\mathrm{d}\gamma}{2\pi}\hat{\mathbf{s}}\hat{\boldsymbol{\rho}}\cdot\mathbf{A} = -\frac{1}{2}\hat{\mathbf{b}}\times\mathbf{A} \tag{101}$$

as desired.

12 Ion Density in Large Aspect-Ratio Tokamak Consider a large aspect-ratio tokamak in which $B \simeq B_0\left(1 - \frac{r\cos\theta}{R_0}\right)$. Suppose that some (fictitious) instability has removed all the trapped ions, so that the ion distribution is approximated by

$$f = \begin{cases} f_M & \lambda < \lambda_c \\ 0 & \lambda > \lambda_c \end{cases} \tag{102}$$

Compute the ion density, n_i, and stress anisotropy $P_{i\parallel} - P_{i\perp}$, in this case.

Solution:

We find

$$n_i = \int \mathrm{d}^3v\ f = \sum_\sigma \int_0^{2\pi} \mathrm{d}\gamma \int_0^{\lambda_c} \int_0^\infty \frac{wB}{m^2|u|} f_M \tag{103}$$

where $\lambda_c = 1/B_M$ where B_M is the maximum of B on a flux surface. This would correspond in our case to $B_0\left(1 + \frac{r}{R_0}\right)$.

Using

$$|u| = \sqrt{\frac{2w}{m}}\sqrt{1 - \lambda B} \tag{104}$$

we can find ($q = w/T$, $T\,\mathrm{d}q = \mathrm{d}w$)

$$n_i = \frac{4\pi B}{m^2}\int_0^{\lambda_c} \mathrm{d}\lambda \int_0^\infty \mathrm{d}w\ \frac{w f_M}{\sqrt{\frac{2w}{m}(1-\lambda B)}} = \frac{4\pi B}{m^2}\sqrt{\frac{m}{2}}\int_0^{\lambda_c} \frac{1}{\sqrt{1-\lambda B}}\int_0^\infty \mathrm{d}w\ \frac{n\sqrt{w}}{\pi^{3/2}\left(\frac{2T}{m}\right)^{3/2}} e^{-w/T} \tag{105}$$

$$= \frac{\cancel{2^{3/2}} n\pi B \cancel{m^{3/2}}}{\cancel{m^{3/2}}\cancel{2^{3/2}}\pi^{3/2}T^{3/2}}\int_0^{\lambda_c} \mathrm{d}\lambda\ \frac{1}{\sqrt{1-\lambda B}}\int_0^\infty \mathrm{d}q\ \sqrt{T^3 q}e^{-q} = \frac{n\pi B \cancel{T^{3/2}}}{\pi^{3/2}\cancel{T^{3/2}}}\int_0^{\lambda_c} \mathrm{d}\lambda\ \frac{1}{\sqrt{1-\lambda B}}\frac{\sqrt{\pi}}{2} \tag{106}$$

$$= \frac{nB}{2}\int_0^{\lambda_c} \mathrm{d}\lambda\ \frac{1}{\sqrt{1-\lambda B}} \tag{107}$$

So using $s = \sqrt{1-\lambda B}$, $\mathrm{d}s = \frac{-B\,\mathrm{d}\lambda}{2\sqrt{1-\lambda B}}$ we find

$$= \frac{nB}{2}\int_1^{\sqrt{1-\lambda_c B}} \mathrm{d}s\ \frac{-2}{B} = -n\left(\sqrt{1-\lambda_c B} - 1\right) \tag{108}$$

and so we get

$$n_i = -n\left(\sqrt{1 - \frac{B}{B_M}} - 1\right) = n\left(\sqrt{1 - \frac{B_0\left(1 - \frac{r\cos\theta}{R_0}\right)}{B_0\left(1 + \frac{r}{R_0}\right)}} - 1\right) \tag{109}$$

$$\boxed{n_i \approx n\left(1 - \sqrt{\frac{r\cos\theta}{R_0}}\right)} \tag{110}$$

Note this is very similar to the magnetic mirror, as there we have the $R_m = \frac{B_M}{B_0}$ which has its inverse show up here.

Next we compute

$$P_\parallel = \int \mathrm{d}^3v \; f m u^2 \tag{111}$$

$$P_\perp = \int \mathrm{d}^3v \; f \mu B \tag{112}$$

We begin with $P_\parallel$ and see

$$P_\parallel = \int \mathrm{d}^3v \; f m u^2 = \sum_\sigma \int_0^{2\pi} \mathrm{d}\gamma \; \int_0^{\lambda_c} \mathrm{d}\lambda \; \int_0^\infty \mathrm{d}w \; \frac{wBmu^2 f_M}{m^2|u|} \tag{113}$$

Using that $u/|u| = \sigma$ we see that this is

$$P_\parallel = \sum_\sigma \int_0^{2\pi} \mathrm{d}\gamma \; \int_0^{\lambda_c} \mathrm{d}\lambda \; \int_0^\infty \mathrm{d}w \; \frac{wBmu\sigma f_M}{m^2} = \int_0^{2\pi} \mathrm{d}\gamma \; \int_0^{\lambda_c} \mathrm{d}\lambda \; \int_0^\infty \mathrm{d}w \; \frac{wBmu f_M}{m^2} \sum_\sigma \sigma \tag{114}$$

There is no other σ dependence and as $\sigma = -1, 1$ we see that

$$\sum_\sigma \sigma = 0 \tag{115}$$

and thus $P_\parallel = 0$.

We are then left with $P_\perp$,

$$P_\perp = \int \mathrm{d}^3v \; f\mu B = \sum_\sigma \int_0^{2\pi} \mathrm{d}\gamma \; \int_0^{\lambda_c} \mathrm{d}\lambda \; \int_0^\infty \frac{wB\mu B}{m^2|u|} f_M = \sum_\sigma \frac{2\pi B^2}{m^2} \int_0^{\lambda_c} \mathrm{d}\lambda \; \frac{\lambda}{\sqrt{1-\lambda B}} \int_0^\infty \frac{w^2}{\sqrt{\frac{2w}{m}}} f_M \tag{116}$$

where I have once again used $|u| = \sqrt{\frac{2w}{m}(1-\lambda B)}$. Plugging in

$$f_M = \frac{n m^{3/2}}{\pi^{3/2} 2^{3/2} T^{3/2}} e^{-w/T} \tag{117}$$

we find

$$P_\perp = \sum_\sigma \frac{2\cancel{\pi} B^2 n}{\cancel{m^2}} \sqrt{\frac{\cancel{m}}{2}} \frac{\cancel{m^{3/2}}}{\cancelto{1/2}{\pi^{3/2}} 2^{3/2} T^{3/2}} \int_0^{\lambda_c} \mathrm{d}\lambda \; \frac{\lambda}{\sqrt{1-\lambda B}} \int_0^\infty \mathrm{d}w \; w^{3/2} e^{-w/T} \tag{118}$$

$$= \sum_\sigma \frac{nB^2}{2\sqrt{\pi} T^{3/2}} \int_0^{\lambda_c} \mathrm{d}\lambda \; \frac{\lambda}{\sqrt{1-\lambda B}} \int_0^\infty \mathrm{d}w \; w^{3/2} e^{-w/T} \tag{119}$$

We then use $u^2 = w/T$, $T^{3/2}u^3 = w^{3/2}$, $2Tu \, \mathrm{d}u = \mathrm{d}w$ to show

$$\int_0^\infty \mathrm{d}w \; w^{3/2} e^{-w/T} = \int_0^\infty \mathrm{d}u \; 2Tu(T^{3/2}u^3) e^{-u^2} = 2T^{5/2} \int_0^\infty \mathrm{d}u \; u^4 e^{-u^2} = \frac{2T^{5/2}(4-1)!!\sqrt{\pi}}{2^{4/2+1}} \tag{120}$$

$$= \frac{2T^{5/2}(3)\sqrt{\pi}}{8} = \frac{3T^{5/2}\sqrt{\pi}}{4} \tag{121}$$

and so

$$P_\perp = \sum_\sigma \frac{nB^2}{2\cancel{\sqrt{\pi}}T^{3/2}} \int_0^{\lambda_c} \mathrm{d}\lambda \, \frac{\lambda}{\sqrt{1-\lambda B}} \frac{3T^{5/2}\cancel{\sqrt{\pi}}}{4} = \sum_\sigma \frac{3nB^2T}{8} \int_0^{\lambda_c} \mathrm{d}\lambda \, \frac{\lambda}{\sqrt{1-\lambda B}} \tag{122}$$

Now we use $u = \sqrt{1-\lambda B}$, $\lambda = \frac{1-u^2}{B}$, $\mathrm{d}u = \frac{-B\,\mathrm{d}\lambda}{2\sqrt{1-\lambda B}}$ and find

$$\int_0^{\lambda_c} \mathrm{d}\lambda \, \frac{\lambda}{\sqrt{1-\lambda B}} = \int_1^{\sqrt{1-\lambda_c B}} \mathrm{d}u \, \frac{-2}{B}\frac{1-u^2}{B} = \frac{-2}{B^2}\left[\left(\sqrt{1-\lambda_c B - 1}\right) - \left(\frac{(1-\lambda_C B)^{3/2}-1}{3}\right)\right] \tag{123}$$

$$= \frac{2}{B^2}\left[\frac{2}{3} + \frac{\sqrt{1-\lambda_c B}\,(1-\lambda_c B - 3)}{3}\right] = \frac{4 - 2\sqrt{1-\lambda_d B}(2+\lambda_c B)}{3B^2} \tag{124}$$

and therefore

$$P_\perp = \sum_\sigma \frac{\cancel{3B^2}nT}{8} \frac{4-2\sqrt{1-\lambda_c B}(2+\lambda_c B)}{\cancel{3B^2}} = \sum_\sigma \frac{nT}{2}\left[1 - \sqrt{1-\lambda_c B}\left(1 + \frac{\lambda_c B}{2}\right)\right] \tag{125}$$

$$= nT\left[1 - \sqrt{1 - \frac{B}{B_M}}\left(1 + \frac{B}{2B_M}\right)\right] \tag{126}$$

$$= nT\left[1 - \sqrt{1 - \frac{B_0(1-\frac{r\cos\theta}{R_0})}{B_0(1+\frac{r}{R_0})}}\left(1 + \frac{B_0\left(1-\frac{r\cos\theta}{R_0}\right)}{2B_0\left(1+\frac{r\cos\theta}{R_0}\right)}\right)\right] \approx nT\left[1 - \sqrt{\frac{r\cos\theta}{R_0}}\left(\frac{1}{2} + \frac{r\cos\theta}{R_0}\right)\right] \tag{127}$$

$$\approx nT\left[1 - \frac{1}{2}\sqrt{\frac{r\cos\theta}{R_0}}\right] \tag{128}$$

Thus, the pressure anisotropy is given by

$$\boxed{P_\parallel - P_\perp \approx nT\left[\frac{1}{2}\sqrt{\frac{r\cos\theta}{R_0}} - 1\right] \approx n_i T \frac{\frac{1}{2}\sqrt{\frac{r\cos\theta}{R_0}} - 1}{1 - \sqrt{\frac{r\cos\theta}{R_0}}} \approx -n_i T\left[1 + \frac{1}{2}\sqrt{\frac{r\cos\theta}{R_0}}\right]} \tag{129}$$

13 Perform Integration Carefully perform each integral to verify (4.184) and (4.185).

$$\int \mathrm{d}^3 v\ f_M = \frac{2n}{\sqrt{\pi}} \int_{-1}^{1} \mathrm{d}\xi\ \int_0^\infty \mathrm{d}\eta\ \eta^2 e^{-\eta^2} \tag{4.184}$$

$$n\mathbf{V}_{\perp 1} = \frac{p}{m\Omega}\hat{\mathbf{b}} \times \left(\nabla \ln p + \frac{e\,\nabla\Phi}{T} \right) \tag{4.185}$$

Solution:

We write out

$$\int \mathrm{d}^3 v\ f_M = \int \mathrm{d}^3 v\ \frac{n}{\pi^{3/2} v_{\mathrm{th}}^3} e^{-v^2/v_{\mathrm{th}}^2} \tag{130}$$

Then use $\cos\theta = \frac{u}{v} \equiv \xi$ in spherical coordinates with $\eta = \frac{v}{v_{\mathrm{th}}}$, $v_{\mathrm{th}}\,\mathrm{d}\eta = \mathrm{d}v$ to find

$$\int \mathrm{d}^3 v\ f_M = \int_0^{2\pi} \mathrm{d}\varphi\ \int_{-1}^{1} \mathrm{d}(\cos\theta)\ \int_0^\infty \mathrm{d}v\ v^2 \frac{n}{\pi^{3/2} v_{\mathrm{th}}^3} e^{-v^2/v_{\mathrm{th}}^2} = \frac{2\pi n}{\pi^{3/2}} \int_{-1}^{1} \mathrm{d}\xi\ \int_0^\infty \mathrm{d}\epsilon\ v_{\mathrm{th}} \frac{\eta^2}{v_{\mathrm{th}}} e^{-\eta^2} \tag{131}$$

$$\int \mathrm{d}^3 v\ f_M = \frac{2n}{\sqrt{\pi}} \underbrace{\int_{-1}^{1} \mathrm{d}\xi}_{2} \underbrace{\int_0^\infty \mathrm{d}\eta\ \eta^2 e^{-\eta^2}}_{\sqrt{\pi}/4} = n \tag{132}$$

as required.

Now for

$$n\mathbf{V}_{\perp 1} = \frac{1}{2\Omega}\hat{\mathbf{b}} \times \int \mathrm{d}^3 v\ s^2\,\nabla f_M \tag{133}$$

$$\nabla f_M = f_M \left[\nabla \ln n + \frac{e\,\nabla\Phi}{T} + \left(\frac{U - e\Phi}{T} - \frac{3}{2} \right) \nabla \ln T \right] \tag{134}$$

while using $s^2 = v_{\mathrm{th}}^2 (1-\xi^2)\eta^2$, $\frac{U-e\Phi}{T} - \frac{3}{2} = \eta^2 - \frac{3}{2}$, $\xi = \frac{u}{v}$, $\eta = \frac{v}{v_{\mathrm{th}}}$, we find

$$n\mathbf{V}_{\perp 1} = \frac{1}{2\Omega}\hat{\mathbf{b}} \times \int \mathrm{d}^3 v\ s^2 f_M \left[\nabla \ln T + \frac{e\,\nabla\Phi}{T} + \left(\eta^2 - \frac{3}{2} \right) \nabla \ln T \right] \tag{135}$$

$$= \frac{n}{2\pi^{3/2}\Omega}\hat{\mathbf{b}} \times \int_0^{2\pi} \mathrm{d}\varphi\ \int_{-1}^{1} \mathrm{d}\xi\ \int_0^\infty \mathrm{d}\eta\ s^2 \eta^2 e^{-\eta^2} \left[\nabla \ln n + \frac{e\,\nabla\Phi}{T} + \left(\eta^2 - \frac{3}{2} \right) \nabla \ln T \right] \tag{136}$$

$$= \frac{2n v_{\mathrm{th}}^2}{2\sqrt{\pi}\Omega}\hat{\mathbf{b}} \times \int_{-1}^{1} \mathrm{d}\xi\ (1-\xi^2) \int_0^\infty \mathrm{d}\eta\ \eta^2 \left[\eta^2 e^{-\eta^2} \left(\nabla \ln n + \frac{e\,\nabla\Phi}{T} - \frac{3}{2}\nabla \ln T \right) + \eta^4 e^{-\eta^2}\,\nabla \ln T \right] \tag{137}$$

$$= \frac{n v_{\mathrm{th}}^2}{\sqrt{\pi}\Omega}\hat{\mathbf{b}} \times \left\{ \left(2 - \frac{2}{3} \right) \left[\left(\nabla \ln n + \frac{e\,\nabla\Phi}{T} - \frac{3}{2}\nabla \ln T \right) \frac{3\sqrt{\pi}}{8} + \nabla \ln T \left(\frac{15\sqrt{\pi}}{16} \right) \right] \right\} \tag{138}$$

$$= \frac{n v_{\mathrm{th}}^2}{\Omega}\hat{\mathbf{b}} \times \left\{ \frac{4}{3} \left[\left(\frac{3}{8}\nabla \ln n + \frac{3}{8}\frac{e\,\nabla\Phi}{T} - \frac{9}{16}\nabla \ln T \right) + \frac{15}{16}\nabla \ln T \right] \right\} \tag{139}$$

$$= \frac{n v_{\text{th}}^2}{\Omega} \hat{\mathbf{b}} \times \left\{ \frac{4}{3} \left[\frac{3}{8} \left(\nabla \ln n + \frac{e\,\nabla\Phi}{T} + \nabla \ln T \right) \right] \right\} \tag{140}$$

$$= \frac{\cancel{2} n T}{m\Omega} \hat{\mathbf{b}} \times \left\{ \frac{\cancel{1}}{\cancel{2}} \left(\nabla (\ln n + \ln T) + \frac{e\,\nabla\Phi}{T} \right) \right\} \tag{141}$$

$$= \frac{p}{\Omega} \hat{\mathbf{b}} \times \left\{ \nabla \ln p + \frac{e\,\nabla\Phi}{T} \right\} \tag{142}$$

as required.

14 Minus Sign in Magnetization Explain in a few sentences the origin of the minus sign in the magnetization, (4.187).

$$\mathbf{M}_1 = -\hat{\mathbf{b}}\frac{p}{eB} \tag{4.187}$$

Solution:

This is due to the diamagnetic nature of plasmas. When there is a magnetic field, it causes gyromotion, and that gyromotion induces a magnetic moment which points against the magnetic field. (That is, positively charged particles orbit the magnetic field lines in a left-handed sense, while electron orbit in a right handed sense). Thus all of the gyromotion together causes a magnetization (from all the magnetic moments) aligned antiparallel to the magnetic field, thus the minus sign.

15 FLR Profile Broadening (FLR is finite Larmor radius) In a magnetized plasma, no fluid variable can vary on a scale narrower than the gyroradius. To show this explicitly, consider the ion guiding-center distribution given by

$$\overline{f}(\mathbf{x},\mathbf{v}) = f_M \frac{n_{\mathrm{gc}}(\mathbf{X})}{n_M} \quad , \tag{143}$$

where f_M is a Maxwellian with constant density n_M and $n_{\mathrm{gc}}(\mathbf{X})$ gives the spatial distribution of ion guiding centers. Assume slab geometry, with uniform magnetic field in the z-direction and spatial variation in the x-direction.

15.a Ion Density Show that the ion density at some point x is generally given by

$$n(x) = \frac{1}{\sqrt{\pi}} \int \mathrm{d}s \; e^{-s^2} n_{\mathrm{gc}}(x + s\rho) \quad , \quad \rho = \frac{v_{\mathrm{th}}}{\Omega} \tag{144}$$

Solution:

We have that $\mathbf{X} = \mathbf{x} - \boldsymbol{\rho}$ and that

$$\overline{n}(\mathbf{X}) = n_{\mathrm{gc}}(\mathbf{X}) \tag{145}$$

We realize that the actual ion density must have

$$n(x) \propto \int \mathrm{d}s \; g(s) n_{\mathrm{gc}}(x + s\rho) \tag{146}$$

with $\rho = v_{\mathrm{th}}/\Omega$, $g(s)$ some weighting function, and s a parameter to sample all of the possible Larmor radii.

Now, we realize that in velocity space, the probability of being any particular speed, v is $e^{-v^2/v_{\mathrm{th}}^2}$, and that the gyroradius of a particle with that particular speed is $\frac{v}{v_{\mathrm{th}}}\rho$. We need the average gyroradius to be ρ as $v = v_{\mathrm{th}}$ is the most probable speed. However, we note that we want to sample "negative" radii since we are in a slab geometry. Thus, we require

$$\rho = \int_{-\infty}^{\infty} \mathrm{d}v \; Ce^{-v^2/v_{\mathrm{th}}^2} \rho \tag{147}$$

We see that that this requires (using $s = v/v_{\mathrm{th}}$)

$$1 = C \int_{-\infty}^{\infty} \mathrm{d}s \; v_{\mathrm{th}} C e^{-s^2} \tag{148}$$

$$1 = v_{\mathrm{th}} C \sqrt{\pi} \tag{149}$$

$$C = \frac{1}{v_{\mathrm{th}}\sqrt{\pi}} \tag{150}$$

Now, to find the average of $n_{\mathrm{gc}}(\mathbf{X})$ we then use

$$n(x) = \frac{1}{v_{\mathrm{th}}\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}v \; e^{-v^2/v_{\mathrm{th}}^2} n_{\mathrm{gc}}(x + \frac{v}{v_{\mathrm{th}}}\rho) \tag{151}$$

$$n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}s \; e^{-s^2} n_{\mathrm{gc}}(x + s\rho) \tag{152}$$

as desired.

15.b Profile Slope not Broadened Show that a region of flat profile ($\mathrm{d}^2n/\mathrm{d}x^2 = 0$) is unaffected by FLR. Thus it is profile-curvature that is broadened, not profile slope.

Solution:

We see that

$$\frac{\mathrm{d}^2 n}{\mathrm{d}x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}s \; e^{-s^2} \frac{\mathrm{d}^2 n_{\mathrm{gc}}(x + s\rho)}{\mathrm{d}x^2} \tag{153}$$

In regions of flat curvature, we have

$$\frac{\mathrm{d}^2 n}{\mathrm{d}x^2} = 0 \tag{154}$$

which implies that $\frac{\mathrm{d}^2 n_{\mathrm{gc}}}{\mathrm{d}x^2} = 0$ so that in this region only

$$n_{\mathrm{gc}} \sim A(x + s\rho) + B \tag{155}$$

Then

$$n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}s \; e^{-s^2} \left(A(x + s\rho) + B\right) = Ax + B \tag{156}$$

as

$$\int_{-\infty}^{\infty} \mathrm{d}s \; e^{-s^2} A s\rho = 0 \tag{157}$$

by the oddness of the integrand in s.

We therefore see that $n(x)$ is completely independent of $s\rho$ (this part yields no contribution to $n(x)$) and hence is independent of having a finite Larmor radius (FLR).

15.c Evaluate the Density

Evaluate $n(x)$ explicitly and exactly in the following two cases:

15.c.1 $n_{\text{gc}}(x) = \sqrt{\pi}\rho\delta(x)$

Solution:

In this case, we have

$$\boxed{n(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}s\ e^{-s^2}\sqrt{\pi}\rho\delta(x+s\rho) = \frac{1}{\sqrt{\pi}}e^{-x^2/\rho^2}\rho = \frac{\rho e^{-x^2/\rho^2}}{\sqrt{\pi}}} \tag{158}$$

15.c.2 $n_{\text{gc}}(x) = n_M e^{-x^2/a^2}$.

Solution:

In this case, we have

$$n(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}s\ e^{-s^2} n_M e^{-(x+s\rho)^2/a^2} = \frac{n_M}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}s\ e^{-(x+s\rho)^2/a^2-s^2} \tag{159}$$

$$= \frac{n_M}{\sqrt{\pi}}e^{-x^2/a^2}\int_{-\infty}^{\infty} \mathrm{d}s\ e^{-(s^2(a^2+\rho^2)+2\rho sx)/a^2} \tag{160}$$

Now let $a^2+\rho^2 = b^2$ and we use that

$$-\frac{b^2}{a^2}(s^2+\frac{2\rho sx}{b^2}) = -\frac{b^2}{a^2}\left(s^2+\frac{2\rho sx}{b^2}\right) - \frac{b^2}{a^2}\left(\frac{\rho^2x^2}{b^4}-\frac{\rho^2x^2}{b^4}\right) \tag{161}$$

$$= -\frac{b^2}{a^2}\left(s^2+\frac{2\rho sx}{b^2}+\frac{\rho^2x^2}{b^4}\right) + \frac{\rho^2x^2}{b^2a^2} = \frac{-b^2}{a^2}\left(s+\frac{\rho x}{b^2}\right)^2 + \frac{\rho^2x^2}{b^2a^2} \tag{162}$$

Thus,

$$n(x) = \frac{n_M}{\sqrt{\pi}}e^{-x^2/a^2+\rho^2x^2/(ba)^2}\int_{-\infty}^{\infty} \mathrm{d}s\ e^{-\frac{b^2}{a^2}\left(s+\frac{\rho x}{b^2}\right)^2} \tag{163}$$

We use that

$$\frac{-x^2}{a^2}\left(1-\frac{\rho^2}{b^2}\right) = -\frac{x^2}{a^2}\left(1-\frac{1}{1+\frac{a^2}{\rho^2}}\right) = \frac{-x^2}{a^2}\left(\frac{\cancel{1}+\frac{a^2}{\rho^2}-\cancel{1}}{1+\frac{a^2}{\rho^2}}\right) = \frac{-x^2}{a^2}\left(\frac{a^2}{\rho^2+a^2}\right) = \frac{-x^2}{\rho^2+a^2} \tag{164}$$

Now we substitute $u = \frac{b}{a}\left(s+\frac{\rho x}{b^2}\right)$ so $\frac{a}{b}\,\mathrm{d}u = \mathrm{d}s$ and find

$$n(x) = \frac{n_M}{\sqrt{\pi}}e^{-x^2/(\rho^2+a^2)}\int_{-\infty}^{\infty} \mathrm{d}u\ \frac{a}{b}e^{-u^2} \tag{165}$$

$$n(x) = \frac{n_M}{\sqrt{\pi}}e^{-x^2/(\rho^2+a^2)}\frac{a\sqrt{\pi}}{b} = \frac{an_M}{b}e^{-x^2/(\rho^2+a^2)} \tag{166}$$

$$\boxed{n(x) = \frac{an_M}{\sqrt{a^2+\rho^2}}e^{-x^2/(\rho^2+a^2)}} \tag{167}$$