1 Radial Distance Between Flux Surfaces and Surface Average Expression Show that the radial distance between two neighboring flux surfaces is given by
$\mathrm{d}r = \mathrm{d}F/|\nabla F|$, where F is any flux label. Then show that the flux surface average can be expressed as

$$\langle A \rangle = \int \mathrm{d}S \frac{A}{|\nabla \mathcal{V}|} \quad , \tag{1}$$

where $\mathrm{d}S$ is the area element on a magnetic surface.

Solution:

Given that F is a flux label, then we may write

$$\mathrm{d}F = \frac{\mathrm{d}F}{\mathrm{d}r}\,\mathrm{d}r = \nabla F \cdot \hat{\mathbf{r}}\,\mathrm{d}r \overset{F=F(r)}{=} |\nabla F|\,\mathrm{d}r \tag{2}$$

$$\mathrm{d}r = \frac{\mathrm{d}F}{|\nabla F|} \quad . \tag{3}$$

Remember that

$$\mathcal{V}(r) = \int_{\mathcal{V}} \mathrm{d}^3x = \int_{\mathcal{V}} \sqrt{g}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\zeta \tag{4}$$

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}r} \equiv \mathcal{V}' = \int \sqrt{g}\,\mathrm{d}\theta\,\mathrm{d}\zeta \quad . \tag{5}$$

Now we have

$$\langle A \rangle = \frac{1}{\mathcal{V}'} \oint \sqrt{g}\,\mathrm{d}\theta\zeta A(\mathbf{x}) = \frac{1}{|\nabla \mathcal{V}|} \oint \mathrm{d}S A(\mathbf{x}) \quad . \tag{6}$$

Now because $\mathcal{V}$ is independent of the surface θ and ζ, as it only depends on the flux label, we may bring it inside of the integral, and because $\mathrm{d}S = \sqrt{g}\,\mathrm{d}\theta\,\mathrm{d}\zeta = \frac{\mathrm{d}\theta\,\mathrm{d}\zeta}{\mathcal{J}}$ (with $\mathcal{J}$ the Jacobian) is the area element on a magnetic surface we omit the circle on the integral as it must be closed.

$$\langle A \rangle = \int \mathrm{d}S \frac{A}{|\nabla \mathcal{V}|} \quad . \tag{7}$$

2 Flux Surface Average Compute the flux surface average of $\nabla\chi \cdot \nabla\theta \times \nabla\zeta$.

Solution:

We note that this is none other than the Jacobian ($\mathcal{J} = \nabla r \cdot \nabla\theta \times \nabla\zeta$) with an extra factor because $\chi = \chi(r)$ so that

$$\nabla\chi \cdot \nabla\theta \times \nabla\zeta = \frac{\mathrm{d}\chi}{\mathrm{d}r} \nabla r \cdot \nabla\theta \times \nabla\zeta = \frac{\mathrm{d}\chi}{\mathrm{d}r} = \frac{\mathrm{d}\chi}{\mathrm{d}r} g^{-1/2} \tag{8}$$

and so

$$\langle \nabla\chi \cdot \nabla\theta \times \nabla\zeta \rangle = \frac{1}{\mathcal{V}'} \oint \sqrt{g}\, \mathrm{d}\theta\, \mathrm{d}\zeta \frac{\mathrm{d}\chi}{\mathrm{d}r} g^{-1/2} \tag{9}$$

$$= \frac{1}{\frac{\mathrm{d}V}{\mathrm{d}r}} \frac{\mathrm{d}\chi}{\mathrm{d}r} \oint \mathrm{d}\theta\, \mathrm{d}\zeta \frac{\mathrm{d}\chi}{\mathrm{d}r} \tag{10}$$

Now we can easily evaluate the integral (and take $\frac{\mathrm{d}\chi}{\mathrm{d}r}$ out as it is independent of θ and ζ) and using some calculus identities we see that

$$\langle \nabla\chi \cdot \nabla\theta \times \nabla\zeta \rangle = \frac{\frac{\mathrm{d}\chi}{\mathrm{d}r}}{\frac{\mathrm{d}V}{\mathrm{d}r}} 4\pi^2 \tag{11}$$

$$\boxed{\langle \nabla\chi \cdot \nabla\theta \times \nabla\zeta \rangle = 4\pi^2 \frac{\mathrm{d}\chi}{\mathrm{d}V} \quad .} \tag{12}$$

3 Periodic Solutions for Differential Equation For what value of the constant C does the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}\theta} = \sin^2\theta - C \tag{13}$$

have periodic solutions?

Solution:

This is a separable equation and so

$$f = \int \sin^2\theta \,\mathrm{d}\theta - \int C \,\mathrm{d}\theta \tag{14}$$

$$= \int \frac{1-\cos(2\theta)}{2}\,\mathrm{d}\theta - C\theta + C_1 \tag{15}$$

$$= \frac{\theta}{2} - \frac{\sin(2\theta)}{4} - C\theta + \underbrace{C_1 + C_2}_{C_3} \tag{16}$$

$$= \theta\left(\frac{1}{2} - C\right) - \frac{\sin(2\theta)}{4} + C_3 \tag{17}$$

where the C_1, C_2, and C_3 are arbitrary constants and C_3 is the constant needed to satisfy boundary conditions. We see that the only way that this is periodic is if the θ term cancels, and so $\boxed{C = \frac{1}{2}}$ is required for periodic solutions.

4 Existence of Hamada Coordinates Show that Hamada coordinates exist, irrespective of force balance, in an axisymmetric system. *Hint:* use symmetry coordinates to evaluate the integral in (3.46).

$$\oint \frac{\mathrm{d}s}{B}\left[(2\pi)^2 - \nabla\mathcal{V}\cdot\nabla\theta\times\nabla\zeta\right] \tag{3.46}$$

Solution:

We note that when

$$I_0 = \oint \frac{\mathrm{d}s}{B} \qquad \alpha = \zeta - q\theta \tag{18}$$

$$\oint I_0\,\mathrm{d}\alpha = I_0\oint \mathrm{d}\alpha \tag{19}$$

is satisfied, then Hamada coordinates exist. As in the book, this is from

$$I_1 = \oint \nabla\mathcal{V}\cdot\nabla\theta\times\nabla\zeta\frac{\mathrm{d}s}{B} \tag{20}$$

with

$$\oint I_0\,\mathrm{d}\alpha = \oint I_1\,\mathrm{d}\alpha = I_1\oint \mathrm{d}\alpha \quad . \tag{21}$$

Now for an axisymmetric system we have $\frac{\partial\mathbf{B}}{\partial\zeta} = 0 \Rightarrow \frac{\partial B}{\partial\zeta} = 0$. So we may write that $\frac{\mathrm{d}s}{B} = \frac{\mathrm{d}s}{B(r,\theta)}$ and $\mathrm{d}\alpha = -q\,\mathrm{d}\theta$ because there is no change along ζ. So we may use that $\mathrm{d}^3x = \chi'\,\mathrm{d}r\,\mathrm{d}\alpha\frac{\mathrm{d}s}{B} = \chi'\,\mathrm{d}r\,\mathrm{d}\alpha\frac{\mathrm{d}s}{B(r,\theta)}$.

Hence, I_0 can only be a function of r, as the θ dependence will be integrated out. Every line must go around the same θ in a loop, and there is no freedom in ζ, so that $\frac{\mathrm{d}s}{B}$ will be the same over the n loops, as all the lines will have the same number of loops past a certain θ and B has no dependence on ζ. So $I_0 = I_0(r)$ and so

$$\oint I_0\,\mathrm{d}\alpha = I_0\oint \mathrm{d}\alpha \tag{22}$$

as desired. This implies

$$\oint \frac{\mathrm{d}s}{B}\left[(2\pi)^2 - \nabla\mathcal{V}\cdot\nabla\theta\times\nabla\zeta\right] = 0 \tag{23}$$

via (21).

5 Show Orthogonality Property of Metric Tensor Show explicitly that (3.52) follows from (3.51).

$$g_{r\zeta} = 0 = g_{\theta\zeta} \quad , \tag{3.52}$$

$$\nabla\zeta_0\cdot\nabla r_0 = 0 = \nabla\zeta_0\cdot\nabla\theta_0 \quad . \tag{3.51}$$

Writing out the metric tensor g^{ij} we find

$$g^{ij} = \begin{bmatrix} \nabla r_0 \cdot \nabla r_0 & \nabla r_0 \cdot \nabla \theta_0 & \nabla r_0 \cdot \nabla \zeta_0 \\ \nabla \theta_0 \cdot \nabla r_0 & \nabla \theta_0 \cdot \nabla \theta_0 & \nabla \theta_0 \cdot \nabla \zeta_0 \\ \nabla \zeta_0 \cdot \nabla r_0 & \nabla \zeta_0 \cdot \nabla \theta_0 & \nabla \zeta_0 \cdot \nabla \zeta_0 \end{bmatrix} = \begin{bmatrix} |\nabla r_0|^2 & \nabla r_0 \cdot \nabla \theta_0 & 0 \\ \nabla r_0 \cdot \nabla \theta_0 & |\nabla \theta_0|^2 & 0 \\ 0 & 0 & |\nabla \zeta_0|^2 \end{bmatrix} \tag{24}$$

We need a matrix such that $g_{ij}g^{ij} = 1$. I'll use Gauss Elimination to find the inverse.

$$\left[\begin{array}{ccc|ccc} |\nabla r_0|^2 & \nabla r_0 \cdot \nabla \theta_0 & 0 & 1 & 0 & 0 \\ \nabla r_0 \cdot \nabla \theta_0 & |\nabla \theta_0|^2 & 0 & 0 & 1 & 0 \\ 0 & 0 & |\nabla \zeta_0|^2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1 \frac{\nabla r_0 \cdot \nabla \theta_0}{|\nabla r_0|^2} + R_2} \left[\begin{array}{ccc|ccc} |\nabla r_0|^2 & \nabla r_0 \cdot \nabla \theta_0 & 0 & 1 & 0 & 0 \\ 0 & |\nabla \theta_0|^2 - \frac{|\nabla r_0 \cdot \nabla \theta_0|^2}{|\nabla r_0|^2} & 0 & -\frac{\nabla r_0 \cdot \nabla \theta_0}{|\nabla r_0|^2} & 1 & 0 \\ 0 & 0 & |\nabla \zeta_0|^2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{{}_2\left(|\nabla \theta_0|^2 + \frac{|\nabla r_0 \cdot \nabla \theta_0|^2}{|\nabla r_0|^2}\right)^{-1} \nabla r_0 \cdot \nabla \theta_0 + R_1} \left[\begin{array}{ccc|ccc} |\nabla r_0|^2 & 0 & 0 & 1 + \left(\frac{|\nabla \theta_0|^2 |\nabla r_0|^2}{\nabla r_0 \cdot \nabla \theta_0} + 1\right)^{-1} & \frac{\nabla r_0 \cdot \nabla \theta_0}{|\nabla \theta_0|^2 + \frac{|\nabla r_0 \cdot \nabla \theta_0|^2}{|\nabla r_0|^2}} & 0 \\ 0 & |\nabla \theta_0|^2 - \frac{|\nabla r_0 \cdot \nabla \theta_0|^2}{|\nabla r_0|^2} & 0 & -\frac{\nabla r_0 \cdot \nabla \theta_0}{|\nabla r_0|^2} & 1 & 0 \\ 0 & 0 & |\nabla \zeta_0|^2 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow g_{ij} = \begin{bmatrix} |\nabla r_0|^{-2} + |\nabla r_0|^{-2} \left(\frac{|\nabla \theta_0|^2 |\nabla r_0|^2}{\nabla r_0 \cdot \nabla \theta_0} + 1\right)^{-1} & |\nabla r_0|^{-2} \frac{\nabla r_0 \cdot \nabla \theta_0}{|\nabla \theta_0|^2 + \frac{|\nabla r_0 \cdot \nabla \theta_0|^2}{|\nabla r_0|^2}} & 0 \\ -\frac{\nabla r_0 \cdot \nabla \theta_0}{|\nabla r_0|^2} \left(|\nabla \theta_0|^2 + \frac{|\nabla r_0 \cdot \nabla \theta_0|^2}{|\nabla r_0|^2}\right)^{-1} & \left(|\nabla \theta_0|^2 + \frac{|\nabla r_0 \cdot \nabla \theta_0|^2}{|\nabla r_0|^2}\right)^{-1} & 0 \\ 0 & 0 & |\nabla \zeta_0|^{-2} \end{bmatrix}$$

And so we can see that $g_{r\zeta} = g_{13} = 0$ and $g_{\theta\zeta} = g_{23} = 0$.

6 Standard Tokamak Estimates The "standard tokamak" – a fictitious but conveniently representative device – has the following parameters:

toroidal field (B_T)	50 kG
major radius (R_0)	300 cm
minor radius (a)	80 cm
safety factor (q)	$q \simeq 1$ (on axis) $q \simeq 3$ (at edge)
central density (n)	10^{14} cm^{-3}
central temperature ($T_i = T_e = T$)	10 keV

Estimate (one significant figure) the following quantities in the standard tokamak:

6.a Toroidal Plasma Current

toroidal plasma current;

Solution:

We have the practical formula (3.147)

$$q(a) = 5a^2 \frac{B_T}{RI_A} \Rightarrow \tag{3.147}$$

$$I_A = \frac{5a^2 B_T}{Rq(a)} \tag{25}$$

where I_A is in amps, but other quantities are in cgs. So

$$I_A = \frac{5(80\text{ cm})^2(5\times 10^4\text{ G})}{(300\text{ cm})(3)} \approx 2\times 10^6\text{ A} = 2\text{MA} \tag{26}$$

6.b Diamagnetic Current Density

diamagnetic current density;

Solution:

We have

$$\mathbf{J}_\perp = \frac{c}{B}\hat{\mathbf{b}}\times \nabla P = \frac{cP'}{B}\hat{\mathbf{b}}\times \nabla r \tag{27}$$

$$\approx \frac{cP'}{B} \approx \frac{cP}{aB_T} \approx \frac{cnk_BT}{aB_T} \tag{28}$$

$$\approx \frac{(3\times 10^{10}\text{ cm/s})(10^{14}\text{ cm}^{-3})(10\text{ keV})}{(80\text{ cm})(5\times 10^4\text{ G})} \tag{29}$$

$$\approx \frac{(3\times 10^{10}\text{ cm/s})(10^{14}\text{ cm}^{-3})(10\text{ keV})(1.6\times 10^{-9}\text{ erg/keV})}{(80\text{ cm})(5\times 10^4\text{ G})} \tag{30}$$

$$\approx 1.2\times 10^{10}\text{ statAmp/cm}^2 \approx 4\times 10^4\text{ A/m}^2 \approx 40\text{ kA/m}^2 \tag{31}$$

6.c Return Current Density

return current density.

Solution:

The return current density is given by (3.114) with $\chi = \frac{\psi_P}{2\pi} = RaB_P = Ra\frac{a}{R}\frac{B_T}{q} = \frac{a^2 B_T}{q}$ and $I = B_T R$, (We note the return current is only the second term, as the first is an integration constant)

$$J_\parallel = \frac{-c}{4\pi}B\frac{\mathrm{d}I}{\mathrm{d}\chi} - c\frac{I}{B}\frac{\mathrm{d}P}{\mathrm{d}\chi} \tag{3.114}$$

$$J_{\text{return}} \approx -c\frac{B_T R}{B}\frac{P}{\frac{a^2 B_T}{q}} \tag{32}$$

$$\approx \frac{-cqRP}{B_T a^2} \tag{33}$$

$$\approx \frac{(3\times 10^{10}\text{ cm/s})(2)(300\text{ cm})(10^{14}\text{ cm}^{-3})(10\text{ keV})(1.6\times 10^{-9}\text{ erg/keV})}{(80\text{ cm})^2(5\times 10^4\text{ G})} \tag{34}$$

$$\approx 9\times 10^{10}\text{ statAmp/cm}^2 \approx 300\text{ KA/m}^2 \tag{35}$$

With a cross section of $\pi a^2 \approx 2\text{ m}^2$ we find the return current to be 600 KA = 0.6 MA. This is a reasonable value given the toroidal current is 2 MA.

7 β and β_P in Standard Tokamak Estimate β and β_P in the standard tokamak.

Solution:

$$\beta = \frac{8\pi \langle P \rangle}{B^2} \approx \frac{4\pi P(r=0)}{B_T^2} \tag{36}$$

$$\approx \frac{4\pi(10^{14}\ \mathrm{cm}^{-3})(10\ \mathrm{keV})(1.6\times 10^{-9}\ \mathrm{erg/keV})}{25\times 10^8\ \mathrm{G}^2} \tag{37}$$

$$\approx \frac{4\pi 1.6\times 10^6\ \mathrm{dyne/cm}^2}{25\times 10^8\ \mathrm{G}^2} \approx .008 = .8\% \tag{38}$$

with $\langle P \rangle = P(r=0)/2$ while

$$\beta_P = \frac{8\pi \langle P \rangle}{B_P^2} = \frac{8\pi \langle P \rangle}{\frac{a^2}{q^2R^2}B_T^2} = \frac{8\pi}{B_T^2}\frac{q^2R^2}{a^2} \approx \beta\frac{q^2(a)R^2}{a^2} \tag{39}$$

$$\approx .008\frac{9(300\ \mathrm{cm})^2}{(80\ \mathrm{cm})^2} \approx .008(126) \approx 1.008 = 108\% \tag{40}$$

8 Thermal Energy in Standard Tokamak Estimate the thermal (plasma) energy in the standard tokamak, in Joules. Assume the volume-averaged density and temperature are about half theri central values. Compare your result to the thermal energy in a cup of coffee and to the electrical energy in a 60 Amp-hr automobile battery.

Solution:

We have the energy as

$$E_p = 2\pi^2 R a^2 \frac{n k_B T}{2} = 10^{20}\ \mathrm{m}^{-3}(\pi^2(3)(.8)^2\ \mathrm{m}^3)(10\ \mathrm{keV})(1.6\times 10^{-19}\ \mathrm{J/keV}) \quad (41)$$

$$E_p \approx 3000\ \mathrm{J} \quad . \quad (42)$$

A cup of coffee let's say is 500 mL at about 70° C. Then its thermal energy as water has a specific heat capacity of about 4.2 J/(gK) would give

$$E_c = mc\Delta T = (500\ \mathrm{g})(4.2\ \mathrm{J/(gK)})(343\ \mathrm{K}) \approx 720000\ \mathrm{J} \quad . \quad (43)$$

(Even if it were a balmy 40° C we'd get 657300 J.)

A battery with 60 Amp-hr. This is

$$E_b = 60\ \mathrm{A\ hr}V = 60\ \mathrm{A}(3600\ \mathrm{s})V = 216000\ \mathrm{C}V \quad (44)$$

and so now it depends on the voltage that an automobile battery is at as $[qV] =$ J. Most likely it is a $V = 12$ V battery so that

$$E_b = 2.6\times 10^6\ \mathrm{J} \quad . \quad (45)$$

In fact the thermal energy in the plasma is far less than these other two examples.

9 Diamagnetic Current Direction Show, using (3.63) and Ampère's law, that $\mathbf{J}_\perp$ has the correct direction to be called a "diamagnetic" current.

$$\mathbf{J}_\perp = \frac{c}{B}\hat{\mathbf{b}} \times \nabla P \tag{3.63}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \Rightarrow \oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{4\pi}{c}\int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \tag{Ampère's Law}$$

Solution:

For some small region set up a local cylindrical coordinate system letting $\mathbf{B}$ be in the z direction so that $\mathbf{J}_\perp$ is in the θ direction. Then because $\hat{\mathbf{b}} \propto \hat{\mathbf{z}}$ while $\nabla P \propto -\hat{\mathbf{r}}$ then

$$\mathbf{J}_\perp \propto \hat{\mathbf{z}} \times (-\hat{\mathbf{r}}) \tag{46}$$

$$\mathbf{J}_\perp \propto -\hat{\boldsymbol{\theta}} \tag{47}$$

Then for

$$\boldsymbol{\nabla} \times \mathbf{B}_D \propto J_\perp \hat{\boldsymbol{\theta}} \propto -\hat{\boldsymbol{\theta}} \tag{48}$$

If one curls one's right hand's fingers in the $-\hat{\boldsymbol{\theta}}$ direction then the thumb will point in the direction of B_D which one can easily verify is in the $-\hat{\mathbf{z}}$ direction. Hence opposing the original $\mathbf{B}$. This is a small enough region that $J_\perp$ is basically constant.

If you do not believe this, then looking at the symmetry of the situation we see that it must be only in the $\hat{\boldsymbol{\theta}}$ direction. Also the $\mathbf{B}_D$ is a function of r only. So

$$-\frac{\partial B_D^z}{\partial r} = -\frac{4\pi}{c}J_\perp \tag{49}$$

$$B_D = \frac{4\pi}{c}J_\perp r + C \tag{50}$$

because we need $B_D = 0$ at $r = R$ at the edge of this "cylinder" we require that $C = -4\pi J_\perp R/c$ and because $R > r$ we see that $B_D < 0$. (We require it to go to zero because outside our little column the B_D should not have any effect).

In more generalized coordinates the same idea would hold, although it would be more complicated to express.

10 Virial Theorem Use the identity

$$\frac{\partial\left(x_\beta A_{\alpha\beta}\right)}{\partial x_\alpha} = A_{\alpha\alpha} + x_\beta \frac{\partial A_{\alpha\beta}}{\partial x_\alpha}$$

for any tensor with Cartesian components $A_{\alpha\beta}$, to deduce the integral relation

$$\int_{\mathcal{V}} \mathrm{d}^3x\ T_{\alpha\alpha} = \int_{\mathcal{S}} \mathrm{d}^2x\ n_\alpha T_{\alpha\beta} x_\beta$$

where $\overleftrightarrow{\mathbf{T}} = \underline{\underline{\mathbf{T}}}$ is the total stress tensor of §3.8 and $\mathcal{V}$ is an arbitrary volume with boundary $\mathcal{S}$. Use this result to conclude that confined plasma equilibrium requires external conductors. See Shafranov (1966).

Solution:

We have using the divergence theorem on the left hand side of the relation then

$$\int_{\mathcal{V}} \frac{\partial\left(x_\beta A_{\alpha\beta}\right)}{\partial x_\alpha}\,\mathrm{d}^3x = \int_{\mathcal{V}} A_{\alpha\alpha}\,\mathrm{d}^3x + \int_{\mathcal{V}} x_\beta \frac{\partial A_{\alpha\beta}}{\partial x_\alpha}\,\mathrm{d}^3x \tag{51}$$

$$\int_{\mathcal{S}} n_\alpha x_\beta A_{\alpha\beta}\,\mathrm{d}^2x = \int_{\mathcal{V}} A_{\alpha\alpha}\,\mathrm{d}^3x + \int_{\mathcal{V}} x_\beta \frac{\partial A_{\alpha\beta}}{\partial x_\alpha}\,\mathrm{d}^3x \tag{52}$$

Now for $T_{\alpha\beta}$ we have $\frac{\partial T_{\alpha\beta}}{\partial x_\alpha} = \boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{T}}} = 0$ and so the last term is zero, and we have

$$\int_{\mathcal{S}} n_\alpha x_\beta T_{\alpha\beta}\,\mathrm{d}^2x = \int_{\mathcal{V}} T_{\alpha\alpha}\,\mathrm{d}^3x \quad . \tag{53}$$

As for the impossibility of a confined plasma, we take a large enough volume that there is no longer any pressure (so far outside the plasma) and also far enough that we can be sure that there are no coils giving any influence, so that B is zero at the large enough distance.

So then the surface term is zero, as the magnetic field due to the plasma will go down as $B \propto \frac{1}{r^2}$ at the very least (more likely $B \propto \frac{1}{r^3}$ as it will be a magnetic dipole far away) and so the surface term goes to zero as we go out to far enough x.

Then we only have ($b_\alpha b_\alpha = 1$ as $b_x^2 + b_y^2 + b_z^2 = |\hat{\mathbf{b}}|^2 = 1$ and $\delta_{\alpha\alpha} = 3$ as $1+1+1=3$ and noting that $T_\perp = P_\perp + \frac{B^2}{8\pi}$ and $T_\parallel = P_\parallel - \frac{B^2}{8\pi}$ refer to components of the tensor)

$$0 = \int_{\mathcal{V}} T_{\alpha\alpha}\,\mathrm{d}^3x = \int_{\mathcal{V}} \left[\left(\delta_{\alpha\alpha} - b_\alpha b_\alpha\right) T_\perp + b_\alpha b_\alpha T_\parallel\right]\,\mathrm{d}^3x \tag{54}$$

$$= \int_{\mathcal{V}} \left[2T_\perp + T_\parallel\right]\,\mathrm{d}^3x = \int_{\mathcal{V}} \left[2P_\perp + 2\frac{B^2}{8\pi} + P_\parallel - \frac{B^2}{8\pi}\right]\,\mathrm{d}^3x \tag{55}$$

$$0 = \int_{\mathcal{V}} \mathrm{d}^3x\ \left[2P_\perp + P_\parallel + \frac{B^2}{8\pi}\right] \quad . \tag{56}$$

Note that $P_\perp \geq 0$, $P_\parallel \geq 0$, and $\frac{B^2}{8\pi} \geq 0$. If we are confining a plasma inside our volume then this quantity must be greater than zero, otherwise there is nothing but vacuum in our volume.

However, we see that the integral of the quantity is zero, and the integrand is non-negative. Hence the integrand is zero, and so we see it's not possible to have a self-confined plasma.

11 Large Aspect-Ratio Flux Coordinates Find approximate symmetry coordinates, the special choice of flux coordinates discussed in §3.4, for shifted-circle geometry considered in §3.12, as follows. Distinguishing the flux and Shafranov coordinates with "f" and "S" subscripts, respectively, show that through $\mathcal{O}(\epsilon)$ we must have

$$r_f = r_S \quad , \quad \theta_f = \theta_S + \epsilon p(\theta_S) \quad , \quad \zeta_f = \zeta_S \quad .$$

Then find the periodic function p by imposing (3.56), dropping $\mathcal{O}(\epsilon^2)$.

$$\sqrt{g_0} = q\chi' \frac{R^2}{I} \quad . \tag{3.56}$$

Solution:

Symmetry coordinates yield $\zeta_f = -\varphi$. Then it is obvious that we must have $\zeta_S = \zeta_f$ by the definition in (3.133).

$$\begin{aligned} R &= R_c(r_S) + r_S \cos\theta_S \quad , \\ \varphi &= -\zeta_S \quad , \\ Z &= r_S \sin\theta_S \quad . \end{aligned} \tag{3.133}$$

It is also clear that $r_f = r_S$ (to order $\mathcal{O}(\epsilon)$) as they both refer to the minor radius. That only leaves θ_S and θ_f. As the other coordinates must correspond to each other, the only direction left is the same in both cases, but there is nothing forcing $\theta_S = \theta_f$. We need a way of connecting these two coordinates.

$$B_T = \frac{I(r)}{R} \approx \frac{I(r)}{R_0 + r_S \cos\theta_S} \approx B_{T0}(r)\left(1 - \frac{r_S}{R_0}\cos\theta_S\right) \tag{57}$$

where $B_{T0}(r) = I/R_0$

Now we also require for flux coordinates that

$$\mathbf{B} \cdot \nabla\zeta = q\,\nabla\zeta_f \cdot \nabla r_f \times \nabla\theta_f = B\frac{I}{R^2} \tag{58}$$

Writing this out explicitly in terms of while evaluating in terms of Shafranov coordinates yields

$$= \frac{q}{R}\hat{\boldsymbol{\zeta}}_s \cdot \left(\left[\frac{\partial r_f}{\partial r_s}\hat{\mathbf{r}} + \frac{\partial r_f}{\partial \theta_s}\hat{\boldsymbol{\theta}}\right] \times \left[\frac{\partial \theta_f}{\partial r_s}\hat{\mathbf{r}} + \frac{\partial \theta_f}{\partial \theta_s}\hat{\boldsymbol{\theta}}\right]\right) \tag{59}$$

$$= \frac{q}{R}\hat{\boldsymbol{\zeta}} \cdot \left[\left(\frac{\partial r_f}{\partial r_s}\frac{\partial \theta_f}{\partial \theta_s} - \frac{\partial r_f}{\partial \theta_s}\frac{\partial \theta_f}{\partial r_s}\right)\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}\right] \tag{60}$$

$$= \frac{q}{R}\left(\frac{\partial r_f}{\partial r_s}\frac{\partial \theta_f}{\partial \theta_s} - \frac{\partial r_f}{\partial \theta_s}\frac{\partial \theta_f}{\partial r_s}\right) = \frac{I}{R^2} \tag{61}$$

Thus (using $r_f = r_s + \epsilon r_{s1}$ and $\theta_f = \theta_s + \epsilon p$ we find

$$\frac{\partial r_f}{\partial r_s}\frac{\partial \theta_f}{\partial \theta_s} - \frac{\partial r_f}{\partial \theta_s}\frac{\partial \theta_f}{\partial r_s} = \frac{I}{qR} \approx \left(1 - \frac{r_s}{R_0}\cos\theta_s\right) \tag{62}$$

$$\left(1 + \epsilon\frac{\partial r_{s1}}{\partial r_s}\right)\left(1 + \epsilon\frac{\partial p}{\partial \theta_s}\right) - (0) = \left(1 - \frac{r_s}{R_0}\cos\theta_s\right) \tag{63}$$

$$\cancel{1} + \epsilon(\frac{\partial r_{s1}}{\partial r_s} + \frac{\partial p}{\partial \theta_s}) = \left(\cancel{1} - \frac{r_s}{R_0}\cos\theta_s\right) \tag{64}$$

We then note that averaging over θ yields

$$r_{s1} = 0 \tag{65}$$

(at least we can choose $r_{s1} = 0$) and hence $r_f = r_s$ to $\mathcal{O}(\epsilon)$. Thus (using $r_s/R_0 \sim \epsilon$)

$$\frac{\partial p}{\partial \theta_s} = -\cos\theta_s \tag{66}$$

$$\boxed{p = -\sin\theta_s} \tag{67}$$

12 Axisymmetric Force Balance Requires Poloidal Field Use (3.78) to show that an axisymmetric system without poloidal field cannot satisfy force balance. It follows that any system lacking poloidal magnetic field must be asymmetric; such devices are called bumpy tori.

$$I(v) = c\Delta P \oint \frac{\mathrm{d}s}{B} = c\Delta P I_0 \tag{3.78}$$

Solution:

We have $\mathbf{J}_\perp = \frac{\mathbf{B}\times\nabla P}{|\mathbf{B}|^2}$. Now, we can see that $\oint \frac{\mathrm{d}s}{B} = \frac{2\pi R}{B_T(r,\theta)}$ as B_T has no ζ dependence. So we see that $q = \frac{\mathbf{B}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\zeta} = \infty$. Now because $\mathbf{J} = \frac{c}{4\pi}\boldsymbol{\nabla}\times\mathbf{B}$, and $\mathbf{B} = \mathbf{B}_T$ then $\mathbf{J} = \mathbf{J}_P$.

So then $\mathbf{J}_\perp = \mathbf{J}_P$. So then $\mathbf{B}\times\nabla P = \mathbf{J}_P$ implies that ∇P is only in the $\hat{\mathbf{r}}$ direction. Because $\mathbf{B}\cdot\nabla P = 0$ even for ∇P in the $\hat{\boldsymbol{\theta}}$ direction as well as the $\hat{\mathbf{r}}$ direction. There is nothing in $\mathbf{J}\times\mathbf{B}$ in the $\hat{\boldsymbol{\theta}}$ direction and so there can be no balance.

As all the field lines do not travel poloidally, then we find that a rational surface can be extended however we want. Now $\mathbf{J}_\perp$ is only in the θ direction and so we see that if $\Delta\mathcal{V}/|\nabla\mathcal{V}|$ is the radial width, then the amount of current flowing across the field line is as given, but we can see that the current is only flowing in one direction, and the previous result shows the failure of force balance as we can choose $\Delta P = 0$ by looking at the current flowing across the line in the radial direction, as this is still a rational surface.

13 Elliptic Geometry In the simplest example of non-circular tokamak geometry, the magnetic surfaces are elliptical in poloidal cross-section (Harris, 1974). [Note that conventional Shafranov geometry becomes slightly elliptical when $\mathcal{O}(\epsilon^2)$-terms are included.] For large aspect-ratio and moderate beta, the corresponding equilibrium is similar to that of Shafranov, the flux-surfaces being approximately described by shifted ellipses.

Examine the shifted-ellipse case, in a manner parallel to §3.12, as follows:

1. Choose coordinates (r_e, θ_e, ζ_e) such that [in the notation of (3.133)],

$$R = R_c(r_e) + r_e \cos\theta_e \quad , \quad \varphi = -\zeta_e \quad , \quad Z = \kappa r_e \sin\theta_e \quad .$$

   Here the elongation parameter κ is a number of order unity.

2. After assuming the poloidal flux depends mainly on r_e, as in (3.137), express the Grad-Shafranov equation in terms of (r_e, θ_e, ζ_e). Neglect $\mathcal{O}(\epsilon^2)$ terms as usual.

$$\chi(\mathbf{x}) = \chi(r_S) + \mathcal{O}(\epsilon^2) \tag{3.137}$$

3. Decompose this result into $\cos(m\theta)$-components, with $m = 0, 1, 2$, and 3. Your $m = 2$ component should have the form

$$\left(\kappa^2 - 1\right)\left[\frac{1}{2}\left(r_e \chi'\right)' - \chi'\right] = 0 \quad ,$$

   requiring, for $\kappa \neq 1$, $\chi \propto r_e^2$. This fixed radial form for the poloidal flux is the salient characteristic of the large aspect-ratio, shifted-ellipse geometry.

Solution:

We begin with

$$\mathbf{B} = I(r)\,\nabla\zeta + \nabla\zeta \times \nabla\chi \tag{68}$$

$$B_T = \frac{I}{R} = \frac{I}{R_c(r_e) + r_e\cos\theta_e} \approx \frac{I}{R_c\left(1 + \frac{r_e}{R_c}\cos\theta_e\right)} \tag{69}$$

$$\approx \frac{I}{R_c}\left(1 - \frac{r_e}{R_c}\cos\theta_e\right) \tag{70}$$

We then note that

$$Z = \kappa r_e \sin\theta_e \tag{71}$$

$$\Rightarrow \nabla Z = \kappa\left(\sin\theta_e\,\nabla r_e + r_e\cos\theta_e\,\nabla\theta_e\right) \tag{72}$$

$$r_e\cos\theta_e\,\nabla\theta_e = \frac{\nabla Z}{\kappa} - \sin\theta_e\,\nabla r_e \tag{73}$$

$$R = R_c(r_e) + r_e\cos\theta_e \tag{74}$$

$$\nabla R = \overset{R_c'\,\nabla r_e}{\cancel{\nabla R_c}} + \nabla r_e\cos\theta_e - r_e\sin\theta_e\,\nabla\theta \tag{75}$$

$$\nabla R - \nabla r_e(R_c' + \cos\theta_e) = -r_e\sin\theta_e\,\nabla\theta_e \tag{76}$$

so taking [(73) + $\cot\theta_e$(76)] and find

$$r_e(\cos\theta_e - \cot\theta_e\sin\theta_e)\,\nabla\theta_e = \cot\theta_e\,\nabla R - \cot\theta_e(R_c' + \cos\theta_e)\,\nabla r_e + \frac{\nabla Z}{\kappa} - \sin\theta_e\,\nabla r_e \tag{77}$$

$$\cancel{r_e(\cos\theta_e - \cos\theta_e)\,\nabla\theta_e} = \cot\theta_e\,\nabla R - \cot\theta_e R_c'\,\nabla r_e - \frac{\cos^2\theta_e}{\sin\theta_e}\,\nabla r_e + \frac{\nabla Z}{\kappa} - \sin\theta_e\,\nabla r_e \tag{78}$$

and so multiplying by $\sin\theta_e$ yields

$$0 = \cos\theta\,\nabla R - \cos\theta_e R_c'\,\nabla r_e - \cos^2\theta_e\,\nabla r_e - \sin^2\theta_e\,\nabla r_e + \frac{\nabla Z}{\kappa} \tag{79}$$

$$0 = \cos\theta\,\nabla R - (1 + \cos\theta_e R_c')\,\nabla r_e + \frac{\nabla Z}{\kappa} \tag{80}$$

$$\nabla r_e = \frac{\cos\theta_e\,\nabla R + \frac{\sin\theta_e}{\kappa}\,\nabla Z}{1 + \cos\theta_e R_c'} \tag{81}$$

We note that the Jacobian is then given by using $\nabla\theta_e = \frac{\nabla Z}{\kappa r_e\cos\theta_e} - \frac{\tan\theta_e}{r_e}\,\nabla r_e$ that

$$\mathcal{J} = \nabla r_e\cdot\nabla\theta_e\times\nabla\zeta_e = \nabla r_e\cdot\left(\frac{\nabla Z\times\nabla\zeta_e}{\kappa r\cos\theta_e} - \tan\theta_e\cancel{\nabla r_e\times\nabla\zeta_e}\right) \tag{82}$$

$$= \left(\frac{\cos\theta_e\,\nabla R + \cancel{\frac{\sin\theta_e}{\kappa}\nabla Z}}{1 + R_c'\cos\theta_e}\right)\cdot\left(\frac{\hat{\mathbf{Z}}\times\hat{\boldsymbol{\zeta}}}{\kappa r R\cos\theta_e}\right) = \frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{R}}}{\kappa r R\,(1 + R_c'\cos\theta_e)} \tag{83}$$

$$= \frac{1}{\kappa r R(1 + R_c'\cos\theta_e)} = \frac{1}{\kappa r R_c\left(1 + \frac{r}{R_c}\cos\theta_e\right)(1 + R_c'\cos\theta_e)} \tag{84}$$

$$= \frac{1}{\kappa r R_c\left(1 + \left(R_c' + \frac{r}{R_c}\right)\cos\theta_e + \mathcal{O}(\epsilon^2)\right)} = \frac{1}{\kappa r R_c\left(1 - \frac{r}{R_0}\Lambda\cos\theta_e + \mathcal{O}(\epsilon^2)\right)} \tag{85}$$

The Grad-Shafranov equation reads

$$R^2\boldsymbol{\nabla}\cdot\left(\frac{\nabla\chi}{R^2}\right) = -I\frac{\mathrm{d}I}{\mathrm{d}\chi} - 4\pi R^2\frac{\mathrm{d}P}{\mathrm{d}\chi} \tag{86}$$

Now we use this can be written as

$$R^2\mathcal{J}\frac{\partial}{\partial\xi^i}\left(\frac{1}{R^2\mathcal{J}}\,\nabla\chi\cdot\nabla\xi^i\right) = -I\frac{\mathrm{d}I}{\mathrm{d}\chi} - 4\pi R^2\frac{\mathrm{d}P}{\mathrm{d}\chi} \tag{87}$$

Which if we write out using $\chi = \chi(r_e)$ so that $\nabla\chi = \chi'\,\nabla r$ we find that this yields

$$R^2\mathcal{J}\left[\frac{\partial}{\partial r_e}\left(\frac{1}{R^2\mathcal{J}}\chi'\,\nabla r_e\cdot\nabla r_e\right) + \frac{\partial}{\partial\theta_e}\left(\frac{1}{R^2\mathcal{J}}\chi'\,\nabla r_e\cdot\nabla\theta_e\right)\right] = -I\frac{\mathrm{d}I}{\mathrm{d}r_e}\frac{1}{\chi'} - 4\pi R^2\frac{\mathrm{d}P}{\mathrm{d}r_e}\frac{1}{\chi'} \tag{88}$$

where the $\nabla r_e\cdot\nabla r_e$ term yields

$$R^2\mathcal{J}\frac{\partial}{\partial r_e}\left(\frac{\chi'}{R^2\mathcal{J}}\frac{\frac{\sin^2\theta_e}{\kappa^2} + \cos^2\theta_e}{(1 + R_c'\cos\theta_e)^2}\right) \tag{89}$$

We note

$$\frac{\sin^2\theta_e + \kappa^2\cos^2\theta_e}{\kappa^2} = \frac{\frac{1-\cos(2\theta_e)}{2} + \kappa^2\frac{1+\cos(2\theta_e)}{2}}{\kappa^2} = \frac{1+\kappa^2+(\kappa^2-1)\cos(2\theta_e)}{2\kappa^2} \tag{90}$$

Now we need to determine $\nabla r_e \cdot \nabla\theta_e$. For this, we construct the Jacobian matrix. Thus we have

$$R = R_c(r_e) + r_e\cos\theta_e \tag{91}$$
$$\varphi = -\zeta_e \tag{92}$$
$$Z \equiv z = \kappa r_e \sin\theta_e \tag{93}$$
$$x = R\cos\varphi \tag{94}$$
$$y = R\sin\varphi \tag{95}$$

yielding

$$\frac{\partial x}{\partial r_e} = \cos\varphi\frac{\partial R}{\partial r_e} = \cos\varphi(R_c' + \cos\theta_e) \tag{96}$$
$$\frac{\partial x}{\partial \theta_e} = \cos\varphi\frac{\partial R}{\partial \theta_e} = -r_e\sin\theta_e\cos\varphi \tag{97}$$
$$\frac{\partial x}{\partial \zeta_e} = R\frac{\partial\cos\varphi}{\partial\zeta_e} = R\frac{\partial\cos\varphi}{\partial\varphi}\overbrace{\frac{\partial\varphi}{\partial\zeta_e}}^{=-1} = R\sin\varphi \tag{98}$$
$$\frac{\partial y}{\partial r_e} = \sin\varphi\frac{\partial R}{\partial r_e} = \sin\varphi(R_c' + \cos\theta_e) \tag{99}$$
$$\frac{\partial y}{\partial \theta_e} = \sin\varphi\frac{\partial R}{\partial \theta_e} = -r_e\sin\theta_e\sin\varphi \tag{100}$$
$$\frac{\partial y}{\partial \zeta_e} = R\frac{\partial\sin\varphi}{\partial\zeta_e} = R\frac{\partial\sin\varphi}{\partial\varphi}\overbrace{\frac{\partial\varphi}{\partial\zeta_e}}^{=-1} = R\cos\varphi \tag{101}$$
$$\frac{\partial Z}{\partial r_e} = \kappa\sin\theta_e \tag{102}$$
$$\frac{\partial Z}{\partial \theta_e} = \kappa r_e\cos\theta_e \tag{103}$$
$$\frac{\partial Z}{\partial \zeta_e} = 0 \tag{104}$$

allowing us to evaluate $g_{ij} = \frac{\partial x^k}{\partial\xi^i}\frac{\partial x^k}{\partial\xi^j}$ so that

$$g_{rr} = \left(\frac{\partial x}{\partial r_e}\right)^2 + \left(\frac{\partial y}{\partial r_e}\right)^2 + \left(\frac{\partial Z}{\partial r_e}\right)^2 \tag{105}$$
$$= \cos^2\varphi(R_c' + \cos\theta_e)^2 + \sin^2\varphi(R_c' + \cos\theta_e)^2 + \kappa^2\sin^2\theta_e \tag{106}$$
$$= \cos^2\theta_e + \kappa^2\sin^2\theta_e + 2R_c'\cos\theta_e + R_c'^2 \tag{107}$$
$$g_{\theta\theta} = \left(\frac{\partial x}{\partial \theta_e}\right)^2 + \left(\frac{\partial y}{\partial \theta_e}\right)^2 + \left(\frac{\partial Z}{\partial \theta_e}\right)^2 \tag{108}$$
$$= r_e^2\sin^2\theta_e\cos^2\varphi + r_e^2\sin^2\theta_e\sin^2\varphi + r_e^2\kappa^2\cos^2\theta = r_e^2(\sin^2\theta_e + \kappa^2\cos^2\theta_e) \tag{109}$$

$$g_{\zeta\zeta} = \left(\frac{\partial x}{\partial \zeta_e}\right)^2 + \left(\frac{\partial y}{\partial \zeta_e}\right)^2 + \left(\frac{\partial Z}{\partial \zeta_e}\right)^2 = R^2 \sin^2\varphi + R^2\cos^2\varphi + 0 = R^2 \tag{110}$$

$$\begin{aligned}
g_{r\theta} = g_{\theta r} &= \frac{\partial x}{\partial \theta_e}\frac{\partial x}{\partial r_e} + \frac{\partial y}{\partial \theta_e}\frac{\partial y}{\partial r_e} + \frac{\partial z}{\partial \theta_e}\frac{\partial z}{\partial r_e} && (111)\\
&= -r_e \sin\theta_e \sin^2\varphi (R_c' + \cos\theta_e) - r_e\sin\theta_e\cos^2\varphi(R_c' + \cos\theta_e) + r_e\kappa^2 \sin\theta_e\cos\theta_e && (112)\\
&= -r_e \sin\theta_e \left((1-\kappa^2)\cos\theta_e + R_c'\right) && (113)\\
g_{\theta\zeta} = g_{\zeta\theta} &= \frac{\partial x}{\partial \theta_e}\frac{\partial x}{\partial \zeta_e} + \frac{\partial y}{\partial \theta_e}\frac{\partial y}{\partial \zeta_e} + \frac{\partial z}{\partial \theta_e}\frac{\partial z}{\partial \zeta_e} && (114)\\
&= -r_e\sin\theta_e\cos\varphi(R\sin\varphi) - r_e\sin\theta_e\sin\varphi(-R\cos\varphi) + 0 = 0 && (115)\\
g_{r\zeta} = g_{\zeta r} &= \frac{\partial x}{\partial \theta_e}\frac{\partial x}{\partial r_e} + \frac{\partial y}{\partial \theta_e}\frac{\partial y}{\partial r_e} + \frac{\partial z}{\partial \theta_e}\frac{\partial z}{\partial r_e} && (116)\\
&= \cos\varphi(R_c' + \cos\theta_e)(R\sin\varphi) + \sin\varphi(R_c' + \cos\theta)(-R\cos\varphi) + 0 = 0 && (117)
\end{aligned}$$

yielding

$$g_{ij} = \begin{bmatrix} g_{rr} & g_{r\theta} & 0 \\ g_{r\theta} & g_{\theta\theta} & 0 \\ 0 & 0 & g_{\zeta\zeta} \end{bmatrix} \tag{118}$$

$$g^{ij} = [g_{ij}]^{-1} = \mathcal{J}^2 \begin{bmatrix} g_{\theta\theta}g_{\zeta\zeta} & -g_{r\theta}g_{\zeta\zeta} & 0 \\ -g_{r\theta}g_{\zeta\zeta} & g_{rr}g_{\theta\theta} & 0 \\ 0 & 0 & g_{rr}g_{\theta\theta} - g_{r\theta}^2 \end{bmatrix} \tag{119}$$

with $1/\mathcal{J}^2 = g_{\zeta\zeta}(g_{rr}g_{\theta\theta} - g_{r\theta}^2)$. Let's show that this leads to the same results we had before. First we see

$$\begin{aligned}
\mathcal{J}^{-2} &= g_{\zeta\zeta}(g_{rr}g_{\theta\theta} - g_{r\theta}^2) && (120)\\
&= R^2\big[(\cos^2\theta_e + \kappa^2\sin^2\theta_e + 2R_c'\cos\theta_e + R_c'^2)r_e^2(\sin^2\theta_e + \kappa^2\cos^2\theta_e) \\
&\qquad - r_e^2\sin^2\theta_e(R_c' + (1-\kappa^2)\cos^2\theta_e)\big] && (121)\\
&= R^2 r_e^2\big[\cancel{\sin^2\theta_e\cos^2\theta_e} + \kappa^2\cos^4\theta_e + \kappa^2\sin^4\theta_e + \cancel{\kappa^4\sin^2\theta_e\cos^2\theta_e} + \cancel{2R-c'\cos\theta_e\sin^2\theta_e} \\
&\qquad + 2R_c'\kappa^2\cos^3\theta_e + \cancel{R_c'^2\sin^2\theta_e} + R_c'^2\kappa^2\cos^2\theta_e - \cancel{R_c'^2\sin^2\theta_e} - \cancel{2R_e'\sin^2\theta_e\cos\theta_e} \\
&\qquad + 2R_c'\kappa^2\sin^2\theta_e\cos\theta_e - \cancel{\sin^2\theta_e\cos^2\theta_e} + 2\kappa^2\sin^2\theta_e\cos^2\theta_e - \cancel{\kappa^4\sin^2\theta_e\cos^2\theta_e}\big] && (122)\\
&= R^2 r_e^2\kappa^2\left[\cos^4\theta_e + \sin^4\theta_e + 2R_c'\cos^3\theta_e + R_c'^2\cos^2\theta_e + 2R_c'\sin^2\theta_e\cos\theta_e + 2\sin^2\theta_e\cos^2\theta_e\right] && (123)\\
&= R^2 r_e^2\kappa^2\left[\left(\cos^2\theta_e + \sin^2\theta_e\right)^2 + R_c'^{\cos}\theta_e\left(2\cos^2\theta_e + 2\sin^2\theta_e\right) + R - c'^2\cos^2\theta_e\right] && (124)\\
&= R^2 r_e^2\kappa^2\left(1 + 2R_c'\cos\theta_e + R_c'^2\cos^2\theta_e\right) && (125)
\end{aligned}$$

Thus, we find

$$\mathcal{J} = \frac{1}{Rr_e\kappa\left(1 + R_c'\cos\theta_e\right)} \tag{126}$$

after Taylor expanding and including only $\mathcal{O}(\epsilon)$ terms, which agrees with our previous result.

Thus we see

$$\nabla r_e \cdot \nabla\theta_e = \frac{-g_{r\theta}g_\zeta\zeta}{R^2r_e^2\kappa^2(1+2R_c'\cos\theta_e)} = \frac{r_e\sin\theta_e(R_c'+(1-\kappa^2)\cos\theta_e)R^2}{R^2r_e^2\kappa^2(1+2R_c'\cos\theta_e)} \tag{127}$$

$$= \frac{\sin\theta_e}{r_e\kappa^2}(R_c'+(1-\kappa^2\cos\theta_e)(1-2R_c'\cos\theta_e) \tag{128}$$

$$= \frac{1}{r_e\kappa^2}(R_c'\sin\theta_e+(1-\kappa^2)\sin\theta\cos\theta(1-2R_c'\cos\theta_e)) \tag{129}$$

$$= \frac{1}{r_e\kappa^2}\left(R_c'\sin\theta_e+\frac{(1-\kappa^2)}{2}\sin(2\theta_e)(1-2R_c'\cos\theta_e)\right) \tag{130}$$

Note in passing that for $\kappa^2 = 1$ we have no $\mathcal{O}(\epsilon^0)$ contribution to the Grad-Shafranov equation.

Thus, looking at only the left-hand side of the Grad-Shafranov equation and simplifying the $\nabla r_e \cdot \nabla r_e$ contribution then yields

$$\frac{1+\kappa^2+(\kappa^2-1)\cos(2\theta_e)}{2\kappa^2}\frac{R^2}{\kappa r_eR_c(1-\frac{r_e}{R_0}\Lambda\cos\theta_e)}\frac{\partial}{\partial r_e}\left(\frac{\chi'}{R^2}\kappa r_eR_c\left(1-\frac{r_e}{R_0}\Lambda\cos\theta_e\right)\right) \tag{131}$$

$$= \frac{1+\kappa^2+(\kappa^2-1)\cos(2\theta_e)}{2\kappa^2}\frac{1}{\kappa\epsilon}\left(1+\epsilon(\Lambda+1)\cos\theta_e\right)\frac{\partial}{\partial r_e}\left(\chi'\kappa\epsilon\left(1-\epsilon(\Lambda+1)\cos\theta_e\right)\right) \tag{132}$$

Taking the zeroth order component yields

$$\frac{1+\kappa^2+(\kappa^2-1)\cos(2\theta_e)}{2\kappa^2}\frac{R_0}{r_e\kappa}\frac{\partial}{\partial r_e}\left(\frac{\kappa r_e\chi'}{R_0}\right) = \frac{1+\kappa^2+(\kappa^2-1)\cos(2\theta_e)}{2\kappa^2}\frac{1}{r_e}(r_e\chi')' \tag{133}$$

Now let's look at the $\nabla r_e \cdot \nabla\theta_e$ term and find

$$\frac{R^2}{\kappa r_eR_c(1-\frac{r_e}{R_0}\Lambda\cos\theta_e)}\frac{\partial}{\partial\theta_e}\left(\frac{\chi' Rr_e\kappa(1-\frac{r_e}{r_0}\Lambda\cos\theta_e)}{R^2}\frac{1}{r_e\kappa^2}\left(R_c'\sin\theta_e+\frac{1-\kappa^2}{2}\sin(2\theta_e)(1-2R_c'\cos\theta_e)\right)\right) \tag{134}$$

$$\frac{R_0^2(1+\frac{r_e}{R_0}\cos\theta_e)^2}{\kappa r_eR_0(1-\frac{r_e}{R_0}\Lambda\cos\theta_e)}\frac{\partial}{\partial\theta_e}\left(\frac{\chi'(1-\frac{r_e}{r_0}\Lambda\cos\theta_e)}{\kappa R_0(1+\frac{r_e}{R_0}\cos\theta_e)}\left(R_c'\sin\theta_e+\frac{1-\kappa^2}{2}\sin(2\theta_e)(1-2R_c'\cos\theta_e)\right)\right) \tag{135}$$

$$\frac{(1+\frac{r_e}{R_0}\cos\theta_e)^2}{\kappa^2r_e(1-\frac{r_e}{R_0}\Lambda\cos\theta_e)}\frac{\partial}{\partial\theta_e}\left(\frac{\chi'(1-\frac{r_e}{r_0}\Lambda\cos\theta_e)}{(1+\frac{r_e}{R_0}\cos\theta_e)}\left(R_c'\sin\theta_e+\frac{1-\kappa^2}{2}\sin(2\theta_e)(1-2R_c'\cos\theta_e)\right)\right) \tag{136}$$

which to $\mathcal{O}(\epsilon^0)$ yields

$$\frac{1}{\kappa^2r_e}\frac{\partial}{\partial\theta_e}\left(\chi'\frac{1-\kappa^2}{2}\sin(2\theta_e)\right) \tag{137}$$

$$= \frac{\chi'(1-\kappa^2)}{2r_e\kappa^2}2\cos(2\theta_e) = \frac{(1-\kappa^2)\chi'}{r_e\kappa^2}\cos(2\theta_e) \tag{138}$$

Thus the entire right hand side of the Grad-Shafranov equation at $\mathcal{O}(\epsilon^0)$ is given by

$$\frac{1+\kappa^2+(\kappa^2-1)\cos(2\theta_e)}{2\kappa^2}\frac{1}{r_e}(r_e\chi')'+\frac{(1-\kappa^2)\chi'}{r_e\kappa^2}\cos(2\theta_e) \tag{139}$$

$$\frac{1+\kappa^2}{2\kappa^2r_e}(r_e\chi')+\frac{\kappa^2-1}{r_e}\left[\frac{1}{2}(r_e\chi')'-\chi'\right]\cos(2\theta_e) \tag{140}$$

The right hand side of the Grad-Shafranov equation will be given by

$$-I\frac{\mathrm{d}I}{\mathrm{d}\chi} - 4\pi R^2\frac{\mathrm{d}P}{\mathrm{d}\chi} \tag{141}$$

$$-I\frac{\mathrm{d}I}{\mathrm{d}\chi} - 4\pi R_0^2(1+\frac{r_e}{R_0}\cos\theta_e)^2\frac{\mathrm{d}P}{\mathrm{d}\chi} \tag{142}$$

and so to zeroth order will yield

$$-I_0\frac{\mathrm{d}I_0}{\mathrm{d}\chi} - 4\pi R_0^2\frac{\mathrm{d}P_0}{\mathrm{d}\chi} \tag{143}$$

where I_0 and P_0 are determined by the profiles chosen accurate to $\mathcal{O}(\epsilon^0)$. We suspect that they have no Fourier components other than $m=0$. Given this, then the $m=0$ component is

$$\frac{1+\kappa^2}{2\kappa^2 r_e}(r_e\chi') = -I_0\frac{\mathrm{d}I_0}{\mathrm{d}\chi} - 4\pi R_0^2\frac{\mathrm{d}P_0}{\mathrm{d}\chi} \tag{144}$$

and the $m=2$ component will be given by

$$\frac{\kappa^2-1}{r_e}\left[\frac{1}{2}(r_e\chi')' - \chi'\right]\cos(2\theta_e) = 0 \tag{145}$$

$$(\kappa^2-1)\left[\frac{1}{2}(r_e\chi')' - \chi'\right] = 0 \tag{146}$$

as stated in the problem statement.

Similarly for the $m=1$ and $m=3$ terms, except that they will come from the $\mathcal{O}(\epsilon)$ equation.

So, first take (132) to $\mathcal{O}(\epsilon)$.

$$\frac{1+\kappa^2+(\kappa^2-1)\cos(2\theta_e)}{2\kappa^2}\left[\frac{(\Lambda+1)}{\kappa}\cos\theta_e\frac{\partial}{\partial r_e}\left(\frac{\kappa r_e\chi'}{R_0}\right) - \frac{R}{\kappa r_e}\frac{\partial}{\partial r_e}\left(\chi'\kappa\frac{r_e^2}{R^2}(\Lambda+1)\cos\theta_e\right)\right] \tag{147}$$

remembering that

$$\cos(2\theta_e)\cos\theta_e = \frac{1}{2}\left(\cos\theta_e + \cos(3\theta_e)\right) \tag{148}$$

then take the same order in (136) (defining $\alpha = \frac{1-\kappa^2}{2}\sin(2\theta_e)$ we get

$$\frac{1+\frac{r_e}{R_0}(2+\Lambda)\cos\theta_e}{\kappa^2 r_e}\frac{\partial}{\partial\theta}\left(\chi'\left(1-\frac{r_e}{R_0}(1+\Lambda)\cos\theta_e\right)(\alpha + R_c'(\sin\theta_e - 2\alpha\cos\theta_e))\right) \tag{149}$$

$$= \frac{\frac{r_e}{R_0}(2+\Lambda)\cos\theta_e}{\kappa^2 r_e}\frac{\partial}{\partial\theta}(\alpha\chi') + \frac{1}{\kappa^2 r_e}\frac{\partial}{\partial\theta}\left(\chi'(R_c'(\sin\theta - 2\alpha\cos\theta_e) - \frac{\alpha r_e}{R_0}(1+\Lambda)\cos\theta_e)\right) \tag{150}$$

$$\begin{aligned} &= \frac{\chi'(1-\kappa^2)(2+\Lambda)\cos\theta_e}{\kappa^2 R_0}\cos(2\theta_e) \\ &\quad + \frac{\chi'}{\kappa^2 r_e}\left(R_c'(\cos\theta_e - \frac{1}{2}\left(\cos\theta_e + 3\cos(3\theta_e)\right)) - \frac{r_e(1+\Lambda)}{2R_0}\left(\cos\theta_e + 3\cos(3\theta_e)\right)\right) \end{aligned} \tag{151}$$

The right hand side we assume at $\mathcal{O}(\epsilon)$ has components with just $\cos\theta_e$. Thus the $m = 1$ component reads

$$\begin{aligned}
&\left[\frac{1+\kappa^2}{2\kappa^2} + \frac{\kappa^2-1}{4\kappa^2}\right]\left(\frac{(1+\Lambda)}{R_0}(r_e\chi')' - \frac{(1+\Lambda)}{\kappa R_0}(r_e^2\chi')'\right) \\
&\quad + \frac{\chi'(1-\kappa^2)(2+\Lambda)}{2\kappa^2 R_0} + \frac{\chi'}{\kappa^2 r_e}\left(\frac{R_c'}{2} - \frac{r_e(1+\Lambda)}{2R_0}\right) \\
&= -I_1\frac{\partial I_1}{\partial\chi} - 8\pi r_e R_0\frac{\mathrm{d}P_1}{\mathrm{d}\chi}
\end{aligned} \tag{152}$$

while the $m = 3$ component will read

$$\begin{aligned}
&\frac{\kappa^2-1}{4\kappa^2}\left[(\Lambda+1)(r_e\chi')' - (1+\Lambda)(r_e^2\chi')'\right] + \frac{\chi'(1-\kappa^2)(2+\Lambda)}{2\kappa^2 R_0} \\
&\quad - \frac{\chi'}{\kappa^2 r_e}\left(\frac{3R_c'}{2} - \frac{3r_e(1+\Lambda)}{R_0}\right) = 0
\end{aligned} \tag{153}$$