1 General Idea

Given a differential equation, $y(x)$ like

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\lambda^2 y \tag{1}$$

and the given solution $y = A\cos(\lambda x) + B\sin(\lambda x)$, it is possible to generate the differential equations for more general solutions, like $x^\alpha(A\cos(\beta x^\gamma) + B\sin(\beta x^\gamma))$.

This is useful for more complicated differential equations as it gives us a simple way to get the answer.

I will compile a list of such generalized differential equations and solutions below, but first let's outline the approach. It is easiest with linear differential equations, though my final example will show it is possible to use this method with nonlinear differential equations.

Start with a linear differential equation of the form

$$\sum_n a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0 \tag{2}$$

with the known solution $y = f(x)$. We then generalize the solution by using the trial solution $f_s(x) = x^\alpha f(x)$ into the differential equation. This will yield

$$\sum_n a_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^\alpha f(x)\right) \tag{3}$$

This will have extra terms compared to (2). In fact we can use that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^\alpha f(x)\right) = \sum_{l=0}^{n}\binom{n}{l}(\alpha)_l x^{\alpha-l}\frac{\mathrm{d}^{n-l}f}{\mathrm{d}x^{n-l}} = \sum_{l=0}^{n}\binom{n}{l}\frac{\alpha!}{(\alpha-l)!}x^{\alpha-l}\frac{\mathrm{d}^{n-l}f}{\mathrm{d}x^{n-l}} \tag{4}$$

with $(\alpha)_l$ the falling factorial and $\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$ corresponding to n choose k. We see that this implies that the $l = 0$ term yields

$$\cancelto{1}{\frac{n!}{0!n!}(\alpha)_0}\, x^\alpha\frac{\mathrm{d}^n f}{\mathrm{d}x^n} \tag{5}$$

Note that we then have

$$\sum_n a_n(x)\sum_{l=0}^{n}\binom{n}{l}(\alpha)_l\frac{\mathrm{d}^{n-l}f}{\mathrm{d}x^{n-l}} = \sum_n a_n\binom{n}{0}(\alpha)_0 x^\alpha\frac{\mathrm{d}^n f}{\mathrm{d}x^n} + \sum_n a_n\sum_{l=1}^{n}\binom{n}{l}(\alpha)_l x^{\alpha-l}\frac{\mathrm{d}^{n-l}f}{\mathrm{d}x^{n-l}} \tag{6}$$

$$= x^\alpha\cancel{\sum_n a_n\frac{\mathrm{d}^n f}{\mathrm{d}x^n}} + \sum_n a_n\sum_{l=1}^{n}\binom{n}{l}(\alpha)_l x^{\alpha-l}\frac{\mathrm{d}^{n-l}f}{\mathrm{d}x^{n-l}} \tag{7}$$

$$= \sum_n a_n\sum_{l=1}^{n}\binom{n}{l}(\alpha)_l x^{\alpha-l}\frac{\mathrm{d}^{n-l}f}{\mathrm{d}x^{n-l}} \tag{8}$$

We can then rewrite the above in the form

$$\sum_{k=0}^{n_{\max}-1} b_k(x) \frac{\mathrm{d}^k y_s}{\mathrm{d}x^k} \tag{9}$$

where the $y_s = x^\alpha f(x)$ is used instead of $\frac{\mathrm{d}^k f}{\mathrm{d}x^k}$. This simply uses the relationships developed above.

And so the differential equation obeyed by $y_s = x^\alpha f(x)$ is given by (where we combine the top equation into a single bottom equation)

$$\sum_n a_n \frac{\mathrm{d}^n y_s}{\mathrm{d}x^n} - \sum_k b_k(x) \frac{\mathrm{d}^k y_s}{\mathrm{d}x^k} = 0 \tag{10}$$

$$\sum_n c_n \frac{\mathrm{d}^n y_s}{\mathrm{d}x^n} = 0 \tag{11}$$

Now we move on to putting in the $f(\beta x^\gamma)$ dependence. Then we simply use the chain rule with $f_s(\beta x^\gamma) \equiv g(t)$ and so $t = \beta x^\gamma$ and so we have

$$\sum_k d_k(t) \frac{\mathrm{d}^k g(t)}{\mathrm{d}t^k} = 0 \tag{12}$$

(where we note that $x^\alpha \to \frac{t^{\alpha/\gamma}}{\beta^{\alpha/\gamma}}$). This is the same as (11) in form.

We then switch the $\frac{\mathrm{d}^l}{\mathrm{d}t^l}$ terms to terms involving $\frac{\mathrm{d}^s}{\mathrm{d}x^s}$ and we have the fully generalized answer.

One can note that one way of using the chain rule is to use

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(g(x)) = \sum \frac{n!}{(m_1!)(1!)^{m_1}(m_2!)(2!)^{m_2}\cdots(m_n!)(n!)^{m_n}} \frac{\mathrm{d}^{(m_1+\cdots+m_n)} f(g)}{\mathrm{d}g^{(m_1+\cdots+m_n)}} \prod_{j=1}^{n} \left(\frac{\mathrm{d}^j g}{\mathrm{d}x^j}\right)^{m_j} \tag{13}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(g(x)) = \sum \frac{n!}{(m_1!)(m_2!)\cdots(m_n!)} \frac{\mathrm{d}^{(m_1+\cdots+m_n)} f(g)}{\mathrm{d}g^{(m_1+\cdots+m_n)}} \prod_{j=1}^{n} \left(\frac{1}{j!}\frac{\mathrm{d}^j g}{\mathrm{d}x^j}\right)^{m_j} \tag{14}$$

with $m_1, \ldots, m_n$ being nonnegative integers satisfying

$$1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n \tag{15}$$

This is called Faà di Bruno's formula.

The following derivations will elucidate the method.

2 Harmonic Oscillator Differential Equation

Let's use

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\lambda y \tag{16}$$

and the given solution $y = A\cos(x) + B\sin(x)$. For simplicity, let's work with $y = \cos(x)$ alone, as everything will work equally well for $B\sin(x)$. We begin by solving for $y_s = x^\alpha y$. First we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(y_s) = \alpha x^{\alpha-1} y + x^\alpha \frac{\mathrm{d}y}{\mathrm{d}x} = \alpha \frac{y_s}{x} + x^\alpha \frac{\mathrm{d}y}{\mathrm{d}x} \tag{17}$$

and so

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}x^2} + \lambda y_s = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^\alpha y) + \lambda x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha x^{\alpha-1} y + x^\alpha \frac{\mathrm{d}y}{\mathrm{d}x}\right) + x^\alpha \lambda y \tag{18}$$

$$= \alpha(\alpha-1)x^{\alpha-2} y + 2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + \underbrace{x^\alpha \frac{\mathrm{d}^2 y}{\mathrm{d}x} + x^\alpha \lambda y}_{x^\alpha\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y\right) = 0} \tag{19}$$

Hence the extra terms are

$$\alpha(\alpha-1)x^{\alpha-2} y + 2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} \tag{20}$$

$$\alpha(\alpha-1)\frac{y_s}{x^2} + \frac{2\alpha}{x}\left(\frac{\mathrm{d}y_s}{\mathrm{d}x} - \alpha\frac{y_s}{x}\right) \tag{21}$$

$$\alpha(\alpha-1)\frac{y_s}{x^2} + \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{2\alpha^2 y_s}{x^2} \tag{22}$$

Hence the differential equation that has the solution $y_s = x^\alpha \cos(\lambda x)$ is given by

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}x^2} + \lambda y_s - \left[\alpha(\alpha-1)\frac{y_s}{x^2} + \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{2\alpha^2 y_s}{x^2}\right] = 0 \tag{23}$$

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}x^2} - \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} + \left(\lambda + \frac{2\alpha^2 - \alpha(\alpha-1)}{x^2}\right) y_s = 0 \tag{24}$$

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}x^2} - \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} + \left(\lambda + \frac{\alpha(\alpha+1)}{x^2}\right) y_s = 0 \tag{25}$$

Now to generalize to $y_t = x^\alpha \cos(\beta x^\gamma)$ we use that we can rewrite this as $y_t = t^{\alpha/\gamma} y_f(t) = x^{\alpha/\gamma} y_s(x)$ with $t = \beta x^\gamma$ (and so $x^\alpha \propto t^{\alpha/\gamma}$) (we only care about the dependence as multiplying by a constant doesn't change the differential equation) and so it satisfies

$$\frac{\mathrm{d}^2 y_f}{\mathrm{d}t^2} - \frac{2\frac{\alpha}{\gamma}}{t}\frac{\mathrm{d}y_f}{\mathrm{d}t} + \left(1 + \frac{\frac{\alpha}{\gamma}(\frac{\alpha}{\gamma}+1)}{t^2}\right) y_f = 0 \tag{26}$$

and now use that $\frac{\mathrm{d}t}{\mathrm{d}x} = \beta\gamma x^{\gamma-1}$ so that

$$\frac{\mathrm{d}y_f}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y_f}{\mathrm{d}x} = \frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} \tag{27}$$

$$\frac{\mathrm{d}^2 y_f}{\mathrm{d}t^2} = \frac{1}{\beta^2\gamma^2 x^{\gamma-1}}\left((1-\gamma)x^{-\gamma}\frac{\mathrm{d}y_f}{\mathrm{d}x} + \frac{1}{x^{\gamma-1}}\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2}\right) \tag{28}$$

and hence we find

$$\frac{\mathrm{d}^2 y_f}{\mathrm{d}t^2} - \frac{2\frac{\alpha}{\gamma}}{t}\frac{\mathrm{d}y_f}{\mathrm{d}t} + \left(1 + \frac{\frac{\alpha}{\gamma}(\frac{\alpha}{\gamma}+1)}{t^2}\right) y_f = 0 \tag{29}$$

$$\frac{1}{\beta^2\gamma^2 x^{2\gamma-2}}\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2} + \frac{1-\gamma}{\beta^2\gamma^2 x^{2\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} - \frac{2\frac{\alpha}{\gamma}}{\beta x^\gamma}\frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} + \left(1 + \frac{\frac{\alpha}{\gamma}\left(\frac{\alpha}{\gamma}+1\right)}{\beta^2 x^{2\gamma}}\right) y_f = 0 \tag{30}$$

$$\frac{1}{\beta^2\gamma^2 x^{2\gamma-2}}\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2} + \frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{(1-\gamma)-2\alpha}{\beta^2\gamma^2 x^{2\gamma-1}}\right) + \left(1 + \frac{\frac{\alpha}{\gamma}\left(\frac{\alpha}{\gamma}+1\right)}{\beta^2 x^{2\gamma}}\right) y_f = 0 \tag{31}$$

$$\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2} + \frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-\gamma-2\alpha}{x}\right) + \left(\beta^2\gamma^2 x^{2\gamma-2} + \frac{\alpha(\alpha+\gamma)}{x^2}\right) y_f = 0 \tag{32}$$

$$\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2} + \frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-\gamma-2\alpha}{x}\right) + \left(\beta^2\gamma^2 x^{2\gamma-2} + \frac{\alpha(\alpha+\gamma)}{x^2}\right) y_f = 0 \tag{33}$$

So we see that the generalized harmonic oscillator differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1-\gamma-2\alpha}{x}\right) + \left(\beta^2\gamma^2 x^{2\gamma-2} + \frac{\alpha(\alpha+\gamma)}{x^2}\right) y = 0 \tag{34}$$

has solutions

$$y = Ax^\alpha\cos(\beta x^\gamma) + Bx^\alpha\sin(\beta x^\gamma) \tag{35}$$

with A and B constants to fit initial conditions.

3 Bessel Differential Equation

The differential equation is given by

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-n^2)y=0 \tag{36}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1-\frac{n^2}{x^2}\right)y=0 \tag{37}$$

with solutions $y=AJ_n(x)+BJ_{-n}(x)$ for non-integer n and $y=AJ_n(x)+CY_n(x)$ for integer n with $Y_n(x)$ sometimes denoted $N_n(x)$ to avoid confusion with spherical harmonics. J_n is the nth order Bessel function of the first kind and $N_n(x)$ is the nth order Bessel function of the second kind with A, B, C and constants that allow fitting of boundary/initial conditions.

We see that

$$\frac{\mathrm{d}y_s}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\alpha y\right)=\alpha x^{\alpha-1}y+x^\alpha\frac{\mathrm{d}y}{\mathrm{d}x} \tag{38}$$

$$\frac{\mathrm{d}^2y_s}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha x^{\alpha-1}y+x^\alpha\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\alpha(\alpha-1)x^{\alpha-2}y+2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x}+x^\alpha\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \tag{39}$$

So let's put in $y_s(x)=x^\alpha y(x)$ into the left hand side of (59) and find

$$\frac{\mathrm{d}^2y_s}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x}+\left(1-\frac{n^2}{x^2}\right)y_s \tag{40}$$

$$=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(x^\alpha y\right)+\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\alpha y\right)+x^\alpha\left(1-\frac{n^2}{x^2}\right)y \tag{41}$$

$$=\left(\alpha(\alpha-1)x^{\alpha-2}y+2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x}+x^\alpha\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\frac{1}{x}\left(\alpha x^{\alpha-1}y+x^\alpha\frac{\mathrm{d}y}{\mathrm{d}x}\right)+x^\alpha\left(1-\frac{n^2}{x^2}\right)y \tag{42}$$

$$\underbrace{x^\alpha\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1-\frac{n^2}{x^2}\right)y\right)}_{=0}+2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x}+x^{\alpha-2}\left(\alpha(\alpha-1)+\alpha\right)y \tag{43}$$

$$=2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x}+x^{\alpha-2}\left(\alpha^2\right)y \tag{44}$$

$$=\frac{2\alpha}{x}\left(x^\alpha\frac{\mathrm{d}y}{\mathrm{d}x}\right)+x^{-2}\alpha^2(x^\alpha y)=\frac{2\alpha}{x}\left(\frac{\mathrm{d}y_s}{\mathrm{d}x}-\alpha\frac{y_s}{x}\right)+\frac{\alpha^2}{x^2}y_s \tag{45}$$

$$=\frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x}+\frac{\alpha^2-2\alpha^2}{x^2}y_s=\frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x}-\frac{\alpha^2}{x^2}y_s \tag{46}$$

Hence the differential equation for $y_s=x^\alpha y$ is given by

$$\frac{\mathrm{d}^2y_s}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x}+\left(1-\frac{n^2}{x^2}\right)y_s-\left[\frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x}-\frac{\alpha^2}{x^2}y_s\right]=0 \tag{47}$$

$$\frac{\mathrm{d}^2y_s}{\mathrm{d}x^2}+\frac{1-2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x}+\left(1+\frac{\alpha^2-n^2}{x^2}\right)y_s=0 \tag{48}$$

Now to generalize to $y_t=x^\alpha y(\beta x^\gamma)$ we use that we can rewrite this as $y_t=t^{\alpha/\gamma}y_f(t)=x^{\alpha/\gamma}y_s(x)$ with $t=\beta x^\gamma$ (and so $x^\alpha\propto t^{\alpha/\gamma}$) (we only care about the dependence as multiplying by a constant doesn't change the differential equation) and so it satisfies

$$\frac{\mathrm{d}^2 y_f}{\mathrm{d}t^2} + \frac{1-2\frac{\alpha}{\gamma}}{t}\frac{\mathrm{d}y_f}{\mathrm{d}t} + \left(1 + \frac{\frac{\alpha^2}{\gamma^2} - n^2}{t^2}\right) y_f = 0 \tag{49}$$

Using that $\frac{\mathrm{d}t}{\mathrm{d}x} = \beta\gamma x^{\gamma-1}$ with $t = \beta x^\gamma$ we have

$$\frac{\mathrm{d}y_f}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y_f}{\mathrm{d}x} = \frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} \tag{50}$$

$$\frac{\mathrm{d}^2 y_f}{\mathrm{d}t^2} = \frac{1}{\beta^2\gamma^2 x^{\gamma-1}}\left((1-\gamma)x^{-\gamma}\frac{\mathrm{d}y_f}{\mathrm{d}x} + \frac{1}{x^{\gamma-1}}\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2}\right) \tag{51}$$

and hence we find

$$\left(\frac{1}{\beta^2\gamma^2 x^{2\gamma-2}}\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2} + \frac{1-\gamma}{\beta^2\gamma^2 x^{2\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x}\right) + \frac{1-2\frac{\alpha}{\gamma}}{\beta x^\gamma}\frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} + \left(1 + \frac{\frac{\alpha^2}{\gamma^2} - n^2}{\beta^2\gamma^2 x^{2\gamma}}\right) y_f = 0 \tag{52}$$

$$\frac{1}{\beta^2\gamma^2 x^{2\gamma-2}}\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2} + \frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-\cancel{\gamma}+\gamma(\cancel{1}-2\frac{\alpha}{\gamma})}{\beta^2\gamma^2 x^{2\gamma-1}}\right) + \left(1 + \frac{\frac{\alpha^2}{\gamma^2} - n^2}{\beta^2\gamma^2 x^{2\gamma}}\right) y_f = 0 \tag{53}$$

$$\frac{1}{\beta^2\gamma^2 x^{2\gamma-2}}\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2} + \frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-2\alpha}{\beta^2\gamma^2 x^{2\gamma-1}}\right) + \left(1 + \frac{\frac{\alpha^2}{\gamma^2} - n^2}{\beta^2\gamma^2 x^{2\gamma}}\right) y_f = 0 \tag{54}$$

$$\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2} + \frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-2\alpha}{x}\right) + \left(\beta^2\gamma^2 x^{2\gamma-2} + \frac{\alpha^2 - n^2\gamma^2}{x^2}\right) y_f = 0 \tag{55}$$

So we see that the generalized Bessel differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1-2\alpha}{x}\right) + \left(\beta^2\gamma^2 x^{2\gamma-2} + \frac{\alpha^2 - n^2\gamma^2}{x^2}\right) y = 0 \tag{56}$$

has solutions

$$y = \begin{cases} Ax^\alpha J_n(\beta x^\gamma) + Bx^\alpha J_{-n}(\beta x^\gamma) & n \text{ is a non-integer} \\ Ax^\alpha J_n(\beta x^\gamma) + Cx^\alpha N_n(\beta x^\gamma) & n \text{ is an integer} \end{cases} \tag{57}$$

with A, B, and C constants to fit initial conditions.

4 Modified Bessel Differential Equation

The differential equation is given by

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - (x^2 + n^2)y = 0 \tag{58}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \left(1 + \frac{n^2}{x^2}\right) y = 0 \tag{59}$$

with solutions $y = AI_n(x) + BI_{-n}(x)$ for non-integer n and $y = AI_n(x) + CK_n(x)$ for integer n . I_n is the nth order modified Bessel function of the first kind and $K_n(x)$ is the nth order modified

Bessel function of the second kind with A, B, C and constants that allow fitting of boundary/initial conditions.

We see that

$$\frac{\mathrm{d}y_s}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^\alpha y\right) = \alpha x^{\alpha-1}y + x^\alpha \frac{\mathrm{d}y}{\mathrm{d}x} \tag{60}$$

$$\frac{\mathrm{d}^2y_s}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha x^{\alpha-1}y + x^\alpha \frac{\mathrm{d}y}{\mathrm{d}x}\right) = \alpha(\alpha-1)x^{\alpha-2}y + 2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + x^\alpha \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \tag{61}$$

So let's put in $y_s(x) = x^\alpha y(x)$ into the left hand side of (59) and find

$$\frac{\mathrm{d}^2y_s}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \left(1+\frac{n^2}{x^2}\right)y_s \tag{62}$$

$$= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(x^\alpha y\right) + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\alpha y\right) - x^\alpha\left(1+\frac{n^2}{x^2}\right)y \tag{63}$$

$$= \left(\alpha(\alpha-1)x^{\alpha-2}y + 2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + x^\alpha\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) + \frac{1}{x}\left(\alpha x^{\alpha-1}y + x^\alpha\frac{\mathrm{d}y}{\mathrm{d}x}\right) - x^\alpha\left(1+\frac{n^2}{x^2}\right)y \tag{64}$$

$$x^\alpha \underbrace{\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \left(1+\frac{n^2}{x^2}\right)y\right)}_{=0} + 2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + x^{\alpha-2}\left(\alpha(\alpha-1)+\alpha\right)y \tag{65}$$

$$= 2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + x^{\alpha-2}\left(\alpha^2\right)y \tag{66}$$

$$= \frac{2\alpha}{x}\left(x^\alpha\frac{\mathrm{d}y}{\mathrm{d}x}\right) + x^{-2}\alpha^2(x^\alpha y) = \frac{2\alpha}{x}\left(\frac{\mathrm{d}y_s}{\mathrm{d}x} - \alpha\frac{y_s}{x}\right) + \frac{\alpha^2}{x^2}y_s \tag{67}$$

$$= \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} + \frac{\alpha^2 - 2\alpha^2}{x^2}y_s = \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha^2}{x^2}y_s \tag{68}$$

Hence the differential equation for $y_s = x^\alpha y$ is given by

$$\frac{\mathrm{d}^2y_s}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \left(1+\frac{n^2}{x^2}\right)y_s - \left[\frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha^2}{x^2}y_s\right] = 0 \tag{69}$$

$$\frac{\mathrm{d}^2y_s}{\mathrm{d}x^2} + \frac{1-2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \left(1+\frac{n^2-\alpha^2}{x^2}\right)y_s = 0 \tag{70}$$

Now to generalize to $y_t = x^\alpha y(\beta x^\gamma)$ we use that we can rewrite this as $y_t = t^{\alpha/\gamma}y_f(t) = x^{\alpha/\gamma}y_s(x)$ with $t = \beta x^\gamma$ (and so $x^\alpha \propto t^{\alpha/\gamma}$) (we only care about the dependence as multiplying by a constant doesn't change the differential equation) and so it satisfies

$$\frac{\mathrm{d}^2y_f}{\mathrm{d}t^2} + \frac{1-2\frac{\alpha}{\gamma}}{t}\frac{\mathrm{d}y_f}{\mathrm{d}t} + \left(1+\frac{\frac{\alpha^2}{\gamma^2}-n^2}{t^2}\right)y_f = 0 \tag{71}$$

Using that $\frac{\mathrm{d}t}{\mathrm{d}x} = \beta\gamma x^{\gamma-1}$ with $t = \beta x^\gamma$ we have

$$\frac{\mathrm{d}y_f}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y_f}{\mathrm{d}x} = \frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} \tag{72}$$

$$\frac{\mathrm{d}^2y_f}{\mathrm{d}t^2} = \frac{1}{\beta^2\gamma^2x^{\gamma-1}}\left((1-\gamma)x^{-\gamma}\frac{\mathrm{d}y_f}{\mathrm{d}x} + \frac{1}{x^{\gamma-1}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2}\right) \tag{73}$$

and hence we find

$$\left(\frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2}+\frac{1-\gamma}{\beta^2\gamma^2x^{2\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x}\right)+\frac{1-2\frac{\alpha}{\gamma}}{\beta x^\gamma}\frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x}-\left(1+\frac{n^2-\frac{\alpha^2}{\gamma^2}}{\beta^2\gamma^2x^{2\gamma}}\right)y_f=0 \tag{74}$$

$$\frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2}+\frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-\cancel{\gamma}+\gamma(\cancel{1}-2\frac{\alpha}{\gamma})}{\beta^2\gamma^2x^{2\gamma-1}}\right)-\left(1+\frac{n^2-\frac{\alpha^2}{\gamma^2}}{\beta^2\gamma^2x^{2\gamma}}\right)y_f=0 \tag{75}$$

$$\frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2}+\frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-2\alpha}{\beta^2\gamma^2x^{2\gamma-1}}\right)-\left(1+\frac{n^2-\frac{\alpha^2}{\gamma^2}}{\beta^2\gamma^2x^{2\gamma}}\right)y_f=0 \tag{76}$$

$$\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2}+\frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-2\alpha}{x}\right)+\left(\beta^2\gamma^2x^{2\gamma-2}+\frac{n^2\gamma^2-\alpha^2}{x^2}\right)y_f=0 \tag{77}$$

So we see that the generalized Bessel differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1-2\alpha}{x}\right)+\left(\beta^2\gamma^2x^{2\gamma-2}+\frac{n^2\gamma^2-\alpha^2}{x^2}\right)y=0 \tag{78}$$

has solutions

$$y=\begin{cases}Ax^\alpha I_n(\beta x^\gamma)+Bx^\alpha I_{-n}(\beta x^\gamma) & n \text{ is a non-integer}\\ Ax^\alpha I_n(\beta x^\gamma)+Cx^\alpha K_n(\beta x^\gamma) & n \text{ is an integer}\end{cases} \tag{79}$$

with A, B, and C constants to fit initial conditions.

5 Airy Differential Equation

Here we have the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - xy = 0 \tag{80}$$

The solutions are $y = A\mathrm{Ai}(x) + B\mathrm{Bi}(x)$ with A and B being integration constants. Following the method we use $y_s = x^\alpha y$ and find using (4) that

$$\frac{\mathrm{d}y_s}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^\alpha y\right) = \alpha x^{\alpha-1} y + x^\alpha \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha}{x} y_s + x^\alpha \frac{\mathrm{d}y}{\mathrm{d}x} \tag{81}$$

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}x^2} = \alpha(\alpha-1)x^{\alpha-2} y + 2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + x^\alpha \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \tag{82}$$

yielding for the left hand side of (80)

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}x^2} - x y_s \tag{83}$$

$$= \left(\alpha(\alpha-1)x^{\alpha-2} y + 2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + x^\alpha \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - x^\alpha (xy) \tag{84}$$

$$= \alpha(\alpha-1)x^{\alpha-2} y + 2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + x^\alpha \underbrace{\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - xy\right)}_{=0} \tag{85}$$

$$\alpha(\alpha-1)x^{\alpha-2} y + 2\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha(\alpha-1)}{x^2} y_s + \frac{2\alpha}{x}\left(x^\alpha \frac{\mathrm{d}y}{\mathrm{d}x}\right) \tag{86}$$

$$= \frac{\alpha(\alpha-1)}{x^2} y_s + \frac{2\alpha}{x}\left(\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha}{x} y_s\right) \tag{87}$$

$$= \frac{\alpha^2 - \alpha - 2\alpha^2}{x^2} y_s + \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} \tag{88}$$

$$= \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha(\alpha+1)}{x^2} y_s \tag{89}$$

And so the differential equation for y_s is given by

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}x^2} - x y_s - \left[\frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha(\alpha+1)}{x^2} y_s\right] = 0 \tag{90}$$

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}x^2} - \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \left(x - \frac{\alpha(\alpha+1)}{x^2}\right) y_s = 0 \tag{91}$$

Now to generalize to $y_t = x^\alpha y(\beta x^\gamma)$ we use that we can rewrite this as $y_t = t^{\alpha/\gamma} y_f(t) = x^{\alpha/\gamma} y_s(x)$ with $t = \beta x^\gamma$ (and so $x^\alpha \propto t^{\alpha/\gamma}$) (we only care about the dependence as multiplying by a constant doesn't change the differential equation) and so it satisfies

$$\frac{\mathrm{d}^2 y_f}{\mathrm{d}t^2} - \frac{2\frac{\alpha}{\gamma}}{t}\frac{\mathrm{d}y_f}{\mathrm{d}t} - \left(t - \frac{\frac{\alpha}{\gamma}(\frac{\alpha}{\gamma}+1)}{t^2}\right) y_f = 0 \tag{92}$$

And so using that $\frac{\mathrm{d}t}{\mathrm{d}x} = \beta\gamma x^{\gamma-1}$ with $t = \beta x^\gamma$ we have

$$\frac{\mathrm{d}y_f}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y_f}{\mathrm{d}x} = \frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} \tag{93}$$

$$\frac{\mathrm{d}^2y_f}{\mathrm{d}t^2} = \frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2} + \frac{1-\gamma}{\beta^2\gamma^2x^{2\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} \tag{94}$$

and hence we find

$$\left(\frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2} + \frac{1-\gamma}{\beta^2\gamma^2x^{2\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x}\right) - \frac{2\frac{\alpha}{\gamma}}{\beta x^\gamma}\left(\frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x}\right) - \left(\beta x^\gamma - \frac{\frac{\alpha}{\gamma}(\frac{\alpha}{\gamma}+1)}{\beta^2x^{2\gamma}}\right)y_f = 0 \tag{95}$$

$$\frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2} + \left(\frac{1-\gamma}{\beta^2\gamma^2x^{2\gamma-1}} - \frac{2\frac{\alpha}{\gamma}}{\beta x^\gamma}\left(\frac{1}{\beta\gamma x^{\gamma-1}}\right)\right)\frac{\mathrm{d}y_f}{\mathrm{d}x} - \left(\beta x^\gamma - \frac{\frac{\alpha}{\gamma}(\frac{\alpha}{\gamma}+1)}{\beta^2x^{2\gamma}}\right)y_f = 0 \tag{96}$$

$$\frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2} + \left(\frac{1-\gamma-2\alpha}{\beta^2\gamma^2x^{2\gamma-1}}\right)\frac{\mathrm{d}y_f}{\mathrm{d}x} - \left(\beta x^\gamma - \frac{\frac{\alpha}{\gamma}(\frac{\alpha}{\gamma}+1)}{\beta^2x^{2\gamma}}\right)y_f = 0 \tag{97}$$

$$\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2} + \left(\frac{1-\gamma-2\alpha}{x}\right)\frac{\mathrm{d}y_f}{\mathrm{d}x} - \left(\beta^3\gamma^2x^{3\gamma-2} - \frac{\alpha(\alpha+\gamma)}{x^2}\right)y_f = 0 \tag{98}$$

So we see that the generalized Airy differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{1-\gamma-2\alpha}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} - \left(\beta^3\gamma^2x^{3\gamma-2} - \frac{\alpha(\alpha+\gamma)}{x^2}\right)y = 0 \tag{99}$$

has solutions

$$y = Ax^\alpha\mathrm{Ai}(\beta x^\gamma) + Bx^\alpha\mathrm{Bi}(\beta x^\gamma) \tag{100}$$

with A and B constants to fit initial conditions.

6 Nonlinear Differential Equation log

Consider

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0 \tag{101}$$

with solution $y = \ln(A + Bx)$ with A and B integration constants. We will find that we can still apply our previous rules and generate a generalized differential equation.

Following the method we use $y_s = x^\alpha y/\delta$. We add in the δ because as it is no longer a linear differential equation we cannot expect $ay(x)$ with a a complex number and $y(x)$ to be a solution to the same differential equation. We find using (4)

$$\frac{\mathrm{d}y_s}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^\alpha y/\delta\right) = \alpha x^{\alpha-1} y/\delta + \frac{x^\alpha}{\delta}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha}{x} y_s + \frac{x^\alpha}{\delta}\frac{\mathrm{d}y}{\mathrm{d}x} \tag{102}$$

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}x^2} = \frac{\alpha(\alpha-1)x^{\alpha-2}y}{\delta} + \frac{2\alpha x^{\alpha-1}}{\delta}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x^\alpha}{\delta}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \tag{103}$$

yielding for the left hand side of (101)

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y_s}{\mathrm{d}x}\right)^2 \tag{104}$$

$$= \frac{x^\alpha}{\delta}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{2\alpha}{\delta}x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha(\alpha-1)}{\delta}x^{\alpha-2}y + \frac{1}{\delta^2}\left(x^\alpha\frac{\mathrm{d}y}{\mathrm{d}x} + \alpha x^{\alpha-1}y\right)^2 \tag{105}$$

$$= \frac{x^\alpha}{\delta}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{2\alpha}{\delta}x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha(\alpha-1)}{\delta}x^{\alpha-2}y + \frac{x^{2\alpha}}{\delta^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{2\alpha}{\delta^2}x^{2\alpha-1}y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha^2}{\delta^2}x^{2\alpha-2}y^2 \tag{106}$$

$$= \frac{x^\alpha}{\delta}\left(\underbrace{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}_{=0} + \left(\frac{x^\alpha}{\delta} - 1\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right) + \frac{2\alpha}{\delta}x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha(\alpha-1)}{\delta}x^{\alpha-2}y + \frac{2\alpha}{\delta^2}x^{2\alpha-1}y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha^2 x^{2\alpha-2}}{\delta^2}y^2 \tag{107}$$

$$= \left(x^{2\alpha} - \delta x^\alpha\right)\left(\frac{1}{\delta}\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{2\alpha}{\delta}x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha(\alpha-1)}{\delta}x^{\alpha-2}y + \frac{2\alpha}{\delta^2}x^{2\alpha-1}y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha^2 x^{2\alpha-2}}{\delta^2}y^2 \tag{108}$$

$$= \left(x^{2\alpha} - x^\alpha\delta\right)\left(\frac{1}{x^\alpha}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha}{x^{\alpha+1}}y_s\right)^2 + \frac{2\alpha}{x}\left(\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha}{x}y_s\right) + \frac{\alpha(\alpha-1)}{x^2}y_s$$
$$+ \frac{2\alpha}{x}y_s\left(\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha}{x}y_s\right) + \frac{\alpha^2}{x^2}y_s^2 \tag{109}$$

$$= (x^{2\alpha} - x^\alpha\delta)\left(\frac{1}{x^{2\alpha}}\left(\frac{\mathrm{d}y_s}{\mathrm{d}x}\right)^2 - \frac{2\alpha}{x^{2\alpha+1}}y_s\frac{\mathrm{d}y_s}{\mathrm{d}x} + \frac{\alpha^2}{x^{2\alpha+2}}y_s^2\right) + \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha(\alpha+1)}{x^2}y_s + \frac{2\alpha}{x}y_s\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha^2}{x^2}y_s^2 \tag{110}$$

$$= \left(\frac{\mathrm{d}y_s}{\mathrm{d}x}\right)^2 - \cancel{\frac{2\alpha}{x}y_s\frac{\mathrm{d}y_s}{\mathrm{d}x}} + \cancel{\frac{\alpha^2}{x^2}y_s^2} - \frac{\delta}{x^\alpha}\left(\frac{\mathrm{d}y_s}{\mathrm{d}x}\right)^2 + \frac{2\alpha\delta}{x^{\alpha+1}}y_s\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha^2\delta}{x^{\alpha+2}}y_s^2$$
$$+ \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha(\alpha+1)}{x^2}y_s + \cancel{\frac{2\alpha}{x}y_s\frac{\mathrm{d}y_s}{\mathrm{d}x}} - \cancel{\frac{\alpha^2}{x^2}y_s^2} \tag{111}$$

and so altogether the left hand side gives

$$\left(\frac{\mathrm{d}y_s}{\mathrm{d}x}\right)^2 - \frac{\delta}{x^\alpha}\left(\frac{\mathrm{d}y_s}{\mathrm{d}x}\right)^2 + \frac{2\alpha\delta}{x^{\alpha+1}}y_s\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha^2\delta}{x^{\alpha+2}}y_s^2 + \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha(\alpha+1)}{x^2}y_s \tag{112}$$

So our differential equation for y_s is

$$\frac{\mathrm{d}^2y_s}{\mathrm{d}x^2} + \cancel{\left(\frac{\mathrm{d}y_s}{\mathrm{d}x}\right)^2} - \left[\cancel{\left(\frac{\mathrm{d}y_s}{\mathrm{d}x}\right)^2} - \frac{\delta}{x^\alpha}\left(\frac{\mathrm{d}y_s}{\mathrm{d}x}\right)^2 + \frac{2\alpha\delta}{x^{\alpha+1}}y_s\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha^2\delta}{x^{\alpha+2}}y_s^2 + \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} - \frac{\alpha(\alpha+1)}{x^2}y_s\right] = 0 \tag{113}$$

$$\frac{\mathrm{d}^2y_s}{\mathrm{d}x^2} + \frac{\delta}{x^\alpha}\left(\frac{\mathrm{d}y_s}{\mathrm{d}x}\right)^2 - \frac{2\alpha\delta}{x^{\alpha+1}}y_s\frac{\mathrm{d}y_s}{\mathrm{d}x} + \frac{\alpha^2\delta}{x^{\alpha+2}}y_s^2 - \frac{2\alpha}{x}\frac{\mathrm{d}y_s}{\mathrm{d}x} + \frac{\alpha(\alpha+1)}{x^2}y_s = 0 \tag{114}$$

Now to generalize to $y_t = x^\alpha y(\beta x^\gamma)$ we use that we can rewrite this as $y_t = \frac{t^{\alpha/\gamma}}{\beta^{\alpha/\gamma}}y_f(t) = \frac{x^{\alpha/\gamma}}{\delta}y_s(x)$ with $t = \beta x^\gamma$. We can no longer ignore the factor $\beta^{-\alpha/\gamma}$ because this is no longer a linear differential equation. So we use that $\delta = \beta^{\alpha/\gamma}$ in our previous equation and $\alpha \to \alpha/\gamma$. We use that $\frac{\mathrm{d}t}{\mathrm{d}x} = \beta\gamma x^{\gamma-1}$ with $t = \beta x^\gamma$ so that we have

$$\frac{\mathrm{d}y_f}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y_f}{\mathrm{d}x} = \frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} \tag{115}$$

$$\frac{\mathrm{d}^2y_f}{\mathrm{d}t^2} = \frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2} + \frac{1-\gamma}{\beta^2\gamma^2x^{2\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} \tag{116}$$

and hence we find

$$\frac{\mathrm{d}^2y_f}{\mathrm{d}t^2} + \frac{\delta}{t^{\alpha/\gamma}}\left(\frac{\mathrm{d}y_f}{\mathrm{d}t}\right)^2 - \frac{2\frac{\alpha}{\gamma}\delta}{t^{\alpha/\gamma+1}}y_f\frac{\mathrm{d}y_f}{\mathrm{d}t} + \frac{\left(\frac{\alpha}{\gamma}\right)^2\delta}{t^{\alpha/\gamma+2}}y_f^2 - \frac{2\frac{\alpha}{\gamma}}{t}\frac{\mathrm{d}y_f}{\mathrm{d}t} + \frac{\frac{\alpha}{\gamma}(\frac{\alpha}{\gamma}+1)}{t^2}y_f = 0 \tag{117}$$

$$\left(\frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2} + \frac{1-\gamma}{\beta^2\gamma^2x^{2\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x}\right) + \frac{\beta^{\alpha/\gamma}}{\beta^{\alpha/\gamma}x^\alpha}\left(\frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x}\right)^2$$
$$- \frac{2\frac{\alpha}{\gamma}\beta^{\alpha/\gamma}}{\beta^{\alpha/\gamma+1}x^{\alpha+\gamma}}y_f\frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} + \frac{\left(\frac{\alpha}{\gamma}\right)^2\beta^{\alpha/\gamma}}{\beta^{\alpha/\gamma+2}x^{\alpha+2\gamma}}y_f^2 - \frac{2\frac{\alpha}{\gamma}}{\beta x^\gamma}\frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} + \frac{\frac{\alpha}{\gamma}(\frac{\alpha}{\gamma}+1)}{\beta^2x^{2\gamma}}y_f = 0 \tag{118}$$

$$\left(\frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2} + \frac{1-\gamma}{\beta^2\gamma^2x^{2\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x}\right) + \frac{1}{x^\alpha}\left(\frac{1}{\beta\gamma x^{\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x}\right)^2$$
$$- \frac{2\frac{\alpha}{\gamma}}{\beta^2\gamma x^{\alpha+2\gamma-1}}y_f\frac{\mathrm{d}y_f}{\mathrm{d}x} + \frac{\left(\frac{\alpha}{\gamma}\right)^2}{\beta^2x^{\alpha+2\gamma}}y_f^2 - \frac{2\alpha}{\beta^2\gamma^2x^{2\gamma-1}}\frac{\mathrm{d}y_f}{\mathrm{d}x} + \frac{\frac{\alpha}{\gamma}(\frac{\alpha}{\gamma}+1)}{\beta^2x^{2\gamma}}y_f = 0 \tag{119}$$

$$\frac{1}{\beta^2\gamma^2x^{2\gamma-2}}\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2} + \frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-\gamma}{\beta^2\gamma^2x^{2\gamma-1}} - \frac{2\alpha}{\beta^2\gamma^2x^{2\gamma-1}}\right) + \frac{1}{\beta^2\gamma^2x^{\alpha+2\gamma-2}}\left(\frac{\mathrm{d}y_f}{\mathrm{d}x}\right)^2 - \frac{2\alpha}{\beta^2\gamma^2x^{\alpha+2\gamma-1}}y_f\frac{\mathrm{d}y_f}{\mathrm{d}x}$$
$$+ \frac{\alpha^2}{\beta^2\gamma^2x^{\alpha+2\gamma}}y_f^2 + \frac{\alpha(\alpha+\gamma)}{\beta^2\gamma^2x^{2\gamma}}y_f = 0 \tag{120}$$

$$\frac{\mathrm{d}^2y_f}{\mathrm{d}x^2} + \frac{1}{x^\alpha}\left(\frac{\mathrm{d}y_f}{\mathrm{d}x}\right)^2 - \frac{2\alpha}{x^{\alpha+1}}y_f\frac{\mathrm{d}y_f}{\mathrm{d}x} + \frac{\alpha^2}{x^{\alpha+2}}y_f^2 + \frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-\gamma-2\alpha}{x}\right) + \frac{\alpha(\alpha+\gamma)}{x^2}y_f = 0 \tag{121}$$

So we see that the generalized logarithm differential equation

$$\frac{\mathrm{d}^2 y_f}{\mathrm{d}x^2} + \frac{1}{x^\alpha}\left(\frac{\mathrm{d}y_f}{\mathrm{d}x}\right)^2 - \frac{2\alpha}{x^{\alpha+1}} y_f \frac{\mathrm{d}y_f}{\mathrm{d}x} + \frac{\alpha^2}{x^{\alpha+2}} y_f^2 + \frac{\mathrm{d}y_f}{\mathrm{d}x}\left(\frac{1-\gamma-2\alpha}{x}\right) + \frac{\alpha(\alpha+\gamma)}{x^2} y_f = 0 \tag{122}$$

has solutions

$$y_f = x^\alpha \ln(A + B\beta x^\gamma) \tag{123}$$

with A and B constants to fit initial conditions.