1 Toroidal Coordinate System Define a toroidal coordinate system (ψ, θ, ζ) by the relations

$$x = \frac{\sqrt{\psi^2 - 1}}{\psi - \cos\theta}\cos\zeta \quad , \quad y = \frac{\sqrt{\psi^2 - 1}}{\psi - \cos\theta}\sin\zeta \quad , \quad z = \frac{\sin\theta}{\psi - \cos\theta} \quad , \tag{1}$$

where ψ ranges from 1 to ∞. Show that the surfaces $\psi =$ constant are axisymmetric tori with a circular poloidal cross section. Compute the metric tensor g_{ij}, and the Jacobian $\mathcal{J}$.

Solution:

The formula for a torus is

$$x(\phi, \xi) = (R + r\cos\phi)\cos\xi \quad , \quad y = (R + r\cos\phi)\sin\xi \quad , \quad z = r\sin\phi \quad . \tag{2}$$

Note that both r (the minor radius) and R (the major radius) are fixed for a particular torus.

So if ψ is a constant we may define $R + r\cos\phi \equiv \dfrac{\sqrt{\psi^2 - 1}}{\psi - \cos\theta}$ and $r\sin\phi \equiv \dfrac{\sin\theta}{\psi - \cos\theta}$ and so to be a torus we need the values of r and R to be consistent. Let's put a parameter a in so that

$$x = a\frac{\sqrt{\psi^2 - 1}}{\psi - \cos\theta}\cos\zeta \quad , \quad y = a\frac{\sqrt{\psi^2 - 1}}{\psi - \cos\theta}\sin\zeta \quad , \quad z = a\frac{\sin\theta}{\psi - \cos\theta} \quad . \tag{3}$$

It will be seen later that this relaxes a requirement between the major and minor radii.

So

$$\left.\begin{array}{ll} r\cos\phi &= \frac{\sqrt{\psi^2-1}}{\psi - \cos\theta} - R = \sqrt{x^2 + y^2} - R \\ r\sin\phi &= \frac{\sin\theta}{\psi - \cos\theta} = z \end{array}\right\} \Rightarrow r^2 = (\sqrt{x^2 + y^2} - R)^2 + z^2 \tag{4}$$

So let's see what this relation gives

$$r^2 = \left(a\frac{\sqrt{\psi^2 - 1}}{\psi - \cos\theta} - R\right)^2 + \frac{a^2\sin^2\theta}{(\psi - \cos\theta)^2} = a^2\frac{\psi^2 - 1}{(\psi - \cos\theta)^2} + R^2 - 2aR\frac{\sqrt{\psi^2 - 1}}{(\psi - \cos\theta)} + \frac{a^2\sin^2\theta}{(\psi - \cos\theta)^2} \tag{5}$$

$$= a^2\frac{\psi^2 - 1 + \sin^2\theta}{(\psi - \cos\theta)^2} + R^2 - 2a\frac{R\sqrt{\psi^2 - 1}}{\psi - \cos\theta} = a^2\frac{(\psi + \cos\theta)\cancel{(\psi - \cos\theta)}}{(\psi - \cos\theta)^{\cancel{2}}} + R^2 - 2a\frac{R\sqrt{\psi^2 - 1}}{\psi - \cos\theta} \tag{6}$$

$$r^2 = \frac{R^2(\psi - \cos\theta) + a^2(\psi + \cos\theta) - 2aR\sqrt{\psi^2 - 1}}{\psi - \cos\theta} \tag{7}$$

$\Rightarrow$

$$0 = \frac{(R^2 - r^2)(\psi - \cos\theta) + a^2(\psi + \cos\theta) - 2aR\sqrt{\psi^2 - 1}}{\psi - \cos\theta} \tag{8}$$

$$0 = (R^2 - r^2)(\psi - \cos\theta) + a^2(\psi + \cos\theta) - 2aR\sqrt{\psi^2 - 1} \tag{9}$$

Now we need to choose this such that R and r are only functions of ψ, which is fixed, if we are to have a torus. That is, if we look at the equation above we can rewrite it as

$$0 = \underbrace{-(R^2 - r^2 - a^2)\cos\theta}_{f(\theta)} + \underbrace{(R^2 - r^2 + a^2) - 2aR\sqrt{\psi^2 - 1}}_{\text{constant}} \tag{10}$$

We see that $f(\theta) = 0$ always if our relationship is to actually hold. Thus we use the solution with $R^2 - r^2 = a^2$ so that this formula holds for all θ. We then find

$$a^2(\psi - \cos\theta + \psi + \cos\theta) - 2aR\sqrt{\psi^2 - 1} = 2(a^2\psi - aR\sqrt{\psi^2 - 1}) = 0 \tag{11}$$

$$\Rightarrow \boxed{R = \frac{a\psi}{\sqrt{\psi^2 - 1}}} \Rightarrow r^2 = \frac{a^2\psi^2}{\psi^2 - 1} - a^2\frac{\psi^2 - 1}{\psi^2 - 1} = \frac{a^2}{\psi^2 - 1} \Rightarrow \boxed{r = \frac{a}{\sqrt{\psi^2 - 1}}}. \tag{12}$$

So we note that the parameter a allows us to have any relationship between major and minor radii that we need. For our specific case we have $a = 1$. Now it is easy to see that for constant ψ that we have axisymmetric tori.

It can also be shown that they are non-intersecting *i.e.* nested tori. This is easy to see when plotting as seen in Figure 1. More formally, the constraint $R^2 - r^2 = a^2$ implies that $R > r$ always and that the tori are nested, as the following argument shows.

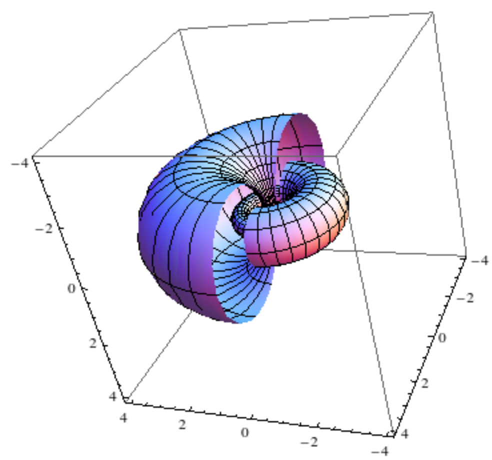


Figure 1: Picture of nested tori for the case described in (1).

To show there are no intersections, let us assume there is an intersection by way of contradiction. Let $a \neq 0$. For there to be an intersection we need

$$R_1 - r_1 = R_2 - r_2 \tag{13}$$

with $R_i^2 - r_i^2 = a^2$ and $R_1 \neq R_2 \Rightarrow r_1 \neq r_2$ satisfied for both $i = 1, 2$. This implies that if we multiply by $R_1 + r_1$ that

$$\overbrace{(R_1 - r_1)(R_1 + r_1)}^{R_1^2 - r_1^2 = a^2} = (R_2 - r_2)(R_1 + r_1) \tag{14}$$

$$a^2 = (R_2 - r_2)(R_1 + r_1) \tag{15}$$

and if we now multiply by $R_2 + r_2$

$$a^2(R_2 + r_2) = \overbrace{(R_2 + r_2)(R_2 - r_2)}^{R_2^2 - r_2^2 = a^2}(R_1 + r_1) \tag{16}$$

$$a^2(R_2 + r_2) = a^2(R_1 + r_1) \tag{17}$$

$$R_1 + r_1 \overset{a \neq 0}{=} R_2 + r_2. \tag{18}$$

So combining (13) and (18) by addition we find

$$R_1 - \cancel{r_1} + R_1 + \cancel{r_1} = R_2 - \cancel{r_2} + R_2 + \cancel{r_2} \tag{19}$$

$$2R_1 = 2R_2 \Rightarrow R_1 = R_2 \tag{20}$$

which contradicts the hypothesis that R_1 and R_2 are different. Hence there are never any intersections.

One can also note that instead of using (3), one could have used

$$x = a\frac{\sinh\tau}{\cosh\tau - \cos\theta}\cos\zeta \quad , \quad y = a\frac{\sinh\tau}{\cosh\tau - \cos\theta}\sin\zeta \quad , \quad z = a\frac{\sin\theta}{\cosh\tau - \cos\theta} \quad , \tag{21}$$

with $\sinh\tau = \sqrt{\psi^2 - 1}$ and $\cosh\tau = \sqrt{1 + \sinh^2\tau} = \psi$ and $\tau \geq 0$. These are what are usually called generalized toroidal coordinates.

Now let's find the metric tensor $g_{ij} = \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}$ where $\xi^1 = \psi$, $\xi^2 = \theta$ and $\xi^3 = \zeta$. Hence

$$\begin{aligned}\frac{\partial x}{\partial \psi} &= \frac{\frac{\psi}{\sqrt{\psi^2-1}}(\psi - \cos\theta) - \sqrt{\psi^2 - 1}}{(\psi - \cos\theta)^2}\cos\zeta = \cos\zeta\frac{\psi(\psi - \cos\theta) - (\psi^2 - 1)}{\sqrt{\psi^2 - 1}(\psi - \cos\theta)^2} \\ &= \frac{1 - \psi\cos\theta}{\sqrt{\psi^2 - 1}(\psi - \cos\theta)^2}\cos\zeta\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\partial y}{\partial \psi} &= \frac{\frac{\psi}{\sqrt{\psi^2-1}}(\psi - \cos\theta) - \sqrt{\psi^2 - 1}}{(\psi - \cos\theta)^2}\sin\zeta = \sin\zeta\frac{\psi(\psi - \cos\theta) - (\psi^2 - 1)}{\sqrt{\psi^2 - 1}(\psi - \cos\theta)^2} \\ &= \frac{1 - \psi\cos\theta}{\sqrt{\psi^2 - 1}(\psi - \cos\theta)^2}\sin\zeta\end{aligned} \tag{23}$$

$$\frac{\partial z}{\partial \psi} = \frac{-\sin\theta}{(\psi - \cos\theta)^2} \tag{24}$$

$$\frac{\partial x}{\partial \theta} = -\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\cos\zeta \tag{25}$$

$$\frac{\partial y}{\partial \theta} = -\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\sin\zeta \tag{26}$$

$$\frac{\partial z}{\partial \theta} = \frac{\cos\theta(\psi-\cos\theta)-\sin^2\theta}{(\psi-\cos\theta)^2} = \frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2} \tag{27}$$

$$\frac{\partial x}{\partial \zeta} = -\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta \tag{28}$$

$$\frac{\partial y}{\partial \zeta} = \frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta \tag{29}$$

$$\frac{\partial z}{\partial \zeta} = 0. \tag{30}$$

So now we need to just calculate each element of the tensor g_{ij} remembering that $g_{ij} = g_{ji}$.

$$g_{11} = \frac{\partial x}{\partial \psi}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \psi}\frac{\partial y}{\partial \psi} + \frac{\partial z}{\partial \psi}\frac{\partial z}{\partial \psi} \tag{31}$$

$$= \frac{(1-\psi\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4}\cos^2\zeta + \frac{(1-\psi\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4}\sin^2\zeta + \frac{\sin^2\theta}{(\psi-\cos\theta)^4} \tag{32}$$

$$= \frac{(1-\psi\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4} + \frac{\sin^2\theta(\psi^2-1)}{(\psi^2-1)(\psi-\cos\theta)^4} = \frac{1+\psi^2\cos^2\theta-2\psi\cos\theta+\psi^2\sin^2\theta-\sin^2\theta}{(\psi^2-1)(\psi-\cos\theta)^4} \tag{33}$$

$$\boxed{g_{11} = \frac{\cos^2\theta-2\psi\cos\theta+\psi^2}{(\psi^2-1)(\psi-\cos\theta)^4} = \frac{(\psi-\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4} = \frac{1}{(\psi^2-1)(\psi-\cos\theta)^2}} \tag{34}$$

$$g_{22} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} \tag{35}$$

$$= \frac{(\psi^2-1)\sin^2\theta}{(\psi-\cos\theta)^4}\cos^2\zeta + \frac{(\psi^2-1)\sin^2\theta}{(\psi-\cos\theta)^4}\sin^2\zeta + \frac{(1-\psi\cos\theta)^2}{(\psi-\cos\theta)^4} \tag{36}$$

$$= \frac{\psi^2\sin^2\theta-\sin^2\theta+1-2\psi\cos\theta+\psi^2\cos^2\theta}{(\psi-\cos\theta)^4} = \frac{\cos^2\theta-2\psi\cos\theta+\psi^2}{(\psi-\cos\theta)^4} \tag{37}$$

$$\boxed{g_{22} = \frac{(\psi-\cos\theta)^2}{(\psi-\cos\theta)^4} = \frac{1}{(\psi-\cos\theta)^2}} \tag{38}$$

$$g_{33} = \frac{\partial x}{\partial \zeta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \zeta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \zeta}\frac{\partial z}{\partial \zeta} \tag{39}$$

$$\boxed{g_{33} = \frac{\psi^2-1}{(\psi-\cos\theta)^2}\sin^2\zeta + \frac{\psi^2-1}{(\psi-\cos\theta)^2}\cos^2\zeta = \frac{\psi^2-1}{(\psi-\cos\theta)^2}} \tag{40}$$

$$g_{12} = \frac{\partial x}{\partial \psi}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \psi}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \psi}\frac{\partial z}{\partial \theta} \tag{41}$$

$$= \left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\cos\zeta\right)\left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\cos\zeta\right)$$

$$
\begin{aligned}
&+\left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\sin\zeta\right)\left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\sin\zeta\right)\\
&+\left(\frac{-\sin\theta}{(\psi-\cos\theta)^2}\right)\left(\frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2}\right)
\end{aligned} \tag{42}
$$

$$
\boxed{g_{12}=\frac{(\psi\cos\theta-1)\sqrt{\psi^2-1}\sin\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^4}-\frac{\sin\theta(\psi\sin\theta\cos\theta-1}{(\psi-\cos\theta)^4}=0} \tag{43}
$$

$$
g_{13}=\frac{\partial x}{\partial\psi}\frac{\partial x}{\partial\zeta}+\frac{\partial y}{\partial\psi}\frac{\partial y}{\partial\zeta}+\frac{\partial z}{\partial\psi}\frac{\partial z}{\partial\zeta} \tag{44}
$$

$$
\begin{aligned}
&=\left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\cos\zeta\right)\left(-\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta\right)\\
&\quad+\left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\sin\zeta\right)\left(\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta\right)\\
&\quad+\left(\frac{-\sin\theta}{(\psi-\cos\theta)^2}\right)(0)
\end{aligned} \tag{45}
$$

$$
\boxed{g_{13}=-\frac{1-\psi\cos\theta}{(\psi-\cos\theta)^3}\sin\zeta\cos\zeta+\frac{1-\psi\cos\theta}{(\psi-\cos\theta)^3}\sin\zeta\cos\zeta=0} \tag{46}
$$

$$
g_{23}=\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\zeta}+\frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\zeta}+\frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\zeta} \tag{47}
$$

$$
\begin{aligned}
&=\left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\cos\zeta\right)\left(-\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta\right)+\left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\sin\zeta\right)\left(\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta\right)\\
&\quad+\left(\frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2}\right)(0)
\end{aligned} \tag{48}
$$

$$
\boxed{g_{23}=\frac{(\psi^2-1)\sin\theta}{(\psi-\cos\theta)^3}\left(\sin\zeta\cos\zeta-\sin\zeta\cos\zeta\right)=0.} \tag{49}
$$

Hence we have altogether

$$
g_{ij}=\begin{bmatrix}\left[(\psi^2-1)(\psi-\cos\theta)^2\right]^{-1} & 0 & 0\\ 0 & (\psi-\cos\theta)^{-2} & 0\\ 0 & 0 & (\psi^2-1)(\psi-\cos\theta)^{-2}\end{bmatrix}. \tag{50}
$$

And so the Jacobian is the inverse of the square root of the determinant of this matrix (i.e., $\sqrt{g}=1/\mathcal{J}$ with g the determinant of g_{ij}). Hence

$$
\mathcal{J}^{-2}=\frac{1}{\cancel{(\psi^2-1)}(\psi-\cos\theta)^2}\frac{\cancel{\psi^2-1}}{(\psi-\cos\theta)^2}\frac{1}{(\psi-\cos\theta)^2}=(\psi-\cos\theta)^{-6}\Rightarrow\mathcal{J}=(\psi-\cos\theta)^3. \tag{51}
$$

Had we kept the a we would have simply retained an a^2 throughout the g_{ij} calculations leading to

$$
\mathcal{J}^{-2}=a^6(\psi-\cos\theta)^{-6}\Rightarrow\mathcal{J}=\frac{(\psi-\cos\theta)^3}{a^3}. \tag{52}
$$

We note that for a having units of meters, then the Jacobian at the very least has the correct units.

Let's look at the case for general toroidal coordinates (τ, θ, ζ), (21)

$$x = a\frac{\sinh\tau}{\cosh\tau - \cos\theta}\cos\zeta \quad , \quad y = a\frac{\sinh\tau}{\cosh\tau - \cos\theta}\sin\zeta \quad , \quad z = a\frac{\sin\theta}{\cosh\tau - \cos\theta} \quad , \tag{21}$$

So then,

$$\begin{aligned}\frac{1}{a}\frac{\partial x}{\partial\tau} &= \frac{\cosh\tau(\cosh\tau - \cos\theta) - \sinh^2\tau}{(\cosh\tau - \cos\theta)^2}\cos\zeta = \frac{\cosh^2\tau - \cosh\tau\cos\theta - \sinh^2\tau}{(\cosh\tau - \cos\theta)^2}\cos\zeta \\ &= \frac{1 - \cosh\tau\cos\theta}{(\cosh\tau - \cos\theta)^2}\cos\zeta\end{aligned} \tag{53}$$

$$\begin{aligned}\frac{1}{a}\frac{\partial y}{\partial\tau} &= \frac{\cosh\tau(\cosh\tau - \cos\theta) - \sinh^2\tau}{(\cosh\tau - \cos\theta)^2}\sin\zeta = \frac{\cosh^2\tau - \cosh\tau\cos\theta - \sinh^2\tau}{(\cosh\tau - \cos\theta)^2}\sin\zeta \\ &= \frac{1 - \cosh\tau\cos\theta}{(\cosh\tau - \cos\theta)^2}\sin\zeta\end{aligned} \tag{54}$$

$$\frac{1}{a}\frac{\partial z}{\partial\tau} = \frac{-\sin\theta\sinh\tau}{(\cosh\tau - \cos\theta)^2} \tag{55}$$

$$\frac{1}{a}\frac{\partial x}{\partial\theta} = \frac{-\sinh\tau\sin\theta}{(\cosh\tau - \cos\theta)^2}\cos\zeta \tag{56}$$

$$\frac{1}{a}\frac{\partial y}{\partial\theta} = \frac{-\sinh\tau\sin\theta}{(\cosh\tau - \cos\theta)^2}\sin\zeta \tag{57}$$

$$\begin{aligned}\frac{1}{a}\frac{\partial z}{\partial\theta} &= \frac{\cos\theta(\cosh\tau - \cos\theta) - \sin^2\theta}{(\cosh\tau - \cos\theta)^2} = \frac{\cos\theta\cosh\tau - (\cos^2\theta + \sin^2\theta)}{(\cosh\tau - \cos\theta)^2} \\ &= \frac{\cos\theta\cosh\tau - 1}{(\cosh\tau - \cos\theta)^2}\end{aligned} \tag{58}$$

$$\frac{1}{a}\frac{\partial x}{\partial\zeta} = -\frac{\sinh\tau}{\cosh\tau - \cos\theta}\sin\zeta \tag{59}$$

$$\frac{1}{a}\frac{\partial y}{\partial\zeta} = \frac{\sinh\tau}{\cosh\tau - \cos\theta}\cos\zeta \tag{60}$$

$$\frac{1}{a}\frac{\partial z}{\partial\zeta} = 0 \tag{61}$$

and thus,

$$\begin{aligned}\frac{g_{11}}{a^2} &= \frac{\partial x}{\partial\tau}\frac{\partial x}{\partial\tau} + \frac{\partial y}{\partial\tau}\frac{\partial y}{\partial\tau} + \frac{\partial z}{\partial\tau}\frac{\partial z}{\partial\tau} \\ &= \left(\frac{1 - \cosh\tau\cos\theta}{(\cosh\tau - \cos\theta)^2}\cos\zeta\right)^2 + \left(\frac{1 - \cosh\tau\cos\theta}{(\cosh\tau - \cos\theta)^2}\sin\zeta\right)^2 + \left(\frac{-\sin\theta\sinh\tau}{(\cosh\tau - \cos\theta)^2}\right)^2 \\ &= \frac{(1 - \cosh\tau\cos\theta)^2 + \sin^2\theta\sinh^2\tau}{(\cosh\tau - \cos\theta)^4} = \frac{1 - 2\cosh\tau\cos\theta + \cosh^2\tau\cos^2\theta + \sin^2\theta\sinh^2\tau}{(\cosh\tau - \cos\theta)^4} \\ &= \frac{1 - 2\cosh\tau\cos\theta + \cosh^2\tau + \sin^2\theta(\sinh^2\tau - \cosh^2\tau)}{(\cosh\tau - \cos\theta)^4} = \frac{1 - 2\cosh\tau\cos\theta + \cosh^2\tau - \sin^2\theta}{(\cosh\tau - \cos\theta)^4} \\ &= \frac{\cos^2\theta - 2\cosh\tau\cos\theta + \cosh^2\tau}{(\cosh\tau - \cos\theta)^4} = \frac{(\cosh\tau - \cos\theta)^2}{(\cosh\tau - \cos\theta)^4} = \frac{1}{(\cosh\tau - \cos\theta)^2}\end{aligned} \tag{62}$$

$$\begin{aligned}
\frac{g_{12}}{a^2} &= \frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \tau}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial \theta} \\
&= \frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta \\
&\quad + \frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta \\
&\quad + \frac{-\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2} \\
&= \frac{(1-\cos\theta\cosh\tau)(\sin\theta\sinh\tau)}{(\cosh\tau-\cos\theta)^4}\left(\cos^2\zeta+\sin^2\zeta-1\right) = 0
\end{aligned} \tag{63}$$

$$\begin{aligned}
\frac{g_{13}}{a^2} &= \frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \tau}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial \zeta} \\
&= -\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\frac{\sinh\tau}{\cosh\tau-\cos\theta}\sin\zeta + \frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta + 0 \\
&= \frac{(1-\cosh\tau\cos\theta)}{(\cosh\tau-\cos\theta)^2}\sin\zeta\cos\zeta\,(-1+1) = 0
\end{aligned} \tag{64}$$

$$\frac{g_{21}}{a^2} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \tau} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \tau} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \tau} = \frac{g_{12}^2}{a^2} = 0 \tag{65}$$

$$\begin{aligned}
\frac{g_{22}}{a^2} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} \\
&= \left(\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\right)^2 + \left(\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\right)^2 + \left(\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2}\right)^2 \\
&= \frac{\sinh^2\tau\sin^2\theta+(1-\cos\theta\cosh\tau)^2}{(\cosh\tau-\cos\theta)^4} = \frac{\sinh^2\tau\sin^2\theta+1-2\cosh\tau\cos\theta+\cos^2\theta\cosh^2\tau}{(\cosh\tau-\cos\theta)^4} \\
&= \frac{\sinh^2\tau\sin^2\theta+(1-\sin^2\theta)\cosh^2\tau+1-2\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^4} = \frac{-\sin^2\theta+\cosh^2\tau+1-2\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^4} \\
&= \frac{\cosh^2\tau-2\cosh\tau\cos\theta+\cos^2\theta}{(\cosh\tau-\cos\theta)^4} = \frac{(\cosh\tau-\cos\theta)^2}{(\cosh\tau-\cos\theta)^4} = \frac{1}{(\cosh\tau-\cos\theta)^2}
\end{aligned} \tag{66}$$

$$\frac{g_{23}}{a^2} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \zeta} = \frac{g_{32}^2}{a^2} = 0 \tag{67}$$

$$\frac{g_{31}}{a^2} = \frac{\partial x}{\partial \zeta}\frac{\partial x}{\partial \tau} + \frac{\partial y}{\partial \zeta}\frac{\partial y}{\partial \tau} + \frac{\partial z}{\partial \zeta}\frac{\partial z}{\partial \tau} = \frac{g_{13}}{a^2} = 0 \tag{68}$$

$$\begin{aligned}
\frac{g_{32}}{a^2} &= \frac{\partial x}{\partial \zeta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \zeta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \zeta}\frac{\partial z}{\partial \theta} \\
&= \frac{-\sinh\tau}{\cosh\tau-\cos\theta}\sin\zeta\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta + \frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta + 0 \\
&= \frac{\sinh^2\tau\sin\theta}{(\cosh\tau-\cos\theta)^3}\,(-1+1) = 0
\end{aligned} \tag{69}$$

$$\begin{aligned} \frac{g_{33}}{a^2} &= \frac{\partial x}{\partial \zeta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \zeta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \zeta}\frac{\partial z}{\partial \zeta} \\ &= \left(-\frac{\sinh \tau}{\cosh \tau - \cos \theta}\sin \zeta\right)^2 + \left(\frac{\sinh \tau}{\cosh \tau - \cos \theta}\cos \zeta\right)^2 + 0 = \frac{\sinh^2 \tau}{(\cosh \tau - \cos \theta)^2} \end{aligned} \tag{70}$$

Thus, we find

$$g_{ij} = \begin{bmatrix} \frac{a^2}{(\cosh \tau - \cos \theta)^2} & 0 & 0 \\ 0 & \frac{a^2}{(\cosh \tau - \cos \theta)^2} & 0 \\ 0 & 0 & \frac{a^2 \sinh^2 \tau}{(\cosh \tau - \cos \theta)^2} \end{bmatrix}. \tag{71}$$

And so the Jacobian is the inverse of the square root of the determinant of this matrix (i.e., $\sqrt{g} = 1/\mathcal{J}$ with g the determinant of g_{ij}). Hence

$$\mathcal{J}^{-2} = \frac{a^6 \sinh^2 \tau}{(\cosh \tau - \cos \theta)^6} \Rightarrow \mathcal{J} = \frac{(\cosh \tau - \cos \theta)^3}{a^3 \sinh \tau} \tag{72}$$

2 Verify Identities Verify (2.30), (2.31), and (2.32).

$$A^i_{,j} \equiv \left(\frac{\partial}{\partial \xi^j}\mathbf{A}\right)^i = \frac{\partial A^i}{\partial \xi^j} + \Gamma^i_{jk}A^k, \tag{2.30}$$

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial \xi^j} + \frac{\partial g_{mj}}{\partial \xi^k} - \frac{\partial g_{jk}}{\partial \xi^m}\right). \tag{2.31}$$

(there is an inconsistency in Hazeltine and Meiss that has a minus sign in the above equation (2.30). This is due to a change in definition from a previous (non-Dover) edition in the contravariant and covariant derivatives) and

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \nabla\xi^m \cdot \frac{\partial}{\partial \xi^j}\left(\frac{\epsilon_{ijk}}{2\mathcal{J}}A^i\,\nabla\xi^j \times \nabla\xi^k\right) = \mathcal{J}\frac{\partial}{\partial \xi^i}\left(\frac{A^i}{\mathcal{J}}\right). \tag{2.32}$$

Solution:

We know that for covariant components

$$\left(\frac{\partial \mathbf{A}}{\partial \xi^j}\right) = A_{i,j}\,\nabla\xi^i = A_{i,j}\frac{\partial \xi^i}{\partial \xi^m} \tag{73}$$

and so for contravariant components we require in analogy

$$\left(\frac{\partial \mathbf{A}}{\partial \xi^j}\right) = A^i_{,j}\frac{\partial \xi^m}{\partial \xi^i} \tag{74}$$

$$\frac{\partial \mathbf{A}}{\partial \xi^j} = \frac{\partial}{\partial \xi^j}\left(A^i\frac{\partial \xi^m}{\partial \xi^i}\right) = \frac{\partial A^i}{\partial \xi^j}\frac{\partial \xi^m}{\partial \xi^i} + A^i\frac{\partial^2 \xi^m}{\partial \xi^j \partial \xi^i} \tag{75}$$

$$= \frac{\partial A^i}{\partial \xi^j}\frac{\partial \xi^m}{\partial \xi^i} + A^i\underbrace{\frac{\partial^2 \xi^m}{\partial x^r \partial x^\ell}\frac{\partial x^\ell}{\partial \xi^j}\frac{\partial x^r}{\partial \xi^i}}_{\Gamma^m_{ji}} \tag{76}$$

$$= \frac{\partial A^i}{\partial \xi^j}\frac{\partial \xi^m}{\partial \xi^i} + A^k\Gamma^m_{jk} \tag{77}$$

Now if we use (104) as an alternate expression for Γ^m_{jk} we see that we can write it as

$$= \frac{\partial A^i}{\partial \xi^j}\frac{\partial \xi^m}{\partial \xi^i} + A^k\frac{\partial \xi^m}{\partial x^\ell}\frac{\partial^2 x^\ell}{\partial \xi^j \partial \xi^k} \tag{78}$$

$$= \frac{\partial A^i}{\partial \xi^j}\frac{\partial \xi^m}{\partial \xi^i} + A^k\frac{\partial \xi^m}{\partial \xi^i}\underbrace{\frac{\partial \xi^i}{\partial x^\ell}\frac{\partial^2 x^\ell}{\partial \xi^j \partial \xi^k}}_{\Gamma^i_{jk}} \tag{79}$$

$$= \frac{\partial A^i}{\partial \xi^j}\frac{\partial \xi^m}{\partial \xi^i} + A^k\frac{\partial \xi^m}{\partial \xi^i}\Gamma^i_{jk} \tag{80}$$

$$= \left(\frac{\partial A^i}{\partial \xi^j} + A^k\Gamma^i_{jk}\right)\frac{\partial \xi^m}{\partial \xi^i} \tag{81}$$

and hence we have

$$A^i_{,j} = \frac{\partial A^i}{\partial \xi^j} + A^k \Gamma^i_{jk}. \tag{82}$$

Now let's prove the relation on Γ^i_{jk}.

$$\frac{\partial}{\partial \xi^i}\left(A^j g_{jk} A^k\right) = A^j_{,i} g_{jk} A^k + A^j g_{jk,i} A^k + A^j g_{jk} A^k_{,i} \tag{83}$$

$$= A_j \frac{\partial A^j}{\partial \xi^i} + A_j \Gamma^j_{i\ell} A^\ell + A^j A^k g_{jk,i} + A_k \frac{\partial A^k}{\partial \xi^i} + A_k \Gamma^k_{i\ell} A^\ell \tag{84}$$

$$= 2\left(A_k \frac{\partial A^k}{\partial \xi^i} + A_k A^j \Gamma^k_{ij}\right) + A^j A^k g_{jk,i} \tag{85}$$

and

$$\frac{\partial}{\partial \xi^i}\left(A^j g_{jk} A^k\right) = A^j_{,i} g_{jk} A^k + A^j g_{jk} A^k_{,i} \tag{86}$$

$$= A^k g_{jk} \frac{\partial A^j}{\partial \xi^i} + A^k g_{jk} \Gamma^j_{i\ell} A^\ell + A^j g_{jk} \frac{\partial A^k}{\partial \xi^i} + A^j g_{jk} \Gamma^k_{i\ell} A^\ell \tag{87}$$

$$= A_j \frac{\partial A^j}{\partial \xi^i} + A_j \Gamma^j_{i\ell} A^\ell + A_k \frac{\partial A^k}{\partial \xi^i} + A_k \Gamma^k_{i\ell} A^\ell \tag{88}$$

$$= 2\left(A_j \frac{\partial A^j}{\partial \xi^i} + A_j A^\ell \Gamma^j_{i\ell}\right). \tag{89}$$

and (85) and (89) should equal each other so

$$g_{jk,i} = 0 \tag{90}$$

$$\Rightarrow$$

$$\frac{\partial g_{jk}}{\partial \xi^i} - g_{mk}\Gamma^m_{ji} - g_{mj}\Gamma^m_{ki} = 0 \tag{91}$$

so permuting the indices and using that both $g_{ij} = g_{ji}$ and $\Gamma^i_{jk} = \Gamma^i_{kj}$ we find

$$\frac{\partial g_{jk}}{\partial \xi^i} = g_{mk}\Gamma^m_{ji} + g_{mj}\Gamma^m_{ki} \tag{92}$$

$$\frac{\partial g_{ij}}{\partial \xi^k} = g_{mi}\Gamma^m_{jk} + g_{mj}\Gamma^m_{ik} \tag{93}$$

$$-\frac{\partial g_{ki}}{\partial \xi^j} = -g_{mk}\Gamma^m_{ij} - g_{mi}\Gamma^m_{kj} \tag{94}$$

$$\Rightarrow$$

$$(92) + (93) + (94) = \frac{\partial g_{jk}}{\partial \xi^i} + \frac{\partial g_{ij}}{\partial \xi^k} - \frac{\partial g_{ki}}{\partial \xi^j} \tag{95}$$

$$= \overbrace{\cancel{g_{mk}\Gamma^m_{ji}}}^{A} + g_{mj}\Gamma^m_{ki} + \overbrace{\cancel{g_{mi}\Gamma^m_{jk}}}^{B} + g_{mj}\Gamma^m_{ik} - \overbrace{\cancel{g_{mk}\Gamma^m_{ij}}}^{A} - \overbrace{\cancel{g_{mi}\Gamma^m_{kj}}}^{B} \tag{96}$$

$$= 2 g_{mj}\Gamma^m_{ik} \tag{97}$$

$$\Rightarrow$$
$$g^{jp}g_{mj}\Gamma^m_{ik} = \frac{g^{jp}}{2}\left(\frac{\partial g_{jk}}{\partial \xi^i} + \frac{\partial g_{ij}}{\partial \xi^k} - \frac{\partial g_{ki}}{\partial \xi^j}\right) \tag{98}$$
$$\delta^p_m\Gamma^m_{ik} = \Gamma^p_{ik}, \tag{99}$$

now letting $p \to i$, $i \to j$, and $j \to m$ then we find

$$\Gamma^i_{jk} = \frac{g^{mi}}{2}\left(\frac{\partial g_{mk}}{\partial \xi^j} + \frac{\partial g_{jm}}{\partial \xi^k} - \frac{\partial g_{kj}}{\partial \xi^m}\right) \tag{100}$$
$$\Gamma^i_{jk} = \frac{g^{im}}{2}\left(\frac{\partial g_{mk}}{\partial \xi^j} + \frac{\partial g_{mj}}{\partial \xi^k} - \frac{\partial g_{jk}}{\partial \xi^m}\right) \tag{101}$$

which is (2.31) as desired.

[Note that this means

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial}{\partial \xi^j}\left[\frac{\partial x^\ell}{\partial \xi^m}\frac{\partial x^\ell}{\partial \xi^k}\right] + \frac{\partial}{\partial \xi^k}\left[\frac{\partial x^\ell}{\partial \xi^m}\frac{\partial x^\ell}{\partial \xi^j}\right] - \frac{\partial}{\partial \xi^m}\left[\frac{\partial x^\ell}{\partial \xi^j}\frac{\partial x^\ell}{\partial \xi^k}\right]\right) \tag{102}$$
$$= \frac{g^{im}}{2}\left(\overbrace{\frac{\partial x^\ell}{\partial \xi^m}\frac{\partial^2 x^\ell}{\partial \xi^j\partial \xi^k}}^{\text{A}} + \overbrace{\frac{\partial x^\ell}{\partial \xi^k}\frac{\partial^2 x^\ell}{\partial \xi^j\partial \xi^m}}^{\text{B}} + \overbrace{\frac{\partial x^\ell}{\partial \xi^m}\frac{\partial^2 x^\ell}{\partial \xi^k\partial \xi^j}}^{\text{A}} + \overbrace{\frac{\partial x^\ell}{\partial \xi^j}\frac{\partial^2 x^\ell}{\partial \xi^k\partial \xi^m}}^{\text{C}} - \overbrace{\frac{\partial x^\ell}{\partial \xi^j}\frac{\partial^2 x^\ell}{\partial \xi^m\partial \xi^k}}^{\text{C}} - \overbrace{\frac{\partial x^\ell}{\partial \xi^k}\frac{\partial^2 x^\ell}{\partial \xi^m\partial \xi^j}}^{\text{B}}\right)$$
$$= g^{im}\frac{\partial x^\ell}{\partial \xi^m}\frac{\partial^2 x^\ell}{\partial \xi^j\partial \xi^k} = \frac{\partial \xi^i}{\partial x^r}\frac{\partial \xi^m}{\partial x^r}\frac{\partial x^\ell}{\partial \xi^m}\frac{\partial^2 x^\ell}{\partial \xi^j\partial \xi^k} = \frac{\partial \xi^i}{\partial x^r}\overbrace{\frac{\partial x^\ell}{\partial x^r}}^{\delta^r_\ell}\frac{\partial^2 x^\ell}{\partial \xi^j\partial \xi^k} \tag{103}$$
$$\Gamma^i_{jk} = \frac{\partial \xi^i}{\partial x^\ell}\frac{\partial^2 x^\ell}{\partial \xi^j\partial \xi^k} \tag{104}$$

which can be used for Γ^i_{jk} rather than the definition given in Hazeltine.] There is an inconsistency in Hazeltine, we should have (and there was in previous editions)

$$A_{i,j} = \frac{\partial A_i}{\partial \xi^j} + A_k\Gamma^k_{ij} \tag{105}$$
$$A^i_{,j} = \frac{\partial A^i}{\partial \xi^j} - A_k\Gamma^i_{jk}. \tag{106}$$

with their definition of Γ^k_{ij}. We see this because here we have

$$\Gamma^k_{ij} = \frac{\partial x^\ell}{\partial \xi^i}\frac{\partial x^m}{\partial \xi^j}\frac{\partial^2 \xi^k}{\partial x^\ell\partial x^m} \tag{107}$$

while in virtually every other book we have

$$\Gamma^{k*}_{ij} = \frac{\partial \xi^k}{\partial x^\ell}\frac{\partial^2 x^\ell}{\partial \xi^i\partial \xi^j} = \frac{\partial \xi^k}{\partial x^\ell}\frac{\partial^2 x^\ell}{\partial \xi^j\partial \xi^i} \quad . \tag{108}$$

We find that

$$\Gamma^k_{ij} = -\Gamma^{k*}_{ij}. \tag{109}$$

The nonequivalency of these two definitions is seen by

$$\frac{\partial \xi^k}{\partial x^\ell}\frac{\partial^2 x^\ell}{\partial \xi^i \partial \xi^j} = \frac{\partial \xi^k}{\partial x^\ell}\frac{\partial}{\partial \xi^j}\left(\frac{\partial x^\ell}{\partial \xi^r}\frac{\partial \xi^r}{\partial x^m}\frac{\partial x^m}{\partial \xi^i}\right) \tag{110}$$

$$= \frac{\partial \xi^k}{\partial x^\ell}\left(\frac{\partial x^\ell}{\partial \xi^r}\frac{\partial \xi^r}{\partial x^m}\frac{\partial^2 x^m}{\partial \xi^j \partial \xi^i} + \frac{\partial x^\ell}{\partial \xi^r}\frac{\partial x^m}{\partial \xi^i}\frac{\partial^2 \xi^r}{\partial \xi^j \partial x^m} + \frac{\partial \xi^r}{\partial x^m}\frac{\partial x^m}{\partial \xi^i}\frac{\partial^2 x^\ell}{\partial \xi^j \partial \xi^r}\right) \tag{111}$$

$$= \frac{\partial \xi^k}{\partial x^\ell}\frac{\partial^2 x^\ell}{\partial \xi^j \partial \xi^i} + \frac{\partial \xi^k}{\partial x^\ell}\frac{\partial x^\ell}{\partial \xi^r}\frac{\partial x^s}{\partial \xi^j}\frac{\partial x^m}{\partial \xi^i}\frac{\partial^2 \xi^r}{\partial x^s \partial x^m} + \frac{\partial \xi^k}{\partial x^\ell}\frac{\partial \xi^r}{\partial x^m}\frac{\partial x^m}{\partial \xi^i}\frac{\partial^2 x^\ell}{\partial \xi^j \partial \xi^r} \tag{112}$$

$$\Rightarrow \tag{113}$$

$$\frac{\partial \xi^k}{\partial x^\ell}\frac{\partial x^\ell}{\partial \xi^r}\frac{\partial x^s}{\partial \xi^j}\frac{\partial x^m}{\partial \xi^i}\frac{\partial^2 \xi^r}{\partial x^s \partial x^m} = -\frac{\partial \xi^r}{\partial x^m}\frac{\partial x^m}{\partial \xi^i}\frac{\partial \xi^k}{\partial x^\ell}\frac{\partial^2 x^\ell}{\partial \xi^j \partial \xi^r} \tag{114}$$

$$\delta^k_r \Gamma^r_{ij} = -\delta^r_i \frac{\partial^2 x^\ell}{\partial \xi^j \partial \xi^r} \tag{115}$$

$$\Gamma^k_{ij} = -\frac{\partial^2 x^\ell}{\partial \xi^j \partial \xi^i} = -\Gamma^{k*}_{ij}. \tag{116}$$

The divergence comes from

$$\nabla \cdot \mathbf{A} = \frac{\partial A^i}{\partial \xi^i} + \Gamma^i_{ik} A^k = \frac{\partial A^i}{\partial \xi^i} + \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial \xi^i} + \frac{\partial g_{mi}}{\partial \xi^k} - \frac{\partial g_{ik}}{\partial \xi^m}\right) A^k \tag{117}$$

Now because of the symmetry $g^{ij} = g^{ji}$ we have

$$g^{im}\frac{\partial g_{mk}}{\partial \xi^i} \overset{i\leftrightarrow m}{=} g^{im}\frac{\partial g_{ik}}{\partial \xi^m} \tag{118}$$

and so (117) reduces to

$$\frac{\partial A^i}{\partial \xi^i} + \frac{1}{2} g^{im}\frac{\partial g_{im}}{\partial \xi^k} A^k. \tag{119}$$

Now using that $(g_{ij})^{-1}$ (*i.e.* the inverse of g_{ij}) is g^{ij}. We also note by linear algebra that

$$(g^{ij})^{-1} = \frac{C^{ij}}{g} \tag{120}$$

$$g = \sum_i g_{ij} C^{ij} \tag{121}$$

where $g = \det g_{ij}$ and C^{ij} is the cofactor matrix of g^{ij}. Looking more closely at this we then find

$$\frac{\partial g}{\partial g_{ij}} = \frac{\partial}{\partial g_{ij}} \sum_i g_{ij} C^{ij} = \sum_i \left[\frac{\partial g_{ij}}{\partial g_{ij}} C^{ij} + g_{ij}\cancel{\frac{\partial C^{ij}}{\partial g_{ij}}}\right] = C^{ij} \tag{122}$$

So that altogether

$$g^{ij} = \frac{1}{g}\frac{\partial g}{\partial g_{ij}} \tag{123}$$

and so (119) reduces to ($g = \mathcal{J}^{-2}$)

$$\frac{\partial A^i}{\partial \xi^i} + \frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial g_{im}}\frac{\partial g_{im}}{\partial \xi^k}A^k = \frac{\partial A^i}{\partial \xi^i} + \frac{1}{2g}\frac{\partial g}{\partial \xi^k}A^k \tag{124}$$

$$\overset{k\to i}{=} \frac{\partial A^i}{\partial \xi^i} + \frac{\mathcal{J}^2}{2}\frac{\partial \mathcal{J}^{-2}}{\partial \xi^i}A^i = \frac{\partial A^i}{\partial \xi^i} + \frac{\mathcal{J}^2}{2}2\mathcal{J}^{-1}\frac{\partial \mathcal{J}^{-1}}{\partial \xi^i}A^i \tag{125}$$

$$= \frac{\partial A^i}{\partial \xi^i} + \mathcal{J}A^i\frac{\partial \mathcal{J}^{-1}}{\partial \xi^i} = \mathcal{J}\frac{\partial}{\partial \xi^i}\left(\frac{A^i}{\mathcal{J}}\right) \tag{126}$$

proving (2.32) as desired.

3 Ordinary, Covariant, and Contravariant Components in Coordinate Systems Compute the ordinary, covariant, and contravariant components of the vector field $\mathbf{A} = \hat{\mathbf{z}}$ in cylindrical coordinates and primitive toroidal coordinates.

$$\begin{aligned} R &= \sqrt{x^2 + y^2} \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \\ Z &= z \end{aligned} \qquad \text{(cylindrical coordinates)}$$

$$\begin{aligned} r &= \sqrt{(R - R_0)^2 + Z^2} \\ \theta &= \tan^{-1}\left(\frac{Z}{R - R_0}\right) \\ \zeta &= -\varphi. \end{aligned} \qquad \text{(toroidal coordinates)}$$

Solution:

Let's begin with cylindrical coordinates. It is simple to see that

$$\hat{\mathbf{z}} = \hat{\mathbf{Z}} + 0\hat{\mathbf{R}} + 0\hat{\boldsymbol{\varphi}} \tag{127}$$

in ordinary vectors.

The contravariant components are given from $A^i = \mathbf{A} \cdot \nabla\xi^i$ so that

$$A^R = \hat{\mathbf{z}} \cdot \nabla R = \hat{\mathbf{z}} \cdot \left(\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}\right) = 0 \tag{128}$$

$$A^\varphi = \hat{\mathbf{z}} \cdot \nabla\varphi = \hat{\mathbf{z}} \cdot \nabla\left[\tan^{-1}\left(\frac{y}{x}\right)\right] \tag{129}$$

$$= \hat{\mathbf{z}} \cdot \left(\hat{\mathbf{x}}\frac{-y}{x^2 + y^2} + \hat{\mathbf{y}}\frac{x}{x^2 + y^2}\right) = 0 \tag{130}$$

$$A^Z = \hat{\mathbf{z}} \cdot \nabla Z = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1. \tag{131}$$

Leading to absolutely no surprises.

I find it easiest to get the covariant quantities by finding the metric tensor g_{ij}. It is easy from the definition of (cylindrical coordinates) to get g^{ij}, which then just needs to be inverted.

$$g^{RR} = \frac{\partial R}{\partial x}\frac{\partial R}{\partial x} + \frac{\partial R}{\partial y}\frac{\partial R}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial R}{\partial z} \tag{132}$$

$$= \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 0 = 1 \tag{133}$$

$$g^{R\varphi} = \frac{\partial R}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial R}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial \varphi}{\partial z} \tag{134}$$

$$= \frac{x}{\sqrt{x^2 + y^2}}\frac{-y}{x^2 + y^2} + \frac{y}{\sqrt{x^2 + y^2}}\frac{x}{x^2 + y^2} + 0 = 0 \tag{135}$$

$$g^{RZ} = \frac{\partial R}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial R}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial Z}{\partial z} = 0 \tag{136}$$

$$g^{\varphi\varphi} = \frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial z}\frac{\partial\varphi}{\partial z} \tag{137}$$

$$= \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} + 0 = \frac{1}{x^2+y^2} \tag{138}$$

$$g^{\varphi Z} = \frac{\partial\varphi}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial\varphi}{\partial z}\frac{\partial Z}{\partial z} = 0 \tag{139}$$

$$g^{ZZ} = \frac{\partial Z}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z}\frac{\partial Z}{\partial z} = 1 \tag{140}$$

Seeing that this is a diagonal matrix then $g_{ij} = \frac{1}{g^{ij}}$ and so the covariant components are

$$A_R = g_{RR}A^R = 0 \tag{141}$$

$$A_\varphi = g_{\varphi\varphi}A^\varphi = 0 \tag{142}$$

$$A_Z = g_{ZZ}A^Z = 1. \tag{143}$$

Again nothing too surprising.

Now let's do the primitive toroidal coordinates.

In ordinary vectors we would simply have

$$\hat{\mathbf{z}} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}. \tag{144}$$

It is easy to see this always points in the $\hat{\mathbf{z}}$ direction by looking at the angles $\theta = 0, \pi/2$.

Now the contravariant components are given by

$$A^r = \hat{\mathbf{z}}\cdot\nabla r = \hat{\mathbf{z}}\cdot\left(\hat{\mathbf{x}}\frac{(R-R_0)\frac{\partial R}{\partial x} + Z\cancel{\frac{\partial Z}{\partial x}}}{\sqrt{(R-R_0)^2+Z^2}} + \hat{\mathbf{y}}\frac{(R-R_0)\frac{\partial R}{\partial y} + Z\cancel{\frac{\partial Z}{\partial y}}}{\sqrt{(R-R_0)^2+Z^2}} + \hat{\mathbf{z}}\frac{(R-R_0)\cancel{\frac{\partial R}{\partial z}} + Z\frac{\partial Z}{\partial z}}{\sqrt{(R-R_0)^2+Z^2}}\right) \tag{145}$$

$$\boxed{A^r = \frac{Z}{\sqrt{(R-R_0)^2+Z^2}} = \frac{Z}{r}} \tag{146}$$

$$A^\theta = \hat{\mathbf{z}}\cdot\nabla\theta = \hat{\mathbf{z}}\cdot\nabla\left[\tan^{-1}\left(\frac{Z}{R-R_0}\right)\right] \tag{147}$$

$$= \hat{\mathbf{z}}\cdot\left(\hat{\mathbf{x}}\frac{(R-R_0)\cancel{\frac{\partial Z}{\partial x}} - Z\frac{\partial(R-R_0)}{\partial x}}{(R-R_0)^2+Z^2} + \hat{\mathbf{y}}\frac{(R-R_0)\cancel{\frac{\partial Z}{\partial y}} - Z\frac{\partial(R-R_0)}{\partial y}}{(R-R_0)^2+Z^2} + \hat{\mathbf{z}}\frac{(R-R_0)\frac{\partial Z}{\partial z} - Z\cancel{\frac{\partial(R-R_0)}{\partial z}}}{(R-R_0)^2+Z^2}\right) \tag{148}$$

$$\boxed{A^\theta = \frac{R-R_0}{(R-R_0)^2+Z^2}} \tag{149}$$

$$\boxed{A^\zeta = \hat{\mathbf{z}}\cdot\nabla\zeta = -\hat{\mathbf{z}}\cdot\nabla\varphi = 0.} \tag{150}$$

Now I still think it's easiest to get covariant components with the metric tensor, although this is more labor-intensive than cylindrical coordinates.

$$g^{rr} = \frac{\partial r}{\partial x}\frac{\partial r}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial r}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial r}{\partial z} \tag{151}$$

$$= \frac{(R-R_0)^2\frac{x^2}{x^2+y^2}}{(R-R_0)^2+Z^2} + \frac{(R-R_0)^2\frac{y^2}{x^2+y^2}}{(R-R_0)^2+Z^2} + \frac{Z^2}{(R-R_0)^2+Z^2} \tag{152}$$

$$\boxed{g^{rr} = \frac{(R-R_0)^2+Z^2}{(R-R_0)^2+Z^2} = 1} \tag{153}$$

$$g^{r\theta} = \frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \theta}{\partial z} \tag{154}$$

$$= \frac{(R-R_0)\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{(R-R_0)^2+Z^2}}\frac{-Z\frac{x}{\sqrt{x^2+y^2}}}{(R-R_0)^2+Z^2} + \frac{(R-R_0)\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{(R-R_0)^2+Z^2}}\frac{-Z\frac{y}{\sqrt{x^2+y^2}}}{(R-R_0)^2+Z^2}$$
$$+ \frac{Z}{\sqrt{(R-R_0)^2+Z^2}}\frac{R-R_0}{(R-R_0)^2+Z^2} \tag{155}$$

$$\boxed{g^{r\theta} = Z(R-R_0)\frac{\frac{-x^2}{x^2+y^2} + \frac{-y^2}{x^2+y^2} + 1}{[(R-R_0)^2+Z^2]^{3/2}} = 0} \tag{156}$$

$$g^{r\zeta} = \frac{\partial r}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \zeta}{\partial z} \tag{157}$$

$$= \frac{(R-R_0)\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{(R-R_0)^2+Z^2}}\frac{y}{\sqrt{x^2+y^2}} + \frac{(R-R_0)\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{(R-R_0)^2+Z^2}}\frac{-x}{\sqrt{x^2+y^2}} + 0 \tag{158}$$

$$\boxed{g^{r\zeta} = \frac{R-R_0}{\sqrt{x^2+y^2}\sqrt{(R-R_0)^2+Z^2}}(xy-yx) = 0} \tag{159}$$

$$g^{\theta\theta} = \frac{\partial \theta}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial y}\frac{\partial \theta}{\partial y} + \frac{\partial \theta}{\partial z}\frac{\partial \theta}{\partial z} \tag{160}$$

$$= \frac{Z^2\frac{x^2}{x^2+y^2}}{[(R-R_0)^2+Z^2]^2} + \frac{Z^2\frac{y^2}{x^2+y^2}}{[(R-R_0)^2+Z^2]^2} + \frac{(R-R_0)^2}{[(R-R_0)^2+Z^2]^2} \tag{161}$$

$$\boxed{g^{\theta\theta} = \frac{(R-R_0)^2+Z^2}{[(R-R_0)^2+Z^2]^2} = \frac{1}{(R-R_0)^2+Z^2}} \tag{162}$$

$$g^{\theta\zeta} = \frac{\partial \theta}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \theta}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial \theta}{\partial z}\frac{\partial \zeta}{\partial z} \tag{163}$$

$$= \frac{-Z\frac{x}{\sqrt{x^2+y^2}}}{(R-R_0)^2+Z^2}\frac{y}{\sqrt{x^2+y^2}} + \frac{-Z\frac{y}{\sqrt{x^2+y^2}}}{(R-R_0)^2+Z^2}\frac{-x}{\sqrt{x^2+y^2}} + 0 \tag{164}$$

$$\boxed{g^{\theta\zeta} = \frac{-Z}{(x^2+y^2)[(R-R_0)^2+Z^2]}(xy-yx) = 0} \tag{165}$$

$$g^{\zeta\zeta} = \frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial \zeta}{\partial z}\frac{\partial \zeta}{\partial z} \tag{166}$$

$$\boxed{g^{\zeta\zeta} = \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} = \frac{1}{R^2}.} \tag{167}$$

Now we see all the off-diagonal elements are again zero so that $g_{ij} = \frac{1}{g^{ij}}$ again. And so the covariant components are given by

$$A_r = g_{rr}A^r = 1\frac{Z}{r} = \frac{Z}{r} \tag{168}$$

$$A_\theta = g_{\theta\theta}A^\theta = \left[(R - R_0)^2 + Z^2\right]\frac{(R - R_0)}{(R - R_0)^2 + Z^2} = R - R_0 \tag{169}$$

$$A_\zeta = g_{\zeta\zeta}A^\zeta = 0. \tag{170}$$

We note that the covariant, contravariant, and ordinary vector components look rather different, but they are consistent with

$$\mathbf{A} = A_i \, \nabla\xi^i \tag{171}$$

4 Curl of Vector Field

Compute the curl of $\mathbf{A} = r\,\nabla\zeta$.

Now we see that the covariant components of A are simply

$$A_r = 0 \quad , \quad , A_\theta = 0 \quad , \quad A_\zeta = r. \tag{172}$$

We have a formula for the contravariant components of $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ with

$$B^i = \mathcal{J}\epsilon_{ijk}\frac{\partial A_k}{\partial \xi^j}. \tag{173}$$

Now from the work of (151) to (167) we see that

$$g = \det g_{ij} = (R-R_0)^2 + Z^2 = r^2 \tag{174}$$

$$\mathcal{J}^2 = g^{-1} \Rightarrow \mathcal{J} = \frac{1}{r}. \tag{175}$$

Now we only need to calculate $\epsilon_{ijk}\frac{\partial A_k}{\partial \xi^j}$ which is extremely simplified thanks to A_ζ being the only nonzero coordinate (let $r \to 1$, $\theta \to 2$, and $\zeta \to 3$ for the Levi-Civita symbol).

$$B^r = \epsilon_{r\theta\zeta}\frac{\partial A_\zeta}{\partial \theta} + \epsilon_{r\zeta\theta}\frac{\partial A_\theta}{\partial \zeta} = \frac{\partial A_\zeta}{\partial \theta} - 0 = \frac{\partial A_\zeta}{\partial \theta} \tag{176}$$

$$B^\theta = \epsilon_{\theta\zeta r}\frac{\partial A_r}{\partial \zeta} + \epsilon_{\theta r\zeta}\frac{\partial A_\zeta}{\partial r} = 0 - \frac{\partial A_\zeta}{\partial r} = -\frac{\partial A_\zeta}{\partial r} \tag{177}$$

$$B^\zeta = \epsilon_{\zeta r\theta}\frac{\partial A_r}{\partial \theta} + \epsilon_{\zeta\theta r}\frac{\partial A_\theta}{\partial r} = 0 - 0 = 0 \tag{178}$$

Given that $A_\zeta = r$ we see that (using the chain rule for partial differentiation which can recycle results from $g^{r\theta}$ in (151) to (167))

$$\frac{\partial A_\zeta}{\partial \theta} = \frac{\partial r}{\partial \theta} = \frac{\partial r}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial \theta} \tag{179}$$

$$= \frac{(R-R_0)\frac{x}{\sqrt{x^2+y^2}}}{r}\frac{r^2}{\frac{-Zx}{\sqrt{x^2+y^2}}} + \frac{(R-R_0)\frac{y}{\sqrt{x^2+y^2}}}{r}\frac{r^2}{\frac{-yx}{\sqrt{x^2+y^2}}} + \frac{Z}{r}\frac{r^2}{(R-R_0)} \tag{180}$$

$$= \frac{(R-R_0)r}{-Z} + \frac{(R-R_0)r}{-Z} + \frac{Zr}{R-R_0} = r\left(\frac{Z}{R-R_0} - \frac{2(R-R_0)}{Z}\right) \tag{181}$$

$$\frac{\partial A_\zeta}{\partial \theta} = r\frac{Z^2 - 2(R-R_0)^2}{Z(R-R_0)} \tag{182}$$

$$\frac{\partial A_\zeta}{\partial r} = \frac{\partial r}{\partial r} = 1. \tag{183}$$

So we find that

$$\boxed{\begin{aligned} B^r &= \left(\frac{1}{r}\right)\left(r\frac{Z^2 - 2(R-R_0)^2}{Z(R-R_0)}\right) = \frac{Z^2 - 2(R-R_0)^2}{Z(R-R_0)} \\ B^\theta &= \frac{1}{r}(-1) = \frac{-1}{r} \\ B^\zeta &= 0 \end{aligned}} \tag{184}$$

5 Show Symplectic Form and $\mathbf{E} \times \mathbf{B}$ drift

Suppose the action is given by

$$S = \int \frac{e}{c}\mathbf{A} \cdot \,\mathrm{d}\mathbf{x} - e\varphi \,\mathrm{d}t \quad . \tag{185}$$

Show that the symplectic form is the antisymmetric matrix $\epsilon_{ijk}B^k/(\mathcal{J}c)$ and that the equations of motion give the $E \times B$ drift.

Solution:

We may rewrite this as

$$S = \int \left[\frac{e}{c}\mathbf{A} \cdot \dot{\mathbf{x}} - e\varphi\right] \,\mathrm{d}t \tag{186}$$

to see that this gives a much better form for integration. We may then interpret our answer as

$$S = \int \left[\mathcal{A} \cdot \dot{\mathbf{x}} - \mathcal{H}\right] \,\mathrm{d}t \tag{187}$$

and so

$$\mathcal{H}(\mathbf{x}) = e\varphi \tag{188}$$

$$\mathcal{A}(\mathbf{x}) = \frac{e}{c}\mathbf{A} \tag{189}$$

which then allows us to use the symplectic form as

$$\omega_{ij} = \frac{\partial \mathcal{A}_j}{\partial x^i} - \frac{\partial \mathcal{A}_i}{\partial x^j} = \epsilon_{km\ell}\frac{\partial \mathcal{A}_\ell}{\partial x^m} \tag{190}$$

where m and ℓ both run through i, j, k with $k \neq i, j$ and $\epsilon_{kij} = 1$. It is easy then to see from

$$\mathcal{B}^i = \mathcal{J}\epsilon_{ijk}\frac{\partial \mathcal{A}_k}{\partial \xi^j} \tag{191}$$

$$\Rightarrow$$

$$\epsilon_{km\ell}\frac{\partial \mathcal{A}_\ell}{\partial x^m} = \frac{\mathcal{B}^k}{\mathcal{J}} \tag{192}$$

$$\epsilon_{kij}\epsilon_{km\ell}\frac{\partial \mathcal{A}_\ell}{\partial x^m} = \epsilon_{kij}\frac{\mathcal{B}^k}{\mathcal{J}} \tag{193}$$

$$(\delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{mj})\frac{\partial \mathcal{A}_\ell}{\partial x^m} = \epsilon_{kij}\frac{\mathcal{B}^k}{\mathcal{J}} \tag{194}$$

$$\frac{\partial \mathcal{A}_j}{\partial x^i} - \frac{\partial \mathcal{A}_i}{\partial x^j} = \epsilon_{ijk}\frac{\mathcal{B}^k}{\mathcal{J}}. \tag{195}$$

Note that the above identity is true regardless of our k which can be shown by using

$$\omega_{ij} = \epsilon_{ijk}\frac{\mathcal{B}^k}{\mathcal{J}} = \epsilon_{ijk}\epsilon_{km\ell}\frac{\mathcal{J}}{\mathcal{J}}\frac{\partial \mathcal{A}_\ell}{\partial x^m} = (\delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{mj})\frac{\partial \mathcal{A}_\ell}{\partial x^m} \tag{196}$$

$$= \frac{\partial \mathcal{A}_j}{\partial x^i} - \frac{\partial \mathcal{A}_i}{\partial x^j}. \tag{197}$$

So that

$$\omega_{ij} = \epsilon_{ijk} \frac{\mathcal{B}^k}{\mathcal{J}} \tag{198}$$

Now because $\mathcal{A} = \frac{e}{c}\mathbf{A}$ then we may interpret (where B^i are the contravariant components of the magnetic field $\mathbf{B}$.)

$$\mathcal{B}^i = \mathcal{J}\epsilon_{ijk} \frac{\partial \mathcal{A}_k}{\partial x^j} \tag{199}$$

$$\mathcal{B}^i = \mathcal{J}\epsilon_{ijk} \frac{\partial A_k}{\partial x^j}\frac{e}{c} = B^i \frac{e}{c}. \tag{200}$$

and so

$$\boxed{\omega_{ij} = \epsilon_{ijk} \frac{eB^k}{\mathcal{J}c}} \tag{201}$$

Which I assume was normalized as

$$\omega_{ij}\dot{x}^j = \frac{\partial \mathcal{H}}{\partial x^i} \tag{202}$$

$$\Rightarrow$$

$$\epsilon_{ijk} \frac{eB^k}{\mathcal{J}c}\dot{x}^j = e\frac{\partial \varphi}{\partial x^i} \tag{203}$$

$$\epsilon_{ijk} \frac{B^k}{\mathcal{J}c}\dot{x}^j = \frac{\partial \varphi}{\partial x^i} \tag{204}$$

In which the electric charge was cancelled out as it was common to both terms.

In any case the equation of motion is then (letting $E_i = -\frac{\partial \varphi}{\partial x^i}$)

$$\epsilon_{ijk} \frac{B^k}{\mathcal{J}c}\dot{x}^j = \frac{\partial \varphi}{\partial x^i} = -E_i \tag{205}$$

$$\epsilon_{ijk}\epsilon_{\ell mi} B^k B_m v^j = -\mathcal{J}c\epsilon_{\ell mi} B_m E^i = \mathcal{J}c\epsilon_{\ell im} E_i B_m \tag{206}$$

$$\left(\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{\ell k}\right) B^k B_m v^j = \mathcal{J}c\epsilon_{\ell im} E_i B_m \tag{207}$$

$$B_m B^m v_\ell - B_\ell B_j v^j = \mathcal{J}c\epsilon_{\ell im} E_i B_m \tag{208}$$

$$B^2\mathbf{v} - \mathbf{B}\left(\mathbf{B}\cdot\mathbf{v}\right) = c\mathbf{E}\times\mathbf{B}, \tag{209}$$

and so we see that in the perpendicular direction to $\mathbf{B}$ we have

$$B^2\mathbf{v}_\perp = c\mathbf{E}\times\mathbf{B} \tag{210}$$

$$\boxed{\mathbf{v}_\perp = \frac{c\mathbf{E}\times\mathbf{B}}{B^2}} \tag{211}$$

as desired.

6 Charged Particle Motion in Magnetic Field Discuss the motion of a charged particle in the magnetic field $\mathbf{B} = x\hat{\mathbf{z}}$. Use of the Hamiltonian (2.48) may prove advantageous. Compare your results to the guiding-center equations, and explain the limitations of the latter.

$$\mathcal{H} = \frac{1}{2m}\left|\mathbf{p} - \frac{e}{c}\mathbf{A}\right|^2 + e\Phi \tag{2.48}$$

Solution:

Let's first find a possible $\mathbf{A}$ from the equation

$$\nabla \times \mathbf{A} = \mathbf{B} = x\hat{\mathbf{z}} : \tag{212}$$
$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0 \tag{213}$$
$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0 \tag{214}$$
$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x \quad . \tag{215}$$

And looking at the last equation we see that

$$A_y - \int \frac{\partial A_x}{\partial y}\,\mathrm{d}x = \frac{x^2}{2} + \text{ const.} \tag{216}$$

So we see that $A_y = \frac{x^2}{2}$ will work just fine. As an aside, I'm bothered by the inconsistent units and so will have $x \to \alpha x$ where αx has the units of magnetic field.

Then we have the Hamiltonian in the form

$$\mathcal{H} = \frac{1}{2m}\left|\mathbf{p} - \frac{e}{c}\mathbf{A}\right|^2 = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2 + \frac{e^2}{c^2}A_y^2 - 2\frac{e}{c}A_y p_y\right) \tag{217}$$
$$\mathcal{H} = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{e^2\alpha^4}{8mc^2}x^4 - \frac{e\alpha^2}{2mc}x^2 p_y \tag{218}$$

Hence we have for Hamilton's equations

$$\dot{x}^i = \frac{\partial \mathcal{H}}{\partial p^i} : \tag{219}$$
$$\dot{x} = \frac{p_x}{m} \tag{220}$$
$$\dot{y} = \frac{p_y}{m} - \frac{e\alpha^2}{2mc}x^2 \tag{221}$$
$$\dot{z} = \frac{p_z}{m} \tag{222}$$

and

$$-\dot{p}^i = \frac{\partial \mathcal{H}}{\partial x^i} : \tag{223}$$
$$\dot{p}_x = \frac{e^2\alpha^4}{2mc^2}x^3 - \frac{e\alpha^2 p_y}{mc}x \tag{224}$$
$$\dot{p}_y = 0 \tag{225}$$
$$\dot{p}_z = 0. \tag{226}$$

The last two equations imply that $p_y = p_{y_0}$ and $p_z = p_{z_0}$ are constants and are whatever their initial value is. The remaining equations then state that

$$\dot{x} = \frac{p_x}{m} \tag{227}$$

$$\dot{p}_x = \frac{e^2\alpha^4}{2mc^2}x^3 - \frac{e\alpha^2 p_{y_0}}{mc}x \tag{228}$$

$$\dot{y} = \frac{p_{y_0}}{m} - \frac{e\alpha^2}{2mc}x^2 \tag{229}$$

$$\dot{z} = \frac{p_{z_0}}{m} \Rightarrow z = \frac{p_{z_0}}{m}t + z_0 \quad . \tag{230}$$

We can get an interesting equation by taking (228) over (227) to find

$$\frac{\frac{\mathrm{d}p_x}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\mathrm{d}p_x}{\mathrm{d}x} = \frac{\frac{e^2\alpha^4}{2mc^2}x^3 - \frac{e\alpha^2 p_{y_0}}{mc}x}{\frac{p_x}{m}} \tag{231}$$

$$= \frac{\overbrace{\frac{e^2\alpha^4}{2c^2}}^{\gamma}x^3 - \overbrace{\frac{e\alpha^2 p_{y_0}}{c}}^{\delta}x}{p_x} \tag{232}$$

$$\int p_x\,\mathrm{d}p = \int \left(\gamma x^3 - \delta x\right)\,\mathrm{d}x \tag{233}$$

$$\frac{1}{2}p_x^2 + \frac{\delta}{2}x^2 - \frac{\gamma}{4}x^4 = \frac{\mathcal{E}}{2} \tag{234}$$

where $\mathcal{E}/2$ is some constant. These give phase space trajectories. Solving for p_x we find

$$p_x = \pm\sqrt{\mathcal{E} - \delta x^2 + \frac{\gamma}{2}x^4}. \tag{235}$$

These can be plotted for various $\mathcal{E}$ yielding Figure 2.

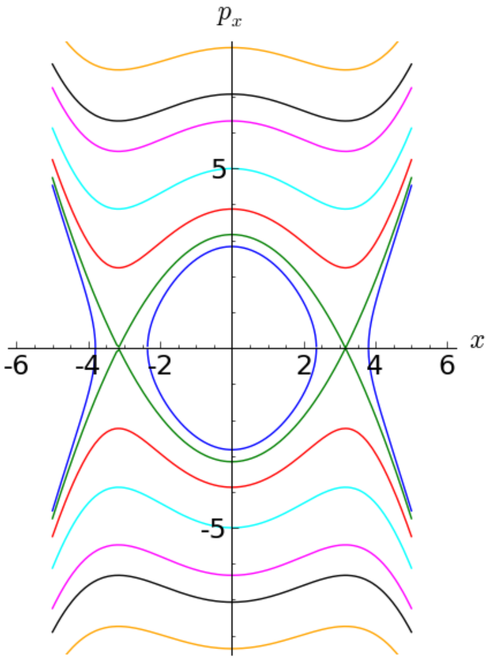


Figure 2: Phase space trajectory for various E for $\delta = 2$ and $\gamma = .1$ so that the separatrix is at $E = \frac{\delta^2}{4\gamma} = \frac{4}{.4} = 10$.

Then we see that there are unbounded trajectories as well as bounded ones because of the separatrix.

Now for $\dot{y}$ we see that if $\frac{p_{y0}}{m} > \frac{e\alpha^2}{2mc}x^2$ for any values then it is possible for y to oscillate. If $\frac{p_{y0}}{m} = 0$ then $\dot{y}$ will simply explode in the negative direction and be completely unbounded. This means that it is possible for y to oscillate if x is bounded, but otherwise will surely explode as well.

We can see this as for a particular p_y (using the same δ and γ as before) we can plot $\dot{y}$-x and $\dot{y}$-p_x for various values of E. These result in figures 3a and 3b. The different E only move the central point of the hyperbolas up or down the $\dot{y}$ axis, just as the different p_y move the parabola's $\dot{y}$ intercept up and down.

The guiding centers state

$$\mathbf{u}_{0\perp} = c\frac{\mathbf{E}\times\mathbf{B}}{B^2} \tag{236}$$

$$m\frac{\mathrm{d}u_{0\parallel}}{\mathrm{d}t} = -\mu\hat{\mathbf{b}}\cdot\nabla B(\mathbf{X}) \tag{237}$$

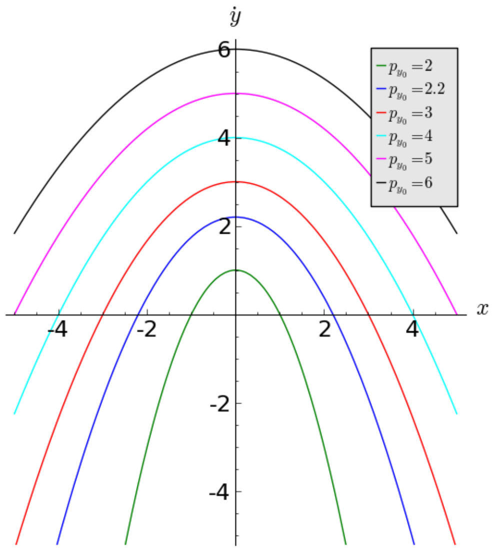
$$\mathbf{u}_{1\perp} = \frac{1}{\Omega B}\mathbf{B}\times\frac{\mathrm{d}\mathbf{u}_0}{\mathrm{d}t} + \frac{\mu}{m\Omega}\hat{\mathbf{b}}\times\nabla B \tag{238}$$

where $\mathbf{u}$ is the guiding center velocity, and $\mathbf{X}$ is the guiding center position. For our given $\mathbf{B}$ with no $\mathbf{E}$ we see that

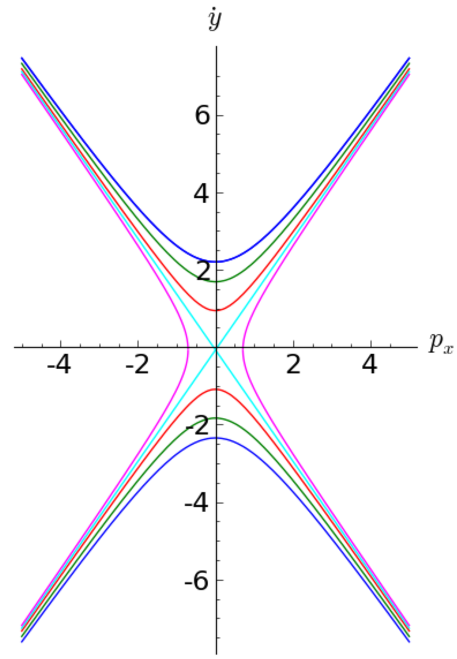
$$\mathbf{u}_{0\perp} = 0 \tag{239}$$

$$\frac{\mathrm{d}u_{0\parallel}}{\mathrm{d}t} = -\mu\hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0 \tag{240}$$

$$u_{0\parallel} = u_0 \tag{241}$$



(a) Plot of x and y for various p_{y_0} and $m = 1$.



(b) Plot for $p_{y_0} = 2.2$, $m = 1$ for various E. We see that the curves morph from parabolas to hyperbolas.

$$\mathbf{u}_{1\perp} = \frac{1}{\Omega B}\mathbf{B} \times \frac{\mathrm{d}\mathbf{u}_0}{\mathrm{d}t} + \frac{\mu}{m\Omega}\hat{\mathbf{b}} \times \nabla B \tag{242}$$

$$= \frac{1}{\Omega B} x\hat{\mathbf{z}} \times u_0\hat{\mathbf{z}} + \frac{\mu}{m\Omega}\hat{\mathbf{z}} \times \hat{\mathbf{x}} = -\frac{\mu}{m\Omega}\hat{\mathbf{y}} \tag{243}$$

and so it really poorly captures the behavior. It does get the parallel direction correct, but fails to capture the $\hat{\mathbf{x}}$ direction and also suggests that $\dot{y}$ is constant while it depends heavily on x. The greatest limitation is that $|\nabla B| = 1$ and so for the region where $x < 1$ we have $\frac{|\nabla B|}{B} = \frac{1}{x} > 1$, and so it is a poor description as the particle isn't necessarily constrained as we have seen above.