Contents

1 Bessel Function Proofs

1.1 Sum of Bessel Functions of First Kind

We begin with the known fact (see section 1.3) that

$$e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(z) \tag{1}$$

i.e., the generating function for $J_n(z)$ is given by $e^{\frac{z}{2}(t-1/t)}$.

Set $t = 1$ and we see

$$1 = e^{\frac{z}{2}\left(1-\frac{1}{1}\right)} = e^{\frac{z}{2}(0)} = \sum_{n=-\infty}^{\infty} 1^n J_n(z) = \sum_{n=-\infty}^{\infty} J_n(z) \tag{2}$$

$$1 = \sum_{n=-\infty}^{\infty} J_n(z) \tag{3}$$

1.2 Sum Of Bessel Functions of First Kind Squared

We begin with the known fact (see section 1.3) that

$$e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(z) \tag{4}$$

i.e., the generating function for $J_n(z)$ is given by $e^{\frac{z}{2}(t-1/t)}$.

Thus we note that

$$e^{\frac{z}{2}\left(t-\frac{1}{t}\right)}e^{\frac{z}{2}\left(-t+\frac{1}{t}\right)} = e^0 = 1 = \left(\sum_{n=-\infty}^{\infty} t^n J_n(z)\right)\left(\sum_{m=-\infty}^{\infty} (-t)^m J_m(z)\right) \tag{5}$$

$$= \left(\sum_{n=-\infty}^{\infty} t^n J_n(z)\right)\left(\sum_{m=-\infty}^{\infty} (t)^m(-1)^m J_m(z)\right) = \left(\sum_{n=-\infty}^{\infty} t^n J_n(z)\right)\left(\sum_{m=-\infty}^{\infty} (t)^m J_{-m}(z)\right) \tag{6}$$

$$= \left(\sum_{n=-\infty}^{\infty} t^n J_n(z)\left(\sum_{m=-\infty}^{\infty} (t)^m J_{-m}(z)\right)\right) = \sum_{n=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty} t^n J_n(z) t^m J_{-m}(z)\right) \tag{7}$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} t^{n+m} J_n(z) J_{-m}(z) \tag{8}$$

So taking only the terms that have t^0 as this must be true order by order (because t is only a formal variable),

$$1 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} t^{n+m} J_n(z) J_{-m}(z)\delta_{n,-m} = \sum_{n=-\infty}^{\infty} J_n^2(z) \tag{9}$$

or altogether

$$\sum_{n=-\infty}^{\infty} J_n^2(z) = 1 \tag{10}$$

1.3 Proof of Bessel Function of First Kind Generating Function

We will prove this by verification.

$$e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = e^{zt/2}e^{-z/(2t)} = \left\{\sum_{n=0}^{\infty}\frac{\left(\frac{zt}{2}\right)^n}{n!}\right\}\left\{\sum_{m=0}^{\infty}\frac{\left(\frac{-z}{2t}\right)^m}{m!}\right\} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{n+m}\frac{(-1)^m t^{n-m}}{n!m!} \tag{11}$$

We now use the substitution $j = n - m$ or $n = j + m$ if you prefer so that

$$= \sum_{j=-\infty}^{\infty}\sum_{\substack{n-m=j\\ m,n\geq 0}}\left(\frac{z}{2}\right)^{n+m}\frac{(-1)^m t^{n-m}}{(n)!m!} = \sum_{j=-\infty}^{\infty}\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m+j}\frac{(-1)^m t^j}{(j+m)!m!} \tag{12}$$

$$= \sum_{j=-\infty}^{\infty} t^j\left(\frac{z}{2}\right)^j\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m}\frac{(-1)^m}{(j+m)!m!} = \sum_{j=-\infty}^{\infty} t^j J_j(z) \tag{13}$$

where the definition

$$J_j(z) = \left(\frac{z}{2}\right)^j\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m}\frac{(-1)^m}{m!(j+m)!} = \left(\frac{z}{2}\right)^j\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m}\frac{(-1)^m}{m!\Gamma(j+m+1)} \tag{14}$$

Thus, we find

$$e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(z) \tag{15}$$

1.3.1 Proof of Expansion of $e^{iz\sin\theta}$

Take the generating function (15) and take $t = e^{i\theta}$. One finds

$$e^{\frac{z}{2}\left(e^{i\theta}-e^{-i\theta}\right)} = \sum_{n=-\infty}^{\infty} e^{in\theta}J_n(z) \tag{16}$$

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} e^{in\theta}J_n(z) \tag{17}$$

where the identity

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \tag{18}$$

was used. Note that taking $t = e^{-i\theta}$ then yields

$$e^{-iz\sin\theta} = \sum_{n=-\infty}^{\infty} e^{-in\theta}J_n(z) \tag{19}$$

unsurprisingly, as this coincides with $\theta \to -\theta$.

2 Calculus Identities

2.1 Flipping a Derivative

Given $\frac{\mathrm{d}f}{\mathrm{d}x} \neq 0$ with $f = f(x)$ we will show

$$\frac{\mathrm{d}x}{\mathrm{d}f} = \frac{1}{\frac{\mathrm{d}f}{\mathrm{d}x}} \tag{20}$$

Let $y = f(x)$ and so by the chain rule

$$1 = \frac{\mathrm{d}y}{\mathrm{d}y} = \frac{\mathrm{d}f}{\mathrm{d}y} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}f} \tag{21}$$

and so

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}f}} \tag{22}$$

Note for $\frac{\mathrm{d}f}{\mathrm{d}x} = 0$ then $\frac{\mathrm{d}x}{\mathrm{d}f} = 0$ as x and f are independent.

2.2 Flipping Partial Derivatives

We will show given $\left(\frac{\partial f}{\partial x}\right)_{x_i} \neq 0$ with $f = f(x, x_i)$ [so that $x = x(f, x_i)$], where x_i are all other variables f depends upon, that

$$\left(\frac{\partial x}{\partial f}\right)_{x_i} = \frac{1}{\left(\frac{\partial f}{\partial x}\right)_{x_i}} \tag{23}$$

Let $z = f(x, x_i)$ be implicitly defined for this function.

Then, (assuming $\left(\frac{\partial x}{\partial f}\right)_{x_i} = \left(\frac{\partial x}{\partial z}\right)_{x_i} \neq 0$; I will omit the subscript x_i indicating variables held constant from now on.)

$$1 = \frac{\partial z}{\partial z} = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial f} \tag{24}$$

Thus,

$$\frac{\partial f}{\partial x} = \frac{1}{\frac{\partial x}{\partial f}}, \quad \frac{\partial x}{\partial f} = \frac{1}{\frac{\partial f}{\partial x}} \tag{25}$$

For $\frac{\partial f}{\partial x} = 0$ then $\frac{\partial x}{\partial f} = 0$ as well as f and x are independent of each other.

2.3 Implicitly Defined Functions Derivatives

Given $f(x_1, x_2, \dots) \equiv f(x_i) = 0$ it will be shown that

$$\frac{\partial x_j}{\partial x_i} = -\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}} \tag{26}$$

with the assumption that $\frac{\partial f}{\partial x_i} \neq 0$ for any x_i.

We first write out the differential of f

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x_i} \mathrm{d}x_i \tag{27}$$

and then use that $x_j = x(x_{i \neq j})$ so that

$$\mathrm{d}x_j = \sum_{i \neq j} \frac{\partial x_j}{\partial x_i} \mathrm{d}x_k \tag{28}$$

Thus,

$$\mathrm{d}f = \sum_{i \neq j} \left(\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial x_i} \right) \mathrm{d}x_i \tag{29}$$

As $f = 0$ then we must have $\mathrm{d}f = 0$ and so for each $\mathrm{d}x_i$

$$\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial x_i} = 0 \tag{30}$$

$$\frac{\partial x_j}{\partial x_i} = -\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}} \tag{31}$$

as desired.

2.4 Chain Rule for Three Variables

Given a relation $f(x, y, z) = 0$ that implicitly defines x, y, z, as $x = x(y, z)$, $y = y(x, z)$, $z = z(x, y)$, let's show that for $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \neq 0$ that

$$\frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -1 \tag{32}$$

which is contrary to naïve expectations.

We use the result of section 2.3 that given $f(x_i) = 0$ that

$$\frac{\partial x_j}{\partial x_i} = -\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}} \tag{31}$$

So we define $f(x, y, z) = 0$ that implicitly defines $x = x(y, z)$, $y = (x, z)$, and $z = (x, y)$. We then have

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = (-1)^3 \left(\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}\right)\left(\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}\right)\left(\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}\right) = -1 \tag{33}$$

as desired.

2.5 Even/Odd Symmetry Implied Derivative Conditions

Let there be a function $f(x, x_i)$ with the symmetry $f(x, x_i) = f(-x, x_i)$ where x and x_i are independent variables of f. Then we have (keeping x_i variables constant)

$$\begin{aligned}\frac{\partial^M f(x,x_i)}{\partial x^M} &= \frac{\partial^M f(-x,x_i)}{\partial x^M} = \frac{\partial^{M-1}}{\partial x^{M-1}}\left(\frac{\partial f(-x,x_i)}{\partial(-x)}\frac{\partial -x}{\partial x}\right) = \frac{\partial^{M-1}}{\partial x^{M-1}}\left(\frac{\partial f(-x,x_i)}{\partial(-x)}(-1)\right) \\ &= (-1)^M \frac{\partial^M f(-x,x_i)}{\partial(-x)^M} = (-1)^M \frac{\partial^M f(-x,x_i)}{\partial(-x)^M}\end{aligned} \tag{34}$$

Taking $\frac{\partial^M f(x,x_i)}{\partial x^M} = g(x, x_i; M)$ we see

$$g(x, x_i; M) = (-1)^M g(-x, x_i; M) \tag{35}$$

and so we see that when M is odd then g is odd and when M is even then g is even.

Alternatively, let there be a function $f(x, x_i)$ with the symmetry $f(x, x_i) = -f(-x, x_i)$ where x and x_i are independent variables of f. Then we have (keeping x_i variables constant)

$$\begin{aligned}\frac{\partial^M f(x,x_i)}{\partial x^M} &= -\frac{\partial^M f(-x,x_i)}{\partial x^M} = -\frac{\partial^{M-1}}{\partial x^{M-1}}\left(\frac{\partial f(-x,x_i)}{\partial(-x)}\frac{\partial -x}{\partial x}\right) = -\frac{\partial^{M-1}}{\partial x^{M-1}}\left(\frac{\partial f(-x,x_i)}{\partial(-x)}(-1)\right) \\ &= -(-1)^M \frac{\partial^M f(-x,x_i)}{\partial(-x)^M} = (-1)^{M+1} \frac{\partial^M f(-x,x_i)}{\partial(-x)^M}\end{aligned} \tag{36}$$

Taking $\frac{\partial^M f(x,x_i)}{\partial x^M} = h(x, x_i; M)$ we see

$$h(x, x_i; M) = (-1)^{M+1} h(-x, x_i; M) \tag{37}$$

and so we see that when M is odd then h is even and when M is even then h is odd.

2.6 Stokes' Theorem, Gauss's Law Corollaries

Let $\mathbf{A}$ be a vector, $\mathbf{c}$ an arbitrary nonzero constant vector, and $\overleftrightarrow{\mathbf{S}}$ a tensor. Take Stokes' Theorem for a surface S (with outward normal $\hat{\mathbf{n}}$ enclosed by a closed curve C [with $\mathrm{d}\boldsymbol{\ell}$ being a line element along the curve C])

$$\int_S \mathrm{d}S \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \mathbf{A} = \oint_C \mathrm{d}\boldsymbol{\ell} \cdot \mathbf{A} \tag{38}$$

and Gauss's Law for a volume V (with outward normal $\hat{\mathbf{n}}$) and enclosing surface S

$$\int_V \mathrm{d}V\ \boldsymbol{\nabla} \cdot \mathbf{A} = \oint_S \mathrm{d}S\ \hat{\mathbf{n}} \cdot \mathbf{A} \tag{39}$$

$$\int_V \mathrm{d}V\ \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{S}} = \oint_S \mathrm{d}S\ \hat{\mathbf{n}} \cdot \overleftrightarrow{\mathbf{S}} \tag{40}$$

Then we have

$$\int_V \mathrm{d}V\ \ \nabla f = \oint_S \mathrm{d}S\ \hat{\mathbf{n}} f \tag{41}$$

$$\int_V \mathrm{d}V\ \boldsymbol{\nabla} \times \mathbf{A} = \oint_S \mathrm{d}S\ \hat{\mathbf{n}} \times \mathbf{A} \tag{42}$$

$$\int_S \mathrm{d}S\ \hat{\mathbf{n}} \times\ \nabla f = \oint_C \mathrm{d}\boldsymbol{\ell}\ f \tag{43}$$

$$\int_S \mathrm{d}S\ (\hat{\mathbf{n}} \times\ \nabla) \times \mathbf{A} = \oint_C \mathrm{d}\boldsymbol{\ell}\ \times \mathbf{A} \tag{44}$$

$$\int_S \mathrm{d}S\ \hat{\mathbf{n}} \cdot (\ \nabla f \times\ \nabla g) = \oint_C \mathrm{d}g\ f = -\oint_C \mathrm{d}f\ g \tag{45}$$

The meaning of $(\hat{\mathbf{n}} \times\ \nabla) \times \mathbf{A}$ is given in index notation as

$$(\hat{\mathbf{n}} \times\ \nabla) \times \mathbf{A} = \epsilon_{lim}\epsilon_{ijk} n_j \partial_k A_m = n_m \partial_l A_m - n_l \partial_m A_m =\ \nabla \mathbf{A} \cdot \hat{\mathbf{n}} - \hat{\mathbf{n}} \boldsymbol{\nabla} \cdot \mathbf{A} \tag{46}$$

2.6.1 First Corollary

We begin with (41). We dot $\mathbf{c}$ into the left side (using $\mathbf{c} \cdot\ \nabla f = \boldsymbol{\nabla} \cdot (f\mathbf{c}) - f \cancel{\boldsymbol{\nabla} \cdot c}$ and defining $\mathbf{G} = f\mathbf{c}$) and find

$$\mathbf{c} \cdot \int_V \mathrm{d}V\ \ \nabla f = \int_V \mathrm{d}V\ \mathbf{c} \cdot\ \nabla f = \int_V \mathrm{d}V\ \boldsymbol{\nabla} \cdot \underbrace{(f\mathbf{c})}_{\mathbf{G}} = \oint_S \mathrm{d}s\ \hat{\mathbf{n}} \cdot \underbrace{f\mathbf{c}}_{\mathbf{G}} = \mathbf{c} \cdot \oint_S \hat{\mathbf{n}} f \tag{47}$$

Thus we can write

$$\hat{\mathbf{c}} \cdot \left(\int_V \mathrm{d}V\ \ \nabla f - \oint_S \hat{\mathbf{n}} f \right) = 0 \tag{48}$$

Since this is true for any arbitrary non-zero vector $\mathbf{c}$, this means that the expression in parentheses must be $\mathbf{0}$ identically.

2.6.2 Second Corollary

Now we handle (42). We dot $\mathbf{c}$ into the left side (using $\mathbf{c}\cdot\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{c}) + \mathbf{A}\cdot\cancel{\boldsymbol{\nabla}\times\mathbf{c}}$, $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B}$ (for vectors $\mathbf{A},\mathbf{B},\mathbf{C}$), and defining $\mathbf{G} = \mathbf{A}\times\mathbf{c}$) and find

$$\begin{aligned}\mathbf{c}\cdot\int_V \mathrm{d}V\ \boldsymbol{\nabla}\times\mathbf{A} &= \int_V \mathrm{d}V\ \mathbf{c}\cdot\boldsymbol{\nabla}\times\mathbf{A} = \int_V \mathrm{d}V\ \boldsymbol{\nabla}\cdot\underbrace{(\mathbf{A}\times\mathbf{c})}_{\mathbf{G}} = \oint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot(\mathbf{A}\times\mathbf{c}) \\ &= \oint_S \mathrm{d}S\ \mathbf{c}\cdot(\hat{\mathbf{n}}\times\mathbf{A}) = \mathbf{c}\cdot\oint_S \mathrm{d}S\ \hat{\mathbf{n}}\times\mathbf{A}\end{aligned} \tag{49}$$

Thus we can write

$$\hat{\mathbf{c}}\cdot\left(\int_V \mathrm{d}V\ \boldsymbol{\nabla}\times\mathbf{A} - \oint_S \mathrm{d}S\ \hat{\mathbf{n}}\times\mathbf{A}\right) = 0 \tag{50}$$

Since this is true for any arbitrary non-zero vector $\mathbf{c}$, this means that the expression in parentheses must be $\mathbf{0}$ identically.

2.6.3 Third Corollary

Now we handle (43). We dot $\mathbf{c}$ into the left side (using $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B}$ (for vectors $\mathbf{A},\mathbf{B},\mathbf{C}$), $\nabla f\times\mathbf{c} = \boldsymbol{\nabla}\times(f\mathbf{c}) - f\cancel{\boldsymbol{\nabla}\times\mathbf{c}}$ and defining $\mathbf{G} = f\mathbf{c}$) and find

$$\begin{aligned}\mathbf{c}\cdot\int_S \mathrm{d}S\ \hat{\mathbf{n}}\times\nabla f &= \int_S \mathrm{d}S\ \mathbf{c}\cdot\hat{\mathbf{n}}\times\nabla f = \int_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\nabla f\times\mathbf{c} = \int_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\underbrace{(f\mathbf{c})}_{\mathbf{G}} \\ &= \oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot\underbrace{(f\mathbf{c})}_{\mathbf{G}} = \mathbf{c}\cdot\oint_C \mathrm{d}\boldsymbol{\ell}\ f\end{aligned} \tag{51}$$

Thus we can write

$$\hat{\mathbf{c}}\cdot\left(\int_S \mathrm{d}S\ \hat{\mathbf{n}}\times\nabla f - \oint_C \mathrm{d}\boldsymbol{\ell}\ f\right) = 0 \tag{52}$$

Since this is true for any arbitrary non-zero vector $\mathbf{c}$, this means that the expression in parentheses must be $\mathbf{0}$ identically.

2.6.4 Fourth Corollary

Now we handle (44). We dot $\mathbf{c}$ into the left side (using $\mathbf{c}\cdot(\hat{\mathbf{n}}\times\nabla)\times\mathbf{A} = \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{c}) - [\hat{\mathbf{n}}\cdot\mathbf{A}\cancel{\boldsymbol{\nabla}\cdot\mathbf{c}} - \hat{\mathbf{n}}\mathbf{A}\cdot\cancel{\nabla}\mathbf{c}]$ and defining $\mathbf{G} = \mathbf{A}\times\mathbf{c}$) and find

$$\begin{aligned}\mathbf{c}\cdot\int_S \mathrm{d}S\ (\hat{\mathbf{n}}\times\nabla)\times\mathbf{A} &= \int_S \mathrm{d}S\ \mathbf{c}\cdot(\hat{\mathbf{n}}\times\nabla)\times\mathbf{A} = \int_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\underbrace{(\mathbf{A}\times\mathbf{c})}_{\mathbf{G}} = \oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot\underbrace{(\mathbf{A}\times\mathbf{c})}_{\mathbf{G}} \\ &= \oint_C \mathrm{d}\boldsymbol{\ell}\ \times\mathbf{A}\cdot\mathbf{c} = \mathbf{c}\cdot\oint_C \mathrm{d}\boldsymbol{\ell}\ \times\mathbf{A}\end{aligned} \tag{53}$$

Thus we can write

$$\hat{\mathbf{c}}\cdot\left(\int_S \mathrm{d}S\ (\hat{\mathbf{n}}\times\nabla)\times\mathbf{A} - \oint_C \mathrm{d}\boldsymbol{\ell}\ \times\mathbf{A}\right) = 0 \tag{54}$$

Since this is true for any arbitrary non-zero vector $\mathbf{c}$, this means that the expression in parentheses must be $\mathbf{0}$ identically.

2.6.5 Fifth Corollary

Now we handle (45). We use that $\nabla f \times \nabla g = \boldsymbol{\nabla} \times (f\,\nabla g) = \nabla f \times \nabla g - f \cancel{\boldsymbol{\nabla} \times \nabla g}$ or $\nabla f \times \nabla g = -\boldsymbol{\nabla} \times (g\,\nabla f) = -\nabla g \times \nabla f + g \cancel{\boldsymbol{\nabla} \times \nabla f}$. Thus defining $\mathbf{G} = f\,\nabla g$ and $\mathbf{H} = -g\,\nabla f$,

$$\int_S \mathrm{d}S\; \hat{\mathbf{n}} \cdot (\nabla f \times \nabla g) = \int_S \mathrm{d}S\; \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \underbrace{(f\,\nabla g)}_{\mathbf{G}} = \oint_C \mathrm{d}\boldsymbol{\ell} \;\cdot \underbrace{f\,\nabla g}_{\mathbf{G}} = \oint_C \mathrm{d}g\; f \tag{55}$$

$$\int_S \mathrm{d}S\; \hat{\mathbf{n}} \cdot (\nabla f \times \nabla g) = \int_S \mathrm{d}S\; \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \underbrace{(-g\,\nabla f)}_{\mathbf{H}} = \oint_C \mathrm{d}\boldsymbol{\ell} \;\cdot \underbrace{-g\,\nabla f}_{\mathbf{H}} = -\oint_C \mathrm{d}f\; g \tag{56}$$

Here, we have used that $\mathrm{d}\boldsymbol{\ell} \cdot \nabla g = \mathrm{d}\boldsymbol{\ell} \cdot \frac{\partial g}{\partial \mathbf{x}} = \underbrace{\mathrm{d}\mathbf{x}}_{\text{on } C} \cdot \frac{\partial g}{\partial \mathbf{x}} = \mathrm{d}g$ where $\mathrm{d}\boldsymbol{\ell}$ is simply $\mathrm{d}\mathbf{x}$ along the curve C. Similarly, $\mathrm{d}\boldsymbol{\ell} \cdot \nabla f = \mathrm{d}\boldsymbol{\ell} \cdot \frac{\partial f}{\partial \mathbf{x}} = \underbrace{\mathrm{d}\mathbf{x}}_{\text{on } C} \cdot \frac{\partial f}{\partial \mathbf{x}} = \mathrm{d}f$ where $\mathrm{d}\boldsymbol{\ell}$ is simply $\mathrm{d}\mathbf{x}$ along the curve C.

2.7 Switching the Constants in Partial Derivatives

This is a slightly more tricky proposition, but is in fact not so terribly difficult. Consider a function $f(x_1(t), x_2(t), \ldots, x_n(t), t)$ and then another equivalent relationship $g(a_1(t), a_2(t), \ldots, a_n(t), t)$. That is $f = g$, and so we have relationships

$$x_n(t) = x_n(a_1, a_2, \ldots, a_n, t) \tag{57}$$

$$a_n(t) = a_n(x_1, x_2, \ldots, x_n, t) \tag{58}$$

for all n. Note that all the x_i are independent of other x_j for $j \neq i$, and similarly a_i is independent of a_j for $j \neq i$. Define $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. For convenience $\mathbf{x}^j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ and analogously $\mathbf{a}^j = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$. Then we can write $f(\mathbf{x}) = f(\mathbf{a}) = g(\mathbf{a}) = g(\mathbf{x})$ as these are all equivalent formulations. Then we can write the differential forms as

$$\mathrm{d}f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)_{\mathbf{x}^i} \mathrm{d}x_i + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} \mathrm{d}t \tag{59}$$

$$\mathrm{d}f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial a_i}\right)_{\mathbf{a}^i} \mathrm{d}a_i + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{a}} \mathrm{d}t \tag{60}$$

$$\mathrm{d}x_i = \sum_{j=1}^{n} \left(\frac{\partial x_i}{\partial a_j}\right)_{\mathbf{a}^j} \mathrm{d}a_j \tag{61}$$

$$\mathrm{d}a_i = \sum_{j=1}^{n} \left(\frac{\partial a_i}{\partial x_j}\right)_{\mathbf{x}^j} \mathrm{d}x_j \tag{62}$$

Substitute (62) into (60) and we then have

$$\mathrm{d}f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial a_i}\right)_{\mathbf{a}^i} \sum_{j=1}^{n} \left(\frac{\partial a_i}{\partial x_j}\right)_{\mathbf{x}^j} \mathrm{d}x_j + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{a}} \mathrm{d}t \tag{63}$$

$$\mathrm{d}f = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial a_i}\right)_{\mathbf{a}^i} \left(\frac{\partial a_i}{\partial x_j}\right)_{\mathbf{x}^j} \mathrm{d}x_j + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{a}} \mathrm{d}t \tag{64}$$

$$\mathrm{d}f = \sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial f}{\partial a_i}\right)_{\mathbf{a}^i}\left(\frac{\partial a_i}{\partial x_j}\right)_{\mathbf{x}^j}\mathrm{d}x_j + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{a}}\mathrm{d}t \tag{65}$$

$$\mathrm{d}f = \left[\left(\frac{\partial \mathbf{a}}{\partial \mathbf{x}}\right)_{\mathbf{x}^j}\cdot\left(\frac{\partial f}{\partial \mathbf{a}}\right)_{\mathbf{a}^i}\right]\cdot \mathrm{d}\mathbf{x} + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{a}}\mathrm{d}t \tag{66}$$

where the last line uses tensor notation and is equivalent to the line before it. If we subtract the original expression from (59) (note that we take $i \to j$ in (59) so we are dealing with the same $\mathrm{d}x_j$ in each case), we see that we get

$$0 = \mathrm{d}f - \mathrm{d}f = \sum_{j=1}^{n}\left[\sum_{i=1}^{n}\left(\frac{\partial f}{\partial a_i}\right)_{\mathbf{a}^i}\left(\frac{\partial a_i}{\partial x_j}\right)_{\mathbf{x}^j} - \left(\frac{\partial f}{\partial x_j}\right)_{\mathbf{x}^j}\right]\mathrm{d}x_j + \left[\left(\frac{\partial f}{\partial t}\right)_{\mathbf{a}} - \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}}\right]\mathrm{d}t \tag{67}$$

because of the independence of the x_i [and t], each differential coefficient must equal zero independently. And so we find

$$\left(\frac{\partial f}{\partial x_j}\right)_{\mathbf{x}^j} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial a_i}\right)_{\mathbf{a}^i}\left(\frac{\partial a_i}{\partial x_j}\right)_{\mathbf{x}^j} \tag{68}$$

$$\left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{a}} \tag{69}$$

There was nothing special about privileging $\mathbf{x}$ and so we could substitute (61) into (59) and we then have

$$\mathrm{d}f = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)_{\mathbf{x}^i}\sum_{j=1}^{n}\left(\frac{\partial x_i}{\partial a_j}\right)_{\mathbf{a}^j}\mathrm{d}a_j + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}}\mathrm{d}t \tag{70}$$

$$\mathrm{d}f = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)_{\mathbf{x}^i}\left(\frac{\partial x_i}{\partial a_j}\right)_{\mathbf{a}^j}\mathrm{d}a_j + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}}\mathrm{d}t \tag{71}$$

$$\mathrm{d}f = \sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)_{\mathbf{x}^i}\left(\frac{\partial x_i}{\partial a_j}\right)_{\mathbf{a}^j}\mathrm{d}a_j + \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}}\mathrm{d}t \tag{72}$$

If we subtract the original expression from (60), we see that we get

$$0 = \mathrm{d}f - \mathrm{d}f = \sum_{j=1}^{n}\left[\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)_{\mathbf{x}^i}\left(\frac{\partial x_i}{\partial a_j}\right)_{\mathbf{a}^j} - \left(\frac{\partial f}{\partial a_j}\right)_{\mathbf{a}^j}\right]\mathrm{d}a_j + \left[\left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} - \left(\frac{\partial f}{\partial t}\right)_{\mathbf{a}}\right]\mathrm{d}t \tag{73}$$

because of the independence of the x_i [and t], each differential coefficient must equal zero independently. And so we find

$$\left(\frac{\partial f}{\partial a_j}\right)_{\mathbf{a}^j} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)_{\mathbf{x}^i}\left(\frac{\partial x_i}{\partial a_j}\right)_{\mathbf{a}^j} \tag{74}$$

$$\left(\frac{\partial f}{\partial t}\right)_{\mathbf{a}} = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} \tag{75}$$

What these say is that if we have a function that we can describe with two sets of variables $\mathbf{x}$ and $\mathbf{a}$, we can change equations with derivatives in $\mathbf{x}$ to derivatives in $\mathbf{a}$ using the rules in (68) and (74) (with the partial time derivatives the exact same holding either $\mathbf{x}$ or $\mathbf{a}$ constant).

In three dimensions with $f(x,y,z) = f(a,b,c)$ this can be written out completely as

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \left(\frac{\partial f}{\partial a}\right)_{b,c}\left(\frac{\partial a}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial b}\right)_{a,c}\left(\frac{\partial b}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial c}\right)_{a,b}\left(\frac{\partial c}{\partial x}\right)_{y,z} \tag{76}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,z} = \left(\frac{\partial f}{\partial a}\right)_{b,c}\left(\frac{\partial a}{\partial y}\right)_{x,z} + \left(\frac{\partial f}{\partial b}\right)_{a,c}\left(\frac{\partial b}{\partial y}\right)_{x,z} + \left(\frac{\partial f}{\partial c}\right)_{a,b}\left(\frac{\partial c}{\partial y}\right)_{x,z} \tag{77}$$

$$\left(\frac{\partial f}{\partial z}\right)_{x,y} = \left(\frac{\partial f}{\partial a}\right)_{b,c}\left(\frac{\partial a}{\partial z}\right)_{x,y} + \left(\frac{\partial f}{\partial b}\right)_{a,c}\left(\frac{\partial b}{\partial z}\right)_{x,y} + \left(\frac{\partial f}{\partial c}\right)_{a,b}\left(\frac{\partial c}{\partial z}\right)_{x,y} \tag{78}$$

or

$$\left(\frac{\partial f}{\partial a}\right)_{b,c} = \left(\frac{\partial f}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial a}\right)_{b,c} + \left(\frac{\partial f}{\partial y}\right)_{x,z}\left(\frac{\partial y}{\partial a}\right)_{b,c} + \left(\frac{\partial f}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial a}\right)_{b,c} \tag{79}$$

$$\left(\frac{\partial f}{\partial b}\right)_{a,c} = \left(\frac{\partial f}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial b}\right)_{a,c} + \left(\frac{\partial f}{\partial y}\right)_{x,z}\left(\frac{\partial y}{\partial b}\right)_{a,c} + \left(\frac{\partial f}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial b}\right)_{a,c} \tag{80}$$

$$\left(\frac{\partial f}{\partial c}\right)_{a,b} = \left(\frac{\partial f}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial c}\right)_{a,b} + \left(\frac{\partial f}{\partial y}\right)_{x,z}\left(\frac{\partial y}{\partial c}\right)_{a,b} + \left(\frac{\partial f}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial c}\right)_{a,b} \tag{81}$$

You should recognize this as simply applying the chain rule, because that is all it is.

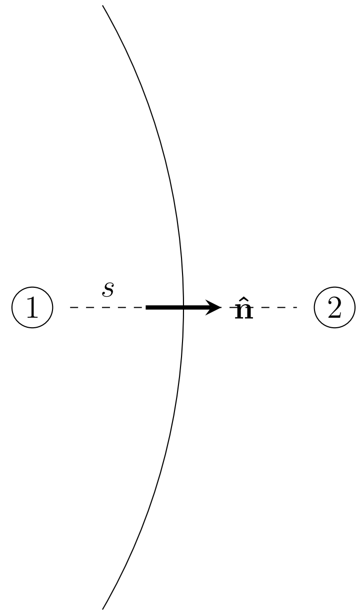

Figure 1: Geometry with parameterized path s as dashed line.

3 Rankine-Hugoniot Conditions for Conservation Laws

Consider a conservation law of the form

$$\frac{\partial \mathbf{u}(\mathbf{x})}{\partial t} + \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{F}}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \tag{82}$$

$$\partial_t u_i + \partial_j F_{ji} = S_i \tag{83}$$

We take region 1 to be the inside and region 2 to be the outside with the normal $\hat{\mathbf{n}}$ pointing from 1 to 2, now suppose we take a path integral along the normal from s_1 to s_2 ($\Delta s \equiv s_2 - s_1 \to 0$) with s parametrizing the path across the 1-2 interface (s is in the $\hat{\mathbf{n}}$ direction). See Figure 1. We then have

$$\int_{s_1}^{s_2} \mathrm{d}s \, \frac{\partial \mathbf{u}(\mathbf{x})}{\partial t} + \int_{s_1}^{s_2} \mathrm{d}s \, \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{F}}(\mathbf{x}) = \int_{s_1}^{s_2} \mathrm{d}s \, \mathbf{S} \tag{84}$$

$$\int_{s_1}^{s_2} \mathrm{d}s \, \partial_t u_i + \int_{s_1}^{s_2} \mathrm{d}s \, \partial_j F_{ji} = \int_{s_1}^{s_2} \mathrm{d}s \, S_i \tag{85}$$

We can parameterize $\mathbf{u}$, $\mathbf{S}$, and $\overleftrightarrow{\mathbf{F}}$ such that they are functions of s (then $\boldsymbol{\nabla}\cdot \to \hat{\mathbf{n}} \cdot \frac{\partial}{\partial s}$). We then see that

$$\int_{s_1}^{s_2} \mathrm{d}s \, \frac{\partial}{\partial t}\mathbf{u}(s) + \int_{s_1}^{s_2} \mathrm{d}s \, \hat{\mathbf{n}} \cdot \frac{\partial}{\partial s}\overleftrightarrow{\mathbf{F}}(s) = \int_{s_1}^{s_2} \mathrm{d}s \, \mathbf{S}(s) \tag{86}$$

$$\int_{s_1}^{s_2} \mathrm{d}s \, \partial_t u_i + \int_{s_1}^{s_2} \mathrm{d}s \, n_j \partial_s F_{ji} = \int_{s_1}^{s_2} \mathrm{d}s \, S_i \tag{87}$$

Now let's integrate from t to a short time later Δt. If the interface is moving at velocity $\mathbf{v}_{\text{int}}$ (which is wholly in the normal direction to the interface) then $\Delta s / \Delta t = \hat{\mathbf{n}} \cdot \mathbf{v}_{\text{int}} = n_j v_{\text{int},j}$. We

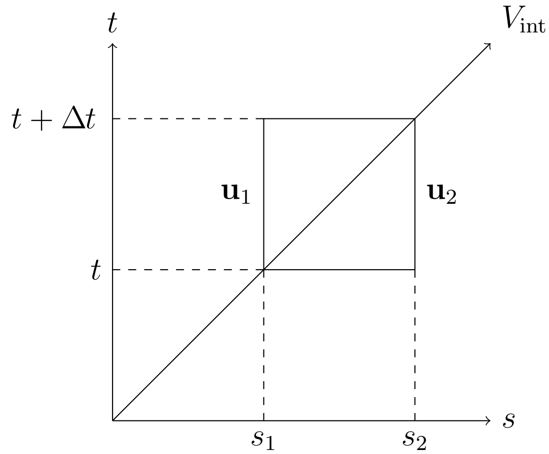

Figure 2: For integrating along s and t we see what values yield 1 quantities and 2 quantities. That is, below V_{int} we have 2 quantities such as $\mathbf{u}_2$ and above the V_{int} line we have 1 quantities such as $\mathbf{u}_1$.

then see

$$\int_{s_1}^{s_2} \mathrm{d}x \int_t^{t+\Delta t} \mathrm{d}t' \, \frac{\partial \mathbf{u}}{\partial t} + \int_t^{t+\Delta t} \mathrm{d}t' \int_{s_1}^{s_2} \mathrm{d}x \; \hat{\mathbf{n}} \cdot \frac{\partial \overleftrightarrow{\mathbf{F}}}{\partial s} = \int_t^{t+\Delta t} \mathrm{d}t' \int_{s_1}^{s_2} \mathrm{d}x \; \mathbf{S} \tag{88}$$

$$\int_{s_1}^{s_2} \mathrm{d}x \int_t^{t+\Delta t} \mathrm{d}t' \; \partial_t u_i + \int_t^{t+\Delta t} \mathrm{d}t' \int_{s_1}^{s_2} \mathrm{d}x \; n_j \partial_s F_{ji} = \int_t^{t+\Delta t} \mathrm{d}t' \int_{s_1}^{s_2} \mathrm{d}x \; S_i \tag{89}$$

Because $\mathbf{S}$ should be a continuous function of s, as $s_1 \to s_2$ gets very small, this contribution becomes negligible and we find

$$\int_{s_1}^{s_2} \mathrm{d}x \; (\mathbf{u}(s, t+\Delta t) - \mathbf{u}(s,t)) + \int_t^{t+\Delta t} \mathrm{d}t' \; \hat{\mathbf{n}} \cdot \left(\overleftrightarrow{\mathbf{F}}(s_2, t) - \overleftrightarrow{\mathbf{F}}(s_1, t) \right) = \int_t^{t+\Delta t} \mathrm{d}t' \; 0 \tag{90}$$

$$\int_{s_1}^{s_2} \mathrm{d}x \; (u_i(s, t+\Delta t) - u_i(s,t)) + \int_t^{t+\Delta t} \mathrm{d}t' \; n_j \cdot (F_{ji}(s_2, t) - F_{ji}(s_1, t)) = \int_t^{t+\Delta t} \mathrm{d}t' \; 0 \tag{91}$$

and so for small enough Δs and Δt we find these to be (see Figure 2)

$$\Delta s \; (\mathbf{u}(s, t+\Delta t) - \mathbf{u}(s,t)) + \Delta t \, \hat{\mathbf{n}} \cdot \left(\overleftrightarrow{\mathbf{F}}(s_2, t) - \overleftrightarrow{\mathbf{F}}(s_1, t) \right) = \mathbf{0} \tag{92}$$

$$\Delta s \; (u_i(s, t+\Delta t) - u_i(s,t)) + \Delta t \, n_j \cdot (F_{ji}(s_2, t) - F_{ji}(s_1, t)) = 0 \tag{93}$$

Dividing by Δt yields with $\Delta s / \Delta t = \mathbf{v}_{\text{int}} \cdot \hat{\mathbf{n}} = n_j v_{\text{int},j}$

$$\frac{\Delta s}{\Delta t} (\mathbf{u}(s, t+\Delta t) - \mathbf{u}(s,t)) + \hat{\mathbf{n}} \cdot \left(\overleftrightarrow{\mathbf{F}}(s_2, t) - \overleftrightarrow{\mathbf{F}}(s_1, t) \right) = \mathbf{0} \tag{94}$$

$$\frac{\Delta s}{\Delta t} (u_i(s, t+\Delta t) - u_i(s,t)) + n_j \cdot (F_{ji}(s_2, t) - F_{ji}(s_1, t)) = 0 \tag{95}$$

We then see that $\mathbf{u}(s, t+\Delta t)$ is $\mathbf{u}_1$ (the limiting value of $\mathbf{u}$ when going to the interface from within 1), and $\mathbf{u}(s,t) = \mathbf{u}_2 = u_{2,i}$ (the limiting value of going to the interface from within 2). Thus, with

$\overleftrightarrow{\mathbf{F}}_1 = F_{1,ji}$ and $\overleftrightarrow{\mathbf{F}}_2 = F_{2,ji}$ defined similarly, we see

$$\left[\mathbf{u}_1\hat{\mathbf{n}}\cdot\mathbf{v}_{\text{int}} - \mathbf{u}_2\hat{\mathbf{n}}\cdot\mathbf{v}_{\text{int}} + \hat{\mathbf{n}}\cdot(\overleftrightarrow{\mathbf{F}}_2 - \overleftrightarrow{\mathbf{F}}_1)\right] = \mathbf{0} \tag{96}$$

$$[u_{1,i}n_j v_{\text{int},j} - u_{2,i}n_j v_{\text{int},j} + n_j(F_{2,ji} - F_{1,ji})] = 0 \tag{97}$$

which using $[\![f]\!] = f_2 - f_1$ yields

$$[\![\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{F}} - \mathbf{u}(\hat{\mathbf{n}}\cdot\mathbf{v}_{\text{int}})]\!] = \mathbf{0} \tag{98}$$

$$\hat{\mathbf{n}}\cdot[\![\overleftrightarrow{\mathbf{F}} - \mathbf{v}_{\text{int}}\mathbf{u}]\!] = \mathbf{0} \tag{99}$$

$$n_j[\![F_{ji} - v_{\text{int},j}u_i]\!] = 0 \tag{100}$$

Note that had we defined $\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{F}} = \partial_j F_{ij}$ then the result would be

$$[\![\overleftrightarrow{\mathbf{F}} - \mathbf{u}\mathbf{v}_{\text{int}}]\!]\cdot\hat{\mathbf{n}} = \mathbf{0} \tag{101}$$

$$n_j[\![F_{ij} - u_i v_{\text{int},j}]\!] = 0 \tag{102}$$

4 Jacobians and Metric Tensors For Common Coordinate Systems

This appendix lists the most useful curvilinear coordinate system properties and transformations. It covers (common) cylindrical coordinates, (plasma) cylindrical coordinates, physicists' spherical coordinates, primitive toroidal coordinates, plasma toroidal coordinates, and general toroidal coordinates.

There are in fact quite a few variations in chosen variables, but I have tried to define a consistent set that are minimally confusing. My common cylindrical coordinates use (r, φ, Z) with r axial distance, φ the azimuthal angle, and Z the axial height. Mathematicians typically use (ρ, θ, z) with ρ axial distance, θ the azimuthal angle, and z the axial height. This notation is fine, but can cause confusion later with spherical coordinates. The plasma toroidal coordinates use (R, Z, ζ) where R is an axial distance, Z is an axial height, and ζ is an azimuthal angle. Note that ζ and φ point in opposite directions so that (R, Z, ζ) and (r, φ, Z) are both right-handed coordinates and the reason for the difference is that the (R, Z, ζ) system can be easily translated into primitive toroidal coordinates ($R \to r$, $Z \to \zeta$, $\zeta \to \theta$). The ISO standard for cylindrical coordinates is (ρ, φ, z).

Physicists' spherical coordinates (r, θ, φ) have r the radius, θ the polar angle, and φ the azimuthal angle. The mathematician's spherical coordinates are also often given by (r, θ, φ) but with θ meaning azimuthal angle and φ the polar angle. This should be avoided as then (r, θ, φ) is not a right-handed system. The logic is that mathematicians' cylindrical uses θ for the azimuth and they want to keep it there. The problems are many because of this lack of uniformity. I will always only use the ISO standard, which is the physicists' notation. Physicists' notation is also the only one consistent with how spherical harmonics are compiled. That is spherical harmonics always use θ as the polar angle, and φ as the azimuthal angle. If you are used to the mathematicians' notation, I would strongly recommend unlearning it and becoming comfortable with the physicists' notation because of the right-handedness and spherical harmonics advantages.

The various toroidal coordinate systems are mostly peculiar to plasma situations, though primitive toroidal coordinates are fairly well known even in mathematics. They use (r, θ, ζ) with r the minor radius, θ is the poloidal angle, and ζ is the toroidal angle. The other types of toroidal coordinates are rarely used, even in plasma physics, and so are listed mostly for completeness.

Note that the metric tensor(s) from coordinate systems (x^1, x^2, x^3) to (ξ^1, ξ^2, ξ^3) are given by the relations

$$g^{ij} = \sum_{k=1}^{3} \frac{\partial \xi^i}{\partial x^k}\frac{\partial \xi^j}{\partial x^k} = \nabla\xi^i \cdot \nabla\xi^j \tag{103}$$

$$g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j} = \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^j} \tag{104}$$

with $\mathbf{x} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ as the position vector. For orthogonal coordinates, off-diagonal $(i \neq j)$ terms should be zero. Also note that $g^{ij} = g^{ji}$ and $g_{ij} = g_{ji}$ and $\sum_{i,j=1}^{3} g_{ij}g^{ij} = \delta_{ij}$.

Also note that (for the $\mathcal{J}$ and J defined below)

$$\mathcal{J} = |\boldsymbol{\mathcal{J}}| = \frac{1}{\nabla\xi^1 \cdot \nabla\xi^2 \times \nabla\xi^3} = \frac{\partial \mathbf{x}}{\partial \xi^1} \cdot \frac{\partial \mathbf{x}}{\partial \xi^2} \times \frac{\partial \mathbf{x}}{\partial \xi^3} \tag{105}$$

$$J = |\mathbf{J}| = \nabla\xi^1 \cdot \nabla\xi^2 \times \nabla\xi^3 = \nabla\xi^2 \cdot \nabla\xi^3 \times \nabla\xi^1 = \nabla\xi^3 \cdot \nabla\xi^1 \times \nabla\xi^2 \tag{106}$$

$$\boldsymbol{\mathcal{J}} \equiv \frac{\partial(x^1, x^2, x^3)}{\partial(\xi^1, \xi^2, \xi^3)} = \begin{bmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^1}{\partial \xi^2} & \frac{\partial x^1}{\partial \xi^3} \\ \frac{\partial x^2}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^2} & \frac{\partial x^2}{\partial \xi^3} \\ \frac{\partial x^3}{\partial \xi^1} & \frac{\partial x^3}{\partial \xi^2} & \frac{\partial x^3}{\partial \xi^3} \end{bmatrix} \tag{107}$$

$$\mathbf{J} = \boldsymbol{\mathcal{J}}^{-1} \equiv \frac{\partial(\xi^1, \xi^2, \xi^3)}{\partial(x^1, x^2, x^3)} = \begin{bmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} & \frac{\partial \xi^1}{\partial x^3} \\ \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^2}{\partial x^2} & \frac{\partial \xi^2}{\partial x^3} \\ \frac{\partial \xi^3}{\partial x^1} & \frac{\partial \xi^3}{\partial x^2} & \frac{\partial \xi^3}{\partial x^3} \end{bmatrix} \tag{108}$$

This follows from the fact that the determinant of a matrix is the volume of the parallelpiped formed by creating vectors from the rows (or columns) of the matrix. The volume of a parallelpiped with vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ pointing from one corner of the parallelepiped has volume $\mathbf{r}_1 \cdot \mathbf{r}_2 \times \mathbf{r}_3$.

Unfortunately, $|\mathcal{J}|$ and $|J|$ are often both called “the Jacobian”, and even more unfortunately Jacobian can refer to the Jacobian matrix rather than the determinant of that matrix as $|\mathcal{J}|$ and $|J|$ are.

It should be noted that for volume element $\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$, the transformed volume element for integration is $|J|\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi^3$.

5 Generic Coordinate Conversion

Here let’s take a coordinate system, (ξ^1, ξ^2, ξ^3) which can be written out in Cartesian coordinates (x, y, z) and assume we know

$$\xi^1 = \xi^1(x, y, z) \tag{109}$$

$$\xi^2 = \xi^2(x, y, z) \tag{110}$$

$$\xi^3 = \xi^3(x, y, z) \tag{111}$$

and assume it is invertible (In other words the Jacobian determinant $|\boldsymbol{\mathcal{J}}| \neq 0$ for this coordinate system transformation)

$$x = x(\xi^1, \xi^2, \xi^3) \tag{112}$$

$$y = y(\xi^1, \xi^2, \xi^3) \tag{113}$$

$$z = z(\xi^1, \xi^2, \xi^3) \tag{114}$$

So we can then find

$$J = \nabla\xi^1 \cdot \nabla\xi^2 \times \nabla\xi^3 = \frac{\partial \xi^1}{\partial \mathbf{x}} \cdot \frac{\partial \xi^2}{\partial \mathbf{x}} \times \frac{\partial \xi^3}{\partial \mathbf{x}} \tag{115}$$

$$\mathcal{J} = \frac{1}{\nabla\xi^1 \cdot \nabla\xi^2 \times \nabla\xi^3} = \frac{\partial \mathbf{x}}{\partial \xi^1} \cdot \frac{\partial \mathbf{x}}{\partial \xi^2} \times \frac{\partial \mathbf{x}}{\partial \xi^3} \tag{116}$$

We can then form the covariant components of the metric tensor $g_{ij} = \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^j}$ with $\mathbf{x} = x\mathbf{x} + y\mathbf{y} + z\mathbf{z}$ a position vector. Note we could write

$$\mathbf{x} = x(\xi^1, \xi^2, \xi^3)\hat{\mathbf{x}} + y(\xi^1, \xi^2, \xi^3)\hat{\mathbf{y}} + z(\xi^1, \xi^2, \xi^3)\hat{\mathbf{z}} \tag{117}$$

and then we would have as components

$$g_{11} = \left(\left(\frac{\partial x(\xi^1, \xi^2, \xi^3)}{\partial \xi^1}\right)_{\xi^2, \xi^3}\right)^2 + \left(\left(\frac{\partial y(\xi^1, \xi^2, \xi^3)}{\partial \xi^1}\right)_{\xi^2, \xi^3}\right)^2 + \left(\left(\frac{\partial z(\xi^1, \xi^2, \xi^3)}{\partial \xi^1}\right)_{\xi^2, \xi^3}\right)^2 \tag{118}$$

$$g_{11} = \left(\frac{\partial x}{\partial \xi^1}\right)^2 + \left(\frac{\partial y}{\partial \xi^1}\right)^2 + \left(\frac{\partial z}{\partial \xi^1}\right)^2 \tag{119}$$

$$\begin{aligned} g_{i'j'} = &\left(\frac{\partial x(\xi^1, \xi^2, \xi^3)}{\partial \xi^{i'}}\right)_{\xi^{j'}, \xi^{k'}} \left(\frac{\partial x(\xi^1, \xi^2, \xi^3)}{\partial \xi^{j'}}\right)_{\xi^{i'}, \xi^{k'}} + \left(\frac{\partial y(\xi^1, \xi^2, \xi^3)}{\partial \xi^{i'}}\right)_{\xi^{j'}, \xi^{k'}} \left(\frac{\partial y(\xi^1, \xi^2, \xi^3)}{\partial \xi^{j'}}\right)_{\xi^{i'}, \xi^{k'}} \\ &+ \left(\frac{\partial z(\xi^1, \xi^2, \xi^3)}{\partial \xi^{i'}}\right)_{\xi^{j'}, \xi^{k'}} \left(\frac{\partial z(\xi^1, \xi^2, \xi^3)}{\partial \xi^{j'}}\right)_{\xi^{i'}, \xi^{k'}} \end{aligned} \tag{120}$$

with the i', j', k' not a sum but an even permutation of $1, 2, 3$. Note that g_{11} is the same, but (118) explicitly shows the objects held constant.

Note that we would find the tangent vector basis (sometimes called the "covariant" vector basis, but remember this is not a great name) as

$$\mathbf{e}_1 = \mathbf{e}_{\xi^1} = \frac{\partial \mathbf{x}}{\partial \xi^1} \tag{121}$$

$$\mathbf{e}_2 = \mathbf{e}_{\xi^2} = \frac{\partial \mathbf{x}}{\partial \xi^2} \tag{122}$$

$$\mathbf{e}_3 = \mathbf{e}_{\xi^3} = \frac{\partial \mathbf{x}}{\partial \xi^3} \tag{123}$$

with $\mathcal{J} = \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3$. Then the tangent-reciprocal vector basis (again, sometimes called the "contravariant" vector basis, but this is a poor name) as

$$\mathbf{e}^1 = \mathbf{e}^{\xi^1} = \nabla \xi^1 \tag{124}$$

$$\mathbf{e}^2 = \mathbf{e}^{\xi^2} = \nabla \xi^2 \tag{125}$$

$$\mathbf{e}^3 = \mathbf{e}^{\xi^3} = \nabla \xi^3 \tag{126}$$

Remember we can use reciprocal relations so that [with (i', j', k') an even cyclic permutation of $(1, 2, 3)$]

$$\mathbf{e}^{i'} = \nabla \xi^{i'} = \frac{\mathbf{e}_{j'} \times \mathbf{e}_{k'}}{\mathbf{e}_{i'} \cdot \mathbf{e}_{j'} \times \mathbf{e}_{k'}} = \frac{\mathbf{e}_{j'} \times \mathbf{e}_{k'}}{\mathcal{J}} = \frac{\frac{\partial \mathbf{x}}{\partial \xi^{j'}} \times \frac{\partial \mathbf{x}}{\partial \xi^{k'}}}{\frac{\partial \mathbf{x}}{\partial \xi^{i'}} \cdot \left(\frac{\partial \mathbf{x}}{\partial \xi^{j'}} \times \frac{\partial \mathbf{x}}{\partial \xi^{k'}}\right)} \tag{??}$$

$$\mathbf{e}_{i'} = \frac{\partial \mathbf{x}}{\partial \xi^{i'}} = \frac{\mathbf{e}^{j'} \times \mathbf{e}^{k'}}{\mathbf{e}^{i'} \cdot \mathbf{e}^{j'} \times \mathbf{e}^{k'}} = \mathcal{J} \mathbf{e}^{j'} \times \mathbf{e}^{k'} = \frac{\nabla \xi^{j'} \times \nabla \xi^{k'}}{\nabla \xi^{i'} \cdot \nabla \xi^{j'} \times \nabla \xi^{k'}} \tag{??}$$

We can then form the contravariant components of the metric tensor $g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j = \nabla\xi^i \cdot \nabla\xi^j$. We can define $x^1 = x$, $x^2 = y$, and $x^3 = z$ for convenience, as well. Note we could write

$$g^{11} = \left(\left(\frac{\partial\xi^1(x,y,z)}{\partial x}\right)_{y,z}\right)^2 + \left(\left(\frac{\partial\xi^1(x,y,z)}{\partial y}\right)_{z,x}\right)^2 + \left(\left(\frac{\partial\xi^1(x,y,z)}{\partial z}\right)_{x,y}\right)^2 \tag{127}$$

$$g^{11} = \left(\frac{\partial\xi^1}{\partial x}\right)^2 + \left(\frac{\partial\xi^1}{\partial y}\right)^2 + \left(\frac{\partial\xi^1}{\partial z}\right)^2 \tag{128}$$

$$\begin{aligned} g^{i'j'} = &\left(\frac{\partial\xi^1(x^1,x^2,x^3)}{\partial x^{i'}}\right)_{x^{j'},x^{k'}} \left(\frac{\partial\xi^1(x^1,x^2,x^3)}{\partial x^{j'}}\right)_{x^{i'},x^{k'}} + \left(\frac{\partial\xi^2(x^1,x^2,x^3)}{\partial x^{i'}}\right)_{x^{j'},x^{k'}} \left(\frac{\partial\xi^2(x^1,x^2,x^3)}{\partial x^{j'}}\right)_{x^{i'},x^{k'}} \\ &+ \left(\frac{\partial\xi^3(x^1,x^2,x^3)}{\partial x^{i'}}\right)_{x^{j'},x^{k'}} \left(\frac{\partial\xi^3(x^1,x^2,x^3)}{\partial x^{j'}}\right)_{x^{i'},x^{k'}} \end{aligned} \tag{129}$$

with the i', j', k' not a sum but an even permutation of $1, 2, 3$.

Finally, I will list the Christoffel symbols via

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial\xi^j} + \frac{\partial g_{jk}}{\partial\xi^i} - \frac{\partial g_{ij}}{\partial\xi^k}\right] \tag{130}$$

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{131}$$

and list the Christoffel symbols one at a time as a matrix. Thus $\Gamma^{k'}_{ij}$ is listed for each k' as a matrix $\mathbf{M}$ with entries M_{ij} given by $\Gamma^{k'}_{ij}$.

6 (Common) Cylindrical Coordinates

We have Cartesian (x, y, z) and cylindrical (r, φ, Z) as our two coordinate systems. ($0 \le r < \infty$, $0 \le \varphi \le 2\pi$, and $-\infty < Z < \infty$)

We use the equations

$$r^2 = x^2 + y^2 \tag{132}$$

$$\tan\varphi = \frac{y}{x} \tag{133}$$

$$Z = z \tag{134}$$

Thus, we find

$$\begin{aligned} r\,\mathrm{d}r &= x\,\mathrm{d}x + y\,\mathrm{d}y \\ \mathrm{d}r &= \frac{x}{r}\,\mathrm{d}x + \frac{y}{r}\,\mathrm{d}y = \cos\varphi\,\mathrm{d}x + \sin\varphi\,\mathrm{d}y \end{aligned} \tag{135}$$

$$\begin{aligned} \sec^2\varphi\,\mathrm{d}\varphi &= \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} \\ \mathrm{d}\varphi &= \cos^2\varphi\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} = \frac{x^2}{x^2+y^2}\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = \frac{\cos\varphi}{r}\,\mathrm{d}y - \frac{\sin\varphi}{r}\,\mathrm{d}x \end{aligned} \tag{136}$$

$$\mathrm{d}Z = \mathrm{d}z \tag{137}$$

and so

$$\mathbf{J} = \frac{\partial(r,\varphi,Z)}{\partial(x,y,z)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\frac{\sin\varphi}{r} & \frac{\cos\varphi}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{138}$$

$$J = \frac{\cos\varphi\cos\varphi}{r} - \frac{\sin\varphi\sin\varphi}{r} = \frac{1}{r} \tag{139}$$

Note that we then have

$$\mathbf{e}^1 = \mathbf{e}^r = \nabla r = \cos\varphi\,\nabla x + \sin\varphi\,\nabla y \tag{140}$$

$$|\nabla r| = 1 \tag{141}$$

$$\mathbf{e}^2 = \mathbf{e}^\varphi = \nabla\varphi = -\frac{\sin\varphi}{r}\nabla x + \frac{\cos\varphi}{r}\nabla y \tag{142}$$

$$|\nabla\varphi| = \sqrt{\frac{\sin^2\varphi + \cos^2\varphi}{r^2}} = \frac{1}{r} \tag{143}$$

$$\mathbf{e}^3 = \mathbf{e}^Z = \nabla Z = \nabla z \tag{144}$$

$$|\nabla Z| = 1 \tag{145}$$

So that

$$\hat{\mathbf{e}}^1 = \hat{\mathbf{e}}^r = \hat{\mathbf{r}} = \cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} \tag{146}$$

$$\hat{\mathbf{e}}^2 = \hat{\mathbf{e}}^\varphi = \hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}} = -\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} \tag{147}$$

$$\hat{\mathbf{e}}^3 = \hat{\mathbf{e}}^Z = \hat{\mathbf{Z}} = \hat{\mathbf{z}} \tag{148}$$

The metric tensor is given by $g^{ij} = \sum_{k=1}^{3} \frac{\partial \xi^i}{\partial x^k}\frac{\partial \xi^j}{\partial x^k}$. Thus

$$\begin{aligned} g^{rr} &= \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2 \\ &= \cos^2\varphi + \sin^2\varphi + 0^2 = 1 \end{aligned} \tag{149}$$

$$\begin{aligned} g^{\varphi\varphi} &= \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \\ &= \frac{\sin^2\varphi}{r^2} + \frac{\cos^2\varphi}{r^2} + 0 = \frac{1}{r^2} \end{aligned} \tag{150}$$

$$\begin{aligned} g^{ZZ} &= \left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2 \\ &= 0 + 0 + 1 = 1 \end{aligned} \tag{151}$$

$$\begin{aligned} g^{r\varphi} &= \frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \varphi}{\partial z} \\ &= \cos\varphi\frac{-\sin\varphi}{r} + \sin\varphi\frac{\cos\varphi}{r} + 0 = 0 \end{aligned} \tag{152}$$

$$\begin{aligned} g^{rZ} &= \frac{\partial r}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial Z}{\partial z} \\ &= \cos\varphi(0) + \sin\varphi(0) + 0(1) = 0 \end{aligned} \tag{153}$$

$$\begin{aligned} g^{\varphi Z} &= \frac{\partial \varphi}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial \varphi}{\partial z}\frac{\partial Z}{\partial z} \\ &= \frac{-\sin\varphi}{r}(0) + \frac{\cos\varphi}{r}(0) + 0(1) = 0 \end{aligned} \tag{154}$$

Thus

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{155}$$

In the other direction we would use

$$x = r\cos\varphi \tag{156}$$
$$y = r\sin\varphi \tag{157}$$
$$z = Z \tag{158}$$

and so

$$\mathrm{d}x = \cos\varphi\,\mathrm{d}r - r\sin\varphi\,\mathrm{d}\varphi \tag{159}$$
$$\mathrm{d}y = \sin\varphi\,\mathrm{d}r + r\cos\varphi\,\mathrm{d}\varphi \tag{160}$$
$$\mathrm{d}z = \mathrm{d}Z \tag{161}$$
$$\mathbf{e}_1 = \mathbf{e}_r = \left(\frac{\partial \mathbf{x}}{\partial r}\right)_{\theta,\varphi} = \cos\varphi\sin\theta\,\nabla x + \sin\varphi\sin\theta\,\nabla y + \cos\theta \tag{162}$$
$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = r\cos\varphi\cos\theta\,\nabla x + r\sin\varphi\cos\theta\,\nabla y - r\sin\theta\,\nabla z \tag{163}$$
$$\mathbf{e}_3 = \mathbf{e}_\varphi = \frac{\partial \mathbf{x}}{\partial \varphi} = -r\sin\varphi\sin\theta\,\nabla x + r\cos\varphi\sin\theta\,\nabla y \tag{164}$$

and so we then have

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(r,\varphi,Z)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial Z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{165}$$
$$\mathcal{J} = r\cos\varphi\cos\varphi + r\sin\varphi\sin\varphi = r \tag{166}$$

Note that we then have

$$\hat{\mathbf{x}} = \cos\varphi\,\nabla r - r\sin\varphi\,\nabla\varphi = \cos\varphi\hat{\mathbf{r}} - \sin\varphi\hat{\boldsymbol{\varphi}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{r}} - \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\varphi}} \tag{167}$$
$$\hat{\mathbf{y}} = \sin\varphi\,\nabla r + r\cos\varphi\,\nabla\varphi = \sin\varphi\hat{\mathbf{r}} + \cos\varphi\hat{\boldsymbol{\varphi}} = \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{r}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\varphi}} \tag{168}$$
$$\hat{\mathbf{z}} = \nabla Z = \hat{\mathbf{Z}} \tag{169}$$

The other metric tensor is given by $g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}$. Thus

$$g_{rr} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \cos^2\varphi + \sin^2\varphi + 0^2 = 1 \tag{170}$$

$$g_{\varphi\varphi} = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 = r^2\sin^2\varphi + r^2\cos^2\varphi + 0 = r^2 \tag{171}$$

$$g_{ZZ} = \left(\frac{\partial x}{\partial Z}\right)^2 + \left(\frac{\partial y}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 = 0 + 0 + 1 = 1 \tag{172}$$

$$g_{r\varphi} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \varphi} = \cos\varphi(-r\sin\varphi) + \sin\varphi(r\cos\varphi) + 0 = 0 \tag{173}$$

$$g_{rZ} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial Z} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial Z} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial Z} = \cos\varphi(0) + \sin\varphi(0) + 0(1) = 0 \tag{174}$$

$$g_{\varphi Z} = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial Z} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial Z} + \frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial Z} = -r\sin\varphi(0) + r\cos\varphi(0) + 0(1) = 0 \tag{175}$$

Thus

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{176}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right] \tag{177}$$

$$\Gamma_{r,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{178}$$

$$\Gamma_{\varphi,ij} = \begin{bmatrix} 0 & r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{179}$$

$$\Gamma_{Z,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{180}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{181}$$

$$\Gamma^r_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{182}$$

$$\Gamma^\varphi_{ij} = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{183}$$

$$\Gamma^Z_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{184}$$

7 (Plasma/Toroidal System) Cylindrical Coordinates

We have Cartesian (x, y, z) and cylindrical (R, Z, ζ) as our two coordinate systems. ($0 \le R < \infty$, $-\infty < Z < \infty$, and $0 \le \zeta \le 2\pi$)

We use the equations

$$R^2 = x^2 + y^2 \tag{185}$$

$$\tan(-\zeta) = \frac{y}{x} \tag{186}$$

$$Z = z \tag{187}$$

Thus, we find

$$\begin{aligned} R\,\mathrm{d}R &= x\,\mathrm{d}x + y\,\mathrm{d}y \\ \mathrm{d}R &= \frac{x}{R}\,\mathrm{d}x + \frac{y}{R}\,\mathrm{d}y = \cos(-\zeta)\,\mathrm{d}x + \sin(-\zeta)\,\mathrm{d}y = \cos\zeta\,\mathrm{d}x - \sin\zeta\,\mathrm{d}y \end{aligned} \tag{188}$$

$$\begin{aligned} -\sec^2\zeta\,\mathrm{d}\zeta &= \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} \\ \mathrm{d}\zeta &= \cos^2\zeta \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2} = \frac{x^2}{x^2+y^2}\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2} = \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2+y^2} \\ &= \frac{\sin(-\zeta)}{R}\,\mathrm{d}x - \frac{\cos(-\zeta)}{R}\,\mathrm{d}y = -\frac{\sin\zeta}{R}\,\mathrm{d}x - \frac{\cos\zeta}{R}\,\mathrm{d}y \end{aligned} \tag{189}$$

$$\mathrm{d}Z = \mathrm{d}z \tag{190}$$

and so

$$\mathbf{J} = \frac{\partial(R, Z, \zeta)}{\partial(x, y, z)} = \begin{bmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\zeta & -\sin\zeta & 0 \\ 0 & 0 & 1 \\ -\frac{\sin\zeta}{R} & -\frac{\cos\zeta}{R} & 0 \end{bmatrix} \tag{191}$$

$$J = -(\frac{-\cos\zeta\cos\zeta}{R} - \frac{\sin\zeta\sin\zeta}{R}) = \frac{1}{R} \tag{192}$$

Note that we then have

$$\mathbf{e}^1 = \mathbf{e}^R = \nabla R = \cos\zeta\,\nabla x - \sin\zeta\,\nabla y \tag{193}$$
$$|\nabla R| = 1 \tag{194}$$
$$\mathbf{e}^2 = \mathbf{e}^\zeta = \nabla\zeta = -\frac{\sin\zeta}{R}\nabla x - \frac{\cos\zeta}{R}\nabla y \tag{195}$$
$$|\nabla\zeta| = \sqrt{\frac{\sin^2\zeta + \cos^2\zeta}{R^2}} = \frac{1}{R} \tag{196}$$
$$\mathbf{e}^3 = \mathbf{e}^Z = \nabla Z = \nabla z \tag{197}$$
$$|\nabla Z| = 1 \tag{198}$$

So that

$$\hat{\mathbf{e}}^1 = \hat{\mathbf{e}}^R = \hat{\mathbf{R}} = \cos\zeta\hat{\mathbf{x}} - \sin\zeta\hat{\mathbf{y}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} \tag{199}$$
$$\hat{\mathbf{e}}^2 = \hat{\mathbf{e}}^\zeta = \hat{\boldsymbol{\zeta}} = -\sin\zeta\hat{\mathbf{x}} - \cos\zeta\hat{\mathbf{y}} = \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} - \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} \tag{200}$$
$$\hat{\mathbf{e}}^3 = \hat{\mathbf{e}}^Z = \hat{\mathbf{Z}} = \hat{\mathbf{z}} \tag{201}$$

The metric tensor is given by $g^{ij} = \sum_{k=1}^{3} \frac{\partial\xi^i}{\partial x^k}\frac{\partial\xi^j}{\partial x^k}$. Thus

$$\begin{aligned} g^{RR} &= \left(\frac{\partial R}{\partial x}\right)^2 + \left(\frac{\partial R}{\partial y}\right)^2 + \left(\frac{\partial R}{\partial z}\right)^2 \\ &= \cos^2\zeta + \sin^2\zeta + 0^2 = 1 \end{aligned} \tag{202}$$

$$\begin{aligned} g^{ZZ} &= \left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2 \\ &= 0 + 0 + 1 = 1 \end{aligned} \tag{203}$$

$$\begin{aligned} g^{\zeta\zeta} &= \left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2 \\ &= \frac{\sin^2\theta}{R^2} + \frac{\cos^2\zeta}{R^2} + 0 = \frac{1}{R^2} \end{aligned} \tag{204}$$

$$\begin{aligned} g^{RZ} &= \frac{\partial R}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial R}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial Z}{\partial z} \\ &= \cos\zeta(0) + \sin\zeta(0) + 0(1) = 0 \end{aligned} \tag{205}$$

$$\begin{aligned} g^{R\zeta} &= \frac{\partial R}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial R}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial \zeta}{\partial z} \\ &= \cos\zeta\frac{-\sin\zeta}{R} - \sin\zeta\frac{-\cos\zeta}{R} + 0 = 0 \end{aligned} \tag{206}$$

$$\begin{aligned} g^{Z\zeta} &= \frac{\partial Z}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial Z}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial Z}{\partial z}\frac{\partial \zeta}{\partial z} \\ &= (0)\frac{-\sin\zeta}{R} + (0)\frac{-\cos\varphi}{R} + (1)0 = 0 \end{aligned} \tag{207}$$

Thus

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{R^2} \end{bmatrix} \tag{208}$$

In the other direction we would use

$$x = R\cos\zeta \tag{209}$$
$$y = -R\sin\zeta \tag{210}$$
$$z = Z \tag{211}$$
$$\mathbf{e}_1 = \mathbf{e}_r = \left(\frac{\partial \mathbf{x}}{\partial R}\right)_{Z,\zeta} = \cos\zeta\,\nabla x - \sin\zeta\,\nabla y \tag{212}$$
$$\mathbf{e}_2 = \mathbf{e}_Z = \frac{\partial \mathbf{x}}{\partial Z} = \nabla z \tag{213}$$
$$\mathbf{e}_3 = \mathbf{e}_\zeta = \frac{\partial \mathbf{x}}{\partial \zeta} = -R\sin\zeta\,\nabla x - R\cos\zeta\,\nabla y \tag{214}$$

and so

$$\mathrm{d}x = \cos\zeta\,\mathrm{d}R - R\sin\zeta\,\mathrm{d}\zeta \tag{215}$$
$$\mathrm{d}y = -\sin\zeta\,\mathrm{d}R - R\cos\zeta\,\mathrm{d}\zeta \tag{216}$$
$$\mathrm{d}z = \mathrm{d}Z \tag{217}$$

and so we then have

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(R,Z,\zeta)} = \begin{bmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial Z} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial Z} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial Z} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \cos\zeta & 0 & -R\sin\zeta \\ -\sin\zeta & 0 & -R\cos\zeta \\ 0 & 1 & 0 \end{bmatrix} \tag{218}$$
$$\mathcal{J} = -(-R\cos\zeta\cos\zeta + R\sin\zeta\sin\zeta) = R \tag{219}$$

Note that we then have

$$\hat{\mathbf{x}} = \cos\zeta\,\nabla R - R\sin\zeta\,\nabla\zeta = \cos\zeta\hat{\mathbf{R}} - \sin\zeta\hat{\boldsymbol{\zeta}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{R}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\zeta}} \tag{220}$$
$$\hat{\mathbf{y}} = -\sin\zeta\,\nabla R - R\cos\zeta\,\nabla\zeta = -\sin\zeta\hat{\mathbf{R}} - \cos\zeta\hat{\boldsymbol{\zeta}} = \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{R}} - \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\zeta}} \tag{221}$$
$$\hat{\mathbf{z}} = \nabla Z = \hat{\mathbf{Z}} \tag{222}$$

The other metric tensor is given by $g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}$. Thus

$$\begin{aligned} g_{RR} &= \left(\frac{\partial x}{\partial R}\right)^2 + \left(\frac{\partial y}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2 \\ &= \cos^2\zeta + \sin^2\zeta + 0^2 = 1 \end{aligned} \tag{223}$$
$$\begin{aligned} g_{ZZ} &= \left(\frac{\partial x}{\partial Z}\right)^2 + \left(\frac{\partial y}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 \\ &= 0 + 0 + 1 = 1 \end{aligned} \tag{224}$$

$$
\begin{aligned}
g_{\zeta\zeta} &= \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2 \\
&= R^2 \sin^2 \zeta + R^2 \cos^2 \zeta + 0 = R^2
\end{aligned} \tag{225}
$$

$$
\begin{aligned}
g_{RZ} &= \frac{\partial x}{\partial R}\frac{\partial x}{\partial Z} + \frac{\partial y}{\partial R}\frac{\partial y}{\partial Z} + \frac{\partial z}{\partial R}\frac{\partial z}{\partial Z} \\
&= \cos\zeta(0) + (-\sin\zeta)(0) + 0(1) = 0
\end{aligned} \tag{226}
$$

$$
\begin{aligned}
g_{R\zeta} &= \frac{\partial x}{\partial R}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial R}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial R}\frac{\partial z}{\partial \zeta} \\
&= \cos\varphi(-R\sin\zeta) - \sin\varphi(-R\cos\zeta) + 0 = 0
\end{aligned} \tag{227}
$$

$$
\begin{aligned}
g_{Z\zeta} &= \frac{\partial x}{\partial Z}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial Z}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial Z}\frac{\partial z}{\partial \zeta} \\
&= (0)(-R\sin\zeta) + (0)(-R\cos\zeta) + (1)0 = 0
\end{aligned} \tag{228}
$$

Thus

$$
g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^2 \end{bmatrix} \tag{229}
$$

Thus we find for the Christoffel symbols that

$$
\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right] \tag{230}
$$

$$
\Gamma_{R,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -R \end{bmatrix} \tag{231}
$$

$$
\Gamma_{Z,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{232}
$$

$$
\Gamma_{\zeta,ij} = \begin{bmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ R & 0 & 0 \end{bmatrix} \tag{233}
$$

and

$$
\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{234}
$$

$$
\Gamma^R_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -R \end{bmatrix} \tag{235}
$$

$$
\Gamma^Z_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{236}
$$

$$
\Gamma^\zeta_{ij} = \begin{bmatrix} 0 & 0 & \frac{1}{R} \\ 0 & 0 & 0 \\ \frac{1}{R} & 0 & 0 \end{bmatrix} \tag{237}
$$

8 (Physicists') Spherical Coordinates

We have Cartesian (x, y, z) and spherical (r, θ, φ) as our two coordinate systems. ($0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$)

We use the equations

$$r^2 = x^2 + y^2 + z^2 \tag{238}$$

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z} \Leftrightarrow \cos\theta = \frac{z}{r} \tag{239}$$

$$\tan\varphi = \frac{y}{x} \tag{240}$$

Thus, we find (using $\frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}\frac{\sqrt{x^2+y^2}}{r} = \cos\varphi\sin\theta$ and similarly for y and that $z = r\cos\theta$ so that $\sqrt{x^2+y^2} = r\sin\theta$)

$$\begin{aligned} r\,\mathrm{d}r &= x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z \\ \mathrm{d}r &= \frac{x}{r}\,\mathrm{d}x + \frac{y}{r}\,\mathrm{d}y = \frac{z}{r}\,\mathrm{d}z = \cos\varphi\sin\theta\,\mathrm{d}x + \sin\varphi\sin\theta\,\mathrm{d}y + \cos\theta\,\mathrm{d}z \end{aligned} \tag{241}$$

$$\begin{aligned} \sec^2\theta\,\mathrm{d}\theta &= \frac{\frac{zx\,\mathrm{d}x + zy\,\mathrm{d}y}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}\,\mathrm{d}z}{z^2} = \frac{x}{z\sqrt{x^2+y^2}}\,\mathrm{d}x + \frac{y}{z\sqrt{x^2+y^2}}\,\mathrm{d}y - \frac{\sqrt{x^2+y^2}}{z^2}\,\mathrm{d}z \\ \mathrm{d}\theta &= \frac{zx}{r^2\sqrt{x^2+y^2}}\,\mathrm{d}x + \frac{zy}{r^2\sqrt{x^2+y^2}}\,\mathrm{d}y - \frac{\sqrt{x^2+y^2}}{r^2}\,\mathrm{d}z \\ \mathrm{d}\theta &= \frac{(r\cos\theta)(r\sin\theta\cos\varphi)}{r^3\sin\theta}\,\mathrm{d}x + \frac{(r\cos\theta)(r\sin\theta\sin\varphi)}{r^3\sin\theta}\,\mathrm{d}y - \frac{r\sin\theta}{r^2}\,\mathrm{d}z \\ &= \frac{\cos\varphi\cos\theta}{r}\,\mathrm{d}x + \frac{\sin\varphi\cos\theta}{r}\,\mathrm{d}y - \frac{\sin\theta}{r}\,\mathrm{d}z \end{aligned} \tag{242}$$

$$\begin{aligned} \sec^2\varphi\,\mathrm{d}\varphi &= \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} \\ \mathrm{d}\varphi &= \cos^2\varphi\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} = \frac{x^2}{x^2+y^2}\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = -\frac{\sin\varphi}{r\sin\theta}\,\mathrm{d}x + \frac{\cos\varphi}{r\sin\theta}\,\mathrm{d}y \end{aligned} \tag{243}$$

and so

$$\mathbf{J} = \frac{\partial(r,\theta,\varphi)}{\partial(x,y,z)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \frac{\cos\varphi\cos\theta}{r} & \frac{\sin\varphi\cos\theta}{r} & -\frac{\sin\theta}{r} \\ \frac{-\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{bmatrix} \tag{244}$$

$$J = \frac{\sin\theta}{r}\left(\cos\varphi\sin\theta\frac{\cos\varphi}{r\sin\theta} - \sin\varphi\sin\theta\frac{-\sin\varphi}{r\sin\theta}\right) + \cos\theta\left(\frac{\cos\varphi\cos\theta}{r}\frac{\cos\varphi}{r\sin\theta} - \frac{\sin\varphi\cos\theta}{r}\frac{-\sin\varphi}{r\sin\theta}\right) \tag{245}$$

$$= \frac{\sin\theta}{r}\frac{\cos^2\varphi + \sin^2\varphi}{r} + \frac{\cos^2\theta}{r^2\sin\theta}\left(\cos^2\varphi + \sin^2\varphi\right) = \frac{\sin^2\theta + \cos^2\theta}{r^2\sin\theta} = \frac{1}{r^2\sin\theta} \tag{246}$$

Note that we then have

$$\mathbf{e}^1 = \mathbf{e}^r = \nabla r = \cos\varphi\sin\theta\,\nabla x + \sin\varphi\sin\theta\,\nabla y + \cos\theta\,\nabla z \tag{247}$$

$$|\nabla r| = 1 \tag{248}$$

$$\mathbf{e}^2 = \mathbf{e}^\theta = \nabla\theta = \frac{\cos\varphi\cos\theta}{r}\,\nabla x + \frac{\sin\varphi\cos\theta}{r}\,\nabla y - \frac{\sin\theta}{r}\,\nabla z \tag{249}$$

$$|\nabla\theta| = \sqrt{\frac{(\cos^2\varphi + \sin^2\varphi)\cos^2\theta + \sin^2\theta}{r^2}} = \sqrt{\frac{1}{r^2}} = \frac{1}{r} \tag{250}$$

$$\mathbf{e}^3 = \mathbf{e}^\varphi = \nabla\varphi = -\frac{\sin\varphi}{r\sin\theta}\,\nabla x + \frac{\cos\varphi}{r\sin\theta}\,\nabla y \tag{251}$$

$$|\nabla\varphi| = \sqrt{\frac{\sin^2\varphi + \cos^2\varphi}{r^2\sin^2\theta}} = \sqrt{\frac{1}{r^2\sin^2\theta}} = \frac{1}{r\sin\theta} \tag{252}$$

So that

$$\hat{\mathbf{e}}^1 = \hat{\mathbf{e}}^r = \hat{\mathbf{r}} = \cos\varphi\sin\theta\hat{\mathbf{x}} + \sin\varphi\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \tag{253}$$

$$\hat{\mathbf{e}}^2 = \hat{\mathbf{e}}^\theta = \hat{\boldsymbol{\theta}} = \cos\varphi\cos\theta\hat{\mathbf{x}} + \sin\varphi\cos\theta\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \tag{254}$$

$$\hat{\mathbf{e}}^3 = \hat{\mathbf{e}}^\varphi = \hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}} \tag{255}$$

$$\tag{256}$$

The metric tensor is given by $g^{ij} = \sum_{k=1}^{3} \frac{\partial\xi^i}{\partial x^k}\frac{\partial\xi^j}{\partial x^k}$. Thus

$$\begin{aligned} g^{rr} &= \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2 \\ &= \cos^2\varphi\sin^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\theta = 1 \end{aligned} \tag{257}$$

$$\begin{aligned} g^{\theta\theta} &= \left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 \\ &= \frac{\cos^2\varphi\cos^2\theta}{r^2} + \frac{\sin^2\varphi\cos^2\theta}{r^2} + \frac{\sin^2\theta}{r^2} = \frac{1}{r^2} \end{aligned} \tag{258}$$

$$\begin{aligned} g^{\varphi\varphi} &= \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \\ &= \frac{\sin^2\varphi}{r^2\sin^2\theta} + \frac{\cos^2\varphi}{r^2\sin^2\theta} = \frac{1}{r^2\sin^2\theta} \end{aligned} \tag{259}$$

$$\begin{aligned} g^{r\theta} &= \frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \theta}{\partial z} \\ &= \cos\varphi\sin\theta\frac{\cos\varphi\cos\theta}{r} + \sin\varphi\sin\theta\frac{\sin\varphi\cos\theta}{r} + \cos\theta\frac{-\sin\theta}{r} = 0 \end{aligned} \tag{260}$$

$$\begin{aligned} g^{r\varphi} &= \frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \varphi}{\partial z} \\ &= \cos\varphi\sin\theta\frac{-\sin\varphi}{r\sin\theta} + \sin\varphi\sin\theta\frac{\cos\varphi}{r\sin\theta} + \cos\theta(0) = 0 \end{aligned} \tag{261}$$

$$\begin{aligned} g^{\theta\varphi} &= \frac{\partial \theta}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \theta}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial \theta}{\partial z}\frac{\partial \varphi}{\partial z} \\ &= \frac{\cos\varphi\cos\theta}{r}\frac{-\sin\varphi}{r\sin\theta} + \frac{\sin\varphi\cos\theta}{r}\frac{\cos\varphi}{r\sin\theta} + \frac{-\sin\theta}{r}(0) = 0 \end{aligned} \tag{262}$$

Thus

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix} \tag{263}$$

In the other direction we would use

$$x = r\cos\varphi\sin\theta \tag{264}$$
$$y = r\sin\varphi\sin\theta \tag{265}$$
$$z = r\cos\theta \tag{266}$$
$$\mathbf{e}_1 = \mathbf{e}_r = \left(\frac{\partial \mathbf{x}}{\partial r}\right)_{\theta,\varphi} = \cos\varphi\sin\theta\,\nabla x - \sin\varphi\sin\theta\,\nabla y \tag{267}$$
$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = r\cos\varphi\cos\theta\,\nabla x + r\sin\varphi\cos\theta\,\nabla y - r\sin\theta\,\nabla z \tag{268}$$
$$\mathbf{e}_3 = \mathbf{e}_\varphi = \frac{\partial \mathbf{x}}{\partial \varphi} = -r\sin\varphi\sin\theta\,\nabla x + r\cos\varphi\sin\theta\,\nabla y \tag{269}$$

and so

$$\mathrm{d}x = \cos\varphi\sin\theta\,\mathrm{d}r + r\cos\varphi\cos\theta\,\mathrm{d}\theta - r\sin\varphi\sin\theta\,\mathrm{d}\varphi \tag{270}$$
$$\mathrm{d}y = \sin\varphi\sin\theta\,\mathrm{d}r + r\sin\varphi\cos\theta\,\mathrm{d}\theta + r\cos\varphi\sin\theta\,\mathrm{d}\varphi \tag{271}$$
$$\mathrm{d}z = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta \tag{272}$$

and so we then have

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \cos\varphi\sin\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} \tag{273}$$

$$\begin{aligned} \mathcal{J} &= \cos\theta\left((r\cos\varphi\cos\theta)(r\cos\varphi\sin\theta) - (-r\sin\varphi\sin\theta)(r\sin\varphi\cos\theta)\right) \\ &\qquad - -r\sin\theta\left((\cos\varphi\sin\theta)(r\cos\varphi\sin\theta) - (-r\sin\varphi\sin\theta)(\sin\varphi\sin\theta)\right) \\ &= r^2\cos^2\theta\sin\theta + r^2\sin^3\theta = r^2\sin\theta \end{aligned} \tag{274}$$

Note that we then have

$$\begin{aligned} \hat{\mathbf{x}} &= \cos\varphi\sin\theta\,\nabla r + r\cos\varphi\cos\theta\,\nabla\theta - r\sin\varphi\sin\theta\,\nabla\varphi \\ &= \cos\varphi\sin\theta\hat{\mathbf{r}} + \cos\varphi\cos\theta\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}} \\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} + \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}} - \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\varphi}} \end{aligned} \tag{275}$$

$$\begin{aligned} \hat{\mathbf{y}} &= \sin\varphi\sin\theta\,\nabla r + r\sin\varphi\cos\theta\,\nabla\theta + r\cos\varphi\sin\theta\,\nabla\varphi \\ &= \sin\varphi\sin\theta\hat{\mathbf{r}} + \sin\varphi\cos\theta\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}} \\ &= \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} + \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\varphi}} \end{aligned} \tag{276}$$

$$\begin{aligned} \hat{\mathbf{z}} &= \cos\theta\,\nabla r - r\sin\theta\,\nabla\theta = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \\ &= \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} - \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}} \end{aligned} \tag{277}$$

The other metric tensor is given by $g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}$. Thus

$$\begin{aligned}
g_{rr} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \\
&= \cos^2\varphi\sin^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\theta = 1
\end{aligned} \tag{278}$$

$$\begin{aligned}
g_{\theta\theta} &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \\
&= r^2\cos^2\varphi\cos^2\theta + r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\theta = r^2
\end{aligned} \tag{279}$$

$$\begin{aligned}
g_{\varphi\varphi} &= \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 \\
&= r^2\sin^2\varphi\sin^2\theta + r^2\cos^2\varphi\sin^2\theta = r^2\sin^2\theta
\end{aligned} \tag{280}$$

$$\begin{aligned}
g_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\
&= \cos\varphi\sin\theta(r\cos\varphi\cos\theta) + \sin\varphi\sin\theta(r\sin\varphi\cos\theta) + \cos\theta(-r\sin\theta) = 0
\end{aligned} \tag{281}$$

$$\begin{aligned}
g_{r\varphi} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \varphi} \\
&= \cos\varphi\sin\theta(-r\sin\varphi\sin\theta) + \sin\varphi\sin\theta(r\cos\varphi\sin\theta) + \cos\theta(0) = 0
\end{aligned} \tag{282}$$

$$\begin{aligned}
g_{\theta\varphi} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \varphi} \\
&= r\cos\varphi\cos\theta(-r\sin\varphi\sin\theta) + r\sin\varphi\cos\theta(r\cos\varphi\sin\theta) + -r\sin\theta(0) = 0
\end{aligned} \tag{283}$$

Thus

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix} \tag{284}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right] \tag{285}$$

$$\Gamma_{r,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{bmatrix} \tag{286}$$

$$\Gamma_{\theta,ij} = \begin{bmatrix} 0 & r & 0 \\ r & 0 & 0 \\ 0 & 0 & r^2\sin\theta\cos\theta \end{bmatrix} \tag{287}$$

$$\Gamma_{\varphi,ij} = \begin{bmatrix} 0 & 0 & r\sin^2\theta \\ 0 & 0 & r^2\sin\theta\cos\theta \\ r\sin^2\theta & r^2\sin\theta\cos\theta & 0 \end{bmatrix} \tag{288}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{289}$$

$$\Gamma^r_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{bmatrix} \tag{290}$$

$$\Gamma^\theta_{ij} = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{291}$$

$$\Gamma^\varphi_{ij} = \begin{bmatrix} 0 & 0 & \frac{1}{r} \\ 0 & 0 & \cot\theta \\ \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \tag{292}$$

9 Primitive Toroidal Coordinates

We have Cartesian (x, y, z) and primitive toroidal coordinates (r, θ, ζ) as our two coordinate systems. ($0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, and $0 \leq \zeta \leq 2\pi$)

We use

$$r^2 = (R - R_0)^2 + z^2 = (\sqrt{x^2+y^2} - R_0)^2 + z^2 \tag{293}$$

$$\tan\theta = \frac{z}{R - R_0} = \frac{z}{\sqrt{x^2+y^2} - R_0} \tag{294}$$

$$\tan(-\zeta) = \frac{y}{x} \tag{295}$$

$$R = \sqrt{x^2+y^2} \tag{296}$$

$$\tag{297}$$

where $\sqrt{x_0^2 + y_0^2} = R_0 > 0$ is a given constant.

Thus, we find

$$\begin{aligned}
\mathrm{d}r &= \frac{2\left(\sqrt{x^2+y^2} - R_0\right)\left(\frac{2x\,\mathrm{d}x + 2y\,\mathrm{d}y}{2\sqrt{x^2+y^2}}\right) + 2z\,\mathrm{d}z}{2\sqrt{\left(\sqrt{x^2+y^2} - R_0\right)^2 + z^2}} = \frac{\left(\sqrt{(x^2+y^2)} - R_0\right)\left(\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{x^2+y^2}}\right) + z\,\mathrm{d}z}{\sqrt{(\sqrt{x^2+y^2} - R_0)^2 + z^2}} \\
&= \frac{(1 - \frac{R_0}{\sqrt{x^2+y^2}})(x\,\mathrm{d}x + y\,\mathrm{d}y) + z\,\mathrm{d}z}{\sqrt{(\sqrt{x^2+y^2} - R_0)^2 + z^2}} \\
&= \frac{(R - R_0)\cos\zeta}{r}\,\mathrm{d}x - \frac{(R - R_0)\sin\zeta}{r}\,\mathrm{d}y + \sin\theta\,\mathrm{d}z \\
&= \cos\theta\cos\zeta\,\mathrm{d}x - \cos\zeta\sin\zeta\,\mathrm{d}y + \sin\theta\,\mathrm{d}z
\end{aligned} \tag{298}$$

$$\sec^2\theta\,\mathrm{d}\theta = \frac{(\sqrt{x^2+y^2} - R_0)\,\mathrm{d}z - z\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{x^2+y^2}}}{(\sqrt{x^2+y^2} - R_0)^2} \tag{299}$$

$$\mathrm{d}\theta = \frac{(\sqrt{x^2+y^2} - R_0)\,\mathrm{d}z - z\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{x^2+y^2}}}{(\sqrt{x^2+y^2} - R_0)^2 + z^2} \tag{300}$$

$$= -\frac{\cos\zeta\sin\theta}{r}\,\mathrm{d}x + \frac{\sin\zeta\sin\theta}{r}\,\mathrm{d}y + \frac{\cos\theta}{r}\,\mathrm{d}z \tag{301}$$

$$\sec^2(-\zeta)(-\,\mathrm{d}\zeta) = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} \tag{302}$$

$$\begin{aligned}\mathrm{d}\zeta &= \cos^2(\zeta)\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2} = \frac{y}{x^2+y^2}\,\mathrm{d}x - \frac{x}{x^2+y^2}\,\mathrm{d}y \\ &= -\frac{\sin\zeta}{R}\,\mathrm{d}x - \frac{\cos\zeta}{R}\,\mathrm{d}y\end{aligned} \tag{303}$$

and so

$$\mathbf{J} = \frac{\partial(r,\theta,\zeta)}{\partial(x,y,z)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\zeta & -\cos\theta\sin\zeta & \sin\theta \\ \frac{-\cos\zeta\sin\theta}{r} & \frac{\sin\zeta\sin\theta}{r} & \frac{\cos\theta}{r} \\ -\frac{\sin\zeta}{R} & -\frac{\cos\zeta}{R} & 0 \end{bmatrix} \tag{304}$$

$$\begin{aligned}J &= \sin\theta\left(\frac{-\cos\zeta\sin\theta}{r}(\frac{-\cos\zeta}{R}) - \frac{\sin\zeta\sin\theta}{r}(\frac{-\sin\zeta}{R})\right) \\ &\quad - \frac{\cos\theta}{r}\left(\cos\theta\cos\zeta(\frac{-\cos\zeta}{R}) - (-\cos\theta\sin\zeta)(\frac{-\sin\zeta}{R})\right) \\ &= \frac{\sin^2\theta}{rR} + \frac{\cos^2\theta}{rR} = \frac{1}{rR}\end{aligned} \tag{305}$$

Note that we then have

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_r = \nabla r = \cos\theta\cos\zeta\,\nabla x - \cos\theta\sin\zeta\,\nabla y + \sin\theta\,\nabla z \tag{306}$$

$$|\nabla r| = 1 \tag{307}$$

$$\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_\theta = \nabla\theta = -\frac{\cos\zeta\sin\theta}{r}\,\nabla x + \frac{\sin\zeta\sin\theta}{r}\,\nabla y + \frac{\cos\theta}{r}\,\nabla z \tag{308}$$

$$|\nabla\theta| = \sqrt{\frac{\cos^2\zeta\sin^2\theta + \sin^2\zeta\sin^2\theta + \cos^2\theta}{r^2}} = \sqrt{\frac{1}{r^2}} = \frac{1}{r} \tag{309}$$

$$\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_\zeta = \nabla\zeta = -\frac{\sin\zeta}{R}\,\nabla x - \frac{\cos\zeta}{R}\,\nabla y \tag{310}$$

$$|\nabla\zeta| = \sqrt{\frac{\sin^2\zeta + \cos^2\zeta}{R^2}} = \sqrt{\frac{1}{R^2}} = \frac{1}{R} \tag{311}$$

So that

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_r = \hat{\mathbf{r}} = \cos\theta\cos\zeta\hat{\mathbf{x}} - \cos\theta\sin\zeta\hat{\mathbf{y}} + \sin\theta\hat{\mathbf{z}} \tag{312}$$

$$\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_\theta = \hat{\boldsymbol{\theta}} = -\cos\zeta\sin\theta\hat{\mathbf{x}} + \sin\zeta\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \tag{313}$$

$$\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_\zeta = \hat{\boldsymbol{\zeta}} = -\sin\zeta\hat{\mathbf{x}} - \cos\zeta\hat{\mathbf{y}} \tag{314}$$

$$\tag{315}$$

The metric tensor is given by $g^{ij} = \sum_{k=1}^{3} \frac{\partial \xi^i}{\partial x^k}\frac{\partial \xi^j}{\partial x^k}$. Thus

$$\begin{aligned}g^{rr} &= \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2 \\ &= \cos^2\theta\cos^2\zeta + \cos^2\theta\sin^2\zeta + \sin^2\theta = 1\end{aligned} \tag{316}$$

$$g^{\theta\theta} = \left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 = \frac{\cos^2\zeta\sin^2\theta}{r^2} + \frac{\sin^2\zeta\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2} = \frac{1}{r^2} \tag{317}$$

$$g^{\zeta\zeta} = \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2 + \left(\frac{\partial\zeta}{\partial z}\right)^2 = \frac{\sin^2\zeta}{R^2} + \frac{\cos^2\zeta}{R^2} = \frac{1}{R^2} \tag{318}$$

$$g^{r\theta} = \frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial\theta}{\partial z} = \cos\zeta\cos\theta\frac{-\cos\zeta\sin\theta}{r} - \sin\zeta\cos\theta\frac{\sin\zeta\sin\theta}{r} + \sin\theta\frac{\cos\theta}{r} = 0 \tag{319}$$

$$g^{r\zeta} = \frac{\partial r}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial\zeta}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial\zeta}{\partial z} = \cos\zeta\cos\theta\frac{-\sin\zeta}{R} - \sin\zeta\cos\theta\frac{-\cos\zeta}{R} + \sin\theta(0) = 0 \tag{320}$$

$$g^{\theta\zeta} = \frac{\partial\theta}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\theta}{\partial y}\frac{\partial\zeta}{\partial y} + \frac{\partial\theta}{\partial z}\frac{\partial\zeta}{\partial z} = \frac{-\cos\zeta\sin\theta}{r}\frac{-\sin\zeta}{R} + \frac{\sin\zeta\sin\theta}{r}\frac{-\cos\zeta}{R} + \frac{\cos\theta}{r}(0) = 0 \tag{321}$$

Thus

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{R^2} \end{bmatrix} \tag{322}$$

In the other direction we would use

$$x = R\cos\zeta \tag{323}$$

$$y = -R\sin\zeta \tag{324}$$

$$z = r\sin\theta \tag{325}$$

$$R - R_0 = r\cos\theta \tag{326}$$

or combining, if we so wish

$$x = (R_0 + r\cos\theta)\cos\zeta \tag{327}$$

$$y = -(R_0 + r\cos\theta)\sin\zeta \tag{328}$$

$$z = r\sin\theta \tag{329}$$

$$\mathbf{e}_1 = \mathbf{e}_R = \left(\frac{\partial\mathbf{x}}{\partial r}\right)_{\theta,\zeta} = \cos\theta\cos\zeta\,\nabla x - \cos\theta\sin\zeta\,\nabla y + \sin\theta\,\nabla z \tag{330}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial\mathbf{x}}{\partial\theta} = -r\sin\theta\cos\zeta\,\nabla x + r\sin\theta\sin\zeta\,\nabla y + r\cos\theta\,\nabla z \tag{331}$$

$$\mathbf{e}_3 = \mathbf{e}_\zeta = \frac{\partial\mathbf{x}}{\partial\zeta} = -(R_0 + r\cos\theta)\sin\zeta\,\nabla x - (R_0 + r\cos\theta)\cos\zeta\,\nabla y \tag{332}$$

and so

$$\begin{aligned}
\mathrm{d}x &= \cos\zeta\,\mathrm{d}R - R\sin\zeta\,\mathrm{d}\zeta && (333)\\
\mathrm{d}y &= -\sin\zeta\,\mathrm{d}R - R\cos\zeta\,\mathrm{d}\zeta && (334)\\
\mathrm{d}z &= \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta && (335)\\
\mathrm{d}R &= \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta && (336)\\
\mathrm{d}x &= \cos\theta\cos\zeta\,\mathrm{d}r - r\sin\theta\cos\zeta\,\mathrm{d}\theta - R\sin\zeta\,\mathrm{d}\zeta && (337)\\
\mathrm{d}y &= -\cos\theta\sin\zeta\,\mathrm{d}r + r\sin\theta\sin\zeta\,\mathrm{d}\theta - R\cos\zeta\,\mathrm{d}\zeta && (338)
\end{aligned}$$

and so we then have

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(r,\theta,\zeta)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\zeta & -r\sin\theta\cos\zeta & -R\sin\zeta \\ -\cos\theta\sin\zeta & r\sin\theta\sin\zeta & -R\cos\zeta \\ \sin\theta & r\cos\theta & 0 \end{bmatrix} \quad (339)$$

$$\begin{aligned}
\mathcal{J} &= \sin\theta\left((-r\sin\theta\cos\zeta)(-R\cos\zeta) - (-R\sin\zeta)(r\sin\theta\sin\zeta)\right)\\
&\quad - r\cos\theta\left(\cos\theta\cos\zeta(-R\cos\zeta) - (-R\sin\zeta)(-\cos\theta\sin\zeta)\right) && (340)\\
&= rR\sin^2\theta + rR\cos^2\theta = rR
\end{aligned}$$

Note that we then have

$$\begin{aligned}
\hat{\mathbf{x}} &= \cos\theta\cos\zeta\,\nabla r - r\sin\theta\cos\zeta\,\nabla\theta - R\sin\zeta\,\nabla\zeta\\
&= \cos\theta\cos\zeta\hat{\mathbf{r}} - \sin\theta\cos\zeta\hat{\boldsymbol{\theta}} - \sin\zeta\hat{\boldsymbol{\zeta}}\\
&= \frac{(R-R_0)x}{rR}\hat{\mathbf{r}} - \frac{zx}{rR}\hat{\boldsymbol{\theta}} + \frac{y}{R}\hat{\boldsymbol{\zeta}} && (341)\\
&= \frac{(\sqrt{x^2+y^2}-R_0)x}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} - \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\zeta}}\\
\hat{\mathbf{y}} &= -\cos\theta\sin\zeta\,\nabla r + r\sin\theta\sin\zeta\,\nabla\theta - R\cos\zeta\,\nabla\zeta\\
&= -\cos\theta\sin\zeta\hat{\mathbf{r}} + \sin\theta\sin\zeta\hat{\boldsymbol{\theta}} - \cos\zeta\hat{\boldsymbol{\zeta}}\\
&= \frac{(R-R_0)y}{rR}\hat{\mathbf{r}} - \frac{yz}{rR}\hat{\boldsymbol{\theta}} - \frac{x}{R}\hat{\boldsymbol{\zeta}} && (342)\\
&= \frac{(\sqrt{x^2+y^2}-R_0)y}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} - \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}} - \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\zeta}}\\
\hat{\mathbf{z}} &= \sin\theta\,\nabla r + r\cos\theta\,\nabla\theta\\
&= \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}\\
&= \frac{z}{r}\hat{\mathbf{r}} + \frac{R-R_0}{r}\hat{\boldsymbol{\theta}} && (343)\\
&= \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} + \frac{\sqrt{x^2+y^2}-R_0}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}}
\end{aligned}$$

The other metric tensor is given by $g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}$. Thus

$$\begin{aligned}
g_{rr} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 && (344)\\
&= \cos^2\theta\cos^2\zeta + \cos^2\theta\sin^2\zeta + \sin^2\theta = 1
\end{aligned}$$

$$\begin{aligned} g_{\theta\theta} &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \\ &= r^2 \sin^2\theta \cos^2\zeta + r^2 \sin^2\theta \sin^2\zeta + r^2\cos^2\theta = r^2 \end{aligned} \tag{345}$$

$$\begin{aligned} g_{\zeta\zeta} &= \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2 \\ &= R^2\sin^2\zeta + R^2\cos^2\zeta + 0 = R^2 \end{aligned} \tag{346}$$

$$\begin{aligned} g_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\ &= \cos\theta\cos\zeta(-r\sin\theta\cos\zeta) - \cos\theta\sin\zeta(r\sin\theta\sin\zeta) + r\sin\theta\cos\theta = 0 \end{aligned} \tag{347}$$

$$\begin{aligned} g_{r\zeta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \zeta} \\ &= \cos\theta\cos\zeta(-R\sin\zeta) + \cos\theta\sin\zeta R\cos\zeta + \sin\theta(0) = 0 \end{aligned} \tag{348}$$

$$\begin{aligned} g_{\theta\zeta} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \zeta} \\ &= -r\sin\theta\cos\zeta(-R\sin\zeta) + r\sin\theta\sin\zeta(-R\cos\zeta) + r\cos\theta(0) = 0 \end{aligned} \tag{349}$$

Thus

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & R^2 \end{bmatrix} \tag{350}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right] \tag{351}$$

$$\Gamma_{r,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -R\cos\theta \end{bmatrix} \tag{352}$$

$$\Gamma_{\theta,ij} = \begin{bmatrix} 0 & r & 0 \\ r & 0 & 0 \\ 0 & 0 & rR\sin\theta \end{bmatrix} \tag{353}$$

$$\Gamma_{\zeta,ij} = \begin{bmatrix} 0 & 0 & R\cos\theta \\ 0 & 0 & -rR\sin\theta \\ R\cos\theta & -rR\sin\theta & 0 \end{bmatrix} \tag{354}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{355}$$

$$\Gamma^r_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -R\cos\theta \end{bmatrix} \tag{356}$$

$$\Gamma^\theta_{ij} = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & \frac{R}{r}\sin\theta \end{bmatrix} \tag{357}$$

$$\Gamma^{\zeta}_{ij} = \begin{bmatrix} 0 & 0 & \frac{\cos\theta}{R} \\ 0 & 0 & -\frac{r\sin\theta}{R} \\ \frac{\cos\theta}{R} & -\frac{r\sin\theta}{R} & 0 \end{bmatrix} \tag{358}$$

10 Plasma Toroidal Coordinates

We have Cartesian (x, y, z) and plasma toroidal coordinates (ψ, θ, ζ) as our two coordinate systems. $(1 < \psi < \infty$, $0 < \theta < 2\pi$, and $0 < \zeta < 2\pi)$

We use

$$x = a\frac{\sqrt{\psi^2 - 1}}{\psi - \cos\theta}\cos\zeta \tag{359}$$

$$y = a\frac{\sqrt{\psi^2 - 1}}{\psi - \cos\theta}\sin\zeta \tag{360}$$

$$z = a\frac{\sin\theta}{\psi - \cos\theta} \tag{361}$$

which means

$$\mathbf{e}_1 = \mathbf{e}_\psi = \left(\frac{\partial \mathbf{x}}{\partial \psi}\right)_{\theta,\zeta} = \frac{a\cos\zeta(1 - \psi\cos\theta)}{\sqrt{\psi^2 - 1}(\psi - \cos\theta)^2}\nabla x + \frac{a\sin\zeta(1 - \psi\cos\theta)}{\sqrt{\psi^2 - 1}(\psi - \cos\theta)^2}\nabla y - \frac{a\sin\theta}{(\psi - \cos\theta)^2}\nabla z \tag{362}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = -\frac{a\sqrt{\psi^2 - 1}\cos\zeta\sin\theta}{(\psi - \cos\theta)^2}\nabla x - \frac{a\sqrt{\psi^2 - 1}\sin\zeta\sin\theta}{(\psi - \cos\theta)^2}\nabla y + \frac{a(\psi\cos\theta - 1)}{(\psi - \cos\theta)^2}\nabla z \tag{363}$$

$$\mathbf{e}_3 = \mathbf{e}_\zeta = \frac{\partial \mathbf{x}}{\partial \zeta} = -\frac{a\sqrt{\psi^2 - 1}}{\psi - \cos\theta}\nabla x + \frac{a\sqrt{\psi^2 - 1}}{\psi - \cos\theta}\nabla y \tag{364}$$

Taking (define $\beta = \frac{z^2}{x^2+y^2}$ and $\gamma = \frac{(1+r^2/a^2)^2}{(1-r^2/a^2)^2}$ where $r^2 = x^2 + y^2 + z^2$)

$$\frac{x^2 + y^2 + z^2}{a^2} = \frac{r^2}{a^2} = \frac{\psi^2 - 1 + \sin^2\theta}{(\psi - \cos\theta)^2} = \frac{\psi^2 - \cos^2\theta}{(\psi - \cos\theta)^2} = \frac{(\psi + \cos\theta)\cancel{(\psi - \cos\theta)}}{(\psi - \cos\theta)^{\cancel{2}}} = \frac{\psi + \cos\theta}{\psi - \cos\theta} \tag{365}$$

$$\psi = \frac{-\cos\theta(1 + \frac{r^2}{a^2})}{1 - \frac{r^2}{a^2}} \Rightarrow \psi^2 = \gamma\cos^2\theta \tag{366}$$

$$\frac{y}{z} = \frac{\sqrt{\psi^2 - 1}}{\sin\theta}\sin\zeta = \frac{\sqrt{\psi^2 - 1}}{\sin\theta}\frac{y}{\sqrt{x^2 + y^2}} \Rightarrow \frac{\sqrt{x^2 + y^2}}{z} = \frac{\sqrt{\psi^2 - 1}}{\sin\theta} \tag{367}$$

$$\sin^2\theta = \frac{z^2}{x^2 + y^2}(\psi^2 - 1) = \beta[(1 - \sin^2\theta)\gamma - 1] \tag{368}$$

$$\sin^2\theta = \frac{\beta(\gamma - 1)}{1 + \gamma\beta} = \frac{4a^2z^2}{(-a^2 + x^2 + y^2)^2 + 2(a^2 + x^2 + y^2)z^2 + z^4} \tag{369}$$

$$\sin\theta = \frac{2az}{\sqrt{(-a^2 + x^2 + y^2)^2 + 2(a^2 + x^2 + y^2)z^2 + z^4}} \tag{370}$$

$$\begin{aligned} \psi^2 &= \left(\frac{1 + r^2/a^2}{1 - r^2/a^2}\right)^2 \left(1 - \frac{4a^2z^2}{(-a^2 + x^2 + y^2)^2 + 2(a^2 + x^2 + y^2)z^2 + z^4}\right) \\ &= \frac{(a^2 + x^2 + y^2 + z^2)^2}{2z^2(a^2 + x^2 + y^2) + (-a^2 + x^2 + y^2)^2 + z^4} \end{aligned} \tag{371}$$

Thus we can rewrite our expressions as the ugly

$$\psi^2 = \frac{\left(a^2+x^2+y^2+z^2\right)^2}{2z^2\left(a^2+x^2+y^2\right)+\left(-a^2+x^2+y^2\right)^2+z^4} \tag{372}$$

$$\sin^2\theta = \frac{\beta(\gamma-1)}{1+\gamma\beta} = \frac{4a^2z^2}{(-a^2+x^2+y^2)^2+2(a^2+x^2+y^2)z^2+z^4} \tag{373}$$

$$\tan\zeta = \frac{y}{x} \tag{374}$$

So we find

$$\mathrm{d}x = \frac{a\cos\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\,\mathrm{d}\psi - \frac{a\sqrt{\psi^2-1}\cos\zeta\sin\theta}{(\psi-\cos\theta)^2}\,\mathrm{d}\theta - \frac{a\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta\,\mathrm{d}\zeta \tag{375}$$

$$\mathrm{d}y = \frac{a\sin\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\,\mathrm{d}\psi - \frac{a\sqrt{\psi^2-1}\sin\zeta\sin\theta}{(\psi-\cos\theta)^2}\,\mathrm{d}\theta + \frac{a\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta\,\mathrm{d}\zeta \tag{376}$$

$$\begin{aligned}\mathrm{d}z &= a\frac{\cos\theta(\psi-\cos\theta)\,\mathrm{d}\theta - \sin\theta(\,\mathrm{d}\psi+\sin\theta\,\mathrm{d}\theta)}{(\psi-\cos\theta)^2} \\ &= -a\frac{\sin\theta}{(\psi-\cos\theta)^2}\,\mathrm{d}\psi + a\frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2}\,\mathrm{d}\theta\end{aligned} \tag{377}$$

We of course then have

$$\begin{aligned}\mathbf{J} = \boldsymbol{\mathcal{J}}^{-1} = \frac{\partial(\psi,\theta,\zeta)}{\partial(x,y,z)} &= \begin{bmatrix} \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} & \frac{\partial\theta}{\partial z} \\ \frac{\partial\zeta}{\partial x} & \frac{\partial\zeta}{\partial y} & \frac{\partial\zeta}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{\psi^2-1}\cos\zeta(1-\psi\cos\theta)}{a} & \frac{\sqrt{\psi^2-1}(1-\psi\cos\theta)\sin\zeta}{a} & -\frac{\left(\psi^2-1\right)\sin\theta}{a} \\ -\frac{\sqrt{\psi^2-1}\cos\zeta\sin\theta}{a} & -\frac{\sqrt{\psi^2-1}\sin\zeta\sin\theta}{a} & \frac{\psi\cos\theta-1}{a} \\ \frac{(\cos\theta-\psi)\sin\zeta}{a\sqrt{\psi^2-1}} & \frac{\cos\zeta(\psi-\cos\theta)}{a\sqrt{\psi^2-1}} & 0 \end{bmatrix}\end{aligned} \tag{378}$$

$$J = \frac{1}{\mathcal{J}} = \frac{(\psi-\cos\theta)^3}{a^3} \tag{379}$$

Because of the ugliness of calculating g^{ij} directly, I use the results of g_{ij} below (414) and invert it to find.

$$g^{ij} = \begin{bmatrix} [(\psi^2-1)(\psi-\cos\theta)^2] & 0 & 0 \\ 0 & (\psi-\cos\theta)^2 & 0 \\ 0 & 0 & (\psi^2-1)(\psi-\cos\theta)^2 \end{bmatrix} \tag{380}$$

We can now note that

$$\mathbf{e}^1 = \mathbf{e}^\psi = \nabla\psi = \frac{\sqrt{\psi^2-1}(1-\psi\cos\theta)\cos\zeta}{a}\nabla x + \frac{\sqrt{\psi^2-1}(1-\psi\cos\theta)\sin\zeta}{a}\nabla y - \frac{\sin\theta(\psi^2-1)}{a}\nabla z \tag{381}$$

$$
\begin{aligned}
|\nabla\psi|^2 &= \frac{(\psi^2-1)(1-\psi\cos\theta)^2\cos^2\zeta+(\psi^2-1)(1-\psi\cos\theta)^2\sin^2\zeta+(\psi^2-1)^2\sin^2\theta}{a^2} \\
&= \frac{(\psi^2-1)(1-\psi\cos\theta)^2+(\psi^2-1)^2\sin^2\theta}{a^2} \\
&= \frac{(\psi^2-1)\left[\cancel{1}-2\psi\cos\theta+\cancel{\psi^2\cos^2\theta}+\psi^2-\cancel{\psi^2\cos^2\theta}-\cancel{1}+\cos^2\theta\right]}{a^2} \\
&= \frac{(\psi^2-1)(\psi-\cos\theta)^2}{a^2}
\end{aligned}
\tag{382}
$$

$$
|\nabla\psi| = \frac{\sqrt{\psi^2-1}(\psi-\cos\theta)}{a} \tag{383}
$$

$$
\mathbf{e}^2=\mathbf{e}^\theta=\nabla\theta=-\frac{\sqrt{\psi^2-1}\cos\zeta\sin\theta}{a}\nabla x+-\frac{\sqrt{\psi^2-1}\sin\zeta\sin\theta}{a}\nabla y+\frac{\psi\cos\theta-1}{a}\nabla z \tag{384}
$$

$$
\begin{aligned}
|\nabla\theta|^2 &= \frac{(\psi^2-1)\cos^2\zeta\sin^2\theta+(\psi^2-1)\sin^2\zeta\sin^2\theta+(1-\psi\cos\theta)^2}{a^2} \\
&= \frac{(\psi^2-1)\sin^2\theta+(1-\psi\cos\theta)^2}{a^2}=\frac{(\psi-\cos\theta)^2}{a^2}
\end{aligned}
\tag{385}
$$

$$
|\nabla\theta| = \frac{\psi-\cos\theta}{a} \tag{386}
$$

$$
\mathbf{e}^3=\mathbf{e}^\zeta=\nabla\zeta=\frac{(\cos\theta-\psi)\sin\zeta}{a\sqrt{\psi^2-1}}\nabla x+\frac{\cos\zeta(\psi-\cos\theta)}{a\sqrt{\psi^2-1}}\nabla y \tag{387}
$$

$$
|\nabla\zeta|^2 = \frac{(\psi-\cos\theta)^2\sin^2\zeta+(\psi-\cos\theta)^2\cos^2\zeta}{a^2(\psi^2-1)}=\frac{(\psi-\cos\theta)^2}{a^2(\psi^2-1)} \tag{388}
$$

$$
|\nabla\zeta| = \frac{\psi-\cos\theta}{a\sqrt{\psi^2-1}} \tag{389}
$$

$$
\boldsymbol{\mathcal{J}}=\mathbf{J}^{-1}=\frac{\partial(x,y,z)}{\partial(\psi,\theta,\zeta)}=\begin{bmatrix} \frac{\partial x}{\partial\psi} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\zeta} \\ \frac{\partial y}{\partial\psi} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\zeta} \\ \frac{\partial z}{\partial\psi} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\zeta} \end{bmatrix}=\begin{bmatrix} \frac{a\cos\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2} & -\frac{a\sqrt{\psi^2-1}\cos\zeta\sin\theta}{(\psi-\cos\theta)^2} & -\frac{a\sqrt{\psi^2-1}\sin\zeta}{\psi-\cos\theta} \\ \frac{a\sin\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2} & -\frac{a\sqrt{\psi^2-1}\sin\zeta\sin\theta}{(\psi-\cos\theta)^2} & \frac{a\sqrt{\psi^2-1}\cos\zeta}{\psi-\cos\theta} \\ \frac{-a\sin\theta}{(\psi-\cos\theta)^2} & \frac{a(\psi\cos\theta-1)}{(\psi-\cos\theta)^2} & 0 \end{bmatrix} \tag{390}
$$

$$
\begin{aligned}
\mathcal{J} &= \frac{-a\sin\theta}{(\psi-\cos\theta)^2}\left(-\frac{a\sqrt{\psi^2-1}\cos\zeta\sin\theta}{(\psi-\cos\theta)^2}\frac{a\sqrt{\psi^2-1}\cos\zeta}{\psi-\cos\theta}-\frac{a\sqrt{\psi^2-1}\sin\zeta}{\psi-\cos\theta}\frac{a\sqrt{\psi^2-1}\sin\zeta\sin\theta}{(\psi-\cos\theta)^2}\right)\\
&\quad-\frac{a(\psi\cos\theta-1)}{(\psi-\cos\theta)^2}\left(\frac{a\cos\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\frac{a\sqrt{\psi^2-1}\cos\zeta}{\psi-\cos\theta}-\frac{-a\sqrt{\psi^2-1}\sin\zeta}{\psi-\cos\theta}\frac{a\sin\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\right)\\
&=\frac{a^3\sin^2\theta(\psi^2-1)}{(\psi-\cos\theta)^5}\left(\cos^2\zeta+\sin^2\zeta\right)\\
&\quad+\frac{a^3(1-\psi\cos\theta)^2}{(\psi-\cos\theta)^5}\left(\cos\zeta^2+\sin^2\zeta\right)\\
&=\frac{a^3}{(\psi-\cos\theta)^5}\left((\psi^2-1)\sin^2\theta+(1-\psi\cos\theta)^2\right)\\
&=\frac{a^3}{(\psi-\cos\theta)^5}\left((\psi^2-1)(1-\cos^2\theta)+1+2\psi\cos\theta+\psi^2\cos^2\theta\right)=\frac{a^3}{(\psi-\cos\theta)^5}\left(\psi^2+2\psi\cos\theta+\cos^2\theta\right)\\
&=\frac{a^3\left(\psi-\cos\theta\right)^2}{(\psi-\cos\theta)^5}=\frac{a^3}{(\psi-\cos\theta)^3}
\end{aligned}
\tag{391}
$$

Note that we then have (using (381) and the following)

$$
\begin{aligned}
\hat{\mathbf{x}} &= \frac{a\cos\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\nabla\psi-\frac{a\sqrt{\psi^2-1}\cos\zeta\sin\theta}{(\psi-\cos\theta)^2}\nabla\theta-\frac{a\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta\,\nabla\zeta\\
&=\frac{\cos\zeta(1-\psi\cos\theta)}{\psi-\cos\theta}\hat{\boldsymbol{\psi}}-\frac{\sqrt{\psi^2-1}\cos\zeta\sin\theta}{\psi-\cos\theta}\hat{\boldsymbol{\theta}}-\sin\zeta\hat{\boldsymbol{\zeta}}
\end{aligned}
\tag{392}
$$

$$
\begin{aligned}
\hat{\mathbf{y}} &= \frac{a\sin\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\nabla\psi-\frac{a\sqrt{\psi^2-1}\sin\zeta\sin\theta}{(\psi-\cos\theta)^2}\nabla\theta+\frac{a\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta\,\nabla\zeta\\
&=\frac{\sin\zeta(1-\psi\cos\theta)}{\psi-\cos\theta}\hat{\boldsymbol{\psi}}-\frac{\sqrt{\psi^2-1}\sin\zeta\sin\theta}{\psi-\cos\theta}\hat{\boldsymbol{\theta}}+\cos\zeta\hat{\boldsymbol{\zeta}}
\end{aligned}
\tag{393}
$$

$$
\begin{aligned}
\hat{\mathbf{z}} &= -a\frac{\sin\theta}{(\psi-\cos\theta)^2}\nabla\psi+a\frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2}\nabla\theta\\
&=-\frac{\sqrt{\psi^2-1}\sin\theta}{\psi-\cos\theta}\hat{\boldsymbol{\psi}}+\frac{\psi\cos\theta-1}{\psi-\cos\theta}\hat{\boldsymbol{\theta}}
\end{aligned}
\tag{394}
$$

The metric tensor is given by $g_{ij}=\sum_{k=1}^{3}\frac{\partial x^k}{\partial\xi^i}\frac{\partial x^k}{\partial\xi^j}$. Thus

$$
g_{\psi\psi}=\frac{\partial x}{\partial\psi}\frac{\partial x}{\partial\psi}+\frac{\partial y}{\partial\psi}\frac{\partial y}{\partial\psi}+\frac{\partial z}{\partial\psi}\frac{\partial z}{\partial\psi} \tag{395}
$$

$$
=\frac{(1-\psi\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4}\cos^2\zeta+\frac{(1-\psi\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4}\sin^2\zeta+\frac{\sin^2\theta}{(\psi-\cos\theta)^4} \tag{396}
$$

$$
=\frac{(1-\psi\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4}+\frac{\sin^2\theta(\psi^2-1)}{(\psi^2-1)(\psi-\cos\theta)^4}=\frac{1+\psi^2\cos^2\theta-2\psi\cos\theta+\psi^2\sin^2\theta-\sin^2\theta}{(\psi^2-1)(\psi-\cos\theta)^4} \tag{397}
$$

$$
\boxed{g_{\psi\psi}=\frac{\cos^2\theta-2\psi\cos\theta+\psi^2}{(\psi^2-1)(\psi-\cos\theta)^4}=\frac{(\psi-\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4}=\frac{1}{(\psi^2-1)(\psi-\cos\theta)^2}} \tag{398}
$$

$$g_{\theta\theta} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} \tag{399}$$

$$= \frac{(\psi^2-1)\sin^2\theta}{(\psi-\cos\theta)^4}\cos^2\zeta + \frac{(\psi^2-1)\sin^2\theta}{(\psi-\cos\theta)^4}\sin^2\zeta + \frac{(1-\psi\cos\theta)^2}{(\psi-\cos\theta)^4} \tag{400}$$

$$= \frac{\psi^2\sin^2\theta - \sin^2\theta + 1 - 2\psi\cos\theta + \psi^2\cos^2\theta}{(\psi-\cos\theta)^4} = \frac{\cos^2\theta - 2\psi\cos\theta + \psi^2}{(\psi-\cos\theta)^4} \tag{401}$$

$$\boxed{g_{\theta\theta} = \frac{(\psi-\cos\theta)^2}{(\psi-\cos\theta)^4} = \frac{1}{(\psi-\cos\theta)^2}} \tag{402}$$

$$g_{\zeta\zeta} = \frac{\partial x}{\partial \zeta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \zeta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \zeta}\frac{\partial z}{\partial \zeta} \tag{403}$$

$$\boxed{g_{\zeta\zeta} = \frac{\psi^2-1}{(\psi-\cos\theta)^2}\sin^2\zeta + \frac{\psi^2-1}{(\psi-\cos\theta)^2}\cos^2\zeta = \frac{\psi^2-1}{(\psi-\cos\theta)^2}} \tag{404}$$

$$g_{\psi\theta} = \frac{\partial x}{\partial \psi}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \psi}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \psi}\frac{\partial z}{\partial \theta} \tag{405}$$

$$\begin{aligned} &= \left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\cos\zeta\right)\left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\cos\zeta\right) \\ &\quad + \left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\sin\zeta\right)\left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\sin\zeta\right) \\ &\quad + \left(\frac{-\sin\theta}{(\psi-\cos\theta)^2}\right)\left(\frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2}\right) \end{aligned} \tag{406}$$

$$\boxed{g_{\psi\theta} = \frac{(\psi\cos\theta-1)\sqrt{\psi^2-1}\sin\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^4} - \frac{\sin\theta(\psi\sin\theta\cos\theta-1}{(\psi-\cos\theta)^4} = 0} \tag{407}$$

$$g_{\psi\zeta} = \frac{\partial x}{\partial \psi}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \psi}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \psi}\frac{\partial z}{\partial \zeta} \tag{408}$$

$$\begin{aligned} &= \left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\cos\zeta\right)\left(-\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta\right) \\ &\quad + \left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\sin\zeta\right)\left(\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta\right) \\ &\quad + \left(\frac{-\sin\theta}{(\psi-\cos\theta)^2}\right)(0) \end{aligned} \tag{409}$$

$$\boxed{g_{\psi\zeta} = -\frac{1-\psi\cos\theta}{(\psi-\cos\theta)^3}\sin\zeta\cos\zeta + \frac{1-\psi\cos\theta}{(\psi-\cos\theta)^3}\sin\zeta\cos\zeta = 0} \tag{410}$$

$$g_{\theta\zeta} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \zeta} \tag{411}$$

$$\begin{aligned} &= \left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\cos\zeta\right)\left(-\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta\right) + \left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\sin\zeta\right)\left(\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta\right) \\ &\quad + \left(\frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2}\right)(0) \end{aligned} \tag{412}$$

$$\boxed{g_{\theta\zeta} = \frac{(\psi^2-1)\sin\theta}{(\psi-\cos\theta)^3}\left(\sin\zeta\cos\zeta - \sin\zeta\cos\zeta\right) = 0.} \tag{413}$$

Hence we have altogether

$$g_{ij} = \begin{bmatrix} \left[(\psi^2-1)(\psi-\cos\theta)^2\right]^{-1} & 0 & 0 \\ 0 & (\psi-\cos\theta)^{-2} & 0 \\ 0 & 0 & (\psi^2-1)(\psi-\cos\theta)^{-2} \end{bmatrix} \tag{414}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right] \tag{415}$$

$$\Gamma_{\psi,ij} = \begin{bmatrix} 0 & \frac{-\sin\theta}{(\psi-\cos\theta)^3(\psi^2-1)} & 0 \\ \frac{-\sin\theta}{(\psi-\cos\theta)^3(\psi^2-1)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{416}$$

$$\Gamma_{\theta,ij} = \begin{bmatrix} \frac{\sin\theta}{(\psi-\cos\theta)^3(\psi^2-1)} & 0 & 0 \\ 0 & \frac{-\sin\theta}{(\psi-\cos\theta)^3} & 0 \\ 0 & 0 & \frac{(\psi^2-1)\sin\theta}{(\psi-\cos\theta)^3} \end{bmatrix} \tag{417}$$

$$\Gamma_{\zeta,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-(\psi^2-1)\sin\theta}{(\psi-\cos\theta)^3} \\ 0 & \frac{-(\psi^2-1)\sin\theta}{(\psi-\cos\theta)^3} & 0 \end{bmatrix} \tag{418}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{419}$$

$$\Gamma^\psi_{ij} = \begin{bmatrix} 0 & \frac{-\sin\theta}{\psi-\cos\theta} & 0 \\ \frac{-\sin\theta}{\psi-\cos\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{420}$$

$$\Gamma^\theta_{ij} = \begin{bmatrix} \frac{\sin\theta}{(\psi-\cos\theta)(\psi^2-1)} & 0 & 0 \\ 0 & \frac{-\sin\theta}{\psi-\cos\theta} & 0 \\ 0 & 0 & \frac{(\psi^2-1)\sin\theta}{\psi-\cos\theta} \end{bmatrix} \tag{421}$$

$$\Gamma^\zeta_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-\sin\theta}{\psi-\cos\theta} \\ 0 & \frac{-\sin\theta}{\psi-\cos\theta} & 0 \end{bmatrix} \tag{422}$$

11 General Toroidal Coordinates

We have Cartesian (x, y, z) and plasma toroidal coordinates (τ, θ, ζ) as our two coordinate systems. $(-\infty < \tau < \infty$, $0 \leq \theta \leq 2\pi$, and $0 \leq \zeta \leq 2\pi)$

We use

$$x = a\frac{\sinh\tau}{\cosh\tau - \cos\theta}\cos\zeta \tag{423}$$

$$y = a\frac{\sinh\tau}{\cosh\tau - \cos\theta}\sin\zeta \tag{424}$$

$$z = a\frac{\sin\theta}{\cosh\tau - \cos\theta} \tag{425}$$

Note that we then have $\sinh\tau = \sqrt{\psi^2 - 1}$ and $\cosh\tau = \sqrt{1 + \sinh^2\tau} = \sqrt{\psi^2} = \psi$ as a connection to our previous coordinates (this would then restrict $0 < \tau < \infty$, which is actually nicer as it removes the $\mathrm{sgn}(\tau)$ functions in some relations).

Thus we can rewrite our expressions as the ugly

$$\cosh^2\tau = \frac{\left(a^2 + x^2 + y^2 + z^2\right)^2}{2z^2\left(a^2 + x^2 + y^2\right) + \left(-a^2 + x^2 + y^2\right)^2 + z^4} \tag{426}$$

$$\sin^2\theta = \frac{\beta(\gamma - 1)}{1 + \gamma\beta} = \frac{4a^2z^2}{(-a^2 + x^2 + y^2)^2 + 2(a^2 + x^2 + y^2)z^2 + z^4} \tag{427}$$

$$\tan\zeta = \frac{y}{x} \tag{428}$$

These are so painfully ugly that we will calculate the Jacobian matrix via determining the results the "other way" first and inverting the matrix.

Note that one can write

$$\rho^2 = x^2 + y^2 \tag{429}$$

$$d_1^2 = (\rho + a)^2 + z^2 \tag{430}$$

$$d_2^2 = (\rho - a)^2 + z^2 \tag{431}$$

$$e^\tau = \frac{d_1}{d_2} \tag{432}$$

$$\cos\theta = \frac{d_1^2 + d_2^2 - 4a^2}{d_1 d_2} \tag{433}$$

So we find

$$\mathrm{d}x = a\frac{1 - \cosh\tau\cos\theta}{(\cosh\tau - \cos\theta)^2}\cos\zeta\,\mathrm{d}\tau - a\frac{\sinh\tau\sin\theta}{(\cosh\tau - \cos\theta)^2}\cos\zeta\,\mathrm{d}\theta - a\frac{\sinh\tau}{\cosh\tau - \cos\theta}\sin\zeta\,\mathrm{d}\zeta \tag{434}$$

$$\mathrm{d}y = a\frac{1 - \cosh\tau\cos\theta}{(\cosh\tau - \cos\theta)^2}\sin\zeta\,\mathrm{d}\tau - a\frac{\sinh\tau\sin\theta}{(\cosh\tau - \cos\theta)^2}\sin\zeta\,\mathrm{d}\theta + a\frac{\sinh\tau}{\cosh\tau - \cos\theta}\cos\zeta\,\mathrm{d}\zeta \tag{435}$$

$$\mathrm{d}z = -a\frac{\sin\theta\sinh\tau}{(\cosh\tau - \cos\theta)^2}\,\mathrm{d}\tau + a\frac{\cos\theta\cosh\tau - 1}{(\cosh\tau - \cos\theta)^2}\,\mathrm{d}\theta \tag{436}$$

which means

$$\mathbf{e}_1 = \mathbf{e}_\tau = \left(\frac{\partial \mathbf{x}}{\partial \psi}\right)_{\theta,\zeta} = a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\theta\,\nabla x + a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\,\nabla y - \frac{a\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\nabla z \tag{437}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = -\frac{a\sinh\tau\sin\theta\cos\zeta}{(\cosh\tau-\cos\theta)^2}\nabla x - \frac{a\sinh\tau\sin\theta\sin\zeta}{(\cosh\tau-\cos\theta)^2}\nabla y + a\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2} \tag{438}$$

$$\mathbf{e}_3 = \mathbf{e}_\zeta = \frac{\partial \mathbf{x}}{\partial \zeta} = -\frac{a\sinh\tau\sin\zeta}{\cosh\tau-\cos\theta}\nabla x + \frac{a\sinh\tau\cos\zeta}{\cosh\tau-\cos\theta} \tag{439}$$

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(\tau,\theta,\zeta)} = \begin{bmatrix} \frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \tau} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \tau} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta & -a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta & -a\frac{\sinh\tau}{\cosh\tau-\cos\theta}\sin\zeta \\ a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta & -a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta & a\frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta \\ -a\frac{\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2} & a\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2} & 0 \end{bmatrix} \tag{440}$$

$$\begin{aligned}
\mathcal{J} &= -a\frac{\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\Bigg(-a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\frac{a\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta \\
&\quad - \frac{a\sinh\tau\sin\zeta}{\cosh\tau-\cos\theta}\frac{a\sinh\tau\sin\theta\sin\zeta}{(\cosh\tau-\cos\theta)^2}\Bigg) \\
&\quad - a\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2}\Bigg(\frac{a(1-\cosh\tau\cos\theta)\cos\zeta}{(\cosh\tau-\cos\theta)^2}\frac{a\sinh\tau\cos\zeta}{\cosh\tau-\cos\theta} \\
&\quad + \frac{a\sinh\tau\sin\zeta}{\cosh\tau-\cos\theta}\frac{a(1-\cosh\tau\cos\theta)\sin\zeta}{(\cosh\tau-\cos\theta)^2}\Bigg) \\
&= \frac{a^3\sin^2\theta\sinh^3\tau}{(\cosh\tau-\cos\theta)^5}\left(\cos^2\zeta+\sin^2\zeta\right) + \frac{a^3(1-\cosh\tau\cos\theta)^2\sinh\tau}{(\cosh\tau-\cos\theta)^5}\left(\cos\zeta^2+\sin^2\zeta\right) \\
&= \frac{a^3\sinh\tau}{(\cosh\tau-\cos\theta)^5}\left((\cosh^2\tau-1)(1-\cos^2\theta)+(1-\cos\theta\cosh\tau)^2\right) \\
&= \frac{a^3\sinh\tau}{(\cosh\tau-\cos\theta)^5}(\cosh\tau-\cos\theta)^2 = \frac{a^3\sinh\tau}{(\cosh\tau-\cos\theta)^3}
\end{aligned} \tag{441}$$

Note that we then have (using (454) and the following equations)

$$\begin{aligned}
\hat{\mathbf{x}} &= a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\,\nabla\tau - a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\,\nabla\theta - a\frac{\sinh\tau}{\cosh\tau-\cos\theta}\sin\zeta\,\nabla\zeta \\
&= \frac{1-\cosh\tau\cos\theta}{\cosh\tau-\cos\theta}\cos\zeta\hat{\boldsymbol{\tau}} - \frac{\sinh\tau\sin\theta}{\cosh\tau-\cos\theta}\cos\zeta\hat{\boldsymbol{\theta}} - \operatorname{sgn}(\tau)\sin\zeta\hat{\boldsymbol{\zeta}}
\end{aligned} \tag{442}$$

$$\begin{aligned}
\hat{\mathbf{y}} &= a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\,\nabla\tau - a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\,\nabla\theta + a\frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta\,\nabla\zeta \\
&= \frac{1-\cosh\tau\cos\theta}{\cosh\tau-\cos\theta}\sin\zeta\hat{\boldsymbol{\tau}} - \frac{\sinh\tau\sin\theta}{\cosh\tau-\cos\theta}\sin\zeta\hat{\boldsymbol{\theta}} + \operatorname{sgn}(\tau)\cos\zeta\hat{\boldsymbol{\zeta}}
\end{aligned} \tag{443}$$

$$\begin{aligned}
\hat{\mathbf{z}} &= -a\frac{\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\nabla\tau + a\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2}\nabla\theta \\
&= -\frac{\sin\theta\sinh\tau}{\cosh\tau-\cos\theta}\hat{\boldsymbol{\tau}} + \frac{\cos\theta\cosh\tau-1}{\cosh\tau-\cos\theta}\hat{\boldsymbol{\theta}}
\end{aligned} \tag{444}$$

The metric tensor is given by $g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}$. Thus

$$\begin{aligned}
\frac{g_{\tau\tau}}{a^2} &= \frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \tau} + \frac{\partial y}{\partial \tau}\frac{\partial y}{\partial \tau} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial \tau} \\
&= \left(\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\right)^2 + \left(\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\right)^2 + \left(\frac{-\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\right)^2 \\
&= \frac{(1-\cosh\tau\cos\theta)^2+\sin^2\theta\sinh^2\tau}{(\cosh\tau-\cos\theta)^4} = \frac{1-2\cosh\tau\cos\theta+\cosh^2\tau\cos^2\theta+\sin^2\theta\sinh^2\tau}{(\cosh\tau-\cos\theta)^4} \\
&= \frac{1-2\cosh\tau\cos\theta+\cosh^2\tau+\sin^2\theta(\sinh^2\tau-\cosh^2\tau)}{(\cosh\tau-\cos\theta)^4} = \frac{1-2\cosh\tau\cos\theta+\cosh^2\tau-\sin^2\theta}{(\cosh\tau-\cos\theta)^4} \\
&= \frac{\cos^2\theta-2\cosh\tau\cos\theta+\cosh^2\tau}{(\cosh\tau-\cos\theta)^4} = \frac{(\cosh\tau-\cos\theta)^2}{(\cosh\tau-\cos\theta)^4} = \frac{1}{(\cosh\tau-\cos\theta)^2}
\end{aligned} \tag{445}$$

$$\begin{aligned}
\frac{g_{\tau\theta}}{a^2} &= \frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \tau}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial \theta} \\
&= \frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta \\
&\quad + \frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta \\
&\quad + \frac{-\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2} \\
&= \frac{(1-\cos\theta\cosh\tau)(\sin\theta\sinh\tau)}{(\cosh\tau-\cos\theta)^4}\left(\cos^2\zeta+\sin^2\zeta-1\right) = 0
\end{aligned} \tag{446}$$

$$\begin{aligned}
\frac{g_{\tau\zeta}}{a^2} &= \frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \tau}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial \zeta} \\
&= -\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\frac{\sinh\tau}{\cosh\tau-\cos\theta}\sin\zeta + \frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta + 0 \\
&= \frac{(1-\cosh\tau\cos\theta)}{(\cosh\tau-\cos\theta)^2}\sin\zeta\cos\zeta\,(-1+1) = 0
\end{aligned} \tag{447}$$

$$\begin{aligned}
\frac{g_{\theta\theta}}{a^2} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} \\
&= \left(\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\right)^2 + \left(\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\right)^2 + \left(\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2}\right)^2 \\
&= \frac{\sinh^2\tau\sin^2\theta+(1-\cos\theta\cosh\tau)^2}{(\cosh\tau-\cos\theta)^4} = \frac{\sinh^2\tau\sin^2\theta+1-2\cosh\tau\cos\theta+\cos^2\theta\cosh^2\tau}{(\cosh\tau-\cos\theta)^4} \\
&= \frac{\sinh^2\tau\sin^2\theta+(1-\sin^2\theta)\cosh^2\tau+1-2\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^4} = \frac{-\sin^2\theta+\cosh^2\tau+1-2\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^4} \\
&= \frac{\cosh^2\tau-2\cosh\tau\cos\theta+\cos^2\theta}{(\cosh\tau-\cos\theta)^4} = \frac{(\cosh\tau-\cos\theta)^2}{(\cosh\tau-\cos\theta)^4} = \frac{1}{(\cosh\tau-\cos\theta)^2}
\end{aligned} \tag{448}$$

$$\frac{g_{\theta\zeta}}{a^2} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \zeta} = \frac{g_{32}^2}{a^2} = 0 \tag{449}$$

$$\begin{aligned}\frac{g_{\zeta\zeta}}{a^2} &= \frac{\partial x}{\partial \zeta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \zeta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \zeta}\frac{\partial z}{\partial \zeta} \\ &= \left(-\frac{\sinh\tau}{\cosh\tau - \cos\theta}\sin\zeta\right)^2 + \left(\frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta\right)^2 + 0 = \frac{\sinh^2\tau}{(\cosh\tau-\cos\theta)^2}\end{aligned} \tag{450}$$

Thus, we find

$$g_{ij} = \begin{bmatrix} \frac{a^2}{(\cosh\tau-\cos\theta)^2} & 0 & 0 \\ 0 & \frac{a^2}{(\cosh\tau-\cos\theta)^2} & 0 \\ 0 & 0 & \frac{a^2\sinh^2\tau}{(\cosh\tau-\cos\theta)^2} \end{bmatrix}. \tag{451}$$

We of course then have

$$\begin{aligned}\mathbf{J} = \boldsymbol{\mathcal{J}}^{-1} = \frac{\partial(\tau,\theta,\zeta)}{\partial(x,y,z)} &= \begin{bmatrix} \frac{\partial\tau}{\partial x} & \frac{\partial\tau}{\partial y} & \frac{\partial\tau}{\partial z} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} & \frac{\partial\theta}{\partial z} \\ \frac{\partial\zeta}{\partial x} & \frac{\partial\zeta}{\partial y} & \frac{\partial\zeta}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\cos\zeta(1-\cos\theta\cosh\tau)}{a} & \frac{\sin\zeta(1-\cos\theta\cosh\tau)}{a} & -\frac{\sin\theta\sinh\tau}{a} \\ -\frac{\cos\zeta\sin\theta\sinh\tau}{a} & -\frac{\sin\zeta\sin\theta\sinh\tau}{a} & \frac{\cos\theta\cosh\tau-1}{a} \\ \frac{(\cos\theta-\cosh\tau)\operatorname{csch}\tau\sin\zeta}{a} & \frac{(\cosh\tau-\cos\theta)\operatorname{csch}\tau\cos\zeta}{a} & 0 \end{bmatrix}\end{aligned} \tag{452}$$

$$J = \frac{1}{\mathcal{J}} = \frac{(\cosh\tau-\cos\theta)^3}{a^3\sinh\tau} \tag{453}$$

This then gives us (utilizing $(1-xy)^2 + (1-x^2)(y^2-1) = (x-y)^2$)

$$\mathbf{e}^1 = \mathbf{e}^\tau = \nabla\tau = \frac{\cos\zeta(1-\cos\theta\cosh\tau)}{a}\nabla x + \frac{\sin\zeta(1-\cos\theta\cosh\tau)}{a}\nabla y + -\frac{\sin\theta\sinh\tau}{a}\nabla z \tag{454}$$

$$\begin{aligned}|\nabla\tau|^2 &= \frac{(1-\cos\theta\cosh\tau)^2\cos^2\zeta + (1-\cos\theta\cosh\tau)^2\sin^2\zeta + \sin^2\theta\sinh^2\tau}{a^2} \\ &= \frac{(1-\cos\theta\cosh\tau)^2 + \sin^2\theta\sinh^2\tau}{a^2} = \frac{(1-\cos\theta\cosh\tau)^2 + (1-\cos^2\theta)(\cosh^2\tau-1)}{a^2} \\ &= \frac{(\cosh\tau-\cos\theta)^2}{a^2}\end{aligned} \tag{455}$$

$$|\nabla\tau| = \frac{\cosh\tau-\cos\theta}{a} \tag{456}$$

$$\mathbf{e}^2 = \mathbf{e}^\theta = \nabla\theta = -\frac{\cos\zeta\sin\theta\sinh\tau}{a}\nabla x + -\frac{\sin\zeta\sin\theta\sinh\tau}{a}\nabla y + \frac{\cos\theta\cosh\tau-1}{a}\nabla z \tag{457}$$

$$\begin{aligned}|\nabla\theta|^2 &= \frac{\sinh^2\tau\sin^2\theta\cos^2\zeta + \sinh^2\tau\sin^2\theta\sin^2\zeta + (1-\cosh\tau\cos\theta)^2}{a^2} \\ &= \frac{\sinh^2\tau\sin^2\theta + (1-\cosh\tau\cos\theta)^2}{a^2} = \frac{(\cosh\tau-\cos\theta)^2}{a^2}\end{aligned} \tag{458}$$

$$|\nabla\theta| = \frac{\cosh\tau-\cos\theta}{a} \tag{459}$$

$$\mathbf{e}^3 = \mathbf{e}^\zeta = \nabla\zeta = \frac{(\cos\theta-\cosh\tau)\operatorname{csch}\tau\sin\zeta}{a}\nabla x + \frac{(\cosh\tau-\cos\theta)\operatorname{csch}\tau\cos\zeta}{a}\nabla y \tag{460}$$

$$|\nabla\zeta|^2 = \frac{(\cosh\tau-\cos\theta)^2\operatorname{csch}^2\tau\sin^2\zeta + (\cosh\tau-\cos\theta)^2\operatorname{csch}^2\tau\cos^2\zeta}{a^2} = \frac{(\cosh\tau-\cos\theta)^2}{a^2\sinh^2\tau} \tag{461}$$

$$|\nabla\zeta| = \frac{\cosh\tau - \cos\theta}{a|\sinh\tau|} \tag{462}$$

Note that

$$g^{ij} = \begin{bmatrix} \frac{(\cosh\tau-\cos\theta)^2}{a^2} & 0 & 0 \\ 0 & \frac{(\cosh\tau-\cos\theta)^2}{a^2} & 0 \\ 0 & 0 & \frac{(\cosh\tau-\cos\theta)^2}{a^2\sinh^2\tau} \end{bmatrix} \tag{463}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial\xi^j} + \frac{\partial g_{jk}}{\partial\xi^i} - \frac{\partial g_{ij}}{\partial\xi^k}\right] \tag{464}$$

$$\Gamma_{\tau,ij} = \begin{bmatrix} \frac{-a^2\sinh^2\tau}{(\cosh\tau-\cos\theta)^3} & \frac{-a^2\sin^2\theta}{(\cosh\tau-\cos\theta)^3} & 0 \\ \frac{-a^2\sin^2\theta}{(\cosh\tau-\cos\theta)^3} & \frac{a^2\sinh^2\tau}{(\cosh\tau-\cos\theta)^3} & 0 \\ 0 & 0 & \frac{a^2\sinh\tau(\cosh\tau\cos\theta-1)}{(\cosh\tau-\cos\theta)^3} \end{bmatrix} \tag{465}$$

$$\Gamma_{\theta,ij} = \begin{bmatrix} \frac{a^2\sin^2\theta}{(\cosh\tau-\cos\theta)^3} & \frac{-a^2\sinh\tau}{(\cosh\tau-\cos\theta)^3} & 0 \\ \frac{-a^2\sinh\tau}{(\cosh\tau-\cos\theta)^3} & \frac{a^2\sin^2\theta}{(\cosh\tau-\cos\theta)^3} & 0 \\ 0 & 0 & \frac{a^2\sin\theta\sinh^2\tau}{(\cosh\tau-\cos\theta)^3} \end{bmatrix} \tag{466}$$

$$\Gamma_{\zeta,ij} = \begin{bmatrix} 0 & 0 & \frac{a^2\sinh\tau(1-\cosh\tau\cos\theta)}{(\cosh\tau-\cos\theta)^3} \\ 0 & 0 & \frac{-a^2\sin\theta\sinh^2\tau}{(\cosh\tau-\cos\theta)^3} \\ \frac{a^2\sinh\tau(1-\cosh\tau\cos\theta)}{(\cosh\tau-\cos\theta)^3} & \frac{-a^2\sin\theta\sinh^2\tau}{(\cosh\tau-\cos\theta)^3} & 0 \end{bmatrix} \tag{467}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{468}$$

$$\Gamma^\tau_{ij} = \begin{bmatrix} \frac{-\sinh\tau}{\cosh\tau-\cos\theta} & \frac{-\sin\theta}{\cosh\tau-\cos\theta} & 0 \\ \frac{-\sin\theta}{\cosh\tau-\cos\theta} & \frac{\sinh\tau}{\cosh\tau-\cos\theta} & 0 \\ 0 & 0 & \frac{\sinh\tau(\cos\theta\cosh\tau-1)}{\cosh\tau-\cos\theta} \end{bmatrix} \tag{469}$$

$$\Gamma^\theta_{ij} = \begin{bmatrix} \frac{\sin\theta}{\cosh\tau-\cos\theta} & \frac{-\sinh\tau}{\cosh\tau-\cos\theta} & 0 \\ \frac{-\sinh\tau}{\cosh\tau-\cos\theta} & \frac{-\sin\theta}{\cosh\tau-\cos\theta} & 0 \\ 0 & 0 & \frac{\sinh\tau\sin\theta}{\cosh\tau-\cos\theta} \end{bmatrix} \tag{470}$$

$$\Gamma^\zeta_{ij} = \begin{bmatrix} 0 & 0 & \frac{1-\cosh\tau\cos\theta}{\sinh\tau(\cosh\tau-\cos\theta)} \\ 0 & 0 & \frac{-\sin\theta}{\cosh\tau-\cos\theta} \\ \frac{1-\cosh\tau\cos\theta}{\sinh\tau(\cosh\tau-\cos\theta)} & \frac{-\sin\theta}{\cosh\tau-\cos\theta} & 0 \end{bmatrix} \tag{471}$$

12 Differential Operators in Coordinate Systems

The following will show the gradient, curl, and divergence of quantities in various coordinate systems. To summarize, for scalar f, vector $\mathbf{A}$, and second order tensor $\overleftrightarrow{\mathbf{T}}$ we find

$$\nabla f = \mathbf{e}^i\frac{\partial f}{\partial\xi^i} \tag{472}$$

$$\nabla\mathbf{A} = \left(\frac{\partial A_k}{\partial \xi^j} - A_i\Gamma^i_{kj}\right)\mathbf{e}^j\mathbf{e}^k \tag{473}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{\mathcal{J}}\frac{\partial}{\partial \xi^i}\left(\mathcal{J}A^i\right) \tag{474}$$

$$(\boldsymbol{\nabla}\times\mathbf{A})^k = \frac{\epsilon^{ijk}}{\mathcal{J}}\frac{\partial A_j}{\partial \xi^i} \tag{475}$$

$$\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}} = \left(\frac{1}{\mathcal{J}}\frac{\partial \mathcal{J}T^{ij}}{\partial \xi^i} + T^{il}\Gamma^j_{il}\right)\mathbf{e}_j \tag{476}$$

$$\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}} = \frac{\epsilon^{ijk}}{\mathcal{J}}\mathbf{e}_k\mathbf{e}^l\left(\frac{\partial T_{jl}}{\partial \xi^i} + T_{ip}\Gamma^p_{jl}\right) \tag{477}$$

I will use that

$$\mathbf{A} = A(1)\hat{\mathbf{e}}^1 + A(2)\hat{\mathbf{e}}^2 + A(3)\hat{\mathbf{e}}^3 \tag{478}$$

$$\overleftrightarrow{\mathbf{T}} = \sum_{i,j=1}^{3} T(i,j)\hat{\mathbf{e}}^i\hat{\mathbf{e}}^j \tag{479}$$

to put vectors and tensors in their standard form (the basis vectors are the normalized tangent-reciprocal basis vectors).

12.1 (Common) Cylindrical Coordinates

We use the right handed coordinates (r, φ, Z). Here $\mathcal{J} = r$.

12.1.1 Gradient

First the gradient of a scalar is found via

$$\begin{aligned}\nabla f &= \mathbf{e}^r\frac{\partial f}{\partial r} + \mathbf{e}^\varphi\frac{\partial f}{\partial \varphi} + \mathbf{e}^Z\frac{\partial f}{\partial Z} \\ &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial Z}\hat{\mathbf{Z}}\end{aligned} \tag{480}$$

The gradient of a vector is given by

$$(\nabla\mathbf{A})(r,r) = (\nabla\mathbf{A})_{rr} = \frac{\partial A_r}{\partial r} = \frac{\partial A(r)}{\partial r} \tag{481}$$

$$(\nabla\mathbf{A})(r,\varphi) = \frac{1}{r}(\nabla\mathbf{A})_{r\varphi} = \frac{1}{r}\frac{\partial A_\varphi}{\partial r} - \frac{A_\varphi}{r^2} = \frac{1}{r}\frac{\partial[rA(\varphi)]}{\partial r} - \frac{A(\varphi)}{r} = \frac{\partial A(\varphi)}{\partial r} \tag{482}$$

$$(\nabla\mathbf{A})(r,Z) = (\nabla\mathbf{A})_{rZ} = \frac{\partial A_Z}{\partial r} = \frac{\partial A(Z)}{\partial r} \tag{483}$$

$$(\nabla\mathbf{A})(\varphi,r) = \frac{1}{r}(\nabla\mathbf{A})_{\varphi r} = \frac{1}{r}\left(\frac{\partial A_r}{\partial \varphi} - \frac{A_\varphi}{r}\right) = \frac{\partial A(r)}{\partial \varphi} - \frac{A(\varphi)}{r} \tag{484}$$

$$(\nabla\mathbf{A})(\varphi,\varphi) = \frac{1}{r^2}(\nabla\mathbf{A})_{\varphi\varphi} = \frac{1}{r^2}\left(rA_r + \frac{\partial A_\varphi}{\partial \varphi}\right) = \frac{1}{r}\frac{\partial A(\varphi)}{\partial \varphi} + \frac{A(r)}{r} \tag{485}$$

$$(\nabla\mathbf{A})(\varphi,Z) = \frac{1}{r}(\nabla\mathbf{A})_{\varphi Z} = \frac{1}{r}\frac{\partial A_Z}{\partial \varphi} = \frac{1}{r}\frac{\partial A(Z)}{\partial \varphi} \tag{486}$$

$$(\nabla\mathbf{A})(Z,r) = (\nabla\mathbf{A})_{Zr} = \frac{\partial A_r}{\partial Z} = \frac{\partial A(r)}{\partial Z} \tag{487}$$

$$(\nabla\mathbf{A})(Z,\varphi) = \frac{1}{r}(\nabla\mathbf{A})_{Z\varphi} = \frac{1}{r}\left(\frac{\partial A_\varphi}{\partial Z}\right) = \frac{\partial A(\varphi)}{\partial Z} \tag{488}$$

$$(\nabla\mathbf{A})(Z,Z) = (\nabla\mathbf{A})_{ZZ} = \frac{\partial A_Z}{\partial Z} = \frac{\partial A(Z)}{\partial Z} \tag{489}$$

As a matrix where rows represent the first index and columns the second index

$$\begin{bmatrix} \frac{\partial A(r)}{\partial r} & \frac{\partial A(\varphi)}{\partial r} & \frac{\partial A(Z)}{\partial r} \\ \frac{1}{r}\frac{\partial A(r)}{\partial \varphi} - \frac{A(\varphi)}{r} & \frac{1}{r}\frac{\partial A(\varphi)}{\partial \varphi} + \frac{A(r)}{r} & \frac{1}{r}\frac{\partial A(Z)}{\partial \phi} \\ \frac{\partial A(r)}{\partial z} & \frac{\partial A(\varphi)}{\partial Z} & \frac{\partial A(Z)}{\partial Z} \end{bmatrix} \tag{490}$$

12.1.2 Divergence

The divergence of a vector is found by

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{1}{r}\frac{\partial(rA^r)}{\partial r} + \frac{1}{r}\frac{\partial(rA^\varphi)}{\partial\varphi} + \frac{1}{r}\frac{\partial(rA^Z)}{\partial Z} \\ &= \frac{1}{r}\frac{\partial(rA(r))}{\partial r} + \frac{1}{r}\frac{\partial A(\varphi)}{\partial\varphi} + \frac{\partial A(Z)}{\partial Z} \end{aligned} \tag{491}$$

The divergence of a second order tensor is found by

$$\begin{aligned} (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(r) = (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^r &= \frac{1}{r}\left(\frac{\partial(rT^{rr})}{\partial r} + \frac{\partial(rT^{\varphi r})}{\partial\varphi} + \frac{\partial(rT^{Zr})}{\partial Z}\right) - rT^{\varphi\varphi} \\ &= \frac{1}{r}\frac{\partial[rT(r,r)]}{\partial r} + \frac{1}{r}\frac{\partial T(\varphi,r)}{\partial\varphi} + \frac{\partial T(Z,r)}{\partial Z} - \frac{T(\varphi,\varphi)}{r} \end{aligned} \tag{492}$$

$$\begin{aligned} (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(\varphi) = r(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^\varphi &= r\frac{1}{r}\left(\frac{\partial(rT^{r\varphi})}{\partial r} + \frac{\partial(rT^{\varphi\varphi})}{\partial\varphi} + \frac{\partial(rT^{Z\varphi})}{\partial Z}\right) + r\frac{T^{r\varphi}+T^{\varphi r}}{r} \\ &= \frac{\partial T(r,\varphi)}{\partial r} + \frac{1}{r}\frac{\partial T(\varphi,\varphi)}{\partial\varphi} + \frac{\partial T(Z,\varphi)}{\partial Z} + \frac{T(r,\varphi)+T(\varphi,r)}{r} \\ &= \frac{1}{r}\frac{\partial[rT(r,\varphi)]}{\partial r} + \frac{1}{r}\frac{\partial T(\varphi,\varphi)}{\partial\varphi} + \frac{\partial T(Z,\varphi)}{\partial Z} + \frac{T(\varphi,r)}{r} \end{aligned} \tag{493}$$

$$\begin{aligned} (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(Z) = (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^Z &= \frac{1}{r}\left(\frac{\partial(rT^{rZ})}{\partial r} + \frac{\partial(rT^{\varphi Z})}{\partial\varphi} + \frac{\partial(rT^{ZZ})}{\partial Z}\right) \\ &= \frac{1}{r}\frac{\partial[rT(r,\varphi)]}{\partial r} + \frac{1}{r}\frac{\partial T(\varphi,Z)}{\partial\varphi} + \frac{\partial T(Z,Z)}{\partial Z} \end{aligned} \tag{494}$$

$$\tag{495}$$

12.1.3 Curl

The curl of a vector is given by

$$\begin{aligned} (\boldsymbol{\nabla}\times\mathbf{A})(r) = (\boldsymbol{\nabla}\times\mathbf{A})^r &= \frac{1}{r}\left(\frac{\partial A_Z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial Z}\right) = \frac{1}{r}\left(\frac{\partial A(Z)}{\partial\varphi} - \frac{\partial[rA(\varphi)]}{\partial Z}\right) \\ &= \frac{1}{r}\frac{\partial A(Z)}{\partial\varphi} - \frac{\partial A(\varphi)}{\partial Z} \end{aligned} \tag{496}$$

$$(\boldsymbol{\nabla}\times\mathbf{A})(\varphi) = r(\boldsymbol{\nabla}\times\mathbf{A})^{\varphi} = \frac{r}{r}\left(\frac{\partial A_r}{\partial Z} - \frac{\partial A_Z}{\partial r}\right) = \left(\frac{\partial A(r)}{\partial Z} - \frac{\partial A(Z)}{\partial r}\right)$$
$$= \frac{\partial A(r)}{\partial Z} - \frac{\partial A(Z)}{\partial r} \tag{497}$$

$$(\boldsymbol{\nabla}\times\mathbf{A})(Z) = (\boldsymbol{\nabla}\times\mathbf{A})^{Z} = \frac{1}{r}\left(\frac{\partial A_\varphi}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right) = \frac{1}{r}\left(\frac{\partial [rA(\varphi)]}{\partial r} - \frac{\partial A(r)}{\partial \varphi}\right)$$
$$= \frac{1}{r}\frac{\partial [rA(\varphi)]}{\partial r} - \frac{1}{r}\frac{\partial A(r)}{\partial \varphi} \tag{498}$$

The curl of a second order tensor is given by

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(r,r) = (\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{r}_{\cdot r} = \frac{1}{r}\left(\frac{\partial T_{Zr}}{\partial \varphi} - \frac{\partial T_{\varphi r}}{\partial Z}\right) - \frac{T_{Z\varphi}}{r^2}$$
$$= \frac{1}{r}\frac{\partial T(Z,r)}{\partial \varphi} - \frac{\partial T(\varphi,r)}{\partial Z} - \frac{T(Z,\varphi)}{r} \tag{499}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(r,\varphi) = \frac{1}{r}(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{r}_{\cdot\varphi} = \frac{1}{r^2}\left(\frac{\partial T_{Z\varphi}}{\partial \varphi} - \frac{\partial T_{\varphi\varphi}}{\partial Z}\right) + \frac{T_{Zr}}{r}$$
$$= \frac{1}{r}\frac{\partial T(Z,\varphi)}{\partial \varphi} - \frac{\partial T(\varphi,\varphi)}{\partial Z} + \frac{T(Z,r)}{r} \tag{500}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(r,Z) = (\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{r}_{\cdot Z} = \frac{1}{r}\left(\frac{\partial T_{ZZ}}{\partial \varphi} - \frac{\partial T_{\varphi Z}}{\partial Z}\right)$$
$$= \frac{1}{r}\frac{\partial T(Z,Z)}{\partial \varphi} - \frac{\partial T(\varphi,Z)}{\partial Z} \tag{501}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(\varphi,r) = r(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{\varphi}_{\cdot r} = r\frac{1}{r}\left(\frac{\partial T_{rr}}{\partial Z} - \frac{\partial T_{Zr}}{\partial r}\right)$$
$$= \frac{\partial T(r,r)}{\partial Z} - \frac{\partial T(Z,r)}{\partial r} \tag{502}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(\varphi,\varphi) = (\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{\varphi}_{\cdot\varphi} = \frac{1}{r}\left(\frac{\partial T_{r\varphi}}{\partial Z} - \frac{\partial T_{Z\varphi}}{\partial r}\right) + \frac{T_{Z\varphi}}{r^2}$$
$$= \frac{\partial T(r,\varphi)}{\partial Z} - \frac{\partial T(Z,\varphi)}{\partial r} + \frac{T(Z,\varphi)}{r} \tag{503}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(\varphi,Z) = r(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{\varphi}_{\cdot Z} = r\frac{1}{r}\left(\frac{\partial T_{rZ}}{\partial Z} - \frac{\partial T_{ZZ}}{\partial r}\right)$$
$$= \frac{\partial T(r,Z)}{\partial Z} - \frac{\partial T(Z,Z)}{\partial r} \tag{504}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(Z,r) = (\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{Z}_{\cdot r} = \frac{1}{r}\left(\frac{\partial T_{\varphi r}}{\partial r} - \frac{\partial T_{rr}}{\partial \varphi}\right) + \frac{T_{r\varphi}}{r^2}$$
$$= \frac{\partial T(\varphi,r)}{\partial r} - \frac{1}{r}\frac{\partial T(r,r)}{\partial \varphi} + \frac{T(r,\varphi)}{r} \tag{505}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(Z,\varphi) = \frac{1}{r}(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{Z}_{\cdot\varphi} = \frac{1}{r^2}\left(\frac{\partial T_{\varphi\varphi}}{\partial r} - \frac{\partial T_{r\varphi}}{\partial \varphi}\right) - \frac{T_{rr}}{r} - \frac{T_{\varphi\varphi}}{r^3}$$
$$= \frac{\partial T(\varphi,\varphi)}{\partial r} - \frac{1}{r}\frac{\partial T(r,\varphi)}{\partial \varphi} - \frac{T(r,r)}{r} - \frac{T(\varphi,\varphi)}{r} \tag{506}$$

$$\begin{aligned}(\boldsymbol{\nabla} \times \overleftrightarrow{\mathbf{T}})(Z, Z) = (\boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{T}})^{Z}_{\cdot Z} &= \frac{1}{r}\left(\frac{\partial T_{\varphi Z}}{\partial r} - \frac{\partial T_{rZ}}{\partial \varphi}\right) \\ &= \frac{\partial T(\varphi, Z)}{\partial r} - \frac{1}{r}\frac{\partial T(r, Z)}{\partial \varphi}\end{aligned} \tag{507}$$

12.2 (Plasma/Toroidal System) Cylindrical Coordinates

We use the right handed coordinates (R, Z, ζ). Here $\mathcal{J} = R$.

12.2.1 Gradient

First the gradient of a scalar is found via

$$\begin{aligned}\nabla f &= \mathbf{e}^R \frac{\partial f}{\partial R} + \mathbf{e}^Z \frac{\partial f}{\partial Z} + \mathbf{e}^\zeta \frac{\partial f}{\partial \zeta} \\ &= \frac{\partial f}{\partial R}\hat{\mathbf{R}} + \frac{\partial f}{\partial Z}\hat{\mathbf{Z}} + \frac{1}{R}\frac{\partial f}{\partial \zeta}\hat{\boldsymbol{\zeta}}\end{aligned} \tag{508}$$

The gradient of a vector is given by

$$(\nabla \mathbf{A})(R,R) = (\nabla \mathbf{A})_{RR} = \frac{\partial A_R}{\partial R} = \frac{\partial A(R)}{\partial R} \tag{509}$$

$$(\nabla \mathbf{A})(R,Z) = \frac{1}{R}(\nabla \mathbf{A})_{RZ} = \frac{\partial A_Z}{\partial R} = \frac{\partial A(Z)}{\partial R} \tag{510}$$

$$(\nabla \mathbf{A})(R,\zeta) = \frac{1}{R}(\nabla \mathbf{A})_{R\zeta} = \frac{1}{R}\left(\frac{\partial A_\zeta}{\partial R} - \frac{A_\zeta}{R}\right) = \frac{1}{R}\frac{\partial [RA(\zeta)]}{\partial R} - \frac{A(\zeta)}{R} = \frac{\partial A(\zeta)}{\partial R} \tag{511}$$

$$(\nabla \mathbf{A})(Z,R) = (\nabla \mathbf{A})_{ZR} = \frac{\partial A_R}{\partial Z} = \frac{\partial A(R)}{\partial Z} \tag{512}$$

$$(\nabla \mathbf{A})(Z,Z) = (\nabla \mathbf{A})_{ZZ} = \frac{\partial A_Z}{\partial Z} = \frac{\partial A(Z)}{\partial Z} \tag{513}$$

$$(\nabla \mathbf{A})(Z,\zeta) = \frac{1}{R}(\nabla \mathbf{A})_{Z\zeta} = \frac{1}{R}\left(\frac{\partial A_\zeta}{\partial Z}\right) = \frac{\partial A(\zeta)}{\partial Z} \tag{514}$$

$$(\nabla \mathbf{A})(\zeta,R) = \frac{1}{R}(\nabla \mathbf{A})_{\zeta R} = \frac{1}{R}\left(\frac{\partial A_R}{\partial \zeta} - \frac{A_\zeta}{R}\right) = \frac{1}{R}\frac{\partial A(R)}{\partial \zeta} - \frac{A(\zeta)}{R} \tag{515}$$

$$(\nabla \mathbf{A})(\zeta,Z) = \frac{1}{R}(\nabla \mathbf{A})_{\zeta Z} = \frac{1}{R}\left(\frac{\partial A_Z}{\partial \zeta}\right) = \frac{1}{R}\frac{\partial A(Z)}{\partial \zeta} \tag{516}$$

$$(\nabla \mathbf{A})(\zeta,\zeta) = \frac{1}{R^2}(\nabla \mathbf{A})_{\zeta\zeta} = \frac{1}{R^2}\left(A_R R + \frac{\partial A_\zeta}{\partial \zeta}\right) = \frac{1}{R}\frac{\partial A(\zeta)}{\partial \zeta} + \frac{A(R)}{R} \tag{517}$$

As a matrix where rows represent the first index and columns the second index

$$\begin{bmatrix} \frac{\partial A(R)}{\partial R} & \frac{\partial A(Z)}{\partial R} & \frac{\partial A(\zeta)}{\partial R} \\ \frac{\partial A(R)}{\partial Z} & \frac{\partial A(Z)}{\partial Z} & \frac{\partial A(\zeta)}{\partial Z} \\ \frac{1}{R}\frac{\partial A(R)}{\partial \zeta} - \frac{A(\varphi)}{R} & \frac{1}{R}\frac{\partial A(Z)}{\partial \zeta} & \frac{1}{R}\frac{\partial A(\zeta)}{\partial \zeta} + \frac{A(R)}{R} \end{bmatrix} \tag{518}$$

12.2.2 Divergence

The divergence of a vector is given by

$$\begin{aligned}\boldsymbol{\nabla} \cdot \mathbf{A} &= \frac{1}{R}\frac{\partial (RA^R)}{\partial R} + \frac{1}{R}\frac{\partial (RA^Z)}{\partial Z} + \frac{1}{R}\frac{\partial (RA^\zeta)}{\partial \zeta} \\ &= \frac{1}{R}\frac{\partial (RA(R))}{\partial R} + \frac{1}{R}\frac{\partial A(Z)}{\partial Z} + \frac{\partial A(\zeta)}{\partial \zeta}\end{aligned} \tag{519}$$

The divergence of a second order tensor is given by

$$\begin{aligned}(\nabla\cdot\overleftrightarrow{\mathbf{T}})(R) = (\nabla\cdot\overleftrightarrow{\mathbf{T}})^R &= \frac{1}{R}\left(\frac{\partial(RT^{RR})}{\partial R} + \frac{\partial(RT^{ZR})}{\partial Z} + \frac{\partial(RT^{\zeta R})}{\partial\zeta}\right) - RT^{\zeta\zeta} \\ &= \frac{1}{R}\frac{\partial(RT(R,R))}{\partial R} + \frac{\partial T(Z,R)}{\partial Z} + \frac{1}{R}\frac{\partial T(\zeta,R)}{\partial\zeta} - \frac{T(\zeta,\zeta)}{R}\end{aligned} \tag{520}$$

$$\begin{aligned}(\nabla\cdot\overleftrightarrow{\mathbf{T}})(Z) = (\nabla\cdot\overleftrightarrow{\mathbf{T}})^Z &= \frac{1}{R}\left(\frac{\partial(RT^{RZ})}{\partial R} + \frac{\partial(RT^{ZZ})}{\partial Z} + \frac{\partial(RT^{\zeta Z})}{\partial\zeta}\right) \\ &= \frac{1}{R}\frac{\partial[RT(R,Z)]}{\partial R} + \frac{\partial T(Z,Z)}{\partial Z} + \frac{1}{R}\frac{\partial T(\zeta,Z)}{\partial\zeta}\end{aligned} \tag{521}$$

$$\begin{aligned}(\nabla\cdot\overleftrightarrow{\mathbf{T}})(\zeta) = R(\nabla\cdot\overleftrightarrow{\mathbf{T}})^\zeta &= R\frac{1}{R}\left(\frac{\partial(RT^{R\zeta})}{\partial R} + \frac{\partial(RT^{Z\zeta})}{\partial Z} + \frac{\partial(RT^{\zeta\zeta})}{\partial\zeta}\right) + R\frac{T^{R\zeta}+T^{\zeta R}}{R} \\ &= \frac{\partial T(R,\zeta)}{\partial R} + \frac{\partial T(Z,\zeta)}{\partial Z} + \frac{1}{R}\frac{\partial T(\zeta,\zeta)}{\partial\zeta} + \frac{T(R,\zeta)+T(\zeta,R)}{R} \\ &= \frac{1}{R}\frac{\partial[RT(R,\zeta)]}{\partial R} + \frac{\partial T(Z,\zeta)}{\partial Z} + \frac{1}{R}\frac{\partial T(\zeta,\zeta)}{\partial\zeta} + \frac{T(\zeta,R)}{R}\end{aligned} \tag{522}$$

12.2.3 Curl

The curl of a vector is given by

$$\begin{aligned}(\nabla\times\mathbf{A})(R) = (\nabla\times\mathbf{A})^R &= \frac{1}{R}\left(\frac{\partial A_\zeta}{\partial Z} - \frac{\partial A_Z}{\partial\zeta}\right) = \frac{1}{R}\left(\frac{\partial[RA(\zeta)]}{\partial Z} - \frac{\partial A(Z)}{\partial\zeta}\right) \\ &= \frac{\partial A(\zeta)}{\partial Z} - \frac{1}{R}\frac{\partial A(Z)}{\partial\zeta}\end{aligned} \tag{523}$$

$$\begin{aligned}(\nabla\times\mathbf{A})(Z) = (\nabla\times\mathbf{A})^Z &= \frac{1}{R}\left(\frac{\partial A_R}{\partial\zeta} - \frac{\partial A_\zeta}{\partial R}\right) \\ &= \frac{1}{R}\frac{\partial A(R)}{\partial\zeta} - \frac{\partial A(\zeta)}{\partial R}\end{aligned} \tag{524}$$

$$\begin{aligned}(\nabla\times\mathbf{A})(\zeta) = R(\nabla\times\mathbf{A})^\zeta &= \frac{1}{R}\left(\frac{\partial A_R}{\partial Z} - \frac{\partial A_Z}{\partial R}\right) \\ &= \frac{1}{R}\left(\frac{\partial A(R)}{\partial Z} - \frac{\partial A(Z)}{\partial R}\right)\end{aligned} \tag{525}$$

The curl of a second order tensor is given by

$$\begin{aligned}(\nabla\times\overleftrightarrow{\mathbf{T}})(R,R) = (\nabla\times\overleftrightarrow{\mathbf{T}})^R_{\cdot R} &= \frac{1}{R}\left(\frac{\partial T_{\zeta R}}{\partial Z} - \frac{\partial T_{ZR}}{\partial\zeta}\right) + \frac{T_{Z\zeta}}{R^2} \\ &= \frac{\partial T(\zeta,R)}{\partial Z} - \frac{1}{R}\frac{\partial T(Z,R)}{\partial\zeta} + \frac{T(Z,\zeta)}{R}\end{aligned} \tag{526}$$

$$\begin{aligned}(\nabla\times\overleftrightarrow{\mathbf{T}})(R,Z) = (\nabla\times\overleftrightarrow{\mathbf{T}})^R_{\cdot Z} &= \frac{1}{R}\left(\frac{\partial T_{\zeta Z}}{\partial Z} - \frac{\partial T_{ZZ}}{\partial\zeta}\right) \\ &= \frac{\partial T(\zeta,Z)}{\partial R} - \frac{1}{R}\frac{\partial T(Z,Z)}{\partial\zeta}\end{aligned} \tag{527}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(R,\zeta) &= \frac{1}{R}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})^{R}_{\cdot\zeta} = \frac{1}{R^2}\left(\frac{\partial T_{\zeta\zeta}}{\partial Z} - \frac{\partial T_{Z\zeta}}{\partial\zeta}\right) - \frac{RT_{ZR}}{R^2} \\ &= \frac{\partial T(\zeta,\zeta)}{\partial Z} - \frac{1}{R}\frac{\partial T(Z,\zeta)}{\partial\zeta} - \frac{T(Z,R)}{R}\end{aligned} \tag{528}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(Z,R) &= (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{Z}_{\cdot R} = \frac{1}{R}\left(\frac{\partial T_{RR}}{\partial\zeta} - \frac{\partial T_{\zeta R}}{\partial R}\right) - \frac{T_{R\zeta}}{R^2} \\ &= \frac{1}{R}\frac{\partial T(R,R)}{\partial\zeta} - \frac{\partial T(\zeta,R)}{\partial R} - \frac{T(R,\zeta)}{R}\end{aligned} \tag{529}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(Z,Z) &= (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{Z}\cdot Z = \frac{1}{R}\left(\frac{\partial T_{RZ}}{\partial\zeta} - \frac{\partial T_{\zeta Z}}{\partial R}\right) \\ &= \frac{1}{R}\frac{\partial T(R,Z)}{\partial\zeta} - \frac{\partial T(\zeta,Z)}{\partial R}\end{aligned} \tag{530}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(Z,\zeta) &= \frac{1}{R}(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{Z}_{\cdot\zeta} = \frac{1}{R^2}\left(\frac{\partial T_{R\zeta}}{\partial\zeta} - \frac{\partial T_{\zeta\zeta}}{\partial R}\right) + \frac{RT_{RR} + \frac{T_{\zeta\zeta}}{R}}{R^2} \\ &= \frac{1}{R}\frac{\partial T(R,\zeta)}{\partial\zeta} - \frac{\partial T(\zeta,\zeta)}{\partial R} + \frac{T(R,R) + T(\zeta,\zeta)}{R}\end{aligned} \tag{531}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\zeta,R) &= R(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\zeta}_{\cdot R} = R\frac{1}{R}\left(\frac{\partial T_{ZR}}{\partial R} - \frac{\partial T_{RR}}{\partial Z}\right) \\ &= \frac{\partial T(Z,R)}{\partial R} - \frac{\partial T(R,R)}{\partial Z}\end{aligned} \tag{532}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\zeta,Z) &= R(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\zeta}_{\cdot Z} = R\frac{1}{R}\left(\frac{\partial T_{ZZ}}{\partial R} - \frac{\partial T_{RZ}}{\partial Z}\right) \\ &= \frac{\partial T(Z,Z)}{\partial R} - \frac{\partial T(R,Z)}{\partial Z}\end{aligned} \tag{533}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\zeta,\zeta) &= (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\zeta}_{\cdot\zeta} = \frac{1}{R}\left(\frac{\partial T_{Z\zeta}}{\partial R} - \frac{\partial T_{R\zeta}}{\partial Z}\right) - \frac{T_{Z\zeta}}{R^2} \\ &= \frac{\partial T(Z,\zeta)}{\partial R} - \frac{\partial T(R,\zeta)}{\partial Z} - \frac{T(Z,\zeta)}{R}\end{aligned} \tag{534}$$

12.3 (Physicists') Spherical Coordinates

We use the right handed coordinates (r, θ, φ). Here $\mathcal{J} = r^2 \sin\theta$.

12.3.1 Gradient

First the gradient of a scalar is found via

$$\begin{aligned}\nabla f &= \mathbf{e}^r \frac{\partial f}{\partial r} + \mathbf{e}^\theta \frac{\partial f}{\partial \theta} + \mathbf{e}^\varphi \frac{\partial f}{\partial \varphi} \\ &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}\end{aligned} \tag{535}$$

The gradient of a vector is given by

$$(\nabla\mathbf{A})(r,r) = (\nabla\mathbf{A})_{rr} = \frac{\partial A_r}{\partial r} = \frac{\partial A(r)}{\partial r} \tag{536}$$

$$(\nabla\mathbf{A})(r,\theta) = \frac{1}{r}(\nabla\mathbf{A})_{r\theta} = \frac{1}{r}\left(\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r}\right) = \frac{1}{r}\frac{\partial[rA(\theta)]}{\partial r} - \frac{A(\theta)}{r} = \frac{\partial A(\theta)}{\partial r} \tag{537}$$

$$(\nabla\mathbf{A})(r,\varphi) = \frac{1}{r\sin\theta}(\nabla\mathbf{A})_{r\varphi} = \frac{1}{r\sin\theta}\left(\frac{\partial A_\varphi}{\partial r} - \frac{A_\varphi}{r}\right) = \frac{1}{r}\frac{\partial[rA(\varphi)]}{\partial r} - \frac{A(\varphi)}{r} = \frac{\partial A(\varphi)}{\partial r} \tag{538}$$

$$(\nabla\mathbf{A})(\theta,r) = \frac{1}{r}(\nabla\mathbf{A})_{\theta r} = \frac{1}{r}\left(\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r}\right) = \frac{1}{r}\frac{\partial A(r)}{\partial \theta} - \frac{A(\theta)}{r} \tag{539}$$

$$(\nabla\mathbf{A})(\theta,\theta) = \frac{1}{r^2}(\nabla\mathbf{A})_{\theta\theta} = \frac{1}{r^2}\left(\frac{\partial A_\theta}{\partial \theta} + A_r r\right) = \frac{1}{r}\frac{\partial A(\theta)}{\partial \theta} + \frac{A(r)}{r} \tag{540}$$

$$\begin{aligned}(\nabla\mathbf{A})(\theta,\varphi) &= \frac{1}{r^2\sin\theta}(\nabla\mathbf{A})_{\theta\varphi} = \frac{1}{r^2\sin\theta}\left(\frac{\partial A_\varphi}{\partial \theta} - \frac{A_\varphi\cos\theta}{\sin\theta}\right) \\ &= \frac{1}{r\sin\theta}\frac{\partial[\sin\theta\, A(\varphi)]}{\partial \theta} - \frac{A(\varphi)\cot\theta}{r}\end{aligned} \tag{541}$$

$$(\nabla\mathbf{A})(\varphi,r) = \frac{1}{r\sin\theta}(\nabla\mathbf{A})_{\varphi r} = \frac{1}{r\sin\theta}\left(\frac{\partial A_r}{\partial \varphi} - \frac{A_\varphi}{r}\right) = \frac{1}{r\sin\theta}\frac{\partial A(r)}{\partial \varphi} - \frac{A(\varphi)}{r} \tag{542}$$

$$(\nabla\mathbf{A})(\varphi,\theta) = \frac{1}{r^2\sin\theta}(\nabla\mathbf{A})_{\varphi\theta} = \frac{1}{r^2\sin\theta}\left(\frac{\partial A_\theta}{\partial \varphi} - A_\varphi\cot\theta\right) = \frac{1}{r\sin\theta}\frac{\partial A(\theta)}{\partial \varphi} - \frac{A(\varphi)\cot\theta}{r} \tag{543}$$

$$\begin{aligned}(\nabla\mathbf{A})(\varphi,\varphi) &= \frac{1}{r^2\sin^2\theta}(\nabla\mathbf{A})_{\varphi\varphi} = \frac{1}{r^2\sin^2\theta}\left(\frac{\partial A_\varphi}{\partial \varphi} + A_r r\sin\theta + A_\theta\sin\theta\cos\theta\right) \\ &= \frac{1}{r\sin\theta}\frac{\partial A(\varphi)}{\partial \varphi} + \frac{A(r)}{r\sin\theta} + \frac{A(\theta)\cot\theta}{r}\end{aligned} \tag{544}$$

As a matrix where rows represent the first index and columns the second index

$$\begin{bmatrix} \frac{\partial A(r)}{\partial r} & \frac{\partial A(\theta)}{\partial r} & \frac{\partial A(\varphi)}{\partial r} \\ \frac{1}{r}\left(\frac{\partial A(r)}{\partial \theta} - A(\theta)\right) & \frac{1}{r}\left(\frac{\partial A(\theta)}{\partial \theta} + A(r)\right) & \frac{1}{r\sin\theta}\left(\frac{\partial[\sin\theta\, A(\varphi)]}{\partial \theta} - A(\varphi)\cos\theta\right) \\ \frac{1}{r\sin\theta}\left(\frac{\partial A(r)}{\partial \varphi} - A(\varphi)\sin\theta\right) & \frac{1}{r\sin\theta}\left(\frac{\partial A(\theta)}{\partial \varphi} - A(\varphi)\cos\theta\right) & \frac{1}{r\sin\theta}\left(\frac{\partial A(\varphi)}{\partial \varphi} + A(r) + A(\theta)\cos\theta\right)\end{bmatrix} \tag{545}$$

12.3.2 Divergence

The divergence of a vector is given by

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{1}{r^2\sin\theta}\frac{\partial(r^2\sin\theta A^r)}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial(r^2\sin\theta A^\theta)}{\partial\theta}+\frac{1}{r^2\sin\theta}\frac{\partial(r^2\sin\theta A^\varphi)}{\partial\varphi}\\ &= \frac{1}{r^2}\frac{\partial(r^2A(r))}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial[\sin\theta A(\theta)]}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A(\varphi)}{\partial\varphi}\end{aligned} \tag{546}$$

The divergence of a second order tensor is given by

$$\begin{aligned}(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(r) = (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^r &= \frac{1}{r^2\sin\theta}\left(\frac{\partial(\mathcal{J}T^{rr})}{\partial r}+\frac{\partial(\mathcal{J}T^{\theta r})}{\partial\theta}+\frac{\partial(\mathcal{J}T^{\varphi r})}{\partial\varphi}-rT^{\theta\theta}\right)-r\sin^2\theta T^{\varphi\varphi}\\ &= \frac{1}{r^2}\frac{\partial[r^2T(r,r)]}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial[\sin\theta T(\theta,r)]}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T(\varphi,r)}{\partial\varphi}-\frac{T(\theta,\theta)+T(\varphi,\varphi)}{r}\end{aligned} \tag{547}$$

$$\begin{aligned}&(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(\theta) = r(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^\theta\\ &= \frac{r}{r^2\sin\theta}\left(\frac{\partial(\mathcal{J}T^{r\theta})}{\partial r}+\frac{\partial(\mathcal{J}T^{\theta\theta})}{\partial\theta}+\frac{\partial(\mathcal{J}T^{\varphi\theta})}{\partial\varphi}\right)+r\frac{T^{r\theta}+T^{\theta r}}{r}-r\sin\theta\cos\theta T^{\varphi\varphi}\\ &= \frac{1}{r}\frac{\partial[rT(r,\theta)]}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial[\sin\theta T(\theta,\theta)]}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T(\varphi,\theta)}{\partial\varphi}+\frac{T(r,\theta)+T(\theta,r)}{r}+\frac{\cot\theta\, T(\varphi,\varphi)}{r}\\ &\quad= \frac{1}{r^2}\frac{\partial[r^2T(r,\theta)]}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial[\sin\theta T(\theta,\theta)]}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T(\varphi,\theta)}{\partial\varphi}+\frac{T(\theta,r)}{r}+\frac{\cot\theta\, T(\varphi,\varphi)}{r}\end{aligned} \tag{548}$$

$$\begin{aligned}&(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(\varphi) = r\sin\theta(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^\varphi\\ &= \frac{r\sin\theta}{r^2\sin\theta}\left(\frac{\partial(\mathcal{J}T^{r\varphi})}{\partial r}+\frac{\partial(\mathcal{J}T^{\theta\varphi})}{\partial\theta}+\frac{\partial(\mathcal{J}T^{\varphi\varphi})}{\partial\varphi}\right)+r\sin\theta\left(\frac{T^{r\varphi}+T^{\varphi r}}{r}+\cot\theta[T^{\theta\varphi}+T^{\varphi\theta}]\right)\\ &= \frac{1}{r}\frac{\partial[rT(r,\varphi)]}{\partial r}+\frac{1}{r}\frac{\partial T(\theta,\varphi)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T(\varphi,\varphi)}{\partial\varphi}+\frac{T(r,\varphi)+T(\varphi,r)}{r}+\cot\theta\frac{T(\theta,\varphi)+T(\varphi,\theta)}{r}\\ &= \frac{1}{r^2}\frac{\partial[r^2T(r,\varphi)]}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial[\sin\theta\, T(\theta,\varphi)]}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T(\varphi,\varphi)}{\partial\varphi}+\frac{T(\varphi,r)}{r}+\cot\theta\frac{T(\varphi,\theta)}{r}\end{aligned} \tag{549}$$

12.3.3 Curl

The curl of a vector is given by

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})(r) = (\boldsymbol{\nabla}\times\mathbf{A})^r &= \frac{1}{r^2\sin\theta}\left(\frac{\partial A_\varphi}{\partial\theta}-\frac{\partial A_\theta}{\partial\varphi}\right)\\ &= \frac{1}{r\sin\theta}\frac{\partial[\sin\theta A(\varphi)]}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial A(\theta)}{\partial\varphi}\end{aligned} \tag{550}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})(\theta) = r(\boldsymbol{\nabla}\times\mathbf{A})^\theta &= r\frac{1}{r^2\sin\theta}\left(\frac{\partial A_r}{\partial\varphi}-\frac{\partial A_\varphi}{\partial r}\right)\\ &= \frac{1}{r\sin\theta}\frac{\partial A(r)}{\partial\varphi}-\frac{1}{r}\frac{\partial[rA(\varphi)]}{\partial r}\end{aligned} \tag{551}$$

$$(\boldsymbol{\nabla}\times\mathbf{A})(\varphi) = r\sin\theta(\boldsymbol{\nabla}\times\mathbf{A})^{\varphi} = \frac{r\sin\theta}{r^2\sin\theta}\left(\frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right) \\ = \frac{1}{r}\frac{\partial[rA(\theta)]}{\partial r} - \frac{1}{r}\frac{\partial A(r)}{\partial\theta} \tag{552}$$

The curl of a second order tensor is given by

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,r) = (\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})^{r}_{\cdot r} = \frac{1}{r^2\sin\theta}\left(\frac{\partial T_{\varphi r}}{\partial\theta} - \frac{\partial T_{\theta r}}{\partial\varphi}\right) + \frac{T_{\theta\varphi} - T_{\varphi\theta}}{r^3\sin\theta} \\ = \frac{1}{r\sin\theta}\frac{\partial[\sin\theta\, T(\varphi,r)]}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial T(\theta,r)}{\partial\varphi} + \frac{T(\theta,\varphi) - T(\varphi,\theta)}{r} \tag{553}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,\theta) = \frac{1}{r}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})^{r}_{\cdot\theta} = \frac{1}{r^3\sin\theta}\left(\frac{\partial T_{\varphi\theta}}{\partial\theta} - \frac{\partial T_{\theta\theta}}{\partial\varphi}\right) + \frac{\cot\theta\, T_{\theta\varphi} + rT_{\varphi r}}{r^3\sin\theta} \\ = \frac{1}{r\sin\theta}\frac{\partial[\sin\theta\, T(\varphi,\theta)]}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial T(\theta,\theta)}{\partial\varphi} + \frac{\cot\theta\, T(\theta,\varphi) + T(\varphi,r)}{r} \tag{554}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,\varphi) = \frac{1}{r\sin\theta}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})^{r}_{\cdot\varphi} \\ = \frac{1}{r^3\sin^2\theta}\left(\frac{\partial T_{\varphi\varphi}}{\partial\theta} - \frac{\partial T_{\theta\varphi}}{\partial\varphi}\right) - \frac{\cot\theta\, T_{\varphi\varphi} + r\sin^2\theta\, T_{\theta r} - \sin\theta\cos\theta\, T_{\theta\theta}}{r^3\sin^2\theta} \\ = \frac{1}{r\sin^2\theta}\frac{\partial[\sin^2\theta\, T(\varphi,\varphi)]}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial T(\theta,\varphi)}{\partial\varphi} - \frac{\cot\theta[T(\varphi,\varphi) + T(\theta,\theta)] + T(\theta,r)}{r} \tag{555}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\theta,r) = r(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot r} = \frac{r}{r^2\sin\theta}\left(\frac{\partial T_{rr}}{\partial\varphi} - \frac{\partial T_{\varphi r}}{\partial r}\right) - r\frac{T_{r\varphi}}{r^3\sin\theta} \\ = \frac{1}{r\sin\theta}\frac{\partial T(r,r)}{\partial\varphi} - \frac{1}{r}\frac{\partial[rT(\varphi,r)]}{\partial r} - \frac{T(r,\varphi)}{r} \tag{556}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\theta,\theta) = (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot\theta} = \frac{1}{r^2\sin\theta}\left(\frac{\partial T_{r\theta}}{\partial\varphi} - \frac{\partial T_{\varphi\theta}}{\partial r}\right) + \frac{\frac{1}{r}T_{\varphi\theta} - \cot\theta\, T_{r\varphi}}{r^2\sin\theta} \\ = \frac{1}{r\sin\theta}\frac{\partial T(r,\theta)}{\partial\varphi} - \frac{1}{r^2}\frac{\partial[r^2T(\varphi,\theta)]}{\partial r} + \frac{T(\varphi,\theta) - \cot\theta\, T(r,\varphi)}{r} \tag{557}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\theta,\varphi) = \frac{1}{\sin\theta}(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot\varphi} = \frac{1}{r^2\sin^2\theta}\left(\frac{\partial T_{r\varphi}}{\partial\varphi} - \frac{\partial T_{\varphi\varphi}}{\partial r}\right) + \frac{\frac{T_{\varphi\varphi}}{r} + r\sin^2\theta\, T_{rr} + \sin\theta\cos\theta\, T_{r\theta}}{r^2\sin^2\theta} \\ = \frac{1}{r\sin\theta}\frac{\partial T(r,\varphi)}{\partial\varphi} - \frac{1}{r^2}\frac{\partial[r^2T(\varphi,\varphi)]}{\partial r} + \frac{T(\varphi,\varphi) + T(r,r) + T(r,\theta)}{r} \tag{558}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\varphi,r) = r\sin\theta(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\varphi}_{\cdot r} = \frac{r\sin\theta}{r^2\sin\theta}\left(\frac{\partial T_{\theta r}}{\partial r} - \frac{\partial T_{rr}}{\partial\theta}\right) + \frac{r\sin\theta\, T_{r\theta}}{r(r^2\sin\theta)} \\ = \frac{1}{r}\frac{\partial[rT(\theta,r)]}{\partial r} - \frac{1}{r}\frac{\partial T(r,r)}{\partial\theta} + \frac{T(r,\theta)}{r} \tag{559}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\varphi,\theta) = \sin\theta(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\varphi}_{\cdot\theta} = \frac{\sin\theta}{r^2\sin\theta}\left(\frac{\partial T_{\theta\theta}}{\partial r} - \frac{\partial T_{r\theta}}{\partial\theta}\right) + \sin\theta\frac{\frac{T_{\theta\theta}}{r} - rT_{rr}}{r^2\sin\theta} \\ = \frac{1}{r^2}\frac{\partial[r^2T(\theta,\theta)]}{\partial r} - \frac{1}{r}\frac{\partial T(r,\theta)}{\partial\theta} + \frac{T(\theta,\theta) - T(r,r)}{r} \tag{560}$$

$$
\begin{aligned}
(\boldsymbol{\nabla} \times \overleftrightarrow{\mathbf{T}})(\varphi, \varphi) = (\boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{T}})^{\varphi}_{\cdot\varphi} &= \frac{1}{r^2 \sin\theta}\left(\frac{\partial T_{\theta\varphi}}{\partial r} - \frac{\partial T_{r\varphi}}{\partial \theta}\right) + \frac{\cot\theta\, T_{r\varphi} - \frac{T_{\theta\varphi}}{r}}{r^2 \sin\theta} \\
&= \frac{1}{r^2}\frac{\partial[r^2 T(\theta,\varphi)]}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial[\sin\theta\, T(r,\varphi)]}{\partial \theta} + \frac{\cot\theta\, T(r,\varphi) - T(\theta,\varphi)}{r}
\end{aligned} \tag{561}
$$

12.4 Primitive Toroidal Coordinates

We use the right handed coordinates (r, θ, ζ). Here $\mathcal{J} = rR = r(R_0 + r\cos\theta)$.

12.4.1 Gradient

First the gradient of a scalar is found via

$$\begin{aligned} \nabla f &= \mathbf{e}^r \frac{\partial f}{\partial r} + \mathbf{e}^\theta \frac{\partial f}{\partial \theta} + \mathbf{e}^\zeta \frac{\partial f}{\partial \zeta} \\ &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{R}\frac{\partial f}{\partial \zeta}\hat{\boldsymbol{\zeta}} \end{aligned} \tag{562}$$

The gradient of a vector is given by

$$(\nabla \mathbf{A})(r,r) = (\nabla \mathbf{A})_{rr} = \frac{\partial A_r}{\partial r} = \frac{\partial A(r)}{\partial r} \tag{563}$$

$$(\nabla \mathbf{A})(r,\theta) = \frac{1}{r}(\nabla \mathbf{A})_{r\theta} = \frac{1}{r}\left(\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r}\right) = \frac{\partial[rA(\theta)]}{\partial r} - \frac{A(\theta)}{r} = \frac{\partial A(\theta)}{\partial r} \tag{564}$$

$$(\nabla \mathbf{A})(r,\zeta) = \frac{1}{R}(\nabla \mathbf{A})_{r\zeta} = \frac{1}{R}\left(\frac{\partial A_\zeta}{\partial r} - \frac{A_\zeta \cos\theta}{R}\right) = \frac{1}{R}\frac{\partial[RA(\zeta)]}{\partial r} - \frac{A(\zeta)\cos\theta}{R} \tag{565}$$

$$(\nabla \mathbf{A})(\theta,r) = \frac{1}{r}(\nabla \mathbf{A})_{\theta r} = \frac{1}{r}\left(\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r}\right) = \frac{1}{r}\frac{\partial A(r)}{\partial \theta} - \frac{A(\theta)}{r} \tag{566}$$

$$(\nabla \mathbf{A})(\theta,\theta) = \frac{1}{r^2}(\nabla \mathbf{A})_{\theta\theta} = \frac{1}{r^2}\left(\frac{\partial A_\theta}{\partial \theta} + A_r r\right) = \frac{1}{r}\frac{\partial A(\theta)}{\partial \theta} + \frac{A(r)}{r} \tag{567}$$

$$\begin{aligned} (\nabla \mathbf{A})(\theta,\zeta) &= \frac{1}{rR}(\nabla \mathbf{A})_{\theta\zeta} = \frac{1}{rR}\left(\frac{\partial A_\zeta}{\partial \theta} - \frac{A_\zeta r\sin\theta}{R}\right) \\ &= \frac{1}{rR}\frac{\partial[RA(\zeta)]}{\partial \theta} - \frac{A(\zeta)\sin\theta}{R} \end{aligned} \tag{568}$$

$$(\nabla \mathbf{A})(\zeta,r) = \frac{1}{R}(\nabla \mathbf{A})_{\zeta r} = \frac{1}{R}\left(\frac{\partial A_r}{\partial \zeta} - \frac{A_\zeta \cos\theta}{R}\right) = \frac{1}{R}\frac{\partial A(r)}{\partial \zeta} - \frac{A(\zeta)\cos\theta}{R} \tag{569}$$

$$(\nabla \mathbf{A})(\zeta,\theta) = \frac{1}{rR}(\nabla \mathbf{A})_{\zeta\theta} = \frac{1}{rR}\left(\frac{\partial A_\theta}{\partial \zeta} - \frac{A_\zeta r\sin\theta}{R}\right) = \frac{1}{R}\frac{\partial A(\theta)}{\partial \zeta} - \frac{A(\zeta)\sin\theta}{R} \tag{570}$$

$$\begin{aligned} (\nabla \mathbf{A})(\zeta,\zeta) &= \frac{1}{R^2}(\nabla \mathbf{A})_{\zeta\zeta} = \frac{1}{R^2}\left(\frac{\partial A_\zeta}{\partial \zeta} + A_r R\cos\theta - \frac{A_\theta R\sin\theta}{r}\right) \\ &= \frac{1}{R}\frac{\partial A(\zeta)}{\partial \zeta} + \frac{A(r)\cos\theta}{R} + \frac{A(\theta)\sin\theta}{R} \end{aligned} \tag{571}$$

As a matrix where rows represent the first index and columns the second index

$$\begin{bmatrix} \frac{\partial A(r)}{\partial r} & \frac{\partial A(\theta)}{\partial r} & \frac{1}{R}\left(\frac{\partial[RA(\zeta)]}{\partial r} - A(\zeta)\cos\theta\right) \\ \frac{1}{r}\left(\frac{\partial A(r)}{\partial \theta} - A(\theta)\right) & \frac{1}{r}\left(\frac{\partial A(\theta)}{\partial \theta} + A(r)\right) & \frac{1}{R}\left(\frac{1}{r}\frac{\partial[RA(\zeta)]}{\partial \theta} - A(\zeta)\sin\theta\right) \\ \frac{1}{R}\left(\frac{\partial A(r)}{\partial \zeta} - A(\zeta)\cos\theta\right) & \frac{1}{R}\left(\frac{\partial A(\theta)}{\partial \zeta} - A(\zeta)\sin\theta\right) & \frac{1}{R}\left(\frac{\partial A(\zeta)}{\partial \zeta} + A(r)\cos\theta + A(\theta)\sin\theta\right) \end{bmatrix} \tag{572}$$

12.4.2 Divergence

The divergence of a vector is given by

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{1}{rR}\frac{\partial(rRA^r)}{\partial r}+\frac{1}{rR}\frac{\partial(rRA^\theta)}{\partial\theta}+\frac{1}{rR}\frac{\partial(rRA^\varphi)}{\partial\varphi}\\ &= \frac{1}{rR}\frac{\partial[rRA(r)]}{\partial r}+\frac{1}{rR}\frac{\partial[RA(\theta)]}{\partial\theta}+\frac{1}{R}\frac{\partial A(\zeta)}{\partial\zeta}\end{aligned} \tag{573}$$

The divergence of a second order tensor is given by

$$\begin{aligned}(\nabla\cdot\overleftrightarrow{\mathbf{T}})(r) = (\nabla\cdot\overleftrightarrow{\mathbf{T}})^r &= \frac{1}{rR}\left(\frac{\partial(\mathcal{J}T^{rr})}{\partial r}+\frac{\partial(\mathcal{J}T^{\theta r})}{\partial\theta}+\frac{\partial(\mathcal{J}T^{\zeta r})}{\partial\zeta}\right)-rT^{\theta\theta}-R\cos\theta\,T^{\zeta\zeta}\\ &= \frac{1}{rR}\frac{\partial[rRT(r,r)]}{\partial r}+\frac{1}{rR}\frac{\partial[RT(\theta,r)]}{\partial\theta}+\frac{1}{R}\frac{\partial T(\zeta,r)}{\partial\zeta}-\frac{T(\theta,\theta)}{r}-\frac{\cos\theta\,T(\zeta\zeta)}{R}\end{aligned} \tag{574}$$

$$\begin{aligned}(\nabla\cdot\overleftrightarrow{\mathbf{T}})(\theta) &= r(\nabla\cdot\overleftrightarrow{\mathbf{T}})^\theta\\ &= \frac{r}{rR}\left(\frac{\partial(\mathcal{J}T^{r\theta})}{\partial r}+\frac{\partial(\mathcal{J}T^{\theta\theta})}{\partial\theta}+\frac{\partial(\mathcal{J}T^{\zeta\theta})}{\partial\zeta}\right)+r\frac{T^{\theta r}+T^{r\theta}}{r}+r\frac{R}{r}\sin\theta T^{\zeta\zeta}\\ &= \frac{1}{R}\frac{\partial[RT(r,\theta)]}{\partial r}+\frac{1}{rR}\frac{\partial[RT(\theta,\theta)]}{\partial\theta}+\frac{1}{R}\frac{\partial T(\zeta,\theta)}{\partial\zeta}+\frac{T(\theta,r)+T(r,\theta)}{r}+\frac{\sin\theta}{R}T(\zeta,\zeta)\end{aligned} \tag{575}$$

$$\begin{aligned}(\nabla\cdot\overleftrightarrow{\mathbf{T}})(\zeta) &= R(\nabla\cdot\overleftrightarrow{\mathbf{T}})^\zeta\\ &= \frac{R}{rR}\left(\frac{\partial(\mathcal{J}T^{r\zeta})}{\partial r}+\frac{\partial(\mathcal{J}T^{\theta\zeta})}{\partial\theta}+\frac{\partial(\mathcal{J}T^{\zeta\zeta})}{\partial\zeta}\right)+R\cos\theta\frac{T^{r\zeta}+T^{\zeta r}}{R}-rR\sin\theta\frac{T^{\theta\zeta}+T^{\zeta\theta}}{R}\\ &= \frac{1}{r}\frac{\partial[rT(r,\zeta)]}{\partial r}+\frac{1}{r}\frac{\partial T(\theta,\zeta)}{\partial\theta}+\frac{1}{R}\frac{\partial T(\zeta,\zeta)}{\partial\zeta}+\cos\theta\frac{T(r,\zeta)+T(\zeta,r)}{R}-\sin\theta\frac{T(\theta,\zeta)+T(\zeta,\theta)}{R}\end{aligned} \tag{576}$$

12.4.3 Curl

The curl of a vector is given by

$$\begin{aligned}(\nabla\times\mathbf{A})(r) = (\nabla\times\mathbf{A})^r &= \frac{1}{rR}\left(\frac{\partial A_\zeta}{\partial\theta}-\frac{\partial A_\theta}{\partial\zeta}\right)\\ &= \frac{1}{rR}\frac{\partial[RA(\zeta)]}{\partial\theta}-\frac{1}{R}\frac{\partial A(\theta)}{\partial\zeta}\end{aligned} \tag{577}$$

$$\begin{aligned}(\nabla\times\mathbf{A})(\theta) = r(\nabla\times\mathbf{A})^\theta &= \frac{r}{rR}\left(\frac{\partial A_r}{\partial\zeta}-\frac{\partial A_\zeta}{\partial r}\right)\\ &= \frac{1}{R}\frac{\partial A(r)}{\partial\zeta}-\frac{1}{R}\frac{\partial[RA(\zeta)]}{\partial r}\end{aligned} \tag{578}$$

$$\begin{aligned}(\nabla\times\mathbf{A})(\zeta) = R(\nabla\times\mathbf{A})^\zeta &= \frac{R}{rR}\left(\frac{\partial A_\theta}{\partial r}-\frac{\partial A_r}{\partial\theta}\right)\\ &= \frac{1}{r}\frac{\partial[rA(\theta)]}{\partial r}-\frac{1}{r}\frac{\partial A(r)}{\partial\theta}\end{aligned} \tag{579}$$

The curl of a second order tensor is given by

$$(\nabla \times \overleftrightarrow{\mathbf{T}})(r,r) = (\nabla \times \overleftrightarrow{\mathbf{T}})^{r}_{\cdot r} = \frac{1}{rR}\left(\frac{\partial T_{\zeta r}}{\partial \theta} - \frac{\partial T_{\theta r}}{\partial \zeta}\right) + \frac{\cos\theta\, T_{\theta\zeta}}{rRR} - \frac{T_{\zeta\theta}}{rrR}$$
$$= \frac{1}{rR}\frac{\partial[RT(\zeta,r)]}{\partial\theta} - \frac{1}{R}\frac{\partial T(\theta,r)}{\partial\zeta} + \frac{\cos\theta\, T(\theta,\zeta)}{R} - \frac{T(\zeta,\theta)}{r} \tag{580}$$

$$(\nabla \times \overleftrightarrow{\mathbf{T}})(r,\theta) = \frac{1}{r}(\nabla \times \overleftrightarrow{\mathbf{T}})^{r}_{\cdot\theta} = \frac{1}{r^2R}\left(\frac{\partial T_{\zeta\theta}}{\partial \theta} - \frac{\partial T_{\theta\theta}}{\partial \zeta}\right) + \frac{T_{\zeta r} r}{r^2R} - \frac{T_{\theta\zeta} r\sin\theta}{Rr^2R}$$
$$= \frac{1}{rR}\frac{\partial[RT(\zeta,\theta)]}{\partial\theta} - \frac{1}{R}\frac{\partial T(\theta,\theta)}{\partial\zeta} + \frac{T(\zeta,r)}{R} - \frac{T(\theta,\zeta)\sin\theta}{R} \tag{581}$$

$$(\nabla \times \overleftrightarrow{\mathbf{T}})(r,\zeta) = \frac{1}{R}(\nabla \times \overleftrightarrow{\mathbf{T}})^{r}_{\cdot\zeta}$$
$$= \frac{1}{rR^2}\left(\frac{\partial T_{\zeta\zeta}}{\partial \theta} - \frac{\partial T_{\theta\zeta}}{\partial \zeta}\right) + \frac{T_{\zeta\zeta} r\sin\theta}{RrR^2} + \frac{T_{\theta\theta}R\sin\theta}{rrR^2} - \frac{T_{\theta r}R\cos\theta}{rR^2}$$
$$= \frac{1}{rR^2}\frac{\partial[R^2T(\zeta,\zeta)]}{\partial\theta} - \frac{1}{R}\frac{\partial T(\theta,\zeta)}{\partial\zeta} + \frac{\sin\theta[T(\zeta,\zeta)\sin\theta + T(\theta,\theta)] - T(\theta,r)\cos\theta}{R} \tag{582}$$

$$(\nabla \times \overleftrightarrow{\mathbf{T}})(\theta,r) = r(\nabla \cdot \overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot r} = \frac{r}{rR}\left(\frac{\partial T_{rr}}{\partial \zeta} - \frac{\partial T_{\zeta r}}{\partial r}\right) - \frac{T_{r\zeta}\cos\theta}{RR}$$
$$= \frac{1}{R}\frac{\partial T(r,r)}{\partial\zeta} - \frac{1}{R}\frac{\partial[RT(\zeta,r)]}{\partial r} - \frac{T(r,\zeta)\cos\theta}{R} \tag{583}$$

$$(\nabla \times \overleftrightarrow{\mathbf{T}})(\theta,\theta) = (\nabla \cdot \overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot\theta} = \frac{1}{rR}\left(\frac{\partial T_{r\theta}}{\partial \zeta} - \frac{\partial T_{\zeta\theta}}{\partial r}\right) + \frac{T_{\zeta\theta}}{rrR} + \frac{T_{r\zeta} r\sin\theta}{RrR}$$
$$= \frac{1}{R}\frac{\partial T(r,\theta)}{\partial\zeta} - \frac{1}{rR}\frac{\partial[rRT(\zeta,\theta)]}{\partial r} + \frac{T(\zeta,\theta)}{r} + \frac{T(r,\zeta)\sin\theta}{R} \tag{584}$$

$$(\nabla \times \overleftrightarrow{\mathbf{T}})(\theta,\zeta) = \frac{r}{R}(\nabla \cdot \overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot\zeta} = \frac{r}{RrR}\left(\frac{\partial T_{r\zeta}}{\partial \zeta} - \frac{\partial T_{\zeta\zeta}}{\partial r}\right) + \frac{T_{\zeta\zeta}\cos\theta}{R^2R} + \frac{T_{rr}R\cos\theta}{R^2} - \frac{T_{r\theta}R\sin\theta}{R^2r}$$
$$= \frac{1}{R}\frac{\partial T(r,\zeta)}{\partial\zeta} - \frac{1}{R^2}\frac{\partial[R^2T(\zeta,\zeta)]}{\partial r} + \frac{[T(\zeta,\zeta) + T(r,r)]\cos\theta - T(r,\theta)\sin\theta}{R} \tag{585}$$

$$(\nabla \times \overleftrightarrow{\mathbf{T}})(\zeta,r) = R(\nabla \cdot \overleftrightarrow{\mathbf{T}})^{\zeta}_{\cdot r} = \frac{R}{rR}\left(\frac{\partial T_{\theta r}}{\partial r} - \frac{\partial T_{rr}}{\partial \theta}\right) + \frac{T_{r\theta}}{rr}$$
$$= \frac{1}{r}\frac{\partial[rT(\theta,r)]}{\partial r} - \frac{1}{r}\frac{\partial T(r,r)}{\partial\theta} + \frac{T(r,\theta)}{r} \tag{586}$$

$$(\nabla \times \overleftrightarrow{\mathbf{T}})(\zeta,\theta) = \frac{R}{r}(\nabla \cdot \overleftrightarrow{\mathbf{T}})^{\zeta}_{\cdot\theta} = \frac{R}{rrR}\left(\frac{\partial T_{\theta\theta}}{\partial r} - \frac{\partial T_{r\theta}}{\partial \theta}\right) - \frac{T_{rr} r}{r^2} - \frac{T_{\theta\theta}}{rr^2}$$
$$= \frac{1}{r^2}\frac{\partial[r^2T(\theta,\theta)]}{\partial r} - \frac{1}{r}\frac{\partial T(r,\theta)}{\partial\theta} - \frac{T(r,r) + T(\theta,\theta)}{r} \tag{587}$$

$$(\nabla \times \overleftrightarrow{\mathbf{T}})(\zeta,\zeta) = (\nabla \cdot \overleftrightarrow{\mathbf{T}})^{\zeta}_{\cdot\zeta} = \frac{1}{rR}\left(\frac{\partial T_{\theta\zeta}}{\partial r} - \frac{\partial T_{r\zeta}}{\partial \theta}\right) - \frac{T_{\theta\zeta}\cos\theta + T_{r\zeta} r\sin\theta}{rRR}$$
$$= \frac{1}{rR}\frac{\partial[rRT(\theta,\zeta)]}{\partial r} - \frac{1}{rR}\frac{\partial[RT(r,\zeta)]}{\partial\theta} - \frac{T(\theta,\zeta)\cos\theta + T(r,\zeta)\sin\theta}{R} \tag{588}$$