A Magnetic Fusion Plasma Physics Primer

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Foreword by K. Bunkers

> *I say most things several times: at first in English, and later with additional devices. If one of these variations makes sense and another is puzzling, then it is safe to continue reading. Truly, when has it ever been unsafe to continue reading? If you encounter an insurmountable obstacle in your reading, walk around it, continue to the next section or chapter with no concern or regret, and never look back.*
>
> — Paul Rosenbaum[1, p. xii]

> *Wisdom is rooted in watching with affection the way people grow.*
>
> — W. W. Sawyer quoting Confucius[2, p. 7]

This is a primer for those wanting to get into magnetic fusion. It is primarily aimed at senior undergraduates and especially new graduate students into the field, with a strong emphasis on explaining the mathematics well, rather than covering the entire breadth of plasma physics. This means that I will let some subjects be covered somewhat quickly and not delve deeply into them. There are plenty of other textbooks that can fill in the areas I do not cover in plasma physics. For what I have explained, though, I have striven to either show my work or cite a work that gives the steps.

With all that said, I will go over many concepts that I wish I had had a stronger background in when entering graduate school. Because this is my textbook, I will also cover subjects that I find intrinsically interesting and so I hope this text will provide a single useful reference for mathematical and physical problem solutions.

The mathematics section should serve as a good reference for many of the tools a plasma physicist (and really, any physicist) uses. While it is by no means complete, it touches on subjects that are often less investigated when doing an undergraduate mathematics, physics, or engineering curricula. However, one can easily go through plasma physics without using the Hamiltonian nature of magnetic field lines or even curvilinear coordinates beyond the cylindrical and primitive toroidal systems. However, I believe it is best to be aware of the type of approximations you can use. I also strongly believe in giving an example with the steps explicitly laid out, and so you will see that I try not to skimp on showing my steps or reasoning.

The second chapter serves as an introduction to plasma physics, but should be considered a *very* brief introduction. I have focused on plasma kinetics and plasma fluid theory in relation to magnetic confinement. Sheath theory and astronomical uses are not covered, and plasma waves are hardly touched upon. This is because I view the plasma chapter as serving a supplementary role to a standard plasma physics textbook. It offers extra explanations and points of view that are often skipped over in many textbooks.

The fusion and economics of fusion chapters serve as a general overview of the challenges and possible promises of thermonuclear fusion. They were the primary reason I wrote this, and so I hope that you do not just skip them. I found that this was very underrepresented in my formal education, and so I hope that it answers questions you may have as to why people think fusion could provide energy to humanity in the future. My personal experience is that people assume that these calculations give the obvious answer that fusion is a good idea. However, fewer than I would have thought have actually thought about where the resource numbers have come from, and to what degree those numbers are optimistic or pessimistic predictions. The questions about practicality are hard to answer, and so it is rather understandable that without a working prototype that these questions are less focused upon.

There are some sections that I think deserve special mention because they are not normally presented to undergraduates, or that if they are, they are presented in a somewhat misleading way. The first is that of the Hamiltonian nature of the magnetic fields in Section 1.10. This helps disprove the commonly held notion that magnetic field lines must be closed. The "derivation" of the Plemelj formula in Section 1.13 shows how these actually come from reasonable definitions (rather than the flawed "proof" using a half-circle integration). The frozen flux theorem uses a generalized Leibniz integral rule that is usually presented without emphasizing that $\boldsymbol{\nabla} \cdot \mathbf{B}$ is playing an enormous role in the proof. I explain this and offer the fuller, more general Leibniz integral rule in Section 2.7.1. A proof for the form of the perturbed normal is presented in Section 2.9.3. I have never found a proof in the literature, though it probably exists somewhere. I still think you should read the other sections of the book, but these particular topics are things I have not seen covered as well elsewhere.

When you write (and publish) your own book you get to write about whatever you like. This is both great and terrible. Great because you can cover what you want. Terrible because you realize that it is hard to say something other people will understand and want to read. I have tried to inject some personality into this textbook, and not have the tone of a "normal" textbook. If you have ever read Boyd's *Chebyshev and Fourier Spectral Methods* or any of W. W. Sawyer's mathematics books (see references in the chapters), you will know what I mean by injecting some personality into the chapters. I hope that this tone does not put you off; in fact, I hope that you find it, at least, a little enjoyable.

There are a couple of other things I'd like to say before you start. Feel free to skip these next parts if you just want to start getting into the physics rather than my opinions.

First, never believe a physicists' history. It is almost always a warped tale that sees the end result as a clear goal that the scientists stepped toward without any backward or sideways steps. Unless a physics textbook gives a reference to interviews, an actual history book, or their personal anecdote of discovery, it is safe to assume that the story is probably wrong or distorted such that it at best sort of correct.[1] Even in this book, apply the same precepts: without a citation to a historical

[1] A good example is Tycho Brahe's Tychonic model where the sun and moon revolve around Earth, and all the other planets around the sun. Many scientists take this to be Tycho wanting to keep the Earth at the center of the universe, but in fact he was simply using that the stars appear to the naked eye to be spheres rather than points (due to atmospheric effects nobody at the time recognized). If you calculate the size of the stars in the Tychonic model, you get them reasonably similar to the sun. If you use a Copernican model, you have the stars extremely far away (due to the lack of parallax), and so the distant stars would have have radii at least ten times as large as the size of our solar system to appear at the size they do on Earth. Tycho thought that our star alone shouldn't be the tiniest dwarf in the universe. Copernicans only response at the time was that God could have made all those stars that enormous. Later there were better arguments to be made against the Tychonic model, but it was not

document, the story is probably distorted.

Second, it is customary to explain how the author sees the chapters of a textbook. The first chapter is entirely skippable if none of the subjects interest you. They are useful, but only when you need them. The plasma chapter is a quick introduction to different plasma physics ideas, usually with a problem to illustrate the method. The fusion chapter is a generic treatment of nuclear fusion with applications to stellar and terrestrial (that is fusion by humans on Earth) fusion. The final chapter goes into more depth into what is necessary for fusion power to be a major industry providing energy for the world. While the chapters build off of each other, they can be read separately without much problem. The two fusion chapters certainly make more sense as read one after another, but it is not strictly necessary.

Finally, I thank you for taking the time to look at this. I did this mostly for my own edification, though I did so hoping it would be of general use to others. Feel free to use any content from the text so long as you properly attribute it.[2] Feel free to contact me with any corrections. I thank all of my friends and colleagues who helped proofread and make suggestions [TODO: list them]. All errors in the text are mine, and mine alone, however. I hope you have an enjoyable read.

Wish you the best,

K. J. Bunkers

2020-03-22

Changes since this foreword:

2020-09-02: Fixed minus sign error in summary formula at the end of appendix 2 2020-09-03: Fixed $K_\nu(z)$ for $z \to \infty$ formula in modified Bessel functions.

some crazy idea.

[2] I have done my very best to ensure that anything I used from others is properly attributed. To the very best of my understanding, I have not needed permission to use any of the public domain images as they are under Creative Common licences that do not require it.

Chapter 1

Mathematical Beginnings

> *"Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions."*
>
> — Felix Klein

This chapter begins with notation, which will be explained within the text as it appears, as well. It then explores coordinate systems and vector and tensor analysis suitable for physics. It is heavily indebted to W. D. D'haeseleer, et. al.'s *Flux Coordinates and Magnetic Field Structure*[9] which serves as an excellent reference for curvilinear coordinate systems and flux coordinates. I think it treats curvilinear coordinates in a way that is less math intensive and more intuitive from a physics perspective. The nature of Fourier series, Fourier transforms, Laplace transforms, ballooning transforms are reviewed and explained. The idea of asymptology, or the study of systems in limits, is then explored briefly. Taylor series are given a special section because of their importance in all physics analyses. Multiple Scale Analysis, a type of perturbation series, and an explanation of linearization follows. The Calculus of Variations is then explained, and its uses in theoretical physics is discussed. Hamiltonian mechanics is developed so that we can discuss the Hamiltonian nature of magnetic field lines. This includes a discussion of Frenet-Serret and Darboux frames, which are of importance for magnetic field line coordinates. The JWKB approximation follows, as it can be used most easily with Hamiltonian systems. Complex contour integration and an explanation of analytic continuation are then given. Finally, I give some of my personal experiences with probability and statistics, and warn you of ways intuition on probability can mislead you.

Other than the Notation Section 1.1 and the useful skill of reading contour plots for Fourier components in Section 1.3.2, all of the material should be viewed as supplementary for plasma physics. This is a good reference or review when you need to use the mathematical operations. The complex contour integration 1.13 is rather brief, because I think other texts adequately explain the ideas.

I have structured this chapter such that notation comes first for easy reference, and then a bunch of topics that I have had to come back to over the course of my career, or that I had questions about and did not want an overly technical explanation. That is, I did not want to have to learn an entire new branch of mathematics.

Curvilinear coordinates are important for analysis of magnetically confined devices, but generally

speaking, unless you are using a simulation code or doing analytic work, you will not have to deal with the flux coordinate systems for complicated calculations. You will use our plasma cylindrical coordinate system (R, Z, ζ) or primitive toroidal coordinates (r, θ, ζ).

Fourier series are used all of the time and this section is worth looking at if you feel rusty with Fourier series or transforms. Interpreting pictures and learning how to see the Fourier structure is an important skill for plasma physics as you will be asked to describe plasma motion and perturbations by their Fourier components. The Laplace transform is touched on, which is important for some theoretical plasma physics calculations. The ballooning transform is useful specifically to plasmas with magnetic flux surfaces.

Taylor series are incredibly useful and ubiquitous in physics. I have collected the multi-dimensional analogues and the complex variables analogue as well. It can be annoying to find or derive these series, and so this makes a good reference. Asymptology is simply a generally useful set of skills for finding solutions when things are in a certain limit. It is best to have some rules and some example problems to see how the ideas are applied.

The section on linearization explains the concepts and goes through how to linearize equations. It is simply the idea of using perturbation series. It shows how one could generalize to smaller contributions, as well. Multiple scale analysis is essentially an extension of this idea that works better in certain circumstances.

The calculus of variations is a useful tool in computations, but also can help in solving some optimization problems in general. It is also good to know how to get results from principles of least action, which will enrich anyone's understanding of physics.

The Hamiltonian form of the magnetic field lines is included mainly because of my own interest in the subject. I go through Hamiltonian physics so that the full machinery that a Hamiltonian gives can be appreciated. It is also a proof that magnetic field lines do not necessarily close on themselves or go to "infinity" as so many textbooks assert.

The Frenet-Serret trajectory formulation (and Darboux frame) goes along with particle and magnetic field line trajectories, and so is included mainly because of my own interest. It can be useful for some plasma calculations if you want to follow a known magnetic field line and understand attributes of the field line structure. In some theoretical plasma stability considerations this frame is very natural.

The JWKB approximation is an important approximation in general. I cover it and explain its connection to Hamiltonians so that you can see how it can be a powerful approximation tool.

Finally, complex integration is covered so we can talk about analytic continuation. Analytic continuation is included to help you through theoretical plasma physics calculations that seem like they should be ill-defined (usually in waves doing complex contour integrals). In addition it serves to explain how we can assign values to what are divergent series [it should always be emphasized that the divergent series does not sum to the assigned value].

1.1 Notations

> *I do not know if any idea ever achieves complete precision. But all that matters for a formal theory, is that the idea is sufficiently precise for what you intend to do with it.*

— W. W. Sawyer[29, p. 145]

This section is simply an overview of the various notation and what they mean. They include general comments and notation specific for this book. Many of the comments will be repeated in the main text, because notation will be commented on as it occurs in addition to the references here.

1.1.1 Functions, Vectors, and Tensors

No theory is kind to us that cheats us of seeing.

— Paul Rosenbaum quoting Henry James[27, p. 215]

I will assume a certain familiarity with common mathematical notations, such as $f(x)$ means a function of the variable x, and $f(x(t), y(t), t)$ (often abbreviated to just $f(x, y, t)$ with assumed dependence of t for other variables) would be a function of x, y, and t with x and y both functions of t. I will use the common convention of using bold face (non-italic) letters to represent vectors (such as $\mathbf{a}$, $\mathbf{A}$). In addition, I will occasionally use bold Greek letters for vectors, as well ($\boldsymbol{\alpha}$, $\boldsymbol{\tau}$, etc.). When handwritten, one sees a variety of notations for vectors including using a blackboard-like font, $\mathbb{A}$, arrows on top $\vec{A}$, or lines below a letter $\underline{A}$ with the arrow notation being the most common. I will also use a typewriter-like font for what I will call a vector array. For me, a vector array is an array of numbers that mathematicians would call a vector, but is not a geometric vector. These will look like `A` or `a`.

When we come to n-polyadics or tensors of order n, the notation becomes far less universal. I will use a double-headed arrow with a bold face letter to denote tensors of order 2, like $\overleftrightarrow{\mathbf{A}}$. In other places this is sometimes instead rendered $\vec{\vec{A}}$ (or sometimes bold face is used $\vec{\vec{\mathbf{A}}}$), and the underline notation also becomes simple $\underline{\underline{A}}$ (sometimes $\underline{\underline{\mathbf{A}}}$). For higher order tensors, the underline and arrow notation are easily extendible, though become increasingly burdensome, with the number of lines corresponding to the order of the tensor.

This is somewhat simplified by using index or Einstein summation notation,[1] though this requires choosing a representation of a vector or tensor, as will be discussed later. Then the number of indices (as superscripts or subscripts) determines the order of the tensor. This is the most common case for higher order tensors. A^i or A_i represents a vector; A^{ij}, A_{ij}, $A^i_{\cdot j}$, $A^{\cdot i}_j$ are second order tensors; A^{ijk}, $A_{ijk,\ldots}$ are third order tensors; etc.

I will also sometimes use the notation $\mathbf{A}$ to indicate a matrix or a higher order tensor, but it will be specifically remarked upon. Using a different font is often used to indicate tensors in textbooks,[2] though I will stick with indices or the double-headed arrow except in limited circumstances.

While limited in use, I will use blackboard bold fonts for four-vectors $\mathbb{A}$ and four-tensors $\overleftrightarrow{\mathbb{A}}$.

To summarize, my notation for functions is standard $f(x(t), t)$, vectors are non-italic bold face characters such as $\mathbf{A}$, second order tensors are shown by $\overleftrightarrow{\mathbf{T}}$, matrices and higher order tensors may be denoted by $\mathbf{M}$. In addition, index or Einstein summation notation may be used where the number of indices determines the order of the tensor.

[1] It is called variously index notation, Einstein notation, the Einstein summation convention, Einstein summation notation, etc.

[2] Using upper case for tensors and lower case for vectors is another convention used in other textbooks.

1.1.2 Differentiation and Integration

In mathematics you don't understand things. You just get used to them.

— John von Neumann

We can now talk about integration and differentiation. Contrary to the actual difficulty of doing the operation, differentiation has a multitude of notations whereas integration has essentially only two.

1.1.2.1 Integration

The common notation for (indefinite) integration of a function $f(x)$ with antiderivative $F(x)$ (and then for a definite integration with $a < b$) is given by

$$F(x) = \int f(x)\,\mathrm{d}x \tag{1.1.1}$$

$$F(b) - F(a) = \int_a^b f(x)\,\mathrm{d}x \tag{1.1.2}$$

I will call this the standard or mathematician's integration notation. As a physicist, it is common to not want exactly the antiderivative, because we want proper boundary conditions enforced (that is $F(x_0) = y_0$ to be enforced). We want to find $F(x)$, and so a dummy integration variable x' is used to find our desired antiderivative. So in some textbooks you will see

$$F(x) = \int^x f(x')\,\mathrm{d}x' \equiv \int_{x_0}^x f(x')\,\mathrm{d}x' \tag{1.1.3}$$

with x_0 often left undefined, but implicitly understood to give the proper boundary conditions.

Notice that integration is an operator and is therefore somewhat odd, in that most notation has operators act on things to the right of the operator. This operator notation puts what is operated on it "inside" of it. For multiple integrations, it is also more difficult to determine which limits go with which variables

$$\int_a^b \int_c^d \int_e^f f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.1.4}$$

where (e, f) bounds are with x, (c, d) bounds are with y, and (a, b) bounds are with z. Despite the logic of this, it is easily confused when doing integration since the integral limits are in the opposite order to the differentials. So physicists often just put the $\mathrm{d}x$ right next to the $\int$. So the above integral would be represented as

$$\int_a^b \mathrm{d}z \int_c^d \mathrm{d}y \int_e^f \mathrm{d}x\ f(x,y,z) \tag{1.1.5}$$

which is much easier to read off the limits. It sometimes requires putting brackets around what is integrated, but this is a small price to pay for the extra clarity on integration order. As we will see, it will also be more logical for some extensions of integration where $\mathrm{d}x$ becomes a vector applied to the integrand.

For this book, I will employ some of my own integration notation. I didn't invent it, but it is not standard. If I write

$$\iiint_{-\infty}^{\infty} \mathrm{d}^3x\ f(x,y,z) \equiv \int_{-\infty}^{\infty} \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}x\ f(x,y,z) = \iiint_{\text{all space}} \mathrm{d}V\ f(x,y,z) \tag{1.1.6}$$

Generalizations are of course also possible with this notation, such as d^2x when only two variables are being used. The $\mathrm{d}V = \mathrm{d}^3x$ here and simply emphasizes the 3D nature of the integral. The reason to avoid using V is that when we do velocity integrals there could be confusion between volume and velocity. I should note that my d^3x literally means a volume in three coordinates and so should not be interpreted as $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. It could be $r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$ for spherical coordinates for example. The same caveat holds for d^2x.

You shouldn't forget that improper integrals are just limits. That is

$$\int_a^{\infty} \mathrm{d}x\ f(x) \equiv \lim_{b\to\infty} \int_a^b \mathrm{d}x\ f(x) \tag{1.1.7}$$

$$\int_{-\infty}^{b} \mathrm{d}x\ f(x) \equiv \lim_{a\to\infty} \int_{-a}^b \mathrm{d}x\ f(x) \tag{1.1.8}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\ f(x) \equiv \lim_{a,b\to\infty} \int_{-a}^b \mathrm{d}x\ f(x) \tag{1.1.9}$$

For the last integral to be defined, we must get the same answer for any way of choosing a and b. This also leads us to the idea of Cauchy principal values. If we have a singularity of $f(x)$ at $a < m < b$, then we define

$$⨍_a^b \mathrm{d}x\ f(x) \equiv \lim_{\epsilon\to 0^+} \left[\int_a^{m-\epsilon} \mathrm{d}x\ f(x) + \int_{m+\epsilon}^b \mathrm{d}x\ f(x)\right] \tag{1.1.10}$$

to be the Cauchy principal value. The 0^+ notation means go to zero from positive values, not from negative values. This works with a and b being $\pm\infty$ using the previous definitions, as well. This is an equal balance of both sides of the singularity, and so sometimes makes the most sense as an interpretaion of an integral with singularity. There is an excess of notation including

$$\begin{aligned} ⨍_a^b \mathrm{d}x\ f(x) &= \mathcal{P}\int_a^b \mathrm{d}x\ f(x) = \mathcal{PV}\int_a^b \mathrm{d}x\ f(x) \\ &= \text{p.v.}\int_a^b \mathrm{d}x\ f(x) = \int_L^* \mathrm{d}x\ f(x) \end{aligned} \tag{1.1.11}$$

and sometimes P, PV, P_v, CPV, or VP are put in front of the integrals. I will confine myself to the line through the integral notation as it is concise and not easy to mistake.

The largest difference that comes about from the physicists' and mathematicians' notation is in how to write Gauss's law for tensors, where the order of dot products matters. This is completely due to the convention of changing from a volume differential $\mathrm{d}^3x = \mathrm{d}V$ to a vector component like surface differential $\mathrm{d}\mathbf{S} = \mathrm{d}S\hat{\mathbf{n}} = \mathrm{d}^2x\hat{\mathbf{n}}$ with outward normal to the surface $\hat{\mathbf{n}}$. Thus,

$$\iint \mathrm{d}\mathbf{S}\ \cdot \overleftrightarrow{\mathbf{A}} = \iint \mathrm{d}^2x\ \hat{\mathbf{n}} \cdot \overleftrightarrow{\mathbf{A}} \neq \iint \overleftrightarrow{\mathbf{A}} \cdot \hat{\mathbf{n}}\,\mathrm{d}^2x = \iint \overleftrightarrow{\mathbf{A}} \cdot\ \mathrm{d}\mathbf{S} \tag{1.1.12}$$

because $\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{A}} \neq \overleftrightarrow{\mathbf{A}}\cdot\hat{\mathbf{n}}$ in general. The mathematician's definition of tensors leads to the unfortunate implication that the divergence operator on a tensor should default to operating on the second index of the tensor. That is [for Cartesian components $x^i \in (x^1, x^2, x^3) = (x, y, z)$]

$$\boldsymbol{\nabla}^{\text{math}} \cdot \overleftrightarrow{\mathbf{A}} \overset{\text{math}}{=} \sum_{i=1}^{3} \frac{\partial}{\partial x^i} A^{ji} \hat{\mathbf{x}}_j \tag{1.1.13}$$

which I find to be unintuitive though it can be made to be consistent. I still much prefer operators acting directly on the objects closest to them on the right. However, the mathematician's definition makes sense for

$$\iiint_V \boldsymbol{\nabla}^{\text{math}} \cdot \overleftrightarrow{\mathbf{A}} \, \mathrm{d}^3x = \iint_{\partial V} \overleftrightarrow{\mathbf{A}} \cdot \, \mathrm{d}\mathbf{S} = \iint_{\partial V} \overleftrightarrow{\mathbf{A}} \cdot \hat{\mathbf{n}} \, \mathrm{d}^2x \tag{1.1.14}$$

with ∂V indicating the boundary of the volume V. This notation will ***never*** be used in this textbook which is why I attached the math superscript. I will instead adhere to the differentials on the left notation

$$\boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{A}} \overset{\text{physics}}{=} \sum_{i=1}^{3} \frac{\partial}{\partial x^i} A^{ij} \hat{\mathbf{x}}_j \tag{1.1.15}$$

So that

$$\iiint_V \mathrm{d}^3x \; \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{A}} = \iint_{\partial V} \mathrm{d}\mathbf{S} \; \cdot \overleftrightarrow{\mathbf{A}} = \iint_{\partial V} \mathrm{d}^2x \; \hat{\mathbf{n}} \cdot \overleftrightarrow{\mathbf{A}} \tag{1.1.16}$$

which seems more intuitive to me, and makes it easy to understand differentiation order on tensors. Remember that order matters for tensors, and so when looking up tensor identities inside integrals it is important to know which convention is being used.

Finally, putting a circle around the stylized s's for the integral indicates an integral over a closed path or surface. So

$$\oint_C \mathrm{d}\boldsymbol{\ell} \; \cdot \mathbf{f}(\mathbf{x}) \tag{1.1.17}$$

$$\oiint_S \mathrm{d}\mathbf{S} \; \cdot \mathbf{f}(\mathbf{x}) \tag{1.1.18}$$

are a closed line integral along a closed path C and a surface integral over the closed surface S, respectively. A closed path connects back on itself, and a closed surface, is a surface that encloses a volume. In other words, if a surface S_1 is closed, then one can find a volume whose surface S_2 coincides with the original surface S_1.

1.1.2.2 Differentiation

Now we can get to the surfeit of notation for derivatives. The problems are many-fold, because despite scant attention paid to the fact there are different types of derivatives, we usually don't think hard about this. When functions are only of a single variable, all these distinctions collapse, and so there is no need to worry. But most functions of interest involve multiple variables, and so it is important to know what the notation is saying.

First, there are two main notations for differentiation: Leibniz and Lagrange notation. Leibniz notation looks like fractions and Lagrange notation has primes (or apostrophes). They both have advantages and disadvantages for displaying different properties. For our prototypical function $f(x)$, the derivative of $f(x)$ is written as $\frac{\mathrm{d}f}{\mathrm{d}x}$ in Leibniz notation and $f'(x)$ in Lagrange notation. If we want to specify the derivative at a certain point a, we see that Lagrange is more concise with $f'(a)$ compared to Leibniz $\frac{\mathrm{d}f}{\mathrm{d}x}|_{x=a}$. However, Leibniz notation provides simpler mnemonics for the chain rule. Suppose $x = x(y)$ and we want $\frac{\mathrm{d}f}{\mathrm{d}y}$, then Leibniz notation suggests

$$\frac{\mathrm{d}f}{\mathrm{d}y} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y} \tag{1.1.19}$$

which makes it look like the $\mathrm{d}x$'s just cancel. This notation is suggestive, even if this cancellation is essentially an abuse of notation, and the rule actually does work this way. The Lagrange notation is messier looking saying

$$f'(x(y)) = f'(x)x'(y) \tag{1.1.20}$$

where one must intuit that $f'(x)$ means take a derivative with respect to x rather than y. That or realize that $f'(x)$ can be read as $f'(y)|_{y=x}$ so that you take the derivative of f with respect to y but replace y with $x(y)$. The nth order derivative is often represented as

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = f^{(n)}(x) \tag{1.1.21}$$

with superscript (n) standing in for n primes in the Lagrange notation.

Note that in physics, Newton dot notation is often used, but it really should only be used for time derivatives (or analogues). This notation is exactly the same as the Lagrange notation, but instead of apostrophes, we put dots over the function. Because it is assumed to be time, if we have $f(t)$ we just write $\dot{f}$ (and not $\dot{f}(t)$ generally, though one can do so to be precise) for $f'(t)$. Multiple dots means multiple derivatives of t, so $\ddot{f} = f''(t)$.

Now, we may get to the more convoluted derivatives of multiple dimensions. Let's take

$$f(x(t), y(t), z(t), t) = f(x^1(t), x^2(t), x^3(t), t)$$

as our typical function. When we take a derivative now, we need to think hard about what is being kept constant, and what is being allowed to vary. Thus, we introduce partial derivatives. Full derivatives have a roman letter d, whereas partial derivatives use ∂ symbols. Generally speaking, only Leibniz notation is used for partial derivatives, though there is another useful mathematician's notation. In the mathematician's notation $f_x = \frac{\partial f}{\partial x}$ and $f_z = \frac{\partial f}{\partial z}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$, etc. I won't use this notation but it can be useful. Problems arise with the mathematician's notation if you need subscripts for other ideas. Thus, if we want to know how f changes holding y, z, and t fixed, then we desire $\frac{\partial f}{\partial x}$. Physicists have often augmented this notation to explicitly say what is fixed, (including having another function $g(x, y, z, t)$ fixed) via

$$\left(\frac{\partial f}{\partial x}\right)_{g(x,y,z,t)=a, h(x,y,z,t)=b, \ldots} \tag{1.1.22}$$

although bare Leibniz partial derivative notation should be assumed to mean

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x}\right)_{y=a, z=b, t=c} \tag{1.1.23}$$

where often a, b, and c are not really specified, and so we simply leave y, z, and t as simply that rather than putting in the values a, b, and c at a specified point. That is y, z, and t are held constant. One does not need the separate variables to be completely independent of each other for partial derivatives. So if we want to know how f changes with time t holding x, y, and z fixed, we would want $\frac{\partial f}{\partial t}$. But we also know that in general x, y, and z may depend on time, and so $\frac{\partial f}{\partial t}$ will not tell us how f actually changes as t changes. That is, we are often interested in the case where we do not want to hold x, y, and z fixed in time as we let t vary. Another way of viewing this is because x, y, and z change as t changes, $\frac{\partial f}{\partial t}$ does not tell us how a function changes as t changes if we are at an actual fixed location. The $\frac{\partial f}{\partial t}$ says what the change would be if you keep your location at $x(t)$, $y(t)$, $z(t)$. In fluid mechanics, $\frac{\partial f}{\partial t}$ answers the question how does f change for fluid parcels if I move with them (for then $x(t)$, $y(t)$, and $z(t)$ are the trajectories of the fluid parcels and the fluid parcel "location" is fixed because we keep track of location changes through time and velocity). If we are standing at a specific location (and so different fluid parcles are flowing past us) and want to know the change in f, we want the total derivative, which recycles "standard" derivative notation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \tag{1.1.24}$$

If we think of $(x, y, z) = \mathbf{x}$, $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}$, and introduce the del operator ∇ for representing a gradient (or using $\frac{\partial}{\partial \mathbf{x}}$ for this as well), then this is written as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = \frac{\partial f}{\partial t} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \nabla f = \frac{\partial f}{\partial t} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \mathbf{x}} \tag{1.1.25}$$

with the last term being a generalization of Leibniz notation that also displays what looks like a simple chain rule.

This total derivative $\frac{\mathrm{d}f}{\mathrm{d}t}$ has a variety of names when we view $x^i(t)$ as position variables and t as time. It is called the full, total, absolute, convective, or advective derivative.[3] Unfortunately, there is no universal name, and convective or advective derivative may in fact be referring only to the $\mathbf{v} \cdot \nabla f$ portion of the total derivative. In fact, there are even more names given to it, such as derivative following the motion, hydrodynamic derivative, Lagrangian derivative,[4] particle derivative, substantial derivative, substantive derivative, or Stokes derivative. I will try to refer only to the total derivative. There is one more factor to consider with the total derivative. If we have multiple species of particles with different velocity vectors, the total derivative notation can be a bit ambiguous. Suppose species s has velocity $\mathbf{v}_s$. Then we can use

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \mathbf{v}_s \cdot \nabla f_s \tag{1.1.26}$$

If, however, we want to use a center-of-mass or some other velocity then it becomes ambiguous what the $\frac{\mathrm{d}f}{\mathrm{d}t}$ is referring to. Thus, we can write

$$\frac{\mathrm{d}_s f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \mathbf{v}_s \cdot \nabla f_s \tag{1.1.27}$$

[3]The difference between convective (from convective flows) and advective (all flow including conduction and convection) transport is also sometimes used so that these two terms may not be interchangeable.

[4]I view this name as especially poor, since the Lagrangian derivative only shows up in an Eulerian specification.

and use as a definition that

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \mathbf{V} \cdot \nabla f_s \tag{1.1.28}$$

with $\mathbf{V}$ referring to a center-of-mass velocity

$$\mathbf{V} = \frac{\sum_s n_s m_s \mathbf{v}_s}{\sum_s n_s m_s} \tag{1.1.29}$$

with m_s being the mass of particles of species s moving at flow velocity $\mathbf{v}_s$ with local number density n_s (remember the flow velocity of a species is not the velocity of the individual particles, but the velocity of a fluid "parcel" of particles over time).

Sometimes a notation of using capital D's is used to indicate the total derivative such as

$$\frac{\mathrm{D}f_s}{\mathrm{D}t} = \frac{\partial f_s}{\partial t} + \mathbf{V} \cdot \nabla f_s \tag{1.1.30}$$

$$\frac{\mathrm{D}_s f_s}{\mathrm{D}_s t} = \frac{\partial f_s}{\partial t} + \mathbf{v}_s \cdot \nabla f_s \tag{1.1.31}$$

I will not use this notation.

As a separate note, many math and physics texts use an italic d (or D) in the derivatives. While this is common, it is a little unfortunate from my personal perspective. Using roman (non-italic) d or D in the derivatives helps keep the notation separate from what it is acting on (in my opinion) and is very easy to implement with today's typesetting software. In any case, you should be aware that it is very common to use dx/dt rather than $\mathrm{d}x/\mathrm{d}t$, but both mean the same thing. It almost never causes confusion which is why it continues to be used.

It is now worth delving a bit into notation for gradients, divergences, and curls in general. I will use $\xi^i = \xi^i(x, y, z)$ and basis vectors as defined in Section 1.2.3, using the associated basis vectors in velocity space [I have added subscript $\mathbf{v}$ to basis vectors in velocity space] with $\sigma^i = \sigma^i(v_x, v_y, v_z)$. The vector $\mathbf{f} = \mathbf{f}(\mathbf{x})$ with the $\mathbf{x}$ standing in for all the x^i, which form a position (geometric) vector. We analogously write $\mathbf{f} = \mathbf{f}(\mathbf{v})$ for a velocity space position vector $\mathbf{v}$. We can also have a scalar $f = f(\mathbf{x})$ or $f = f(\mathbf{v})$.

$$\frac{\partial f}{\partial \mathbf{x}} = \nabla f \equiv \sum_i \mathbf{e}_i \frac{\partial f}{\partial \xi^i} \tag{1.1.32}$$

$$\frac{\partial f}{\partial \mathbf{v}} = \nabla_{\mathbf{v}} f \equiv \sum_i \mathbf{e}_{\mathbf{v},i} \frac{\partial f}{\partial \sigma^i} \tag{1.1.33}$$

$$\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{f} = \boldsymbol{\nabla} \cdot \mathbf{f} \equiv \sum_i \frac{1}{\mathcal{J}} \frac{\partial (\mathcal{J} f^i)}{\partial \xi^i} \tag{1.1.34}$$

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{f} = \boldsymbol{\nabla}_{\mathbf{v}} \cdot \mathbf{f} \equiv \sum_i \frac{1}{\mathcal{J}} \frac{\partial (\mathcal{J} f^i)}{\partial \sigma^i} \tag{1.1.35}$$

$$\frac{\partial}{\partial \mathbf{x}} \times \mathbf{f} = \boldsymbol{\nabla} \times \mathbf{f} = \sum_{i,j} \frac{\partial f_i}{\partial \xi^j} \mathbf{e}^j \times \mathbf{e}^i \tag{1.1.36}$$

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{f} = \boldsymbol{\nabla}_{\mathbf{v}} \times \mathbf{f} = \sum_{i,j} \frac{\partial f^i}{\partial \sigma^j} \mathbf{e}^j_{\mathbf{v}} \times \mathbf{e}^i_{\mathbf{v}} \tag{1.1.37}$$

with representations of the vectors as described in Section 1.2.5. The $\partial/\partial\mathbf{x}$ notation is often more useful than ∇ because it specifies what the derivative is with respect to, and it naturally suggests chain rule relations with tensors that are true.

1.1.2.3 Basis Vector Terminology

As a quick reminder if you are not delving into the curvilinear coordinates section, $\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial \xi^i}$ are called tangent basis vectors, and $\mathbf{e}^i = \frac{\partial \xi^i}{\partial \mathbf{x}} = \nabla \xi^i$ are tangent-reciprocal basis vectors (often shortened to reciprocal basis vectors with an assumption that they are reciprocal to the tangent basis set). One can remember this via $\mathbf{e}_i$ has a lower index and so ξ^i is in the "denominator", whereas $\mathbf{e}^i$ has an upper index and so ξ^i is in the "numerator". Superscript indices on tensor components mean a contravariant representation (both "super" and "contra" have an "r") and subscript indices on tensor components mean a covariant representation (neither "sub" nor "co" contain an "r").

Unfortunately, there is no common terminology for "standard" vector representations. That is there is no terminology with unit vector bases rather than tangent or reciprocal bases. In this text, we normalize the $\mathbf{e}_i$ or $\mathbf{e}^i$ with a hat, so "standard" vector representations will have $\hat{\mathbf{e}}_i$ or $\hat{\mathbf{e}}^i$. In addition, I use $\hat{\mathbf{x}}_i = \hat{\mathbf{x}}^i$ for the Cartesian unit vectors with $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}$, $\hat{\mathbf{x}}_2 = \hat{\mathbf{y}}$, and $\hat{\mathbf{x}}_3 = \hat{\mathbf{z}}$. Many math texts prefer $\hat{\mathbf{i}} = \hat{\mathbf{x}}$, $\hat{\mathbf{j}} = \hat{\mathbf{y}}$, and $\hat{\mathbf{k}} = \hat{\mathbf{z}}$ for historical reasons. There is no clear advantage to the ijk notation over the xyz notation other than being widespread in math. The ijk notation requires remembering that i is associated with x, etc., which is a small burden, but mostly it becomes annoying when we wish to put indices on things. For then we have to continually explain the unit vectors are different that the index sums or worry that a missed hat will cause confusion.

1.1.3 Big and Little O Notation

Nota res mala, optima (a known evil thing is best).

— Erasmus

Finally, a comment on notation for approximations. It is common to employ "big O" and "little O" notation.[5] Non-mathematicians tend to be fairly cavalier in this notation, but it usually does not cause much confusion. For big O notation I will use $\mathcal{O}(\cdot)$. Little O notation uses $o(\cdot)$. This notation is used to show that asymptotic expansions are "accurate" in the relevant sense. So for some $k < \infty$ we say

$$f(x) = \mathcal{O}\big(g(x)\big) \text{ as } x \to a \Leftrightarrow \limsup_{x\to a} \left|\frac{f(x)}{g(x)}\right| \leq k \tag{1.1.38}$$

with lim sup meaning the supremum limit which is the "largest" possible limiting value of the function. Note that this means the (usual, i.e., not a supremum type) limit itself may not be defined. If you don't like this lim sup definition we can alternatively say that we can always find positive real numbers M and δ such that

$$f(x) = \mathcal{O}\big(g(x)\big) \text{ as } x \to a \Leftrightarrow |f(x)| \leq M g(x) \text{ when } 0 < |x - a| < \delta \tag{1.1.39}$$

[5] These are the letter O and not zeros.

If a is unspecified, then it will imply $a = 0$ here. Note how this does not mean $|f(x) - g(x)| \to 0$. We often write the error in series this way. So for

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \mathcal{O}(x^4) \tag{1.1.40}$$

Here this should be interpreted as for some k

$$f(x) = \exp(x) - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) \tag{1.1.41}$$

$$\limsup_{x\to 0} \left| \frac{\exp(x) - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)}{x^4} \right| = k \tag{1.1.42}$$

$$\tag{1.1.43}$$

or for any $\delta > 0$ there is a non-negative M such that

$$|f(x)| \leq Mg(x) \text{ when } 0 < |x - a| < \delta \tag{1.1.44}$$

This can be read as saying that the error in the approximation is limited by a value near kx^4 as x gets small.

We can note that the big O notation satisfies

$$f_1 = \mathcal{O}(g_1) \quad , \quad f_2 = \mathcal{O}(g_2) \Rightarrow f_1 f_2 = \mathcal{O}(g_1 g_2) \tag{1.1.45}$$

$$\Rightarrow f_1 + f_2 = \mathcal{O}(\max(g_1, g_2)) \tag{1.1.46}$$

$$\Rightarrow f_1 \mathcal{O}(g_1) = \mathcal{O}(f_1 g_1) \tag{1.1.47}$$

Little O notation is a stronger statement and not generally used outside of mathematical contexts. It says (for $g(x) \neq 0$)

$$f(x) = o\big(g(x)\big) \text{ as } x \to a \Leftrightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = 0 \tag{1.1.48}$$

If $g(x)$ is a problem (it equals zero but perhaps $f(x)$ does not in the limit) then one is forced into saying for the case $x \to \infty$ that for every ϵ there exists a constant N such that

$$f(x) = o\big(g(x)\big) \; x \to \infty \Leftrightarrow |f(x)| \leq \epsilon g(x) \tag{1.1.49}$$

for all $x \geq N$. In the case $x \to a$, we say that for every constant ϵ, there exists a $\delta > 0$ such that

$$|f(x)| \leq \epsilon g(x) \text{ when } 0 < |x - a| < \delta \tag{1.1.50}$$

Little O notation satisfies

$$f_1 = o(g_1) \quad , \quad f_2 = o(g_2) \quad , \quad f_1 = o(g_1) \Rightarrow k f_1 = o(g_1) \tag{1.1.51}$$

$$\Rightarrow f_1 f_2 = o(g_1 g_2) \tag{1.1.52}$$

$$f_1 = o(g_1) \quad , \quad g_1 = o(h_1) \Rightarrow f_1 = o(h_1) \tag{1.1.53}$$

Mathematically, big O notation does not guarantee we actually have a good approximation, though when it is applied it almost always is. For example

$$\sin(x) = x + \mathcal{O}(x) \tag{1.1.54}$$

but we generally would write

$$\sin(x) = x + \mathcal{O}(x^2) \tag{1.1.55}$$

to give the most information on our approximation. One can think of big O and little O notation as saying the relative error $\frac{|f(x)-g(x)|}{|f(x)|}$ (rather than the absolute error $|f(x) - g(x)|$) in approximating $f(x)$ with $g(x)$ gets very small.

1.1.4 Differentials

> *If civilizations capable of supporting mathematics continue, two things appear almost certain — that practical methods, such as Lebesgue integration, will remain as a permanent possession of ours, and that the theories of infinity, used to justify such applications, will continue to fluctuate as the centuries pass.*
>
> — W. W. Sawyer[28, pp. 153–154]

Differentials and differential forms are related concepts that often serve as useful mnemonic to incredibly useful analysis tool for functions that are used in physics. In some ways, they are generalized derivatives. It is common to use a differential of a function in physics. Given a function $f(x,y,z)$, we write

$$\mathrm{d}f = \left(\frac{\partial f}{\partial x}\right)_{y,z}\mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)_{x,z}\mathrm{d}y + \left(\frac{\partial f}{\partial z}\right)_{x,y}\mathrm{d}z \tag{1.1.56}$$

From a physicist's perspective, this is a shortcut to writing out the change in f is the change in f due to x alone times the change in x which is $\mathrm{d}x$, plus the change in f due to y alone times the change in y which is $\mathrm{d}y$, plus the change in f due to z alone times the change in z which is $\mathrm{d}z$. It is possible to mathematically make this a rigorous statement with hyperreal numbers and "nonstandard analysis",[6] but this should really be seen as a shortcut to not have to write out partial derivatives over and over for each variable. It so happens that if we view the $\mathrm{d}x$ operator as a gradient, we can translate this into

$$\nabla f = \left(\frac{\partial f}{\partial x}\right)_{y,z}\nabla x + \left(\frac{\partial f}{\partial y}\right)_{x,z}\nabla y + \left(\frac{\partial f}{\partial z}\right)_{x,y}\nabla z \tag{1.1.57}$$

and it is quite possible, to use this connection to derive results. In addition one can find the partial derivatives by saying we "divide through" by $\mathrm{d}x$ which means we keep y and z constant, and so

$$\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{y,z} = \left(\frac{\partial f}{\partial x}\right)_{y,z}\frac{\mathrm{d}x}{\mathrm{d}x} + \left(\frac{\partial f}{\partial y}\right)_{x,z}\frac{\cancel{\mathrm{d}y}}{\cancel{\mathrm{d}x}} + \left(\frac{\partial f}{\partial z}\right)_{x,y}\frac{\cancel{\mathrm{d}z}}{\cancel{\mathrm{d}x}} \tag{1.1.58}$$

where we realize that left hand side is not actually the full differential but a partial differential because we kept y and z constant. This can also work as a mnemonic.

In fact, let's explore this just a bit more, and we will find the useful concept of a differential form. Differential forms are important because they restrict our view to certain functions that have really convenient properties. We note that $\nabla x_i = \hat{\mathbf{x}}_i$ and $\mathrm{d}x$ fulfill the same function, and that this justifies $\nabla\nabla x_i = \mathrm{d}\,\mathrm{d}x_i = 0$, just as with differential forms. What if we looked at $\nabla\nabla f$? Let's define

$$\nabla f = f_x\,\nabla x + f_y\,\nabla y + f_z\,\nabla z \tag{1.1.59}$$

$$\begin{aligned}\nabla\nabla f &= \nabla f_x\,\nabla x + \nabla f_y\,\nabla y + \nabla f_z\,\nabla z\\ &= \frac{\partial f_x}{\partial x}\nabla x\,\nabla x + \frac{\partial f_x}{\partial y}\nabla y\,\nabla x + \frac{\partial f_x}{\partial z}\nabla z\,\nabla x\\ &+ \frac{\partial f_y}{\partial x}\nabla x\,\nabla y + \frac{\partial f_y}{\partial y}\nabla y\,\nabla y + \frac{\partial f_y}{\partial z}\nabla z\,\nabla y\\ &+ \frac{\partial f_z}{\partial x}\nabla x\,\nabla z + \frac{\partial f_z}{\partial y}\nabla y\,\nabla z + \frac{\partial f_z}{\partial z}\nabla z\,\nabla z\end{aligned} \tag{1.1.60}$$

[6] Nonstandard analysis refers to rigorous calculus with infinitesimals.

Now in order for $\nabla\nabla f$ to be well defined, we must have f satisfying certain properties, such as

$$\frac{\partial f_x}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f_y}{\partial x} \tag{1.1.61}$$

$$\frac{\partial f_y}{\partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial f_z}{\partial y} \tag{1.1.62}$$

$$\frac{\partial f_z}{\partial x} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial f_x}{\partial z} \tag{1.1.63}$$

With this, it is clear that $\nabla\nabla f$ is a symmetric tensor. It is also clear that there are no restrictions on $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, or $\frac{\partial^2 f}{\partial z^2}$ other than that they exist. Given this, we see that $\nabla\nabla f$ is a totally symmetric tensor. We'd like to ignore the diagonal entries, so we create an antisymmetric tensor from the components that have restrictions on them from (1.1.61)-(1.1.63), where I will define $A(\overleftrightarrow{\mathbf{T}})$ as

$$\begin{aligned} A(\nabla\nabla f) = {} & \nabla x \nabla x + \frac{\partial f_x}{\partial y} \nabla y \nabla x + \frac{\partial f_x}{\partial z} \nabla z \nabla x \\ & - \frac{\partial f_x}{\partial y} \nabla x \nabla y + 0 \nabla y \nabla y + \frac{\partial f_y}{\partial z} \nabla z \nabla y \\ & - \frac{\partial f_x}{\partial z} \nabla x \nabla z + - \frac{\partial f_y}{\partial z} \nabla y \nabla z + 0 \nabla z \nabla z \end{aligned} \tag{1.1.64}$$

However, this has a lot of duplicate information in the entries since it is antisymmetric. We could instead only look at the upper diagonal entries, recognizing that the lower diagonal entries are simply the negative of the upper diagonal entries. That is we recognize that $A(\nabla\nabla f) + A(\nabla\nabla f)^\intercal = \mathbf{0}$.

With differential forms, we simplify this notation. Instead of writing $\nabla x \nabla y$, we write $\mathrm{d}x \wedge \mathrm{d}y$ and we simply interpret $\mathrm{d}y \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}y$ because we recognize that we are imposing this antisymmetry condition on the constructed tensor above. This is a bit like saying $A(\nabla\nabla f)^\intercal = -A(\nabla\nabla f)$. However with differential forms we simply enforce $\mathrm{d}\,\mathrm{d}f = \sum_{i,j=1}^{3} \frac{\partial f_{x_j}}{\partial x_i} \mathrm{d}x_i \wedge \mathrm{d}x_j = 0$ via the antisymmetric nature of our wedge product $\wedge$, often called the exterior product.[7] Thus we have

$$\mathrm{d}\,\mathrm{d}f = 0 = \sum_{i,j=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j \tag{1.1.65}$$

which enforces that our function satisfy $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ naturally for us rather than having to deal with the entire tensor $\nabla\nabla f$ and making sure that f is a suitable function. What we have actually done is ensure that we have a conservative vector ∇f, and called it a differential 1-form. This differential 1-form is simply a perfect differential (or exact differential), which is a restatement that it can be integrated by only considering the endpoints because there is no path dependency when integrating such a function. You can note this is a very special kind of function. Just like analytic functions are a special type of complex function, differential forms are a special type of differentiable/integrable function.

If we go to higher dimensions, we might want to construct such perfect differentials for vectors themselves, and higher order tensors. For a vector $\mathbf{A}$, we can construct $\nabla\mathbf{A}$. Here $\mathbf{A} = \sum_i A_i \hat{\mathbf{x}}_i$

[7] In fact, our dd operation is called the exterior derivative.

which we easily see can be translated into a differential 1-form as $A = \sum_i A_i \, \mathrm{d}x_i$ and so

$$\nabla \mathbf{A} = \sum_i \nabla A_i \hat{\mathbf{x}}_i = \sum_{i,j} \frac{\partial A_i}{\partial x_j} \hat{\mathbf{x}}_j \hat{\mathbf{x}}_i \tag{1.1.66}$$

which can be translated into a differential 2-form. First we recognize that we can write the above in a matrix form. If we care only about the antisymmetric part, then we can translate this part into the differential 2-form

$$\mathrm{d}\mathbf{A} = \sum_{i,j} \frac{\partial A_i}{\partial x_j} \, \mathrm{d}x_j \wedge \mathrm{d}x_i \tag{1.1.67}$$

Note that we are losing information about the components $\frac{\partial A_i}{\partial x_i}$ which form the diagonal of $\nabla \mathbf{A}$. From a differential form point of view, these components are not important because differential 2-forms are a special type of function, and they do not worry about the symmetric parts. Indeed, differential forms can be viewed as the constructed antisymmetric tensors (like that of (1.1.64)). Clearly in order for the A_i to form a perfect differential, we will require $\mathrm{d}\,\mathrm{d}A = 0$ because we want $\frac{\partial^2 A_i}{\partial x_j \partial x_k} = \frac{\partial^2 A_i}{\partial x_k \partial x_j}$, just as we did for f by the exactly analogous reasoning. That is we can write out

$$\nabla \nabla \mathbf{A} = \nabla \sum_i \nabla A_i \hat{\mathbf{x}}_i = \sum_i \nabla \nabla A_i \hat{\mathbf{x}}_i = \sum_{i,j} \nabla \left(\frac{\partial A_i}{\partial x_j} \hat{\mathbf{x}}_j \right) \hat{\mathbf{x}}_i = \tag{1.1.68}$$

$$= \sum_{i,j,k} \frac{\partial (\frac{\partial A_i}{\partial x_j} \hat{\mathbf{x}}_k)}{\partial x_k} \hat{\mathbf{x}}_j \hat{\mathbf{x}}_i = \sum_{i,j,k} \left(\frac{\partial}{\partial x_k} \frac{\partial A_i}{\partial x_j} \right) \hat{\mathbf{x}}_k \hat{\mathbf{x}}_j \hat{\mathbf{x}}_i \tag{1.1.69}$$

We can again realize that what we want is that $\frac{\partial^2 A_i}{\partial x_k \partial x_j} = \frac{\partial^2 A_i}{\partial x_j \partial x_k}$ is something we desire for our function to be a perfect differential (integrable). Thus we could form $A(\nabla \nabla \mathbf{A})$, and use that it is an antisymmetric matrix (so that flipping any two vector components leads to a minus sign). Alternatively, we can encode this information more concisely using the wedge product and the properties of differential forms that the sum over all indices equals zero. This is again saying that $\nabla \mathbf{A}$ is a perfect differential and so could be integrated only caring about the endpoints.[8] We can perform this for any tensor order n, making the same translation into differential n-forms and differential $n+1$ forms. With our differential forms, however, the enforcement of our tensors to being perfect differentials is automatically enforced. Thus, differential forms force us to work only with special types of functions, ones that are guaranteed to have a conservative form.

So now we can write out the "vector components", actually a differential 1-form, as $\alpha = \sum_i \alpha_i \, \mathrm{d}x_i$ and we automatically can form the perfect differential second order tensor, called a differential 2-form via

$$\mathrm{d}\alpha = \sum_i \mathrm{d}\alpha_i \wedge \mathrm{d}x_i = \sum_{i,j} \frac{\partial \alpha_i}{\partial x_j} \, \mathrm{d}x_j \wedge \mathrm{d}x_i \tag{1.1.70}$$

[8] For the interested reader, I have been fairly careless in this description, as there are subtleties in making differential forms more rigorous. For example, a differential form is usually considered to be in the dual space of a corresponding vector. Consult a mathematics book on differential forms such as **??** if you want a more rigorous examination. Note that for ease of presentation, I considered Cartesian vector representations, as well.

for any number of dimensions. Differential k-forms are formed in an exactly analogous manner. We always have $\mathrm{d}\,\mathrm{d}\alpha = 0$ for any k-form α as previously stated. We want to only work with perfect differentials and this enforces it. The only other rule, which is unintuitive at first, is

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta \tag{1.1.71}$$

for α a k-form. This property is easy to understand for $\alpha = \alpha_i \,\mathrm{d}x_i$ and $\beta = \beta_j \,\mathrm{d}x_j$, as

$$\begin{aligned}
\mathrm{d}(\alpha \wedge \beta) &= \mathrm{d}\alpha_i\beta_j \,\mathrm{d}x_i \wedge \mathrm{d}x_j = \sum_k \frac{\partial \alpha_i \beta_j}{\partial x_k} \,\mathrm{d}x_k \wedge \mathrm{d}x_i \wedge \mathrm{d}x_j \\
&= \sum_k \left(\beta_j \frac{\partial \alpha_i}{\partial x_k} + \alpha_i \frac{\partial \beta_j}{\partial x_k} \right) \mathrm{d}x_k \wedge \mathrm{d}x_i \wedge \mathrm{d}x_j \\
&= \sum_k \left(\beta_j \frac{\partial \alpha_i}{\partial x_k} \,\mathrm{d}x_k \wedge \mathrm{d}x_i \wedge \mathrm{d}x_j - \alpha_i \frac{\partial \beta_j}{\partial x_k} \,\mathrm{d}x_i \wedge \mathrm{d}x_k \wedge \mathrm{d}x_j \right) \\
&= \mathrm{d}\alpha \wedge \beta - \alpha \wedge \mathrm{d}\beta
\end{aligned} \tag{1.1.72}$$

whereas if k is even, then we will have to flip two wedge products which leaves the sign unchanged.

The important thing to recognize is that the differential form notation makes it easy to form perfect differentials (that is functions that can be integrated in the exact same way that it is easy to integrate a derivative). When we restrict ourselves to no wedge products, however, things are analogous to using gradients. Differential forms make it possible to write down proofs of things that are somewhat more complicated to work out in general for tensors in a tidy way. For example Gauss's theorem and Stokes theorem are easily proven with generalizations for multiple dimensions via differential forms. Since $\mathbb{1} : \nabla\mathbf{A} = \boldsymbol{\nabla} \cdot \mathbf{A}$ and $\mathbb{1} \overset{\times}{\cdot} \nabla\mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A}$ this should not be all that surprising. For then we can have perfect differentials for $\nabla\mathbf{A}$ by seeing how the exterior derivative acts on certain differential forms.

For example, clearly for a 0-form α, then $\mathrm{d}\alpha$ corresponds to the gradient of a scalar. For a 1-form β in three dimensions, then $\mathrm{d}\beta$ looks like a curl. We could then interpret $\mathrm{d}\,\mathrm{d}\alpha = 0$ as the curl of a gradient being zero. For γ a 2-form in three dimensions, then we see

$$\gamma = \sum_{ij} \gamma_{ij} \,\mathrm{d}x_i \wedge \mathrm{d}x_j \tag{1.1.73}$$

$$\mathrm{d}\gamma = \sum_{ij} \frac{\partial \gamma_{ij}}{\partial x_k} \,\mathrm{d}x_k \wedge \mathrm{d}x_i \wedge \mathrm{d}x_j = \left[\frac{\partial}{\partial x_3} (\gamma_{12} - \gamma_{21}) + \frac{\partial}{\partial x_1} (\gamma_{23} - \gamma_{32}) + \frac{\partial}{\partial x_2} (\gamma_{31} - \gamma_{13}) \right] \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \tag{1.1.74}$$

$$= 2 \left(\frac{\partial \gamma_{23}}{\partial x_1} + \frac{\partial \gamma_{31}}{\partial x_2} + \frac{\partial \gamma_{12}}{\partial x_3} \right) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \tag{1.1.75}$$

which is reminiscent of a divergence. Indeed, we can interpret $\mathrm{d}\,\mathrm{d}\beta = 0$ as the divergence of a curl being zero.

1.1.4.1 Flipping Derivatives

Let's now prove a couple of interesting differentiation properties that appear to "flip" derivatives. First, that for full derivatives we actually can just "flip" the derivative and get what you might

expect. Consider $f(x)$ with $\frac{\mathrm{d}f}{\mathrm{d}x} \neq 0$.

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}f}} \tag{1.1.76}$$

Then we can write $f(x) = y$ and apply the chain rule.

$$1 = \frac{\mathrm{d}y}{\mathrm{d}y} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}f} \tag{1.1.77}$$

which yields the desired relation. Note that if $\frac{\mathrm{d}f}{\mathrm{d}x} = 0$ then $\frac{\mathrm{d}x}{\mathrm{d}f} = 0$ since f and x are independent of each other then.

Now let's look at flipping a partial derivative. We write $f(x_j) = z$ as before. We must also have $\left(\frac{\partial z}{\partial x_j}\right)_{x_{i\neq j}} = \left(\frac{\partial f}{\partial x_j}\right)_{x_{i\neq j}}$ so that $\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\mathrm{d}z}{\mathrm{d}f} = 1$. Indeed, we can replace z by f and vice versa wherever they appear. Then consider

$$1 = \frac{\mathrm{d}f}{\mathrm{d}z} = \left(\frac{\partial f}{\partial x_j}\right)_{x_{i\neq j}} \left(\frac{\partial x_j}{\partial z}\right)_{x_{i\neq j}} \tag{1.1.78}$$

$$1 = \left(\frac{\partial f}{\partial x_j}\right)_{x_{i\neq j}} \left(\frac{\partial x_j}{\partial f}\right)_{x_{i\neq j}} \tag{1.1.79}$$

$$\left(\frac{\partial f}{\partial x_j}\right)_{x_{i\neq j}} = \frac{1}{\left(\frac{\partial x_j}{\partial f}\right)_{x_{i\neq j}}} \tag{1.1.80}$$

This rule is less useful than it might appear since usually we don't want to hold the same things constant when flipping a partial derivative, but it can still come in handy sometimes.

Let's now look at the relationship between x_j in an implicit function $f(x_j) = 0$. If we don't have an equation for $f(x_j)$, then we can always consider forcing $f(x_j) = z$ then consider the new function $g(x_j, z) = 0$. We could then add z to the x_j group. We will just assume we have $f(x_j) = 0$ from the onset for this proof. This means that the x_j are not all independent since we can move an x_j to the right-hand side of the $f(x_j) = 0$ relation and solve for it in terms of the $x_{i\neq j}$[9] Again we assume $\left(\frac{\partial f}{\partial x_j}\right)_{x_{i\neq j}} \neq 0$ for all j.

We then have ($x_{i\neq j}$ is all x_i except that the x_j one)

$$\mathrm{d}f = \sum_j \left(\frac{\partial f}{\partial x_j}\right)_{x_{i\neq j}} \mathrm{d}x_j \tag{1.1.81}$$

We also have via the implicit relations that

$$\mathrm{d}x_j = \sum_{i\neq j} \left(\frac{\partial x_j}{\partial x_i}\right)_{x_{i\neq j}} \mathrm{d}x_i \tag{1.1.82}$$

[9] In practice this may be impossible to do in a convenient way, but it is always possible to actually do for a well-defined function.

Let's replace x_k in the above relation and we have

$$\mathrm{d}f = 0 = \left(\frac{\partial f}{\partial x_k}\right)_{x_{i\neq k}} \mathrm{d}x_k + \sum_{j\neq k}\left(\frac{\partial f}{\partial x_j}\right)_{x_{i\neq j}} \mathrm{d}x_j \tag{1.1.83}$$

$$= \left(\frac{\partial f}{\partial x_k}\right)_{x_{i\neq k}} \sum_{i\neq k}\left(\frac{\partial x_1}{\partial x_i}\right)_{x_{i\neq k}} \mathrm{d}x_i + \sum_{j\neq k}\left(\frac{\partial f}{\partial x_j}\right)_{x_{j\neq k}} \mathrm{d}x_j \tag{1.1.84}$$

which simply says (rewriting the sum over i to a sum over j)

$$0 = \sum_{j\neq k}\left[\left(\frac{\partial f}{\partial x_k}\right)_{x_{j\neq k}}\left(\frac{\partial x_k}{\partial x_j}\right)_{x_{j\neq k}} + \left(\frac{\partial f}{\partial x_j}\right)_{x_{j\neq k}}\right]\mathrm{d}x_j \tag{1.1.85}$$

All of these $\mathrm{d}x_j$ are independent of each other within this formulation (that is, each coefficient of each of our $\mathrm{d}x_j$ with $j \neq k$ are independent now)[10] so that each coefficient of $\mathrm{d}x_j$ must equal zero. So

$$\left(\frac{\partial f}{\partial x_k}\right)_{x_{i\neq k}}\left(\frac{\partial x_k}{\partial x_j}\right)_{x_{j\neq k}} + \left(\frac{\partial f}{\partial x_j}\right)_{x_{j\neq k}} = 0 \tag{1.1.86}$$

$$\left(\frac{\partial x_k}{\partial x_j}\right)_{x_{j\neq k}} = -\frac{\left(\frac{\partial f}{\partial x_j}\right)_{x_{j\neq k}}}{\left(\frac{\partial f}{\partial x_k}\right)_{x_{j\neq k}}} \tag{1.1.87}$$

which is the somewhat surprising relationship between variables in an implicitly defined function. You may not have expected the $-$ sign.

If we then consider a function $f(x_j) = f(x, y, z) = 0$ we can see this as implicitly defining $x_j = x_j(x_i)$ for each j. For $\frac{\partial f}{\partial x_j} \neq 0$ we have

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = (-1)\left(\frac{\cancel{\left(\frac{\partial f}{\partial y}\right)_{x,z}}}{\left(\frac{\partial f}{\partial x}\right)_{y,z}}\right)(-1)\left(\frac{\cancel{\left(\frac{\partial f}{\partial z}\right)_{x,y}}}{\cancel{\left(\frac{\partial f}{\partial y}\right)_{x,z}}}\right)(-1)\left(\frac{\cancel{\left(\frac{\partial f}{\partial x}\right)_{y,z}}}{\cancel{\left(\frac{\partial f}{\partial z}\right)_{x,y}}}\right) = -1 \tag{1.1.88}$$

which will often even surprise your mathematician friends.

[10] This is simply saying that we have that the $\mathrm{d}x_i$ are all independent now because we used that $x_k = x_k(x_i)$. The previous possible interrelationships between the x_j were eliminated by putting in the expression for x_k in terms of the x_i.

1.1.5 Summary of Notation

> *The invention of things comes more easily to the human mind than words, which accounts for the use of so many unsuitable terms and half-baked expressions.*
>
> — Alexis de Tocqueville[30, p. 184]

I have summarized my notation in the following tables, Tables 1.1-1.4.

Operation	Here	Other
Single-Variable Function	$f(t)$	
Multi-Variable Function	$f(x,y,z)$ or $f(x(t),y(t),z(t),t)$	
Vector	$\mathbf{A}, \mathbf{a}, \boldsymbol{\alpha}$	$\vec{A}, \vec{a}, \vec{\alpha}$, $\underline{A}, \underline{a}, \underline{\alpha}, \mathbb{A}$
Vector function	$\mathbf{A}(\mathbf{x}) = \mathbf{A}(x^i,t) = \mathbf{A}(x,y,z)$	$\vec{A}(\vec{x},t)$, $\underline{A}(\underline{x},t)$
Second Order Tensor	$\overleftrightarrow{\mathbf{T}}$, $\overleftrightarrow{\mathbf{s}}$ or $T_{ij}, T^{ij}, T^i_{\cdot j}, T^{\cdot i}_j$	$\vec{\vec{A}}$, $\underline{\underline{A}}$
General Tensor	T, s or $T^{i_1\ldots i_n}_{j_1\ldots j_n}$	
Matrix	M, m	$\underline{\underline{M}}$
Four-Vector	$\mathbb{A}$	
Second Order Four-Tensor	$\overleftrightarrow{\mathbb{A}}$	
Indefinite Integral	$\int \mathrm{d}x\ f(x) \int^x \mathrm{d}x'\ f(x')$	$\int f(x)\,\mathrm{d}x$, $\int^x f(x')\,\mathrm{d}x'$
Definite Integral	$\int_a^b \mathrm{d}x\ f(x)$	$\int_a^b f(x)\,\mathrm{d}x$
Multiple Integration	$\int_a^b \mathrm{d}x \int_c^d \mathrm{d}y\ f(x,y)$	$\int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$
Volume Differential	$\mathrm{d}V = \mathrm{d}^3x$	$\mathrm{d}\mathbf{x}$, dVol
Volume Integral	$\iiint_V \mathrm{d}^3x\ f(\mathbf{x}) = \iiint_V \mathrm{d}V\ f(\mathbf{x})$	$\iiint_V \mathrm{d}\mathbf{x}\ f(\mathbf{x})$

Table 1.1: My notation ("Here") with common alternative notations ("Other") for mathematical and physical operations. Note that upper and lower case letters are not used to distinguish different objects in my general notation.

Operation	Here	Other
Cauchy Principal Integral	$\fint_a^b \mathrm{d}x\ f(x)$	$\mathcal{P}, \mathcal{PV}$, p.v., CPV $PV, VP,$ or, P in front of integral
Surface Integral	$\iint_S \mathrm{d}^2x\ \mathbf{n}\cdot\mathbf{f}(\mathbf{x}) = \iint_S \mathrm{d}S\ \mathbf{n}\cdot\mathbf{f}(\mathbf{x})$	
Surface Integral (cont.)	$\iint_S \mathrm{d}\mathbf{S}\ \cdot\mathbf{f}(\mathbf{x})$	
Closed Surface Integral	$\oiint_S \mathrm{d}^2x\ \mathbf{n}\cdot\mathbf{f}(\mathbf{x}) = \oiint_S \mathrm{d}S\ \mathbf{n}\cdot\mathbf{f}(\mathbf{x})$	
Closed Surface Integral (cont.)	$\oiint_S \mathrm{d}\mathbf{S}\ \cdot\mathbf{f}(\mathbf{x})$	
Line Integral	$\int_C \mathrm{d}\boldsymbol{\ell}\ \cdot\mathbf{f} = \int_C \mathrm{d}\mathbf{x}\ \cdot\mathbf{f}$	
Closed Line Integral	$\oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot\mathbf{f} = \oint_C \mathrm{d}\mathbf{x}\ \cdot\mathbf{f}$	
First Derivative	$\frac{\mathrm{d}f}{\mathrm{d}t} = f'(t) = \dot{f}$	f_t, $\frac{df}{dt}$
nth Derivative	$\frac{\mathrm{d}^n f}{\mathrm{d}t^n} = f^{(n)}(t)$	$\frac{d^n f}{dt^n}$
Partial Derivative	$\frac{\partial f(\mathbf{x},t)}{\partial t} = \left(\frac{\partial f(\mathbf{x},t)}{\partial t}\right)_{\mathbf{x}}$	$f_t(\mathbf{x},t), f_{,t}, f_{;t}$
Total Derivative	$\frac{\mathrm{d}f(\mathbf{x},t)}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t}\cdot\frac{\partial f}{\partial \mathbf{x}}$ or $\frac{\mathrm{d}_s f(\mathbf{x})}{\mathrm{d}t}$	$\frac{\mathrm{D}f(\mathbf{x},t)}{\mathrm{D}t} = \frac{Df(\mathbf{x},t)}{Dt}$
Gradient operator	$\nabla f = \frac{\partial f}{\partial \mathbf{x}}$ or $\mathbf{e}^i\partial_i f$	$\mathrm{grad}(\mathbf{f})$
Divergence operator	$\boldsymbol{\nabla}\cdot\mathbf{f} = \frac{\partial}{\partial \mathbf{x}}\cdot\mathbf{f}$ or $\partial_i(\mathcal{J}f_i)\mathcal{J}$	$\nabla\cdot\mathbf{f}, \mathrm{div}(\mathbf{f})$
Convergence operator (rare)	Not used	$\mathrm{con}(\mathbf{f}) = -\boldsymbol{\nabla}\cdot\mathbf{f}$
Curl operator	$\boldsymbol{\nabla}\times\mathbf{f} = \frac{\partial}{\partial \mathbf{x}}\times\mathbf{f}$ or $\mathbf{e}_k\epsilon_{ijk}\partial_i(f_j)/\mathcal{J}$	$\nabla\times\mathbf{f}, \mathrm{curl}\,\mathbf{f}, \mathrm{rot}(\mathbf{f})$

Table 1.2: My notation ("Here") with common alternative notations ("Other") for mathematical and physical operations. Note that I use the comma notation for covariant derivatives rather than partial derivatives.

Operation	Here	Other
Velocity Gradient operator	$\nabla_{\mathbf{v}} f = \dfrac{\partial f}{\partial \mathbf{v}}$	
Velocity Divergence operator	$\boldsymbol{\nabla}_{\mathbf{v}} \cdot \mathbf{f} = \dfrac{\partial}{\partial \mathbf{v}} \cdot \mathbf{f}$	$\nabla_{\mathbf{v}} \cdot \mathbf{f}$
Velocity Curl operator	$\boldsymbol{\nabla}_{\mathbf{v}} \times \mathbf{f} = \dfrac{\partial}{\partial \mathbf{v}} \times \mathbf{f}$	$\nabla_{\mathbf{v}} \times \mathbf{f}$
Four-Gradient	$\Box = \mathbf{e}^{\mu} \dfrac{\partial}{\partial \xi^{\mu}} = \mathbf{e}^{0} \dfrac{\partial}{\partial \xi^{0}} + \mathbf{e}^{i} \dfrac{\partial}{\partial \xi^{i}}$	
d'Alembertian Operator	$\Box^2 f = \Box \cdot \Box f$	
Dot Product	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{AB}, \mathrm{dot}(\mathbf{A}, \mathbf{B})$
Cross Product	$\mathbf{A} \times \mathbf{B}$	$[\mathbf{AB}], [\mathbf{A}, \mathbf{B}], \mathbf{A} \wedge \mathbf{B}, \mathrm{cross}(\mathbf{A}, \mathbf{B})$
Asymptotic Expansion Order x^n	$\mathcal{O}(x^n)$	
Einstein Summation	$a_i b_{ij} c_{jk} d_k = \sum_{i,j,k} a_i b_{ij} c_{jk} d_k$	
Even Cyclic Permutation Indices	$(i', j', k') = \begin{cases} (1,2,3) \\ (3,1,2) \\ (2,3,1) \end{cases}$	
Tangent Basis Vectors	$\mathbf{e}_i = \dfrac{\partial \mathbf{x}}{\partial \xi^i}$	
Tangent Basis Unit Vectors	$\hat{\mathbf{e}}_i = \dfrac{\partial \mathbf{x}}{\partial \xi^i} \Big/ \left\lvert \dfrac{\partial \mathbf{x}}{\partial \xi^i} \right\rvert$	
Tangent-Reciprocal Basis Vectors	$\mathbf{e}^i = \dfrac{\partial \xi^i}{\partial \mathbf{x}} = \nabla \xi^i$	
Tangent-Reciprocal Basis Unit Vectors	$\hat{\mathbf{e}}^i = \dfrac{\partial \xi^i}{\partial \mathbf{x}} \Big/ \left\lvert \dfrac{\partial \xi^i}{\partial \mathbf{x}} \right\rvert = \dfrac{\nabla \xi^i}{\lvert \nabla \xi^i \rvert}$	
Cartesian Unit Vectors	$\hat{\mathbf{x}}^i = \hat{\mathbf{x}}_i$, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$	$\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$
Dyad	$\mathbf{AB}$	$\mathbf{A} \otimes \mathbf{B}$

Table 1.3: My notation ("Here") with common alternative notations ("Other") for mathematical and physical operations. The (i', j', k') is for indices, and the prime indicates no summation in these formulas. Also Greek indices indicate summation from 0 to 3 whereas Roman indices indicate summation from 1 to 3 so the d'Alembertian operator is for 4D spacetime.

Operation	Here	Other
Double Dot Product	$\mathbf{AB} : \mathbf{CD} = (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$	$\mathbf{AB} : \mathbf{CD} = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D})$
Cross Dot Product	$\mathbf{AB} \overset{\times}{\cdot} \mathbf{CD} = (\mathbf{A} \times \mathbf{C})(\mathbf{B} \cdot \mathbf{D})$	$\mathbf{AB} \overset{\times}{\cdot} \mathbf{CD} = (\mathbf{A} \times \mathbf{D})(\mathbf{B} \cdot \mathbf{D})$
Dot Cross Product	$\mathbf{AB} \overset{\cdot}{\times} \mathbf{CD} = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \times \mathbf{D})$	$\mathbf{AB} \overset{\cdot}{\times} \mathbf{CD} = (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \times \mathbf{C})$
Cross Cross Product	$\mathbf{AB} \overset{\times}{\times} \mathbf{CD} = (\mathbf{A} \times \mathbf{C})(\mathbf{B} \times \mathbf{D})$	$\mathbf{AB} \overset{\times}{\times} \mathbf{CD} = (\mathbf{A} \times \mathbf{D})(\mathbf{B} \times \mathbf{C})$
Covariant Derivative	$\left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)^k = V^k_{;j}$	$V^k_{,j}, V^k_{\|\|j}$
Covariant Derivative	$\left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)_k = V_{k;j}$	$V_{k,j}, V_{k\|\|j}$
Frenet-Serret Quantities	$\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}$	$\mathbf{T}, \mathbf{N}, \mathbf{B}, \mathbf{t}, \mathbf{n}, \mathbf{b}, \hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}$
Magnetic Frenet-Serret	$\hat{\mathbf{b}}, \hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{\beta}}$	$\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}$
Vector Array	$\mathtt{A}, \mathtt{a}, \mathtt{z}, \mathtt{x}$	
Complex Conjugation	$\bar{z}$, z^*	

Table 1.4: My notation ("Here") with common alternative notations ("Other") for mathematical and physical operations. I use the semicolon for covariant differentiation rather than a comma (which is sometimes used for partial differentiation) or a pipe | or double pipe ||.

1.2 Coordinate Systems and Vectors

> *It is conceivable (though presumably not desirable) that young children could be led to associate the symbol (2,3) with the idea of 2 cats and 3 dogs.*
>
> — W. W. Sawyer[28, p. 28]

This section is a broad overview of curvilinear coordinate systems. It is based on the fact that we deal with a single Euclidean space[11] when we use physics, and so roots itself to vectors as geometric objects with particular representations given a coordinate system for our Euclidean space. Given a vector basis set, that is a set of vectors that allow us to point to any location in our space,[12] we will show it has a natural reciprocal basis, which then allows us to represent all vectors in the space in a very concise form. Note that the basis and reciprocal basis can have their constituent vectors change in space so long as at each location the vector basis can still span the entire space (that is, they can represent a vector "pointing" to anywhere else within the space we are using). This leads naturally to contravariant and covariant components of vectors and so the tangent basis and its reciprocal basis, the tangent-reciprocal basis are then emphasized and developed for any coordinate choice,[13] regardless of whether the basis set is orthonormal. We will also explore extensions beyond three dimensions. A brief explanation of Einstein summation notation is shown with its translation

[11] This is not true in general relativity, but even then one uses Euclidean space ideas locally. In any case, I am trying to say that in classical physics, the dual space is not really relevant, except insofar as it is a useful for simplifying operations.

[12] We usually say the basis spans the space.

[13] Any, meaning, any admissible coordinate system. You must choose coordinates that do allow you to specify locations everywhere unambiguously.

capabilities from abstract geometric vectors to a Cartesian representation and back, which allows us to find generic vector identities quickly. Generalizations of vectors to polyadics and tensors are explained and some important relations for these in curvilinear coordinates are derived. Finally, some extra properties of geometric vectors that are of use in physical situations are examined, with different types of geometric vectors appearing under certain coordinate transformations.

1.2.1 An Analogy

> *Analogy pervades all our thinking, our everyday speech and our trivial conclusions as well as artistic ways of expression and the highest scientific achievements.*
>
> — George Pólya[26, p. 37]

What is a vector? Do you think you could give a good definition? Generally, we have an intuitive idea of what a vector is. Usually something like a direction and magnitude. Unfortunately, there is a certain vagueness about vectors due to the word being used in math and physics for different ideas. Below we will deal with the specifics of what this means (physics usually deals with what I will call "geometric vectors" while math has a more general definition). For now, I would like to explain why it is important to recognize that the language around vectors can be confusing with an analogy to real numbers. This analogy will help us realize that the notation for something, and what that something is, are different things, even though we often talk as if they are interchangeable.

If I ask you if 10.1 is a real number, what would you say? Most likely you will say yes. If I say "10.1" is a real number, what now? If you understand the quotations as indicating the symbols "1", "0", "." and "1", then of course that representation is not a real number. It is a representation of a real number but not the number itself! (What the number itself is, is a problem for philosophy. I would just say it is an abstract object or pattern rather than a single physical thing.) This is why we must always be careful about using the word "is"[14] when referring to such objects. We are using conventions to represent the actual object corresponding to the actual number[15] ten and one-tenth when writing 10.1. This representation could just as easily be binary, in which case the real number being represented would be two and one-half.

Why bring this up? This may all seem obvious. The problem comes when we turn to other ideas. It is burdensome to say that 10.1 is the representation of a real number, and so we shorten it to "10.1 is a real number". It is implicitly understood we are referring to the idea rather than to the representation, just as when we use words.[16] This does not usually cause any confusion, and so we continue our daily lives content and not thinking about the fact that the number seems to be in some way more primitive and basic than the representation we use. When it comes to vectors, a question arises of whether the representation actually is the vector. That is if I write a vector $\mathbf{X}$ representing a position at $x = 1$, $y = 1$, and $z = 1$ in a particular coordinate system, is

$$\mathbf{X} \stackrel{?}{=} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\intercal} \tag{1.2.1}$$

[14] You always have to ensure you are using the same meaning of "is" when you're being precise. The meaning of "is" has even led a U.S. President into trouble.

[15] Whatever an actual number is.

[16] This convention is easy to see in natural speech. If someone asks for a dollar, and you hand them a sheet of paper with the word "dollar" on it, they will not be satisfied.

Or is $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\intercal$ simply a representation of the vector $\mathbf{X}$ in a specific coordinate system? If you think it is just a representation, then you can speak of vectors in different coordinate systems being represented differently, but that they all refer to the same vector. If you think that the array of values is the vector, then you are forced to say that vectors are changing in different coordinate systems. This is just a way of saying it matters whether you consider a vector to be an array of numbers or to be a geometric object that can be represented by an array of numbers. This is exactly analogous to asking if the real number ten and one-half is 10.5, or if that is simply its representation in base 10. For in base 8, we have ten and one-half as 12.4, in base 16 it would be $a.8$, etc. Yet, we know that all of these refer to the same number.

This book will take the view that vectors are geometric objects represented in coordinate systems, but that any single representation is not *the* vector, but *a* representation of the vector. Just as 10.1 is not ten and one tenth, but a representation of ten and one tenth in base 10. This will clear up some confusions, and seems more true to the sense that vector is used in physics. In mathematics textbooks, it may make sense to make the array of numbers a "vector", in which case, different terms and nomenclature may be more enlightening. I will refer to such "vectors" as vector arrays to emphasize they are not geometric, and have no such geometric properties.

One of the best advantages of considering vectors as abstract geometric objects is that if we relate an abstract vector to a specific representation (say Cartesian coordinates) and perform operations on the representation, then translate back into abstract vector notation, we will have proved the vector relation for any representation. To see why it is so, we need only remind ourselves that if all specific representations are equal for the abstract object, and if we can translate from a specific representation back to the abstract, then it must be true for any other specific representation. This may sound difficult, but in practice it is straightforward reasoning. Suppose you have a specific coordinate system for a vector and that you can relate it to the abstract representation without a coordinate system. If you then know that there are vector identities in the abstract representation system, these abstract representations when translated into any specific coordinate representation must also be true. So you can use them there, and then translate back into your specific coordinate system. Because this is a requirement, it means that if you have two equivalent forms in a specific coordinate system, and both can be translated into the abstract vector representation, then the two abstract vector representations must identically be equal.

Thus, if we have an equation for a specific coordinate representation but can translate both sides into the abstract representation, we know the abstract representation identity is true, and can be used for any other specific coordinate representation. The real trick is in learning the translation from abstract to a specific representation and back.

To make this less abstract and more concrete for ourselves, let's go back to our real number example. Suppose we have real numbers a, b, and c with the operations of addition $+$ and multiplication $\cdot$. Suppose we only know how to add and multiply integers (and implicitly fractions). We wish to "prove" $a \cdot (b + c) = a \cdot b + a \cdot c$ for any representation of a, b, and c. One way of doing this is to represent a, b, and c, as numbers in decimal notation (all of a_i, b_i, c_i are integers in $[0, 9]$).

$$a = \sum_i a_i (10)^i \tag{1.2.2}$$

$$b = \sum_i b_i (10)^i \tag{1.2.3}$$

$$c = \sum_i c_i (10)^i \tag{1.2.4}$$

$$b + c = \sum_i (b_i + c_i)(10)^i \tag{1.2.5}$$

$$a \cdot b = \left(\sum_i a_i 10^i \right) \left(\sum_j b_j 10^j \right) = \sum_{i,j} a_i b_j 10^{i+j} \tag{1.2.6}$$

Note that the above doesn't tell us what the decimal representation for $a + b$ actually is, because $a_i + b_i$ is not necessarily in $[0, 9]$. If we view these as the way to translate between abstract and a specific representation then we can consider $a \cdot (b + c)$.

$$\begin{aligned} a \cdot (b + c) &= \left(\sum_i a_i (10^i) \right) \left[\sum_j (b_j + c_j)(10)^j \right] = \sum_{i,j} a_i (b_j + c_j) 10^{i+j} \\ &= \sum_{i,j} a_i b_j 10^{i+j} + \sum_{i,j} a_i c_j 10^{i+j} = a \cdot b + a \cdot c \end{aligned} \tag{1.2.7}$$

Which via our translation rules shows that we must have $a \cdot (b + c) = a \cdot b + a \cdot c$ even though we only proved it for a base 10 representation. For completeness, the steps of combining the sums is via

$$\left(\sum_i a_i 10^i \right) \left(\sum_j b_j 10^j \right) = \sum_i \left[a_i 10^i \sum_j (b_j 10^j) \right] = \sum_i \left[\sum_j (a_i b_j 10^{i+j}) \right] \equiv \sum_{i,j} a_i b_j 10^{i+j} \tag{1.2.8}$$

where we use that we can bring constants into the summations twice. That is, the $\sum_j b_j 10^j$ is a constant with respect to the sum over i, and $a_i 10^i$ is a constant with respect to the sum over j.

This is of course an overcomplicated way of "proving"[17] the result of associativity, but it gives us one way of seeing why it must be true. All our specific multiplications and additions were done with integers a_i, b_i, and c_i, rather than with real numbers (other than multiplying by the factor 10^i when i is negative, which introduces fractions). The intermediate steps depend on the specific representation, but because we know the abstract version and can translate, we know that the first term $a \cdot (b + c)$ and the last $a \cdot b + a \cdot c$ are equal for any representation (that is, base).

1.2.2 Coordinate Systems

Appearances are a glimpse of the obscure.

— Anthony Gottlieb quoting Anaxagoras[16, p. 94]

Let's now start getting into what exactly is meant by the terms vector and coordinate system for this text. We will begin with what may seem very simple, the coordinate systems. First, we must deal with what a coordinate system is. Let's first consider a simple Cartesian grid in two dimensions (See Figure 1.1). A coordinate system is then simply a way of assigning an origin and thus unambiguously giving a position on a plane. The simplest way is to have two orthogonal

[17] It is not really a proof, since I have not gone through the details rigorously.

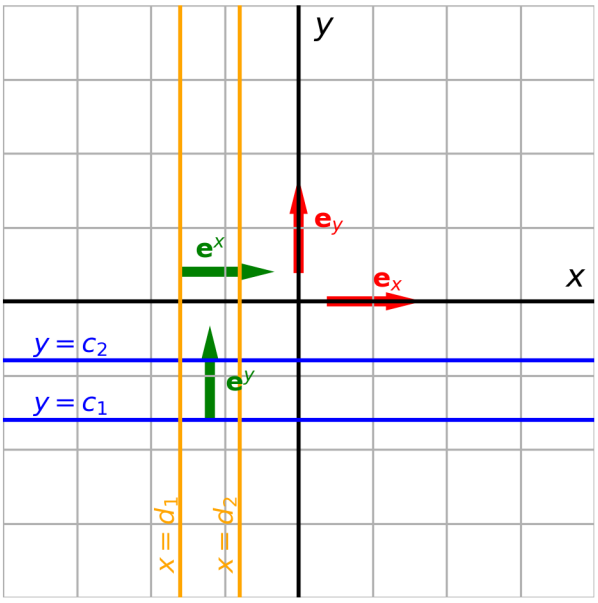

Figure 1.1: A simple Cartesian grid. Note that the lines are all perpendicular, and so a direction in on coordinate direction has no projection onto the other direction. Here $c_2 > c_1$ and $d_2 > d_1$.

directions (orthogonal, perpendicular, and normal[18] are all synonymous. That is, the projection of one axis direction on the other is zero; there is no overlap in the axes' directions). Projection here simply means that were I to take a step along one coordinate axis, would this necessarily involve taking any movement in a direction along another coordinate axis.[19] If it does, then they are not orthogonal. If I can move along the coordinate axis and this would not involve movement in a direction along another coordinate axis, then that coordinate axis is orthogonal to the others. This orthogonality is of course not necessary for coordinate systems, but it is almost always useful.

The important thing is that a coordinate system can specify a position in a space unambiguously. So a coordinate system is often specified as $(n_1, n_2, \cdots, n_j)$ where the j are different (though not necessarily orthogonal) directions. If the j are orthogonal, however, then it makes life much easier when trying to formulate equations for the coordinate system, as we will see.

For example, in Figure 1.2, the two axes are not independent/orthogonal. This opens up different ways of measuring positions. Say the axes are ξ^1 and ξ^2. We can either measure a direction by going along a coordinate's axis (say ξ^1 by fixing ξ_2 and following ξ^1 as it varies) or by going perpendicular to the axis. We will call the vectors that go along coordinate axes $\mathbf{e}_{\xi^1}$ and $\mathbf{e}_{\xi^2}$. This corresponds to the tangent basis because it goes along the axis. Alternatively, we can form the

[18] Normal comes from Latin *normalis* "forming a right angle" which comes from carpenter squares. So normal as perpendicular is the original meaning.

[19] This is saying given vectors pointing along a coordinate axes, say $\mathbf{e}_i$, then the projection of $\mathbf{e}_1$ upon $\mathbf{e}_2$ is given by $\text{Proj}_{\mathbf{e}_2}(\mathbf{e}_1) = \mathbf{e}_2 \dfrac{\mathbf{e}_1 \cdot \mathbf{e}_2}{\mathbf{e}_2 \cdot \mathbf{e}_2}$ where I have used the norm is the same as the square root of the dot product between two vectors.

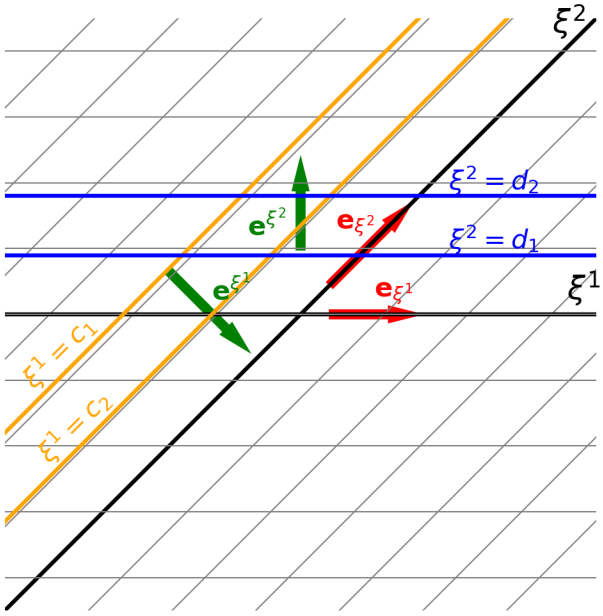

Figure 1.2: A slanted grid with horizontal coordinate $\xi^1 = x - y$ and "vertical" coordinate $\xi^2 = y$ with x and y Cartesian coordinates. There is clearly overlap of directions between $\mathbf{e}^{\xi^1}$ and $\mathbf{e}^{\xi^2}$. Note that $\mathbf{e}_{\xi^1}$ and $\mathbf{e}_{\xi^2}$ also overlap and point in different directions than $\mathbf{e}^{\xi^1}$ and $\mathbf{e}^{\xi^2}$. Here $c_2 > c_1$ and $d_2 > d_1$. Finally, note that one must do this analysis at every point generally. Here our basis vectors are not changing direction throughout space so we have an easier job.

surfaces (or lines for us in 2D) by letting ξ^1 vary with ξ^2 held constant and following the normal to this surface (line) [call it $\mathbf{e}^{\xi^1}$]. Similarly, we can let ξ^2 vary with ξ^1 held constant and follow the normal to this surface (line) [call it $\mathbf{e}^{\xi^2}$]. This corresponds to the tangent-reciprocal basis. We see that the tangent-reciprocal and tangent basis vectors (even for the same associated coordinate) do not necessarily point in the same direction. While not necessarily apparent geometrically we will also require a tangent-reciprocal basis to have a dot product of zero with all vectors in the tangent basis except their corresponding one (So $\mathbf{e}^{\xi^1} \cdot \mathbf{e}_{\xi^1} = \mathbf{e}^{\xi^2} \cdot \mathbf{e}_{\xi^2} = 1$ and $\mathbf{e}^{\xi^1} \cdot \mathbf{e}_{\xi^2} = \mathbf{e}^{\xi^2} \cdot \mathbf{e}_{\xi_1} = 0$). In 2D this is very easy to do. Set $\xi^1 = c_1$ and let ξ^2 vary. This will form a curve. If you go along the curve you are in the tangent basis (determining the tangent ξ_2 direction). If you form many curves for different c_i you can find the normal direction to the curves and you are using the tangent-reciprocal basis (determining the tangent-reciprocal ξ_2 direction).

This brings us to geometric vectors,[20] which are in a sense coordinate-free. All this means, is that when you are given such a vector, you don't need to worry about writing the equations for a given coordinate system. This is advantageous because certain vector identities allow us to derive generally true equations which then can be adapted to specific useful coordinate systems for the problem at hand. This is why it is useful to not call the specific array of numbers the vector.

Some of the geometric properties I am speaking of are translating the origin does not change what the vector represents and a proper rotation[21] of the vector does not change what the vector represents. In mathematics and computer science textbooks you may have a vector defined as an object over a field F and scalars S such that with $u, v, w \in F$ (and $0 \in F$) and $a, b \in S$

$$u + (v + w) = (u + v) + w \tag{1.2.9}$$
$$u + v = v + u \tag{1.2.10}$$
$$v + 0 = v \tag{1.2.11}$$
$$v + (-v) = 0 \quad \text{with } -v \in F \tag{1.2.12}$$
$$a(bv) = (ab)v \tag{1.2.13}$$
$$(1)v = v \quad \text{with } 1 \in S \tag{1.2.14}$$
$$a(u + v) = au + av \tag{1.2.15}$$
$$(a + b)u = au + bu \tag{1.2.16}$$

These types of vectors do not have the same properties as geometric vectors, because they are more general. I will call these "vector arrays" rather than vectors to make this clear. When I use them, I will use the font `a` or `A` which is a typewriter-like font to indicate they only have the above properties.

Geometric vectors have additional properties to vector arrays. For example, when converting coordinate systems for geometric vectors one can find a general transformation law. It is also worth briefly explaining a vector as a magnitude and direction. Let $\mathbf{V}$ be a geometric vector represented in Cartesian coordinates as $V_x = (V_{x,1}, V_{x,2}, \ldots, V_{x,n})$, and let $y_i = (x_1, x_2, \ldots, x_n)$ be a set of coordinates y in terms of the Cartesian coordinates x_j. Then y_i is a position vector. We can define a magnitude or norm for a vector via $|V_x| \equiv \sqrt{V_x \cdot V_x} = \sqrt{\sum_{i=1}^{n} (V_{x,i})^2}$. The direction

[20] Sometimes people call these Euclidean vectors, but I stay with geometric to point to the fact that these vectors have geometric properties

[21] A rotation without any mirror reflection. Mathematicians consider mirror reflections to be an improper rotation. Most people would not call mirror reflection any type of rotation.

can then be determined by the relative contributions of the $V_{x,i}$. We then can say that a vector has a magnitude and a direction in this sense.

The rotation and translation properties of the general vector arrays are not going to satisfy geometric vector properties in general. Those vector arrays, for example, can contain quantities where it does not make sense to move the origin system because the elements cannot mix. For example, the quantity (x, y) with $x, y \in \mathbb{R}$ and scalars in $\mathbb{R}$ would qualify as a vector if x represented money earned by a company and y represented temperature at a specific location on Earth.[22] However, if we had an example $(15, 20)$ it would not make sense to "change coordinates" (for example by rotating and translating our coordinate system) so that we now have $(3, 0)$. The new coordinates would be a mix of the original variables, but there is no connection between them like there are between spatial directions. This new coordinate system would be complicated and nonsensical since there aren't general properties (such as symmetries for the space spanning money and temperature) that there are for space (rotational and translation symmetries).

So now let's deal with vectors that we usually care about as physicists, three dimensional geometric vectors. I will use boldface for vectors such as $\mathbf{V}$ or $\mathbf{v}$. For a unit vector (a vector whose magnitude is 1 in the system we are using), I will append a hat, so $\hat{\mathbf{V}}$ or $\hat{\mathbf{v}}$.

I am going to quickly talk about dot and cross products because I suspect you already know what they are. A dot product between two (three dimensional) vectors (either geometric vectors or vector arrays) is given by

$$\mathbf{V} \cdot \mathbf{W} = |\mathbf{V}||\mathbf{W}| \cos\theta = \sum_{i=1}^{3} V^i \mathbf{e}_i \cdot V^i \mathbf{e}^i \tag{1.2.17}$$

where $\mathbf{e}_i$ is a basis vector for a representation of $\mathbf{V}$. For vector arrays we can enforce $\mathbf{e}_i = \hat{\mathbf{e}}_i$ for orthogonal unit vectors. The dot product's $\cos\theta$ interpretation is the angle between the two vectors. Also, $|\mathbf{V}|$ is the norm or magnitude of $\mathbf{V}$ and is given by $\sqrt{\mathbf{V} \cdot \mathbf{V}}$. For vector arrays we can generalize to N rather than three. The angle interpretation can still hold, though its physical meaning may be ephemeral.

The cross product is peculiar to three dimensions only. We write for $\hat{\mathbf{n}}$ perpendicular to both $\mathbf{V}$ and $\mathbf{W}$ with the angle between $\mathbf{V}$ and $\mathbf{W}$ being θ again

$$\mathbf{V} \times \mathbf{W} = \hat{\mathbf{n}}|\mathbf{V}||\mathbf{W}| \sin\theta \tag{1.2.18}$$

For a Cartesian representation we have

$$\mathbf{V} \cdot \mathbf{W} = \sum_{i,j=1}^{3} V_i \mathbf{x}_i \cdot W_j \mathbf{x}_j = \sum_{i,j=1}^{3} \delta_{ij} V_i W_j = \sum_{i=1}^{3} V_i W_i \tag{1.2.19}$$

In general we have

$$\mathbf{V} \times \mathbf{W} = \sum_{i,j=1}^{3} V^i W^j \mathbf{x}_i \times \mathbf{x}_j \tag{1.2.20}$$

[22] If you are not familiar with $\mathbb{R}$, this means the real numbers so $x \in \mathbb{R}$ means x is an element of the real numbers. In normal-speak, x is a real number. The "blackboard" font is used for other common number systems, with $\mathbb{Z}$ meaning the integers, $\mathbb{C}$ meaning the complex numbers, and $\mathbb{R}^n$ meaning a vector array of real numbers of dimension n.

with the Cartesian unit vectors ($\mathbf{x}_i$) satisfying

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{y}} \times \hat{\mathbf{x}} = \hat{\mathbf{z}} = \hat{\mathbf{x}}_3 \tag{1.2.21}$$
$$\hat{\mathbf{y}} \times \hat{\mathbf{z}} = -\hat{\mathbf{z}} \times \hat{\mathbf{y}} = \hat{\mathbf{x}} = \hat{\mathbf{x}}_1 \tag{1.2.22}$$
$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = -\hat{\mathbf{x}} \times \hat{\mathbf{z}} = \hat{\mathbf{y}} = \hat{\mathbf{x}}_2 \tag{1.2.23}$$

This can be represented with the Levi-Civita tensor for Cartesian vector representations as

$$\mathbf{V} \times \mathbf{W} = \sum_{i,j=1} \epsilon_{ijk} V_i W_j \tag{1.2.24}$$

When we get to curvilinear coordinates we will explore the more general representations possible for geometric vectors.

Now we need to think about spanning a space. Mathematics tells us that for an N dimensional space we need at least N vectors. For us, we then need three linearly independent vectors (that is they actually have components pointing in three directions). To find out if we have independent vectors, take three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and we can find

$$A = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{a} \tag{1.2.25}$$

In three dimensions we are lucky and the cross and dot product provide us an easy way to find the volume (here A) spanned by the three vectors. If $A = 0$, then there are not three independent directions. If it is non-zero, then we can use these three vectors to span all of 3D space (that is, we can get to any coordinate in 3D space with combinations of these three vectors). In N dimensions, we are forced to represent the vectors $\mathbf{a}_i$ in Cartesian form $\mathbf{a}_i = \sum_{j=1}^{N} a_i(j)\mathbf{x}_j$ and find the determinant of the matrix

$$\mathbf{A} \equiv \begin{bmatrix} a_1(1) & a_2(1) & \cdots & a_N(1) \\ a_1(2) & a_2(2) & \cdots & a_N(2) \\ \vdots & \vdots & \ddots & \vdots \\ a_1(N) & a_2(N) & \cdots & a_N(N) \end{bmatrix} \tag{1.2.26}$$

If $A = \det(\mathbf{A})$ is not equal to zero, then the vectors $\mathbf{a}_i$ are linearly independent.

Now, we can normalize each vector so that $A = 1$ by using

$$\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}| \tag{1.2.27}$$
$$\hat{\mathbf{b}} = \mathbf{b}/|\mathbf{b}| \tag{1.2.28}$$
$$\hat{\mathbf{c}} = \mathbf{c}/|\mathbf{c}| \tag{1.2.29}$$

where $|\mathbf{v}|$ is the magnitude of vector $\mathbf{v}$. This is useful because then $|\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = |\hat{\mathbf{c}}| = 1$ and $A = 1$ if all are unit vectors. Generally speaking, unit basis vectors are often written with hats such as $\hat{\mathbf{e}}_i$ or $\hat{\mathbf{e}}^i$ with $i \in [1, 2, \ldots, N]$ for N dimensions. When we are referring to vectors at this point, it is not important whether it is a subscript of superscript, but when we learn about contravariant and covariant components of vectors, care will need to be taken, as we should stay within usual convention to minimize confusion.

Now when we are given a basis vector set that is not singular (that is $A \neq 0$), you can form a reciprocal basis vector set. For Cartesian vectors, this is hard to understand because the basis

vector and reciprocal basis set are the same vectors. Given the (three dimensional) basis set above of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, the reciprocal set (subscript r) is given by

$$\mathbf{a}_r = \frac{\mathbf{b}\times\mathbf{c}}{A} \tag{1.2.30}$$

$$\mathbf{b}_r = \frac{\mathbf{c}\times\mathbf{a}}{A} \tag{1.2.31}$$

$$\mathbf{c}_r = \frac{\mathbf{a}\times\mathbf{b}}{A} \tag{1.2.32}$$

One can then check that

$$\begin{aligned} \mathbf{a}_r\cdot\mathbf{a} = \mathbf{b}_r\cdot\mathbf{b} = \mathbf{c}_r\cdot\mathbf{c} = 1 \\ \mathbf{a}_r\cdot\mathbf{b} = \mathbf{a}_r\cdot\mathbf{c} = 0 \\ \mathbf{b}_r\cdot\mathbf{a} = \mathbf{b}_r\cdot\mathbf{c} = 0 \\ \mathbf{c}_r\cdot\mathbf{a} = \mathbf{c}_r\cdot\mathbf{a} = 0 \end{aligned} \tag{1.2.33}$$

Now the usefulness of a set of reciprocal vectors is that they can be used to write any vector $\mathbf{V}$ very simply.

$$\mathbf{V} = (\mathbf{V}\cdot\mathbf{a})\mathbf{a}_r + (\mathbf{V}\cdot\mathbf{b})\mathbf{b}_r + (\mathbf{V}\cdot\mathbf{c})\mathbf{c}_r \tag{1.2.34}$$

$$\mathbf{V} = (\mathbf{V}\cdot\mathbf{a}_r)\mathbf{a} + (\mathbf{V}\cdot\mathbf{b}_r)\mathbf{b} + (\mathbf{V}\cdot\mathbf{c}_r)\mathbf{c} \tag{1.2.35}$$

For example, the Cartesian basis vectors are usually denoted

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{x}} = \hat{\mathbf{i}} \tag{1.2.36}$$

$$\hat{\mathbf{e}}_2 = \hat{\mathbf{y}} = \hat{\mathbf{j}} \tag{1.2.37}$$

$$\hat{\mathbf{e}}_3 = \hat{\mathbf{z}} = \hat{\mathbf{k}} \tag{1.2.38}$$

Note that $\hat{\mathbf{x}}_r = \hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$, and so on, so that indeed, the reciprocal set is just the normal Cartesian vectors we are used to. Thus, we can write any vector $\mathbf{V}$ as

$$\mathbf{V} = V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}} + V_z\hat{\mathbf{z}} = V_1\hat{\mathbf{e}}_1 + V_2\hat{\mathbf{e}}_2 + V_3\hat{\mathbf{e}}_3 \tag{1.2.39}$$

$$V_i = \mathbf{V}\cdot\hat{\mathbf{e}}_i \tag{1.2.40}$$

Remember that the above is general. If we choose a basis set that is the tangent basis, then we have the interesting relationships with the tangent-reciprocal basis that are more fully developed below.

Even with our simple slanted example, we saw that the tangent-reciprocal and tangent basis do not have to coincide. We will now consider three dimensions and more complicated coordinates.

1.2.3 Curvilinear Coordinate Systems

> *Unfortunately, the change of coordinates alters the coefficients of the differential equation by inserting "metric factors" that depend on the mapping. The transformation of first derivatives is messy but not too bad. The transformation of second derivatives is very messy. The transformation of third and fourth derivatives — the programmer shudders and turns away in horror.*
>
> — J. P. Boyd[5, p. 69]

Let's now look at the more interesting and difficult case of a non-Cartesian coordinate system. We will base everything off of Euclidean space, and so things will be implicitly referenced to the Cartesian coordinates $(x, y, z) = (x^1, x^2, x^3)$.[23] Suppose we wish to use some arbitrary set of coordinates (ξ^1, ξ^2, ξ^3). We want these to be expressible in terms of Cartesian coordinates

$$\xi^1 = \xi^1(x^1, x^2, x^3) \tag{1.2.41}$$
$$\xi^2 = \xi^2(x^1, x^2, x^3) \tag{1.2.42}$$
$$\xi^3 = \xi^3(x^1, x^2, x^3) \tag{1.2.43}$$

and if these uniquely point out coordinates, this needs to be a one-to-one transformation, so that we can find

$$x^1 = x^1(\xi^1, \xi^2, \xi^3) \tag{1.2.44}$$
$$x^2 = x^2(\xi^1, \xi^2, \xi^3) \tag{1.2.45}$$
$$x^3 = x^3(\xi^1, \xi^2, \xi^3) \tag{1.2.46}$$

One can test if the set of transformations ξ^i are one-to-one via calculating the Jacobian matrix

$$\boldsymbol{\mathcal{J}} \equiv \frac{\partial(x^1, x^2, x^3)}{\partial(\xi^1, \xi^2, \xi^3)} = \begin{bmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^1}{\partial \xi^2} & \frac{\partial x^1}{\partial \xi^3} \\ \frac{\partial x^2}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^2} & \frac{\partial x^2}{\partial \xi^3} \\ \frac{\partial x^3}{\partial \xi^1} & \frac{\partial x^3}{\partial \xi^2} & \frac{\partial x^3}{\partial \xi^3} \end{bmatrix} \tag{1.2.47}$$

and seeing if the determinant is everywhere non-zero.[24] The Jacobian determinant is then

$$\mathcal{J} = |\boldsymbol{\mathcal{J}}| \equiv \left| \frac{\partial(x^1, x^2, x^3)}{\partial(\xi^1, \xi^2, \xi^3)} \right| = \begin{vmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^1}{\partial \xi^2} & \frac{\partial x^1}{\partial \xi^3} \\ \frac{\partial x^2}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^2} & \frac{\partial x^2}{\partial \xi^3} \\ \frac{\partial x^3}{\partial \xi^1} & \frac{\partial x^3}{\partial \xi^2} & \frac{\partial x^3}{\partial \xi^3} \end{vmatrix} \tag{1.2.48}$$

with the conventional definition of matrix determinants. In addition, we will redefine (ξ^1, ξ^2, ξ^3) so that they ensure $\mathcal{J} > 0$. This can be done for $\mathcal{J} < 0$ by switching two labels, such as $2 \leftrightarrow 3$, so that the new coordinate system is $(\zeta^1, \zeta^2, \zeta^3) = (\xi^1, \xi^3, \xi^2)$. Similarly, we can define the inverse Jacobian

$$\mathbf{J} = \boldsymbol{\mathcal{J}}^{-1} \equiv \frac{\partial(\xi^1, \xi^2, \xi^3)}{\partial(x^1, x^2, x^3)} = \begin{bmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} & \frac{\partial \xi^1}{\partial x^3} \\ \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^2}{\partial x^2} & \frac{\partial \xi^2}{\partial x^3} \\ \frac{\partial \xi^3}{\partial x^1} & \frac{\partial \xi^3}{\partial x^2} & \frac{\partial \xi^3}{\partial x^3} \end{bmatrix} \tag{1.2.49}$$

The inverse Jacobian determinant is then

$$J = |\mathbf{J}| = 1/\mathcal{J} \equiv \left| \frac{\partial(\xi^1, \xi^2, \xi^3)}{\partial(x^1, x^2, x^3)} \right| = \begin{vmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} & \frac{\partial \xi^1}{\partial x^3} \\ \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^2}{\partial x^2} & \frac{\partial \xi^2}{\partial x^3} \\ \frac{\partial \xi^3}{\partial x^1} & \frac{\partial \xi^3}{\partial x^2} & \frac{\partial \xi^3}{\partial x^3} \end{vmatrix} \tag{1.2.50}$$

[23] It's probably time to justify the superscript notation. It could be easily confused with powers, but in practice it is not. The reason to use superscripts is because there are few other places to put indices. Thus we must be careful if we want the square of a coordinate and it is safer to put the coordinate in parentheses when taking powers, such as $(x^2)^2$ for the square of coordinate x^2.

[24] If there are points that are singularities, you can use the system as long as you avoid those points. Similarly, if you only care about a certain region of space, you only need the Jacobian determinant non-zero in the region you care about.

Note that there is always confusion in any comparison of texts about the direction of the Jacobian and the inverse Jacobian. As long as one is consistent, it does not actually matter, but I prefer the Jacobian determinant to be in the numerator when calculating integrals, so that

$$\iiint \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3\,F(x^1,x^2,x^3) = \iiint \mathrm{d}\xi^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi^3\,\mathcal{J}F(\xi^1,\xi^2,\xi^3) \tag{1.2.51}$$

Some authors prefer to call the Jacobian determinant what I call the inverse Jacobian determinant. Even worse, there is no convention on whether the "Jacobian" refers to the matrix itself, or the determinant of the matrix. In this text, Jacobian is reserved for the matrix and Jacobian determinant is used for the determinant of the matrix. But readers be warned, the Jacobian determinant is used far more often in plasma literature so Jacobian usually means Jacobian determinant there.

It can be shown fairly easily by the definition of a determinant and our later definitions of basis vectors that

$$\mathcal{J} = \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 \tag{1.2.52}$$

$$J = \mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3 \tag{1.2.53}$$

Now, we can use the ξ^i and plot them as curves on the Cartesian coordinate plane, just as we did in 2D. We can then define basis vectors such that the basis vector $\mathbf{e}_i$ points along the coordinate curve of ξ^i. This means that $\mathbf{e}_i = \mathbf{e}_i(x^1,x^2,x^3)$ and so is not necessarily constant in space. They are also not necessarily unit vectors, hence the lack of a hat. Such basis vectors are usually called coordinate basis vectors or tangent basis vectors. Now if we want these basis vectors to point along the coordinates, we want them to be tangent to the curves defined by the ξ^i. Luckily, we know from calculus how to find the tangent curve for a given function, the derivative. Thus, if we have a given position vector $\mathbf{x}$ we have

$$\mathbf{e}_i = \left(\frac{\partial \mathbf{x}}{\partial \xi^i}\right)_{\xi^k|^{k\neq i}} = \frac{\partial \mathbf{x}}{\partial \xi^i} \tag{1.2.54}$$

for each $i \in [1,2,3]$. Because we chose our transformation to have non-zero Jacobian, we can now show that $\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 \neq 0$ because

$$\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = \frac{\partial \mathbf{x}}{\partial \xi^1} \cdot \frac{\partial \mathbf{x}}{\partial \xi^2} \times \frac{\partial \mathbf{x}}{\partial \xi^3} = \left|\frac{\partial(x^1,x^2,x^3)}{\partial(\xi^1,\xi^2,\xi^3)}\right| = \mathcal{J}(\mathbf{x}) \neq 0 \tag{1.2.55}$$

where $\mathbf{x}$'s Cartesian components are x^1,x^2,x^3 [and we know for all x^i the Jacobian determinant should be non-zero (or at least over the regions we care about, and we can note the equivalence to the Jacobian determinant calculation above (1.2.48))].

These tangent basis vectors do not need to have the same units, can be dimensionless or not, and do not need to be orthogonal. We can still define unit vectors via (there is no summation in the following equation)

$$\hat{\mathbf{e}}_i = \frac{\mathbf{e}_i}{|\mathbf{e}_i|} = \frac{\partial \mathbf{x}}{\partial \xi^i} \Big/ \left|\frac{\partial \mathbf{x}}{\partial \xi^i}\right| = \frac{\mathbf{e}_i}{h_i} \tag{1.2.56}$$

with scale factors h_i defined by

$$h_i = \left|\frac{\partial \mathbf{x}}{\partial \xi^i}\right| \tag{1.2.57}$$

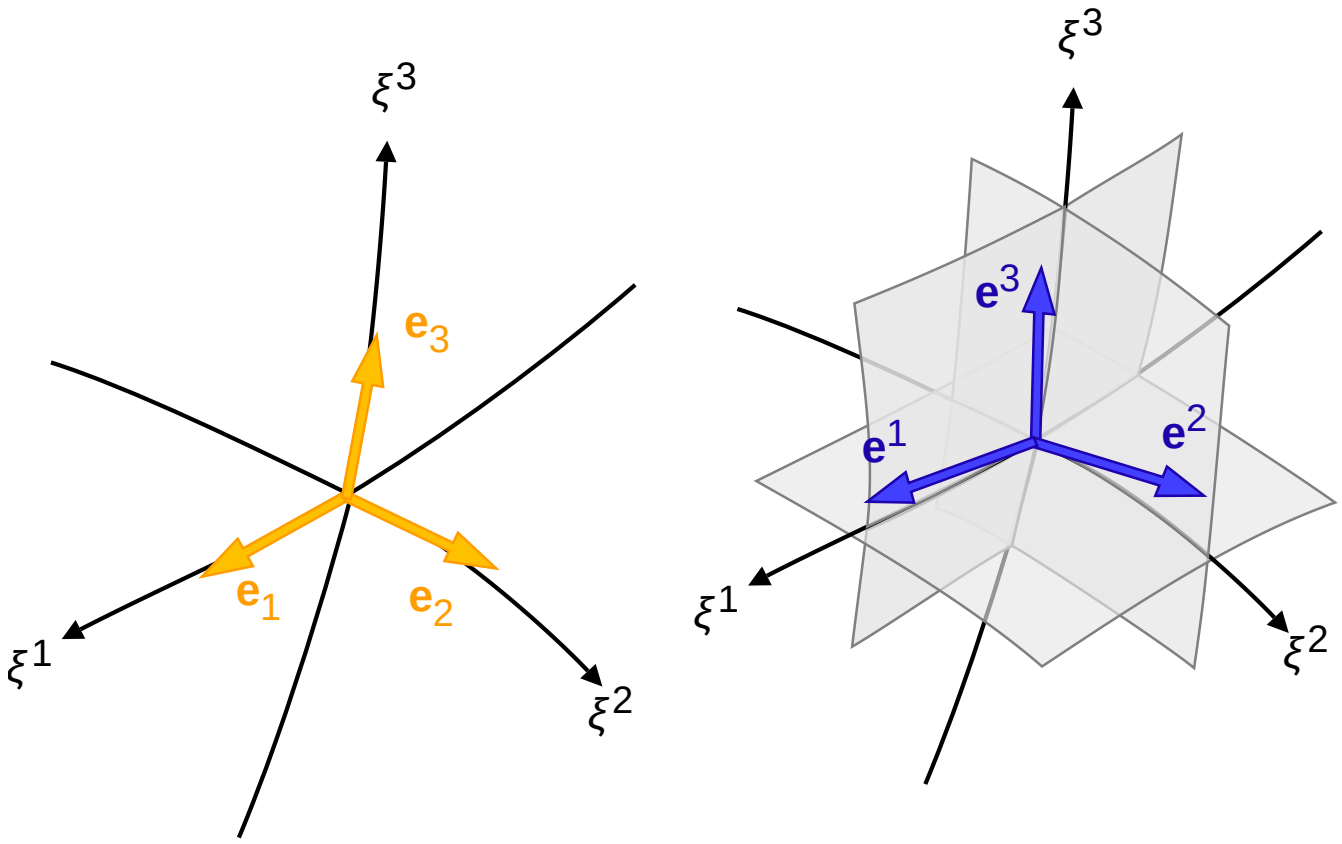

Figure 1.3: This shows the two ways of creating a vector basis. The left plot shows the tangent vector basis (along coordinate axes) and the right plot shows the reciprocal basis (perpendicular to coordinate surfaces). The image is a slightly modified version of an image created by Maschen on 2012-09-02, under a CC0 1.0 license [found on Wikipedia].

We have chosen to use subscripts for these tangent basis vectors. Later this will be justified by convention.

We can now use the alternative method of using the ξ^i to define coordinates. We can view each ξ^i as defining a set of contours. For example, we can form a surface for ξ^1 by keeping it constant and allowing ξ^2 and ξ^3 to vary. We can then use these constant surface values for the ξ^i to label each surface, and we can then use the gradient (that is follow the steepest descent or ascent) to define a direction on each surface. These new directions will be perpendicular to the other coordinate lines, and hence perpendicular to the corresponding other tangent basis vectors (ensuring that $\mathbf{e}_i \cdot \mathbf{e}^j = 0$ for $i \neq j$). See Figure 1.3 for an example of what is happening in a simplified picture.

A particularly simple way to find the gradient directions is to consider $\mathrm{d}\mathbf{x}$. If we want to relate this differential to changes in our coordinates ξ^i we can use that the gradient along the vector to find what small changes would be. This is typically written

$$\mathrm{d}\xi^i = \nabla\xi^i \cdot \mathrm{d}\mathbf{x} \tag{1.2.58}$$

Suppose we would like to do the opposite, though. We can then use the chain rule on $\mathbf{x}(\xi^1, \xi^2, \xi^3)$

$$\mathrm{d}\mathbf{x} = \sum_{i=1}^{3} \frac{\partial \mathbf{x}}{\partial \xi^i} \mathrm{d}\xi^i \tag{1.2.59}$$

We can use that $\frac{\partial \mathbf{x}}{\partial \xi^i} = \mathbf{e}_i$. Then

$$\mathrm{d}\xi^i = \nabla\xi^i \cdot \mathrm{d}\mathbf{x} = \nabla\xi^i \cdot \sum_{j=1}^{3} \mathbf{e}_j \, \mathrm{d}\xi^j \tag{1.2.60}$$

where I have relabeled the i in the summation to j so that there is no confusion. It then also makes it clear that

$$\nabla\xi^i \cdot \mathbf{e}_j = \delta^i_j \tag{1.2.61}$$

as otherwise $\mathrm{d}\xi^i \neq \mathrm{d}\xi^i$ which is nonsensical. We then can see that $\nabla\xi^i$ are the reciprocal vectors to the $\mathbf{e}_j$ and so we identify

$$\mathbf{e}^i = \nabla\xi^i \tag{1.2.62}$$

Note that this derivation is basically just using that the gradient points normal to contour surfaces.

Alternatively, we can recognize with our notation and generalized chain rules that

$$\nabla\xi^i \cdot \mathbf{e}_j = \frac{\partial \xi^i}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial \xi^j} = \frac{\partial \xi^i}{\partial \xi^j} = \delta^i_j \tag{1.2.63}$$

So long as you are comfortable with the generalized chain rule, this is a much clearer method of showing that the tangent-reciprocal basis is indeed the reciprocal basis to the tangent basis.

This means that we can use the relationships previously stated to find the reciprocal vectors from the basis vectors and vice versa. If we consider (i', j', k') to be an even permutation of (1,2,3).[25]

$$\mathbf{e}^{i'} = \nabla\xi^{i'} = \frac{\mathbf{e}_{j'} \times \mathbf{e}_{k'}}{\mathbf{e}_{i'} \cdot \mathbf{e}_{j'} \times \mathbf{e}_{k'}} = \frac{\mathbf{e}_{j'} \times \mathbf{e}_{k'}}{\mathcal{J}} = \frac{\frac{\partial \mathbf{x}}{\partial \xi^{j'}} \times \frac{\partial \mathbf{x}}{\partial \xi^{k'}}}{\frac{\partial \mathbf{x}}{\partial \xi^{i'}} \cdot \left(\frac{\partial \mathbf{x}}{\partial \xi^{j'}} \times \frac{\partial \mathbf{x}}{\partial \xi^{k'}} \right)} \tag{1.2.64}$$

$$\mathbf{e}_{i'} = \frac{\partial \mathbf{x}}{\partial \xi^{i'}} = \frac{\mathbf{e}^{j'} \times \mathbf{e}^{k'}}{\mathbf{e}^{i'} \cdot \mathbf{e}^{j'} \times \mathbf{e}^{k'}} = \frac{\mathbf{e}^{j'} \times \mathbf{e}^{k'}}{J} = \frac{\nabla\xi^{j'} \times \nabla\xi^{k'}}{\nabla\xi^{i'} \cdot \nabla\xi^{j'} \times \nabla\xi^{k'}} \tag{1.2.65}$$

Finally, it is easy to confuse the tangent basis and the reciprocal basis vectors. One helpful mnemonic is that for $\mathbf{e}^i$ the i is on top, so $\frac{\partial \xi^i}{\partial \mathbf{x}}$ has the ξ^i on top, while for $\mathbf{e}_i$ the i is on bottom so $\frac{\partial \mathbf{x}}{\partial \xi^i}$ has ξ^i on the bottom.

1.2.4 Contravariant and Covariant "Vectors"

Despite the section name, I will argue that we should not label vectors as contravariant or covariant when talking about coordinate transformations unless we are very clear about what is meant.

Generally a "contravariant vector" is defined by its transformation properties. When I say contravariant vector, I mean a vector viewed as an array of numbers. This is most easily illuminated by introducing a change in scale for a specific vector. Suppose instead of measuring length in

[25] This means that there exists a way we can read the numbers left to right as 1,2,3 so long as we view it cyclically. That is, the number on the right can be considered to the left of the leftmost number. Alternatively, write 1, 2, 3 in a circle increasing counterclockwise. Then (i', j', k') are an even permutation if you can read i', j', and k' counterclockwise along the 1, 2, 3 circle.

meters, we now measure in millimeters. If we look only at the numbers, then for a vector pointing to a particular position, it will suddenly increase by 1000. Thus, when we multiply our scale by 0.001, the contravariant vector is multiplied by 1000, hence the vectors vary against ("contra") the scale change.

On the other hand, we could consider a vector that represents the gradient of the temperature. Then this "covariant vector" (again, meaning a vector viewed as an array of numbers) will increase by 1000 when we do the scale change and so varies with ("co") the scale change.

Mathematically, if we consider a vector to be its components, then the above makes perfect sense. We can then write a "contravariant vector" [written as V^i with $i \in (1,2,3)$] as one that varies from initial coordinate ζ^i to ζ'^i via

$$V'^i = \sum_{j=1}^{3} \frac{\partial \zeta'^i}{\partial \zeta^j} V^j \tag{1.2.66}$$

where as a "covariant vector" ∇f [written $V_i = \partial f/\partial \zeta^i$ for $i \in (1,2,3)$]

$$V'_i = \frac{\partial f}{\partial \zeta'^i} = \sum_{j=1}^{3} \frac{\partial \zeta^j}{\partial \zeta'^i} \frac{\partial f}{\partial \zeta^j} = \sum_{j=1}^{3} \frac{\partial \zeta^j}{\partial \zeta'^i} V_j \tag{1.2.67}$$

Remember that if you consider a vector to be an array of numbers, that is, the vector components themselves, then vectors may change. But we want the vectors themselves to not depend any specific coordinates. We want them to be a general representation. In the next section we will explain why we should only talk of vector components being contravariant or covariant, rather than of vectors. As I have emphasized, in certain mathematical contexts, it makes sense to talk of vectors as essentially arrays, but for us it will introduce confusion. If you want to get into the mathematics of dual spaces, it is a rich environment that can help prove useful relations, but it is not what we are concerned with here. In a real physical sense, we are only dealing with one Euclidean space, so the dual space formalism is extraneous. In the end, we want our vectors to have geometric properties and so we don't want the "vectors" to change when changing coordinate systems (that is, changing representations).

1.2.5 Contravariant and Covariant Components of Vectors

We can now consider covariant and contravariant components of vectors, which is my preferred interpretation and what will be adopted from now on in this text. There is some confusion about these terms, but on a simplistic level, you can remember that components with superscript indices are contravariant and that components with subscript indices are covariant.

It is important to keep in mind the word "components" above. In this formulation or interpretation, it does not make sense to talk of covariant and contravariant "vectors" as we did in the previous section because the vectors are geometric objects that do not change; just their representations are changed when changing coordinate systems.[26] We can still care about properties of our specific representation of geometric vectors for the specific coordinate systems. Thus, we care about the

[26] An example in mathematics where it may make sense to call vectors contravariant or covariant is when one can find a dual space to the vector space one is interested in. These cases consider arrays of numbers to be vectors, however.

components of the vector and how they transform. The vector, then, is the primary geometric object, and so cannot be called contravariant or covariant.

So what do contravariant and covariant mean? Consider a general vector $\mathbf{V}$. Then we could write

$$\mathbf{V} = \sum_{i=1}^{3} (\mathbf{V} \cdot \mathbf{e}_i)\mathbf{e}^i = \sum_{i=1}^{3} V_i \mathbf{e}^i = \sum_{i=1}^{3} V_i \, \nabla \xi^i \tag{1.2.68}$$

$$\mathbf{V} = \sum_{i=1}^{3} (\mathbf{V} \cdot \mathbf{e}^i)\mathbf{e}_i = \sum_{i=1}^{3} V^i \mathbf{e}_i = \sum_{i=1}^{3} V^i \frac{\partial \mathbf{x}}{\partial \xi^i} \tag{1.2.69}$$

$$V_i \equiv \mathbf{V} \cdot \mathbf{e}_i \tag{1.2.70}$$

$$V^i \equiv \mathbf{V} \cdot \mathbf{e}^i \tag{1.2.71}$$

Then (1.2.68) is a covariant representation and (1.2.69) is a contravariant representation. The covariant representation uses the tangent basis vectors where as the contravariant representation uses the tangent-reciprocal basis vectors. It is important to note that the components of the vector V_i or V^i need not have any specific units, and do not have to have the same units as the vector itself. This is because the basis vectors can have units. As we will see, we will use the previous definitions (1.2.66) and (1.2.67), for contravariant and covariant components, respectively.

It is important that we recognize the basis vectors should not be labeled as covariant or contravariant because as vectors they can be represented in a covariant or contravariant way. That is, given $\mathbf{e}_j$ which is one of the tangent-reciprocal basis set vectors (being the reciprocal basis to the tangent basis set), then it's covariant representation will be

$$\mathbf{e}_j = \sum_{i=1}^{3} (\mathbf{e}_j \cdot \mathbf{e}_i)\mathbf{e}^i = \sum_{i=1}^{3} (\mathbf{e}_j)_i \mathbf{e}^i \equiv \sum_{i=1}^{3} g_{ji}\mathbf{e}^i \tag{1.2.72}$$

using the notation of $(\mathbf{e}_j)_i = g_{ij}$. And, of course, the contravariant representation will be

$$\mathbf{e}_j = \sum_{i=1}^{3} (\mathbf{e}_j \cdot \mathbf{e}^i)\mathbf{e}_i = \sum_{i=1}^{3} (\mathbf{e}_j)^i \mathbf{e}_i = \mathbf{e}_j \tag{1.2.73}$$

which is quite simple (we must have $(\mathbf{e}_j)^i = \delta^i_j$ or else we do not have tangent and reciprocal bases). Sometimes people will call the $\mathbf{e}_i$ the covariant basis vectors, because they pair with the contravariant vector components and have a subscript. This should be avoided in our formulation because the basis vectors are vectors[27] and hence could be represented with a contravariant or a covariant representation. In fact, one could just as justifiably argue that they should be called the contravariant basis vectors since the contravariant representation of a tangent basis vector $\mathbf{e}_1 = 1\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3$ is much simpler than the covariant representation $\mathbf{e}_1 = g_{11}\mathbf{e}^1 + g_{12}\mathbf{e}^2 + g_{13}\mathbf{e}^3$! In fact, sometimes people claim the $\mathrm{d}\mathbf{x}$ is naturally contravariant because its contravariant representation is very simple. Therefore it is a contravariant vector. Following this logic, then what are usually called "covariant basis vectors" should actually be the contravariant basis vectors. Hence, we ignore the issue altogether and call them tangent basis vectors (though you should be aware that tangent basis vectors are often called the covariant vectors or covariant basis vectors). Similarly, we refer only to the tangent-reciprocal basis vectors (again, you should be aware other texts refer to reciprocal basis vectors as contravariant vectors or contravariant basis vectors).

[27] It should be obvious, but sometimes tautologies are useful.

Now, let's explore the $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ and $g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j$ coefficients. We remember that for any generic vector $\mathbf{V}$ that

$$\mathbf{V} = \sum_{i=1}^{3} \mathbf{V} \cdot \mathbf{e}_i \mathbf{e}^i = \sum_{i=1}^{3} \mathbf{V} \cdot \mathbf{e}^i \mathbf{e}_i \tag{1.2.74}$$

$$V_i = \mathbf{V} \cdot \mathbf{e}_i \tag{1.2.75}$$

$$V^i = \mathbf{V} \cdot \mathbf{e}^i \tag{1.2.76}$$

Thus

$$V_i = \mathbf{V} \cdot \mathbf{e}_i = \sum_{j=1}^{3} V^j \mathbf{e}_j \cdot \mathbf{e}_i = \sum_{j=1}^{3} g_{ji} V^j = \sum_{j=1}^{3} g_{ij} V^j \tag{1.2.77}$$

$$V^i = \mathbf{V} \cdot \mathbf{e}^i = \sum_{j=1}^{3} V_j \mathbf{e}^j \cdot \mathbf{e}^i = \sum_{j=1}^{3} g^{ji} V_j = \sum_{j=1}^{3} g^{ij} V_j \tag{1.2.78}$$

so that we have a simple rule of changing contravariant components into covariant components and vice versa. We can also use the expression for $\mathbf{V}$ with the basis vectors themselves to write

$$\mathbf{e}_i = \sum_{j=1}^{3} \mathbf{e}_i \cdot \mathbf{e}_j \mathbf{e}^j = \sum_{j=1}^{3} g_{ij} \mathbf{e}^j \tag{1.2.79}$$

$$\mathbf{e}^i = \sum_{j=1}^{3} \mathbf{e}^i \cdot \mathbf{e}^j \mathbf{e}_j = \sum_{j=1}^{3} g^{ij} \mathbf{e}_j \tag{1.2.80}$$

Thus, if we take

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j = \sum_{k=1}^{3} g_{ik} \mathbf{e}^k \cdot \mathbf{e}^j = \sum_{k=1}^{3} g_{ik} g^{kj} \tag{1.2.81}$$

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i = \sum_{k=1}^{3} g^{ik} \mathbf{e}_k \cdot \mathbf{e}_j = \sum_{k=1}^{3} g^{ik} g_{kj} \tag{1.2.82}$$

Which gives use $\sum_k g_{ik} g^{kj} = \delta_i^j \equiv g_j^i$ and $\sum_k g^{ik} g_{kj} = \delta_j^i = \delta_i^j \equiv g_j^i = g_i^j$. If we view $\mathbf{g}$ as the matrix form of g_{ik} and $\mathbf{G}$ as the matrix form of g^{kj} then these statements mean

$$\mathbf{gG} = \mathbf{Gg} = \mathbb{1} \tag{1.2.83}$$

$$\mathbf{g} = [\mathbf{G}]^{-1} \tag{1.2.84}$$

$$g \equiv \det(\mathbf{g}) \Rightarrow g^{-1} = \det(\mathbf{G}) \tag{1.2.85}$$

so that we can find these g_{ij} and g^{ij} metric components by forming a matrix from one of those and inverting the matrix. The determinants are inverses by the properties of their respective matrices. As a reminder, we can write

$$g_{ij} = \sum_{i,j=1}^{3} \mathbf{e}_i \cdot \mathbf{e}_j = \sum_{i,j=1}^{3} \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^j} \tag{1.2.86}$$

$$g^{ij} = \sum_{i,j=1}^{3} \mathbf{e}^i \cdot \mathbf{e}^j = \sum_{i,j=1}^{3} \frac{\partial \xi^i}{\partial \mathbf{x}} \cdot \frac{\partial \xi^j}{\partial \mathbf{x}} = \sum_{i,j=1}^{3} \nabla \xi^i \cdot \nabla \xi^j \tag{1.2.87}$$

If we write the tensor $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}$ in matrix form, then we can think of ourselves as taking the determinant of the product of two of the same matrices

$$\begin{aligned}\det(\mathbf{g}) &= \det\left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} \cdot \frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right) = \det\left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right) \det\left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right) \\ &= \left(\frac{\partial \mathbf{x}}{\partial \xi^1} \cdot \frac{\partial \mathbf{x}}{\partial \xi^2} \times \frac{\partial \mathbf{x}}{\partial \xi^3}\right)^2 = (\mathcal{J})^2\end{aligned} \tag{1.2.88}$$

which means $g = \sqrt{\mathcal{J}}$. Thus we see the connection between the Jacobian and the metric coefficients. The analogous result holds for the inverse

$$\begin{aligned}\det(\mathbf{G}) &= \det\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}} \cdot \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}\right) = \det\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}\right) \det\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}\right) \\ &= \left(\frac{\partial \xi^1}{\partial \mathbf{x}} \cdot \frac{\partial \xi^2}{\partial \mathbf{x}} \times \frac{\partial \xi^3}{\partial \mathbf{x}}\right)^2 = (J)^2 = (\mathcal{J})^{-2}\end{aligned} \tag{1.2.89}$$

This makes sense of seeing vectors as geometric objects. Why do contravariant components vary that way? Because it keeps the vector object from changing. For if we changed our tangent basis from ξ^i to ξ'^i, we require

$$\mathbf{V} = \sum_{i=1}^{3} V^i \mathbf{e}_i = \sum_{i=1}^{3} V'^i \mathbf{e}'_i \tag{1.2.90}$$

Remember that $\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial \xi^i}$ and $\mathbf{e}'_i = \frac{\partial \mathbf{x}}{\partial \xi'^i}$ so

$$\sum_{i=1}^{3} V^i \mathbf{e}_i = \sum_{i=1}^{3} V^i \frac{\partial \mathbf{x}}{\partial \xi^i} = \sum_{i,j=1}^{3} V^i \frac{\partial \xi'^j}{\partial \xi^i} \frac{\partial \mathbf{x}}{\partial \xi'^j} = \sum_{i,j=1}^{3} V^i \frac{\partial \xi'^j}{\partial \xi^i} \mathbf{e}'_j = \sum_{j=1}^{3} V'^j \mathbf{e}'_j \tag{1.2.91}$$

We know that this must be true for each component j, so

$$\sum_{i=1}^{3} V^i \frac{\partial \xi'^j}{\partial \xi^i} = V'^j \tag{1.2.92}$$

just as we required in (1.2.66). This is the transparent justification from our geometric vector point of view.

Similarly, consider the reciprocal basis representation

$$\mathbf{V} = \sum_{i=1}^{3} V_i \mathbf{e}^i = \sum_{i=1}^{3} V'_i \mathbf{e}'^i \tag{1.2.93}$$

$$\sum_{i=1}^{3} V_i \mathbf{e}_i = \sum_{i=1}^{3} V_i \frac{\partial \xi^i}{\partial \mathbf{x}} = \sum_{i,j=1}^{3} V_i \frac{\partial \xi^i}{\partial \xi'^j} \frac{\partial \xi'^j}{\partial \mathbf{x}} = \sum_{i,j=1}^{3} V_i \frac{\partial \xi^i}{\partial \xi'^j} \mathbf{e}'^j = \sum_{j=1}^{3} V'_j \mathbf{e}'^j \tag{1.2.94}$$

and so this must be true for each component j giving

$$\sum_{i=1}^{3} V_i \frac{\partial \xi^i}{\partial \xi'^j} = V'_j \tag{1.2.95}$$

the same as our proposed law in (1.2.67), except now the justification is transparent.

While it may feel as if I am preaching when I say vectors are not covariant or contravariant, it is important to realize this to avoid confusion when comparing what I say to other texts. Vectors (in this text) have covariant and contravariant representations where the components are covariant or contravariant. When using the tangent basis vectors, the components are contravariant, and when using the reciprocal basis vectors, the components are covariant. The contravariant components are superscripted and the covariant are subscripted by convention (this is universally followed, and the mnemonic "contra" and "super" both contain an "r" whereas neither "co" nor "sub" do is often used to remember how these go together).

1.2.6 Multiple Dimensions

> *There may be three dimensions in this room and five next door. As a professional mathematician, I have no idea; I can only ask some competent physicist to instruct me in the facts.*
>
> — G. H. Hardy

> *Mathematically, there is no obvious reason for stopping at 3 rather than any other number. In this universe, North is perpendicular to East, and up is perpendicular to both, but we cannot find a fourth direction perpendicular to all three. There is however no reason why a universe should not exist with four or five or six dimensions. We have got used to three, but that is not a reason.*
>
> — W. W. Sawyer[29, p. 68]

You may be wondering about four-dimensional or higher cases. This is of special importance for special relativity or general relativity. All of the previous machinery works with only minor changes, other than the reciprocal basis definition. Given a basis $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ the reciprocal basis $\{\mathbf{b}^1, \cdots, \mathbf{b}^n\}$ satisfies

$$\mathbf{b}_i \cdot \mathbf{b}^j = \delta_i^j \tag{1.2.96}$$

In our case, by employing two different basis vector sets, we can simply use the normal dot product. The signs will be taken care of in g_{ij} and g^{ij} when considering purely contravariant or purely covariant component dot product forms. Were we to generalize to complex values we would write $(\mathbf{b}_i)^* \cdot \mathbf{b}^j = \mathbf{b}_i \cdot (\mathbf{b}^j)^* = \delta_i^j$ and $(\mathbf{b}^i)^* \cdot \mathbf{b}_j = \mathbf{b}^i \cdot (\mathbf{b}_j)^* = \delta_j^i$. Note that again this says nothing about $\mathbf{b}_i \cdot \mathbf{b}_j$ which we say is g_{ij}. We use $\mathbf{b}_i(j)$ to mean the jth component of $\mathbf{b}_i$ in the Cartesian vector representation of $\mathbf{b}_i$ (so $\mathbf{b}_i(j) = \mathbf{b}_i \cdot \hat{\mathbf{x}}_j$). Then we can form two $n \times n$ matrices

$$\mathbf{B}_b = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix} \equiv \begin{bmatrix} \mathbf{b}_1(1) & \cdots & \mathbf{b}_n(1) \\ \vdots & \vdots & \vdots \\ \mathbf{b}_1(n) & \cdots & \mathbf{b}_n(n) \end{bmatrix} \tag{1.2.97}$$

$$\mathbf{B}^b = \begin{bmatrix} \mathbf{b}^1 & \cdots & \mathbf{b}^n \end{bmatrix} \equiv \begin{bmatrix} \mathbf{b}^1(1) & \cdots & \mathbf{b}^n(1) \\ \vdots & \vdots & \vdots \\ \mathbf{b}^1(n) & \cdots & \mathbf{b}^n(n) \end{bmatrix} \tag{1.2.98}$$

Then by our definitions of reciprocal basis we must have

$$(\mathbf{B}_b)^\intercal \mathbf{B}^b = (\mathbf{B}^b)^\intercal \mathbf{B}_b = \mathbb{1} \tag{1.2.99}$$

$$\mathbf{B}_b = [(\mathbf{B}^b)^\intercal]^{-1} \tag{1.2.100}$$

$$\mathbf{B}^b = [(\mathbf{B}_b)^\intercal]^{-1} \tag{1.2.101}$$

Thus, given any basis vectors we can figure out the form of the reciprocal basis by inverting the transpose of the basis vector matrix. Usually inverting a 4×4 matrix is painful, but it is not difficult and one can then define the reciprocal basis.

Given a basis, we can form the reciprocal basis one at a time, as well. To find reciprocal basis $\mathbf{b}^j$ we use

$$(\mathbf{B}_b)^\intercal \mathbf{b}^j = \mathbf{1}_j \tag{1.2.102}$$

where $\mathbf{1}_j$ represents a column vector with 1 in the jth row and zeros elsewhere. We can then find the components of $\mathbf{b}^j$ through a solve or through Cramer's rule which says

$$\mathbf{b}^j(i) = \frac{\det(\mathbf{B}_b^\intercal)_i}{\det \mathbf{B}_b^\intercal} \tag{1.2.103}$$

where $(\mathbf{B}_b^\intercal)_i$ means replace the ith column of $(\mathbf{B}_b^\intercal)$ with $\mathbf{b}^j$.

As an example, consider special relativity in Cartesian coordinates. We have four dimensions, with ct[28] as a new dimension of spacetime. There are two ways to generalize. One can let i go from 0 to 3 or from 1 to 4. I will choose 0 to 3, so that $\mathbf{e}_0$ points in the time direction (actually the ct direction though I will just choose $c = 1$ for convenience) chosen so that $\mathbf{e}_0 \cdot \mathbf{e}_0 = -1$. This means this is not the Cartesian orthonormal basis! This would cause problems if we didn't already have the machinery to deal with non-orthogonal bases already developed. With this choice, we have to think about how to develop this. We can either change the dot product definition or we are forced to introduce the imaginary unit i. With i, we have the advantage of using the standard dot product and so we can construct our reciprocal basis with the reciprocal matrix approach above. It is more common, I think, to define a new dot product as

$$\mathbb{X} \cdot \mathbb{Y} = \sum_{i=1}^{3} X_i Y^i - X_0 Y^0 = \sum_{i=1}^{3} g^{ij} X_i Y_j - g^{00} X_0 Y_0 = \sum_{i=1}^{3} g_{ij} X^i Y^j - g_{00} X^0 Y^0 = X^i Y_i - X^0 Y_0 \tag{1.2.104}$$

It is conventional to use Greek letter indices to indicate a sum over all four dimensions and Latin letters for sums over 1 through 3. So

$$\mathbb{X} \cdot \mathbb{Y} \equiv X_\mu Y^\mu \equiv X_i Y^i - X_0 Y^0 \tag{1.2.105}$$

is the conventional notation. We then need to modify our approach slightly. We begin with the tangent basis and start with $\mathbf{e}_0$. We use the underlying Cartesian coordinates $\hat{\mathbf{x}}_i$ for the other

[28] Note that we could just use t as for our system we don't require our basis vectors to have the same units, but we would then find the components always include a factor of c.

components since clearly $\mathbf{e}_i = \hat{\mathbf{x}}_i = \frac{\partial \mathbb{X}}{\partial x^i}$. We want $\mathbf{e}_0 = \hat{\mathbf{x}}_0$ where we think of this as $\mathbf{e}_0 = \frac{\partial \mathbb{X}}{\partial x^0} \equiv \hat{\mathbf{x}}_0$. That is, we define the tangent basis in the normal way and find

$$\mathbf{e}_0 = \hat{\mathbf{x}}_0 \tag{1.2.106}$$
$$\mathbf{e}_1 = \hat{\mathbf{x}}_1 \tag{1.2.107}$$
$$\mathbf{e}_2 = \hat{\mathbf{x}}_2 \tag{1.2.108}$$
$$\mathbf{e}_3 = \hat{\mathbf{x}}_3 \tag{1.2.109}$$

which I will call the standard basis (even though $\hat{\mathbf{x}}_0$ is technically not a unit vector with our defined dot product). We then want (use i' is one of either 1, 2, or 3) [we expand $\mathbf{e}^i$ in terms of $\mathbf{e}_j$ with $\mathbf{e}^i \cdot \mathbf{e}_j = \mathbf{e}^i \cdot \mathbf{x}_j = (e^i)_j$]

$$\mathbf{e}_0 \cdot \mathbf{e}^0 = \hat{\mathbf{x}}_0 \cdot \left[(e^0)_0\hat{\mathbf{x}}_0 + (e^0)_1\hat{\mathbf{x}}_1 + (e^0)_2\hat{\mathbf{x}}_2 + (e^0)_3\hat{\mathbf{x}}_3\right] = -(e^0)_0 \tag{1.2.110}$$
$$\mathbf{e}_0 \cdot \mathbf{e}^{i'} = \hat{\mathbf{x}}_0 \cdot \left[(e^{i'})_0\hat{\mathbf{x}}_0 + (e^{i'})_1\hat{\mathbf{x}}_1 + (e^{i'})_2\hat{\mathbf{x}}_2 + (e^{i'})_3\hat{\mathbf{x}}_3\right] = -(e^{i'})_0 \tag{1.2.111}$$

Thus for $\mathbf{e}^0 \cdot \mathbf{e}_0 = 1$ we need $-(e^0)_0 = 1$ or $(e^0)_0 = -1$ with $(e^{i'})_0 = 0$. Note how this process is exactly the same as inverting our matrix before, but now we can't represent it as simply matrix multiplication. Similarly we have

$$\mathbf{e}_{i'} \cdot \mathbf{e}^0 = \hat{\mathbf{x}}_{i'} \cdot [-\hat{\mathbf{x}}_0] = 0 \tag{1.2.112}$$
$$\mathbf{e}_1 \cdot \mathbf{e}^{j'} = \hat{\mathbf{x}}_1 \cdot \left[(e^{j'})_0\hat{\mathbf{x}}_0 + (e^{j'})_1\hat{\mathbf{x}}_1 + (e^{j'})_2\hat{\mathbf{x}}_2 + (e^{j'})_3\hat{\mathbf{x}}_3\right] = (e^{j'})_1 = \delta_1^{j'} \tag{1.2.113}$$
$$\mathbf{e}_2 \cdot \mathbf{e}^{j'} = \hat{\mathbf{x}}_2 \cdot \left[(e^{j'})_0\hat{\mathbf{x}}_0 + (e^{j'})_1\hat{\mathbf{x}}_1 + (e^{j'})_2\hat{\mathbf{x}}_2 + (e^{j'})_3\hat{\mathbf{x}}_3\right] = (e^{j'})_2 = \delta_2^{j'} \tag{1.2.114}$$
$$\mathbf{e}_3 \cdot \mathbf{e}^{j'} = \hat{\mathbf{x}}_3 \cdot \left[(e^{j'})_0\hat{\mathbf{x}}_0 + (e^{j'})_1\hat{\mathbf{x}}_1 + (e^{j'})_2\hat{\mathbf{x}}_2 + (e^{j'})_3\hat{\mathbf{x}}_3\right] = (e^{j'})_3 = \delta_3^{j'} \tag{1.2.115}$$

Clearly this implies that the tangent-reciprocal basis must satisfy

$$\mathbf{e}^0 = -\hat{\mathbf{x}}_0 \tag{1.2.116}$$
$$\mathbf{e}^1 = \hat{\mathbf{x}}_1 \tag{1.2.117}$$
$$\mathbf{e}^2 = \hat{\mathbf{x}}_2 \tag{1.2.118}$$
$$\mathbf{e}^3 = \hat{\mathbf{x}}_3 \tag{1.2.119}$$

which is the tangent-reciprocal basis of the standard tangent basis. Thus our original, standard basis and reciprocal basis can be used to represent vectors in special relativity with our special dot product.

Another method is to introduce operator $\mathsf{F}(\mathbf{x})$ which switches the sign on the first row of the matrix or vector it operates on.[29] Clearly the inverse is $\mathsf{F}^{-1} = \mathsf{F}$ as this switches the sign twice. Thus

$$(\mathbf{B}_b)^\intercal \mathsf{F}(\mathbf{B}^b) = (\mathbf{B}^b)^\intercal \mathsf{F}(\mathbf{B}_b) = \mathbb{1} \tag{1.2.120}$$
$$\mathbf{B}_b = \mathsf{F}[(\mathbf{B}^b)^\intercal]^{-1} \tag{1.2.121}$$
$$\mathbf{B}^b = \mathsf{F}[(\mathbf{B}_b)^\intercal]^{-1} \tag{1.2.122}$$

[29] F can be represented as a matrix. It's easier to just switch the first row's sign, though. As a matrix, F is the same as $\mathbb{1}$ but with the furthest upper-left 1 switched with a -1. If we had decided on the other sign convention, then F would switch the sign of the lower three rows of the matrix and the matrix is the same as $-\mathbb{1}$ except the furthest upper left has a $+1$ instead of a -1.

In this case we use

$$\mathbf{B}_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.2.123}$$

$$[(\mathbf{B}_b)^\intercal]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.2.124}$$

$$\mathbf{B}^b = \mathbf{F}([(\mathbf{B}_b)^\intercal]^{-1}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.2.125}$$

which reproduces our previous argument's results.

It is worth looking at the road less traveled. In this case, we keep our non-time Cartesian vectors real, so we don't worry about complex conjugates. What if we had introduced complex values and kept our original dot product? We would find that we must have $\mathbf{e}_0 = i\hat{\mathbf{x}}_0$ and so

$$\mathbf{B}_b = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.2.126}$$

$$\mathbf{B}^b = [(B_b)^\intercal]^{-1} = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.2.127}$$

and so we have

$$\mathbf{e}_0 = i\hat{\mathbf{x}}_0 \tag{1.2.128}$$
$$\mathbf{e}_1 = \hat{\mathbf{x}}_1 \tag{1.2.129}$$
$$\mathbf{e}_2 = \hat{\mathbf{x}}_2 \tag{1.2.130}$$
$$\mathbf{e}_3 = \hat{\mathbf{x}}_3 \tag{1.2.131}$$
$$\mathbf{e}^0 = -i\hat{\mathbf{x}}_0 \tag{1.2.132}$$
$$\mathbf{e}^1 = \hat{\mathbf{x}}_1 \tag{1.2.133}$$
$$\mathbf{e}^2 = \hat{\mathbf{x}}_2 \tag{1.2.134}$$
$$\mathbf{e}^3 = \hat{\mathbf{x}}_3 \tag{1.2.135}$$

This may feel more natural and is easier, but is rarely what is chosen. Four vectors use the modified dot product route, instead. With the operator $\mathbf{F}$, it isn't much of a change in difficulty and there are no worries about the interpretation of i.

One final note. I chose a signature $(-+++)$ for my dot product, but one often chooses $(+---)$. With $(+---)$ the energy-momentum vector "squared" yields the rest mass, whereas with mine,

you get the negative of the rest mass. My signature is often called the East Coast,[30] spacelike, relativity, Pauli, or mostly pluses convention. The $(+---)$ is often called the West Coast, timelike, particle physics, Landau-Lifshitz, or mostly minuses convention.

1.2.7 Index and Einstein Notation Summation

It is now worth looking at simplifying some of our notation. There is a common practice of using repeated indices to indicate a summation. Therefore we write

$$\mathbf{V}\cdot\mathbf{W}=\sum_{i=1}^{3}\sum_{j=1}^{3}V_iW^j(\mathbf{e}^i\cdot\mathbf{e}_j)=V_iW^j\mathbf{e}^i\cdot\mathbf{e}_j=V_iW^i=\sum_{i=1}^{3}\sum_{j=1}^{3}V^iW_j(\mathbf{e}_i\cdot\mathbf{e}^j)=V^iW_j\mathbf{e}_i\cdot\mathbf{e}^j=V^iW_i \tag{1.2.136}$$

It is easier to omit the summations, and the notation is unambiguous when only two indices are repeated. It is important to realize that when using the summation convention, that indices can only be repeated twice without ambiguity. This summation convention is sometimes called Einstein notation or Einstein summation notation.[31] Any two repeated indices (which are summed) are called dummy indices or dummy variables (because we can switch any of the dummy symbols used without changing the meaning of the term). Any index that is not repeated is a non-dummy, or free index. It can only be changed if you change all the corresponding indices on all sides of the equation on each term. As an example in $A_{ijk}B_{jil}$, k and l are non dummy indices and i and j are dummy indices. Thus we can write $A_{ijk}B_{jil}=A_{mnk}B_{nml}$. But if we have $A_{ijk}B_{jil}=C_kD_l$ but we cannot write ~~$A_{ijk}B_{jil}=C_mD_l$~~. We could write $A_{ijm}B_{jin}=C_mD_n$, however.

For the dot product, we could also define $g^{ij}=\mathbf{e}^i\cdot\mathbf{e}^j$ and $g_{ij}=\mathbf{e}_i\cdot\mathbf{e}_j$ with $g^i_j=\delta^i_j$ the Kronecker delta (that is, it is equal to one only when i and j are the same). We will call g_{ij} and g^{ij} the metric coefficients and later show that indeed they are metric tensor coefficients. Then other equivalent dot product forms would be

$$\mathbf{V}\cdot\mathbf{W}=g^{ij}V_iW^j=g^{ij}V^iW_j=g_{ij}V^iW_j=g_{ij}V_iW^j \tag{1.2.137}$$

For $\sqrt{\mathbf{V}\cdot\mathbf{V}}\equiv|\mathbf{V}|$ we define this as the magnitude of vector $\mathbf{V}$. If you are worried I haven't defined dot products for vectors, then remember we can always write $\mathbf{e}_i$ and $\mathbf{e}^i$ in terms of Cartesian $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ and use the known relations for dot products and cross products to determine the values. We have the same properties of the normal dot product where $\mathbf{A}\cdot\mathbf{B}=|\mathbf{A}||\mathbf{B}|\cos\theta_{\mathbf{AB}}$ where $\theta_{\mathbf{AB}}$ is the angle between the two vectors. Thus we can use dot products to test for orthogonality.

It is also useful to define a cross product using a symbol called the Levi-Civita symbol ϵ^{ijk} or ϵ_{ijk}[32]

[30] The East Coast and West Coast convention names come from the US. East Coast schools like Princeton and Harvard mostly used the East Coast convention, while West Coast schools in California mostly used the West Coast convention in the past. I was once told that you could remember the West Coast metric signature because it has the + on the left (West) in the signature $(+---)$ and the West Coast is warm. One could make the mnemonic that the side of the pluses corresponds with the left (West) and right (East) coasts.

[31] Sometimes it is called index notation, but at other times index notation means there is no implied sum. Generally repeated indices imply summation unless otherwise noted.

[32] The upper or lower indices have no meaning at this point. We can make the Levi-Civita symbol into a third order tensor so that it would correspond to contravariant and covariant components, but I will not do so. It is just convenient to have both so that it appears to act like a regular tensor in many of our formula.

such that

$$\epsilon^{ijk} = \epsilon_{ijk} = \begin{cases} 1 & i,j,k \text{ form a even permutation of (1,2,3)} \\ -1 & i,j,k \text{ form an odd permutation of (1,2,3)} \\ 0 & \text{otherwise} \end{cases} \tag{1.2.138}$$

Then a cross product can be written as

$$\mathbf{V} \times \mathbf{W} = V_i \mathbf{e}^i \times W_j \mathbf{e}^j = V^i \mathbf{e}_i \times W_j \mathbf{e}_j \tag{1.2.139}$$

which via our reciprocal relations (1.2.64)-(1.2.65) means (the primes on i,j,k indicating cyclical relations)

$$(\mathbf{V} \times \mathbf{W})^{k'} = \mathcal{J}(V_{i'} W_{j'} - V_{j'} W_{i'}) \mathbf{e}^{k'} \tag{1.2.140}$$

$$(\mathbf{V} \times \mathbf{W})_{k'} = J(V^{i'} W^{j'} - V^{j'} W^{i'}) \mathbf{e}_{k'} \tag{1.2.141}$$

and we can summarize this with the Levi-Civita symbol more nicely as

$$(\mathbf{V} \times \mathbf{W})^k = \mathcal{J} \epsilon^{ijk} V_i W_j \tag{1.2.142}$$

$$(\mathbf{V} \times \mathbf{W})_k = J \epsilon_{ijk} V^i W^j \tag{1.2.143}$$

remembering that $\mathcal{J} = J^{-1}$. Finally, we can again use $\mathbf{V} \times \mathbf{W} = \hat{\mathbf{n}} |\mathbf{V}| |\mathbf{W}| \sin \theta_{\mathbf{VW}}$ where $\hat{\mathbf{n}}$ is a unit vector pointing perpendicular to $\mathbf{V}$ and $\mathbf{W}$ in a right-handed fashion, and $\theta_{\mathbf{VW}}$ is the angle between $\mathbf{V}$ and $\mathbf{W}$.

1.2.8 Parallel and Perpendicular Vector Directions

One useful way of decomposing a vector is into components parallel and perpendicular to a specific direction. Let's consider a unit vector $\hat{\mathbf{b}}$ and a general vector $\mathbf{V}$. We can then decompose the vector $\mathbf{V}$ along and perpendicular to $\hat{\mathbf{b}}$. This is usually denoted by $\mathbf{V}_\parallel$ and $\mathbf{V}_\perp$ for the parallel and perpendicular components. We find that we can project the part of $\mathbf{V}$ along $\hat{\mathbf{b}}$ to find

$$\mathbf{V}_\parallel \equiv V_\parallel \hat{\mathbf{b}} = \mathbf{V} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} \tag{1.2.144}$$

The rest of the vector is the perpendicular portion, so

$$\mathbf{V}_\perp = \mathbf{V} - \mathbf{V}_\parallel \tag{1.2.145}$$

A more convenient representation is

$$\mathbf{V}_\perp = -\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{V}) = (\hat{\mathbf{b}} \times \mathbf{V}) \times \hat{\mathbf{b}} \tag{1.2.146}$$

For this we use (for vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$)

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{1.2.147}$$

so that

$$\mathbf{V}_\perp = \mathbf{V}(\hat{\mathbf{b}} \cdot \hat{\mathbf{b}}) - \mathbf{V} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} = \mathbf{V} - \mathbf{V}_\parallel \tag{1.2.148}$$

Thus, the decomposition can be written

$$\mathbf{V} = \mathbf{V}_{\parallel} + \mathbf{V}_{\perp} = V_{\parallel}\hat{\mathbf{b}} - \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{V}) \tag{1.2.149}$$

We can also note that for another vector $\mathbf{W}$ that

$$\mathbf{V}_{\perp} \cdot \mathbf{W} = \mathbf{V}_{\perp} \cdot \left(\mathbf{W}_{\parallel} + \mathbf{W}_{\perp}\right) = \mathbf{V}_{\perp} \cdot \mathbf{W}_{\perp} = (\mathbf{V} - \mathbf{V}_{\parallel}) \cdot \mathbf{W}_{\perp} = \mathbf{V} \cdot \mathbf{W}_{\perp} \tag{1.2.150}$$

$$\mathbf{V}_{\parallel} \cdot \mathbf{W} = \mathbf{V}_{\parallel} \cdot (\mathbf{W}_{\parallel} + \mathbf{W}_{\perp}) = \mathbf{V}_{\parallel} \cdot \mathbf{W}_{\parallel} = \left(\mathbf{V}_{\parallel} + \mathbf{V}_{\perp}\right) \cdot \mathbf{W}_{\parallel} = \mathbf{V} \cdot \mathbf{W}_{\parallel} \tag{1.2.151}$$

Thus, we can move parallels and perpendiculars across vectors when doing dot products.

It is also often convenient to define gradients in directions analogously. Then for a scalar function f we have

$$\nabla f = \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla f - \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \nabla f) = \nabla_{\parallel} f + \nabla_{\perp} f \tag{1.2.152}$$

In addition, sometimes people define parallel and perpendicular divergences so that (with l along $\hat{\mathbf{b}}$)

$$\boldsymbol{\nabla}_{\parallel} \cdot \mathbf{V} \equiv \boldsymbol{\nabla} \cdot \mathbf{V}_{\parallel} = \boldsymbol{\nabla} \cdot \mathbf{V} - \boldsymbol{\nabla}_{\perp} \cdot \mathbf{V} = \boldsymbol{\nabla} \cdot (V_{\parallel}\hat{\mathbf{b}}) = \frac{\partial V_{\parallel}}{\partial l} \tag{1.2.153}$$

$$\boldsymbol{\nabla}_{\perp} \cdot \mathbf{V} \equiv \boldsymbol{\nabla} \cdot (\mathbf{V}_{\perp}) \tag{1.2.154}$$

It is then obvious that

$$\boldsymbol{\nabla}_{\perp} \cdot \mathbf{V}_{\parallel} = \boldsymbol{\nabla}_{\parallel} \cdot \mathbf{V}_{\perp} = 0 \tag{1.2.155}$$

The above definitions also have the added benefit of meaning that we can move the parallels and perpendiculars across vectors in dot products when using del.

For the case when $\boldsymbol{\nabla} \cdot \mathbf{V} = 0$ we further have

$$0 = \boldsymbol{\nabla} \cdot (\mathbf{V}_{\parallel} + \mathbf{V}_{\perp}) = \boldsymbol{\nabla}_{\parallel} \cdot \mathbf{V}_{\parallel} + \cancel{\boldsymbol{\nabla}_{\perp} \cdot \mathbf{V}_{\parallel}} + \cancel{\boldsymbol{\nabla}_{\parallel} \cdot \mathbf{V}_{\perp}} + \boldsymbol{\nabla}_{\perp} \cdot \mathbf{V}_{\perp} \tag{1.2.156}$$

$$\boldsymbol{\nabla}_{\parallel} \cdot \mathbf{V}_{\parallel} = -\boldsymbol{\nabla}_{\perp} \cdot \mathbf{V}_{\perp} \tag{1.2.157}$$

Note that (1.2.157) only holds when $\boldsymbol{\nabla} \cdot \mathbf{V} = 0$.

1.2.8.1 Transport Parallel, Cross, and Perpendicular Components

It is sometimes useful to write some formula such that we consider a parallel, a perpendicular, and a "cross" direction. One should always remember in this case that while the cross direction is necessarily perpendicular to the perpendicular direction, this does not mean that we can decompose the vector into a parallel, perpendicular, and cross direction in the way we did before! The definition of a cross component is usually given with the cross component using a subscript wedge, $\wedge$. The reason to pick out a cross direction is that the perpendicular and the cross direction can have different behavior, for example in the Braginskii closure for the stress tensor. In these cases it is important to make sure each component of the vector actually follows the correct physics. What happens is that we define

$$\mathbf{q} = \mathbf{q}_{\parallel} + \mathbf{q}_{\wedge} + \mathbf{q}_{\perp} \tag{1.2.158}$$

but then this comes from defining components of a distribution function as

$$f(\mathbf{v}) = f_{\parallel}(\mathbf{v}) + f_{\wedge}(\mathbf{v}) + f_{\perp}(\mathbf{v}) = \alpha_{\parallel}\mathbf{v}_{\parallel} + \alpha_{\wedge}\mathbf{b}\times\mathbf{v}_{\perp} + \alpha_{\perp}\mathbf{v}_{\perp} \tag{1.2.159}$$

Then if $\mathbf{q}$ comes from the moment of $\mathbf{Q}(\mathbf{v})$ of the distribution function we define

$$\mathbf{q}_{\parallel} = \iiint_{-\infty}^{\infty} \mathrm{d}^3 v\ f_{\parallel}\mathbf{Q}(\mathbf{v}) \tag{1.2.160}$$

$$\mathbf{q}_{\wedge} = \iiint_{-\infty}^{\infty} \mathrm{d}^3 v\ f_{\wedge}\mathbf{Q}(\mathbf{v}) \tag{1.2.161}$$

$$\mathbf{q}_{\perp} = \iiint_{-\infty}^{\infty} \mathrm{d}^3 v\ f_{\perp}\mathbf{Q}(\mathbf{v}) \tag{1.2.162}$$

In this case the notation has a separate meaning from our decomposition above. One might expect $\mathbf{q}_{\wedge} \stackrel{?}{=} \mathbf{b}\times\mathbf{q}$. But if there were so, the notation would need to be consistent with our previous definitions (1.2.144)-(1.2.146). This means we would require

$$\mathbf{b}\times\mathbf{q} = \cancel{\mathbf{b}\times\mathbf{q}_{\parallel}} + \overbrace{\mathbf{b}\times\mathbf{q}_{\wedge}}^{=-\mathbf{q}_{\perp}} + \mathbf{b}\times\mathbf{q}_{\perp} \tag{1.2.163}$$

$$\overbrace{\mathbf{b}\times(\mathbf{b}\times\mathbf{q})}^{-\mathbf{q}_{\perp}} = -\mathbf{b}\times\mathbf{q}_{\perp} + \overbrace{\mathbf{b}\times\mathbf{b}\times\mathbf{q}_{\perp}}^{-\mathbf{q}_{\perp}} \tag{1.2.164}$$

$$0 = -\mathbf{b}\times\mathbf{q}_{\perp}$$

which would mean $\mathbf{q}_{\perp}$ is parallel to $\mathbf{b}$ or $\mathbf{0}$, an enormous contradiction! Thus, when looking at transport literature, be aware of the danger of interpreting parallel $\parallel$ and perpendicular $\perp$ directions relative to the cross direction. In these cases it is defined by components of a distribution function rather than purely in terms of a vector direction. It is important to just look at the definitions. For example, the flow velocity $\mathbf{u}$ given by moment $\mathbf{v}$ of the first order electron distribution function f (the zeroth order is a Maxwellian) for a Lorentz plasma model such as considered in Helander[21] yields

$$n_e\mathbf{u}_{\parallel} = \frac{32}{3\pi}\frac{\tau_{ei}}{m_e}\left(\nabla_{\parallel}p_e + n_e e E_{\parallel} + \frac{3}{2}n_e k_B\,\nabla_{\parallel}T_e\right) \tag{1.2.165}$$

$$n_e\mathbf{u}_{\wedge} = \frac{\mathbf{b}\times\nabla p_e}{m_e\Omega_e} + n_e\frac{\mathbf{E}\times\mathbf{B}}{B^2} \tag{1.2.166}$$

$$n_e\mathbf{u}_{\perp} = -\frac{1}{m_e\Omega_e^2\tau_{ei}}\left(\nabla_{\perp}p_e + n_e e\mathbf{E}_{\perp} - \frac{3}{2}n_e k_B\,\nabla_{\perp}T_e\right) \tag{1.2.167}$$

You can then confirm for yourself that $\mathbf{u}_{\wedge}\cdot\mathbf{u}_{\perp} \neq 0$.

If we had $\vee$ (representing $\wedge$ as a separate decomposition from perpendicular and parallel) as a cross component based purely off of $\mathbf{b}$ we would require

$$\mathbf{u}_{\vee} = \mathbf{b}\times\mathbf{u} \tag{1.2.168}$$

$$\mathbf{u}_{\perp} = -\mathbf{b}\times(\mathbf{b}\times\mathbf{u}) \tag{1.2.169}$$

$$\mathbf{u}_{\vee}\cdot\mathbf{u}_{\perp} = 0 \tag{1.2.170}$$

Thus we see that the cross component $\mathbf{q}_{\wedge}$ for vector $\mathbf{q}$ does not have the simple interpretation of simply $\mathbf{b}\times\mathbf{q}$.

1.2.9 Tensors

Now that we have looked at vectors, it is worth delving into generalizations of vectors. In order to treat them as objects similar to vectors, we will have to explain some of the subtleties and requirements for writing tensors in unambiguous ways.

First of all, what is a tensor. It is an object that transforms similarly to vectors under coordinate transformations, but describes more complex geometric information than simply direction for quantities. For example if you have a surface of a cube, a stress tensor can tell you the stress in a particular direction along the surface when applying a force per area in specified directions on the cube. We can begin to see the properties of tensors by looking at dyads, a specific type of tensor. A dyad is an object that is essentially two vectors pushed together into a single object. That is, it takes two basis vectors to give it its direction. Suppose we have two different vectors $\mathbf{V}$ and $\mathbf{W}$. Then the dyad is written

$$\mathbf{VW} \equiv \mathbf{V} \otimes \mathbf{W} \tag{1.2.171}$$

when using a basis set (and its reciprocal basis set) we can identify four possible representations of the dyad

$$\mathbf{VW} = V^i W^j \mathbf{e}_i \mathbf{e}_j = V^i W_j \mathbf{e}_i \mathbf{e}^j = V_i W^j \mathbf{e}^i \mathbf{e}_j = V_i W_j \mathbf{e}^i \mathbf{e}^j \tag{1.2.172}$$

It is often convenient to then define $\overleftrightarrow{\mathbf{S}} \equiv \sum_j \mathbf{V}_j \mathbf{W}_j$ as a second order tensor (vectors are first order tensors). If the tensor is made of a single dyad $\mathbf{VW}$ we can write it as

$$\overleftrightarrow{\mathbf{S}} = \mathbf{VW} = S^{ij} \mathbf{e}_i \mathbf{e}_j = S^i_j \mathbf{e}_i \mathbf{e}^j = S^j_i \mathbf{e}^i \mathbf{e}_j = S_{ij} \mathbf{e}^i \mathbf{e}^j \tag{1.2.173}$$

We could of course extend to triads and so on by "adding" more vectors, but I will not delve into it, as the same principles apply as before.[33] There is also a question of whether all second order tensors can be represented as dyads. I hinted at this by writing a sum of dyads, and the answer is no. Dyads cannot form all second order tensors. If we expand with Cartesian vectors with a matrix notation we can see why fairly quickly[34]. Let $\mathbf{A}$ and $\mathbf{B}$ be the vectors forming dyad $\mathbf{AB}$ and $\mathbf{BA}$

$$\begin{aligned}
\mathbf{AB} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}})(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) && (1.2.174)\\
&= A_xB_x\hat{\mathbf{x}}\hat{\mathbf{x}} + A_xB_y\hat{\mathbf{x}}\hat{\mathbf{y}} + A_xB_z\hat{\mathbf{x}}\hat{\mathbf{z}} \\
&\quad + A_yB_x\hat{\mathbf{y}}\hat{\mathbf{x}} + A_yB_y\hat{\mathbf{y}}\hat{\mathbf{y}} + A_yB_z\hat{\mathbf{y}}\hat{\mathbf{z}} && (1.2.175)\\
&\quad + A_zB_x\hat{\mathbf{z}}\hat{\mathbf{x}} + A_zB_y\hat{\mathbf{z}}\hat{\mathbf{y}} + A_zB_z\hat{\mathbf{z}}\hat{\mathbf{z}} \\
&= \begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\end{bmatrix} \begin{bmatrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{bmatrix} \begin{bmatrix}\hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}}\end{bmatrix} && (1.2.176)\\
\mathbf{BA} &= (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) && (1.2.177)
\end{aligned}$$

[33] The terminology uses Greek number roots, so vectors could be considered monads, then dyads, triads, tetrads, pentads, etc. Scalars could then be considered 0-ads or medenads or oudenads. When we get to explaining polyadics the same notation applies but now with -adic instead of -ad.

[34] Note that this matrix form is mnemonic, as the unit vectors inside a matrix would require far more motivation to be rigorous

$$\begin{aligned}
&= B_xA_x\hat{\mathbf{x}}\hat{\mathbf{x}} + B_xA_y\hat{\mathbf{x}}\hat{\mathbf{y}} + B_xA_z\hat{\mathbf{x}}\hat{\mathbf{z}} \\
&\quad + B_yA_x\hat{\mathbf{y}}\hat{\mathbf{x}} + B_yA_y\hat{\mathbf{y}}\hat{\mathbf{y}} + B_yA_z\hat{\mathbf{y}}\hat{\mathbf{z}} \qquad (1.2.178)\\
&\quad + B_zA_x\hat{\mathbf{z}}\hat{\mathbf{x}} + B_zA_y\hat{\mathbf{z}}\hat{\mathbf{y}} + B_zA_z\hat{\mathbf{z}}\hat{\mathbf{z}} \\
&= \begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\end{bmatrix} \begin{bmatrix} B_xA_x & B_xA_y & B_xA_z \\ B_yA_x & B_yA_y & B_yA_z \\ B_zA_x & B_zA_y & B_zA_z \end{bmatrix} \begin{bmatrix}\hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}}\end{bmatrix} \qquad (1.2.179)
\end{aligned}$$

Thus we see the sense in which $(\mathbf{AB})^\intercal = \mathbf{BA}$. If we ignore the two vectors of vectors on the sides of the matrix, then we can identify the central matrix as the matrix representation of our dyad. We also see that it will not in general be possible to choose 6 numbers such that we can form the nine entries making up a second order tensor. It turns out that one can express a tensor as a sum of dyads (called a dyadic), however. The same holds true for triads, tetrads, etc. with the sum of them then being called triadics, tetradics, etc.

One of the most important things to remember when using tensors is that we must be very careful of order. In the above representations I was very careful to include the basis vectors in a specific order. This is because if we use a dot product with a dyad/tensor we no longer can use commutativity. That is for $\overleftrightarrow{\mathbf{S}} = \mathbf{VW}$

$$\mathbf{V}\cdot\overleftrightarrow{\mathbf{S}} = (\mathbf{V}\cdot\mathbf{V})\mathbf{W} \neq \overleftrightarrow{\mathbf{S}}\cdot\mathbf{V} = \mathbf{V}(\mathbf{W}\cdot\mathbf{V}) \qquad (1.2.180)$$

Thus we see it is generally true that $\mathbf{V}\cdot\overleftrightarrow{\mathbf{S}} \neq \overleftrightarrow{\mathbf{S}}\cdot\mathbf{V}$, though in specific cases it may be true for a specific $\overleftrightarrow{\mathbf{S}}$ or $\mathbf{V}$.

Now we run into the same problems of "contravariant", "covariant", and "mixed" tensors. If we view a second order tensor as its components, then it makes sense to talk of covariant tensors and such, but if we view the tensor as the geometrical object, then it is the representation of the tensor that is covariant, contravariant, or mixed. With our new freedom, we can also introduce new operators to act on tensors. An important new one is often called the double dot operator $\mathbf{:}$. This is defined by

$$\overleftrightarrow{\mathbf{T}}\mathbf{:}\mathbf{VW} = \mathbf{W}\cdot\overleftrightarrow{\mathbf{T}}\cdot\mathbf{V} \qquad (1.2.181)$$

(this is the most common plasma physics definition, some literature will use $\overleftrightarrow{\mathbf{T}}\mathbf{:}\mathbf{VW} = \mathbf{V}\cdot\overleftrightarrow{\mathbf{T}}\cdot\mathbf{W}$ but this will never be used here). We can make similar definitions for two tensors. We can also define a unit tensor such that $\mathbb{1}\cdot\overleftrightarrow{\mathbf{T}} = \overleftrightarrow{\mathbf{T}}\cdot\mathbb{1} = \overleftrightarrow{\mathbf{T}}$ and $\mathbb{1}\cdot\mathbf{V} = \mathbf{V}\cdot\mathbb{1} = \mathbf{V}$. Then the trace of a tensor (the sum of its diagonal elements in matrix form) is given by

$$\mathrm{Tr}\,\overleftrightarrow{\mathbf{T}} = \mathbb{1}\mathbf{:}\overleftrightarrow{\mathbf{T}} \qquad (1.2.182)$$

In addition we can define the $\overset{\times}{\cdot}$, $\overset{\cdot}{\times}$, and $\overset{\times}{\times}$ operators between two dyads or tensors. We have

$$\mathbf{VW}\overset{\times}{\cdot}\mathbf{YZ} = (\mathbf{V}\times\mathbf{Y})(\mathbf{W}\cdot\mathbf{Z}) \qquad (1.2.183)$$

$$\overleftrightarrow{\mathbf{T}}\overset{\times}{\cdot}\mathbf{VW} = (-\mathbf{V}\times\overleftrightarrow{\mathbf{T}})\cdot\mathbf{W} \qquad (1.2.184)$$

$$\mathbf{VW}\overset{\cdot}{\times}\mathbf{YZ} = (\mathbf{V}\cdot\mathbf{Y})(\mathbf{W}\times\mathbf{Z}) \qquad (1.2.185)$$

$$\overset{\leftrightarrow}{\mathbf{T}} \overset{\cdot}{\times} \mathbf{VW} = \mathbf{V} \cdot \overset{\leftrightarrow}{\mathbf{T}} \times \mathbf{W} \tag{1.2.186}$$

$$\mathbf{VW} \overset{\times}{\times} \mathbf{YZ} = (\mathbf{V} \times \mathbf{Y})(\mathbf{W} \times \mathbf{Z}) \tag{1.2.187}$$

$$\overset{\leftrightarrow}{\mathbf{T}} \overset{\times}{\times} \mathbf{VW} = -\mathbf{V} \times \overset{\leftrightarrow}{\mathbf{T}} \times \mathbf{W} \tag{1.2.188}$$

Note that with these definitions

$$\mathbf{VW} \overset{\times}{\cdot} \mathbf{YZ} = -\mathbf{YZ} \overset{\times}{\cdot} \mathbf{VW} = -(\mathbf{Y} \times \mathbf{V})(\mathbf{Z} \cdot \mathbf{W}) \tag{B.89}$$

$$\mathbf{VW} \overset{\cdot}{\times} \mathbf{YZ} = -\mathbf{YZ} \overset{\cdot}{\times} \mathbf{VW} = -(\mathbf{Y} \cdot \mathbf{V})(\mathbf{Z} \times \mathbf{W}) \tag{B.90}$$

$$\mathbf{VW} \overset{\times}{\times} \mathbf{YZ} = \mathbf{YZ} \overset{\times}{\times} \mathbf{VW} = (\mathbf{Y} \times \mathbf{V})(\mathbf{Z} \times \mathbf{W}) \tag{B.91}$$

In addition, we can consider the forms of tensors or dyads. We define the transpose of a dyad or tensor with $\intercal$ such that

$$(\mathbf{VW})^\intercal = \mathbf{WV} \tag{1.2.189}$$

Thus we define the symmetric and antisymmetric parts of an order two tensor or dyad via

$$\overset{\leftrightarrow}{\mathbf{T}}_\mathrm{S} = \frac{\overset{\leftrightarrow}{\mathbf{T}} + \overset{\leftrightarrow}{\mathbf{T}}^\intercal}{2} \tag{1.2.190}$$

$$(\mathbf{VW})_\mathrm{S} = \frac{\mathbf{VW} + \mathbf{WV}}{2} \tag{1.2.191}$$

$$\overset{\leftrightarrow}{\mathbf{T}}_\mathrm{A} = \frac{\overset{\leftrightarrow}{\mathbf{T}} - \overset{\leftrightarrow}{\mathbf{T}}^\intercal}{2} \tag{1.2.192}$$

$$(\mathbf{VW})_\mathrm{A} = \frac{\mathbf{VW} - \mathbf{WV}}{2} \tag{1.2.193}$$

When we get to tensors of higher order, we have to consider making it symmetric (or antisymmetric) index by index. That is for an nth order tensor

$$\mathbf{T}_\mathrm{S} = \frac{1}{n!} \sum_{i_k} T_{i_{k1},i_{k2},\ldots,i_{kn}} \mathbf{e}^{i_{k1}} \mathbf{e}^{i_{k2}} \cdots \mathbf{e}^{i_{kn}} \equiv T_{(i_1,i_2,\ldots,i_n)} \tag{1.2.194}$$

$$\mathbf{T}_\mathrm{A} = \frac{1}{n!} \sum_{b_1,b_2,\ldots,b_n} \delta^{b_1,b_2,\ldots,b_n}_{i_1,i_2,\ldots,i_n} T_{b_1,b_2,\ldots,b_n} \mathbf{e}^{b_1} \mathbf{e}^{b_2} \cdots \mathbf{e}^{b_n} \equiv T_{[i_1,i_2,\ldots,i_n]} \tag{1.2.195}$$

when considering a specific representation. Here the generalized Kronecker delta, is used and the sum over i_k means sum over all possible combinations of the indices. This notation is cumbersome without index notation, but the $(i_1, \ldots, i_n)$ and $[i_1, \ldots, i_n]$ notation can be extended by writing $T_{i_1,[i_2,\ldots,i_n]}$, for example, to only antisymmetrize on indices i_2 through i_n. Indeed, we can do this for any representation $T^{[i_1,\cdots}_{\cdots,i_n]}$. There is simply no easy way to represent this in general with our abstract notation beyond putting the A or S subscripts to indicate antisymmetric or symmetric.

Now we consider operators that contract dyads or tensors or create dyads or tensors from vectors. One is called the vector operation which contracts a second order tensor or dyad down to a vector. We define

$$\mathrm{vec}\,(\mathbf{VW}) = (\mathbf{VW})_\times = \mathbf{V} \times \mathbf{W} \tag{1.2.196}$$

with the analogous definition for the tensor notation. Another operator we can call the dyad operator, as it takes a vector and changes it into a dyad. It is defined by

$$\text{dyad}(\mathbf{V}) = -\frac{1}{2}\mathbb{1} \times \mathbf{V} = -\frac{1}{2}\mathbf{V} \times \mathbb{1} \tag{1.2.197}$$

Now the important thing to note is that

$$\begin{aligned} \text{dyad}(\text{vec}(\mathbf{VW})) &= -\frac{1}{2}\mathbb{1} \times (\mathbf{VW})_\times = -\frac{1}{2}\mathbb{1} \times (\mathbf{V} \times \mathbf{W}) = \frac{-1}{2}[(\mathbb{1} \cdot \mathbf{W})\mathbf{V} - (\mathbb{1} \cdot \mathbf{V})\mathbf{W}] \\ &= \frac{\mathbf{VW} - \mathbf{WV}}{2} = (\mathbf{VW})_\text{A} \end{aligned} \tag{1.2.198}$$

$$\text{dyad}(\text{vec}(\overleftrightarrow{\mathbf{T}})) = \overleftrightarrow{\mathbf{T}}_\text{A} \tag{1.2.199}$$

The BAC-CAB rule for $\mathbb{1} \times (\mathbf{V} \times \mathbf{W}) = (\mathbf{V} \times \mathbf{W}) \times \mathbb{1}$ is not obvious and you should check it yourself with Einstein summation notation.

Finally, the identity

$$\text{vec}(\text{dyad}(\mathbf{V})) = \text{vec}(-\frac{1}{2}\mathbb{1} \times \mathbf{V}) = (-\frac{1}{2}\mathbb{1} \times \mathbf{V})_\times = \mathbf{V} \tag{1.2.200}$$

is what one might naively expect. Again, showing it for yourself with index notation is a good idea (you will use $\delta_{ii} = 3$).

We can find the covariant T_{ij}, contravariant T^{ij}, and mixed representations ($T^{j}_{\cdot i}$ and $T^{i}_{\cdot j}$ where the $\cdot$ in front of an index indicates that the other index is the first one) via the use of the double dot product with tangent and reciprocal basis vectors

$$T_{ij} = \overleftrightarrow{\mathbf{T}} : \mathbf{e}^j\mathbf{e}^i \tag{1.2.201}$$

$$T^{ij} = \overleftrightarrow{\mathbf{T}} : \mathbf{e}_j\mathbf{e}_i \tag{1.2.202}$$

$$T^{i}_{\cdot j} = \overleftrightarrow{\mathbf{T}} : \mathbf{e}^j\mathbf{e}_i \tag{1.2.203}$$

$$T^{\cdot i}_{j} = \overleftrightarrow{\mathbf{T}} : \mathbf{e}_j\mathbf{e}^i \tag{1.2.204}$$

These follow the same rules for transformation as vectors of the same name, namely

$$T'^{ij} = \frac{\partial\zeta'^i}{\partial\zeta^k}\frac{\partial\zeta'^j}{\partial\zeta^l}T^{kl} \tag{1.2.205}$$

$$T'_{ij} = \frac{\partial\zeta^k}{\partial\zeta'^i}\frac{\partial\zeta^l}{\partial\zeta'^j}T_{kl} \tag{1.2.206}$$

$$T'^{i}_{\cdot j} = \frac{\partial\zeta'^i}{\partial\zeta^k}\frac{\partial\zeta^l}{\partial\zeta'^j}T^{k}_{\cdot l} \tag{1.2.207}$$

$$T'^{\cdot i}_{j} = \frac{\partial\zeta^l}{\partial\boldsymbol{\prime}\zeta^j}\frac{\partial\zeta'^i}{\partial\zeta^k}T^{\cdot k}_{l} \tag{1.2.208}$$

For higher order objects we can use multiple dot products to form the covariant, contravariant and mixed representations. Note that when the components of $\overleftrightarrow{\mathbf{T}}$ are represented as purely contravariant they are called tensor components of type $(2,0)$, when they are purely covariant the components are called type $(0,2)$, and for either mixed representation they are type $(1,1)$. This also generalized so that tensor components of type (p,q) have p upper indices (so p tangent basis vectors) and q lower indices (so q reciprocal basis vectors).

With this diversion, let's now look at tensor calculus with these objects.

1.2.10 Tensor Calculus

Tensor calculus identities are summarized in many books, but here we will focus on giving meaning to some of the symbols I have been using before. For example, the del or nabla operator ∇. This symbol is a representation of a certain type of derivative for vector and tensor quantities. In Cartesian coordinates one can remember its form via associating $\nabla \doteq \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$ so that the differential operators act on whatever is to the right of the del. Thus for a general scalar function f we would have

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{y}} \tag{1.2.209}$$

The problem with defining it this way is that in other coordinate systems you cannot write ∇ as just derivatives and the associated coordinate direction. What should ∇ be thought of as, then? We should recognize it as an operator mnemonic for three different operations, taking a gradient, taking the divergence, and taking the curl of objects. I will make this a bit more clear by using $\boldsymbol{\nabla}\cdot$ and $\boldsymbol{\nabla}\times$ for divergence and curl as a subtle reminder that these are three different types of operators and the use of a single symbol is for convenience. In addition, I will use a less-often used notation that is extremely convenient. As we saw above, I would use $\frac{\partial}{\partial \mathbf{x}} \equiv \nabla$ with $\frac{\partial}{\partial \mathbf{x}}\cdot$ and $\frac{\partial}{\partial \mathbf{x}}\times$ as the divergence and curl, respectively. This is often useful because we can then easily see that $\frac{\partial \mathbf{x}}{\partial \mathbf{x}} = \nabla\mathbf{x} = \mathbb{1}$ and that the chain rule becomes easier to recognize.

1.2.10.1 Gradient

So now let's consider the gradient. Let's focus on scalars first. The gradient is a type of derivative, so we can think of it in general curvilinear coordinates as looking at the change along each coordinate direction. Thus

$$\nabla f = \frac{\partial f}{\partial \mathbf{x}} = \frac{\partial f}{\partial \xi^i}\frac{\partial \xi^i}{\partial \mathbf{x}} = \frac{\partial f}{\partial \xi^i}\nabla\xi^i = \frac{\partial f}{\partial \xi^i}\mathbf{e}^i \tag{1.2.210}$$

Thus, the covariant representation of ∇f is especially simple. That is

$$(\nabla f) = (\nabla f \cdot \mathbf{e}_i)\mathbf{e}^i = \frac{\partial f}{\partial \xi^j}\mathbf{e}^j \cdot \mathbf{e}_i\mathbf{e}^i = \frac{\partial f}{\partial \xi^j}\delta^j_i\mathbf{e}^i = \frac{\partial f}{\partial \xi^i}\mathbf{e}^i \tag{1.2.211}$$

$$(\nabla f)_i = \frac{\partial f}{\partial \xi^i} \tag{1.2.212}$$

The contravariant representation is

$$\nabla f = \nabla f \cdot \mathbf{e}^i\mathbf{e}_i = \frac{\partial f}{\partial \xi^j}\mathbf{e}^j \cdot \mathbf{e}^i\mathbf{e}_i = \frac{\partial f}{\partial \xi^j}g^{ij}\mathbf{e}_i \tag{1.2.213}$$

$$(\nabla f)^i = \frac{\partial f}{\partial \xi^j}g^{ij} \tag{1.2.214}$$

where we remember $g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j$ (which is clearly symmetric) and $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$, and finally $g^i_j = \delta^i_j = \mathbf{e}^i \cdot \mathbf{e}_j$.

We can then move on to vectors. Given a vector $\mathbf{V}$ we then have

$$\begin{aligned}\nabla\mathbf{V} &= \frac{\partial\mathbf{V}}{\partial\mathbf{x}} = \frac{\partial\xi^i}{\partial\mathbf{x}}\frac{\partial\mathbf{V}}{\partial\xi^i} = \nabla\xi^i\frac{\partial\mathbf{V}}{\partial\xi^i} = \mathbf{e}^i\frac{\partial\mathbf{V}}{\partial\xi^i} \\ &= \mathbf{e}^i\frac{\partial}{\partial\xi^i}\left[V^j\mathbf{e}_j\right] = \frac{\partial V^j}{\partial\xi^i}\mathbf{e}^i\mathbf{e}_j + V^j\mathbf{e}^i\frac{\partial\mathbf{e}_j}{\partial\xi^i} \\ &= \mathbf{e}^i\frac{\partial}{\partial\xi^i}\left[V_j\mathbf{e}^j\right] = \frac{\partial V_j}{\partial\xi^i}\mathbf{e}^i\mathbf{e}^j + V_j\mathbf{e}^i\frac{\partial\mathbf{e}^j}{\partial\xi^i}\end{aligned} \tag{1.2.215}$$

It is important to note that to keep with our convention we must put the $\nabla\xi^i = \mathbf{e}^i$ on the left so that when doing $\mathbf{A}\cdot\nabla\mathbf{V}$ we have the components of $\mathbf{A}$ dotted into the gradient and not into the $\mathbf{V}$. As this quantity is a second order tensor, we can of course use whatever mixed, contravariant, or covariant representation that we wish. The complexities occur because $\frac{\partial\mathbf{e}^i}{\partial\xi^j}$ and $\frac{\partial\mathbf{e}_i}{\partial\xi^j}$ are not known until we decide upon what the ξ^i are. Both $\mathbf{e}^i$ and $\mathbf{e}_i$ generally change in space so we cannot take them outside of the derivative term. We will consider the gradient of a vector in more detail in Section 1.2.10.5. For now we use the Christoffel symbols of the first and second kind

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial\xi^j} + \frac{\partial g_{jk}}{\partial\xi^i} - \frac{\partial g_{ij}}{\partial\xi^k}\right] \tag{1.2.216}$$

$$\Gamma^k_{ij} = \frac{g^{kl}}{2}\left[\frac{\partial g_{il}}{\partial\xi^j} + \frac{\partial g_{jl}}{\partial\xi^i} - \frac{\partial g_{ij}}{\partial\xi^l}\right] \tag{1.2.217}$$

and write

$$(\nabla\mathbf{V})^{\cdot k}_{j} \equiv \left(\frac{\partial\mathbf{V}}{\partial\xi^j}\right)^k\mathbf{e}^j\mathbf{e}_k = \left[\frac{\partial V^k}{\partial\xi^j} + V^i\Gamma^k_{ij}\right]\mathbf{e}^j\mathbf{e}_k \tag{1.2.218}$$

$$(\nabla\mathbf{V})_{jk} \equiv \left(\frac{\partial\mathbf{V}}{\partial\xi^j}\right)_k\mathbf{e}^j\mathbf{e}^k = \left[\frac{\partial V_k}{\partial\xi^j} - V_i\Gamma^i_{kj}\right]\mathbf{e}^j\mathbf{e}^k \tag{1.2.219}$$

$$(\nabla\mathbf{V})^{jk} \equiv \left(\frac{\partial\mathbf{V}}{\partial\xi^j}\right)^k g^{jl}\mathbf{e}_l\mathbf{e}_k = \left[\frac{\partial V^k}{\partial\xi^j}g^{jl} + V^i g^{jl}\Gamma^k_{ij}\right]\mathbf{e}_l\mathbf{e}_k \tag{1.2.220}$$

$$(\nabla\mathbf{V})^{j}_{\cdot k} \equiv \left(\frac{\partial\mathbf{V}}{\partial\xi^j}\right)_k g^{jl}\mathbf{e}_l\mathbf{e}^k = \left[\frac{\partial V_k}{\partial\xi^j}g^{jl} - V_i g^{jl}\Gamma^i_{kj}\right]\mathbf{e}_l\mathbf{e}^k \tag{1.2.221}$$

1.2.10.2 Curl

We will use that the curl of a gradient is zero and that the curl of a scalar times a vector can be decomposed. Then we can write

$$\boldsymbol{\nabla}\times\mathbf{e}^i = \boldsymbol{\nabla}\times\nabla\xi^i = 0 \tag{1.2.222}$$

$$\boldsymbol{\nabla}\times f\mathbf{V} = \nabla f\times\mathbf{V} + f\boldsymbol{\nabla}\times\mathbf{V} \tag{1.2.223}$$

Thus for a vector represented covariantly, we have

$$\boldsymbol{\nabla}\times\mathbf{V} = \frac{\partial}{\partial\mathbf{x}}\times\mathbf{V} = \boldsymbol{\nabla}\times\left(V_i\mathbf{e}^i\right) = \cancel{V_i\boldsymbol{\nabla}\times\mathbf{e}^i} + \nabla V_i\times\mathbf{e}^i = \frac{\partial V_i}{\partial\xi^j}\nabla\xi^j\times\mathbf{e}^i = \frac{\partial V_i}{\partial\xi^j}\mathbf{e}^j\times\mathbf{e}^i \tag{1.2.224}$$

Note that but i and j are dummy variables because they are summed over. Using the reciprocal vector relations (1.2.64)-(1.2.65) with the Levi-Civita symbol ϵ^{ijk} we then have

$$\boldsymbol{\nabla}\times\mathbf{V} = \frac{\epsilon^{ijk}}{\mathcal{J}}\frac{\partial V_j}{\partial\xi^i}\mathbf{e}_k \tag{1.2.225}$$

via

$$\mathbf{e}^i \times \mathbf{e}^j = \begin{cases} 0 & i = j \\ \mathbf{e}_k/\mathcal{J} & \text{if } (i,j,k) \text{ are an even permutation of (1,2,3)} \\ -\mathbf{e}_k/\mathcal{J} & \text{if } (i,j,k) \text{ are an odd permutation of (1,2,3)} \end{cases} \tag{1.2.226}$$

For tensors we can use the same sort of calculation. We have when the tensor is represented covariantly that (using (B.156))

$$\begin{aligned} \boldsymbol{\nabla} \times \overset{\leftrightarrow}{\mathbf{T}} &= \boldsymbol{\nabla} \times (T_{ij}\mathbf{e}^i\mathbf{e}^j) = \boldsymbol{\nabla} \times (T_{ij}\mathbf{e}^i)\mathbf{e}^j - T_{ij}\mathbf{e}^i \times \nabla\mathbf{e}^j \\ &= \frac{\partial T_{ij}}{\partial \xi^k}\mathbf{e}^k \times \mathbf{e}^i\mathbf{e}^j - T_{ij}\mathbf{e}^i \times \mathbf{e}^k\frac{\partial \mathbf{e}^j}{\partial \xi^k} \\ &= \frac{\partial T_{jl}}{\partial \xi^i}\mathbf{e}^i \times \mathbf{e}^j\mathbf{e}^l - T_{il}\mathbf{e}^i \times \mathbf{e}^j\frac{\partial \mathbf{e}^l}{\partial \xi^j} \\ &= \frac{\epsilon^{ijk}}{\mathcal{J}}\left(\frac{\partial T_{jl}}{\partial \xi^i}\mathbf{e}_k\mathbf{e}^l - T_{il}\frac{\partial \mathbf{e}^l}{\partial \xi^j}\right) \\ &= \frac{\epsilon^{ijk}}{\mathcal{J}}\mathbf{e}_k\left(\frac{\partial T_{jl}}{\partial \xi^i}\mathbf{e}^l + T_{il}\Gamma^l_{jp}\mathbf{e}^p\right) = \frac{\epsilon^{ijk}}{\mathcal{J}}\mathbf{e}_k\mathbf{e}^l\left(\frac{\partial T_{jl}}{\partial \xi^i} + T_{ip}\Gamma^p_{jl}\right) \\ &= \frac{\epsilon^{ijk}}{\mathcal{J}}\mathbf{e}_k\left(\frac{\partial T_{jl}}{\partial \xi^i}\mathbf{e}^l + T_{il}g^{pr}\Gamma^l_{jp}\mathbf{e}_r\right) \end{aligned} \tag{1.2.227}$$

While not very simple, this is not a terribly difficult form to deal with. Some other representations of $\overset{\leftrightarrow}{\mathbf{T}}$ yield nice forms, as well. If we use $\overset{\leftrightarrow}{\mathbf{T}} = T_i^{\cdot j}\mathbf{e}^i\mathbf{e}_j$ we find

$$\begin{aligned} \boldsymbol{\nabla} \times \overset{\leftrightarrow}{\mathbf{T}} &= \frac{\epsilon^{ijk}}{\mathcal{J}}\mathbf{e}_k\mathbf{e}_l\left(\frac{\partial T_j^{\cdot l}}{\partial \xi^i} - T_i^{\cdot p}\Gamma^p_{jl}\right) \\ \boldsymbol{\nabla} \times \overset{\leftrightarrow}{\mathbf{T}} &= \frac{\epsilon^{ijk}}{\mathcal{J}}\mathbf{e}_k\left(\frac{\partial T_j^{\cdot l}}{\partial \xi^i}\mathbf{e}_l - T_i^{\cdot p}\Gamma_{p,jl}\mathbf{e}^l\right) \end{aligned} \tag{1.2.228}$$

In Cartesian coordinates any formula will yield

$$\boldsymbol{\nabla} \times \overset{\leftrightarrow}{\mathbf{T}} = \epsilon^{ijk}\frac{\partial T_{jl}}{\partial x^i}\hat{\mathbf{x}}_k\hat{\mathbf{x}}^l = \epsilon_{ijk}\frac{\partial T_{jl}}{\partial x^i}\hat{\mathbf{x}}_k\hat{\mathbf{x}}_l = \epsilon_{ijk}\frac{\partial T_{kl}}{\partial x^j}\hat{\mathbf{x}}_i\hat{\mathbf{x}}_l \tag{1.2.229}$$

which is what we would get assuming that $\boldsymbol{\nabla}\times$ acts on the first index of $\overset{\leftrightarrow}{\mathbf{T}}$. One should be wary that in some literature the curl acts on the second index. I will not even write down such a thing so as not to cause any confusion.

1.2.10.3 Divergence

Now let's consider the divergence. This can be applied to tensors or vectors. First we'll consider vectors. Then we can use that the divergence of a curl is zero, so that with

$$\mathbf{e}_{i'} = \mathcal{J}\,\nabla\xi^{j'} \times \nabla\xi^{k'} = \mathcal{J}\mathbf{e}^{j'} \times \mathbf{e}^{k'} \tag{1.2.230}$$

$$\nabla\xi^{j'} \times \nabla\xi^{k'} = \boldsymbol{\nabla} \times \left(\xi^{j'}\,\nabla\xi^{k'}\right) \tag{1.2.231}$$

$$\mathbf{e}_{i'}/\mathcal{J} = \mathbf{e}^{j'} \times \mathbf{e}^{k'} = \boldsymbol{\nabla} \times \left(\xi^{j'}\,\nabla\xi^{k'}\right) \tag{1.2.232}$$

Thus, if we take the divergence, we get zero and hence

$$\boldsymbol{\nabla}\cdot(\mathbf{e}_{i'}/\mathcal{J})=0 \tag{1.2.233}$$

and so

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{V}=\boldsymbol{\nabla}\cdot(V^j\mathbf{e}_j)&=\boldsymbol{\nabla}\cdot\left(\left[\mathcal{J}V^j\right]\frac{\mathbf{e}_j}{\mathcal{J}}\right)=\mathcal{J}V^j\boldsymbol{\nabla}\cdot\cancel{\left(\frac{\mathbf{e}_j}{\mathcal{J}}\right)}+\frac{\mathbf{e}_j}{\mathcal{J}}\cdot\nabla\left(\mathcal{J}V^j\right)\\&=\frac{\mathbf{e}_j}{\mathcal{J}}\cdot\frac{\partial(\mathcal{J}V^j)}{\partial\xi^k}\mathbf{e}^k=\frac{\delta^k_j}{\mathcal{J}}\frac{\partial\left(\mathcal{J}V^j\right)}{\partial\xi^k}=\frac{1}{\mathcal{J}}\frac{\partial(\mathcal{J}V^j)}{\partial\xi^j}\end{aligned} \tag{1.2.234}$$

In addition, there is another form that is sometimes useful. As we have shown above $\nabla\mathbf{V}=\nabla\xi^i\frac{\partial\mathbf{V}}{\partial\xi^i}$ therefore

$$\mathbb{1}:\nabla\mathbf{V}=\boldsymbol{\nabla}\cdot\mathbf{V}=\nabla\xi^i\cdot\frac{\partial\mathbf{V}}{\partial\xi^i}=\frac{\partial\mathbf{V}}{\partial\xi^i}\cdot\nabla\xi^i=\frac{\partial\mathbf{V}}{\partial\xi^i}\cdot\mathbf{e}^i \tag{1.2.235}$$

And we can use for a tensor that (with (B.67)).

$$\begin{aligned}\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}&=\boldsymbol{\nabla}\cdot(T^{ij}\mathbf{e}_i\mathbf{e}_j)=\boldsymbol{\nabla}\cdot(T^{ij}\mathbf{e}_i)\mathbf{e}_j+T^{ij}\mathbf{e}_i\cdot\nabla\mathbf{e}_j\\&=\frac{1}{\mathcal{J}}\frac{\partial(\mathcal{J}T^{ij})}{\partial\xi^i}\mathbf{e}_j+T^{ij}\mathbf{e}_i\cdot\mathbf{e}^k\frac{\partial\mathbf{e}_j}{\partial\xi^k}=\frac{1}{\mathcal{J}}\frac{\partial(\mathcal{J}T^{ij})}{\partial\xi^i}\mathbf{e}_j+T^{ij}\delta^k_i\Gamma^l_{jk}\mathbf{e}_l\\&=\frac{1}{\mathcal{J}}\frac{\partial(\mathcal{J}T^{ij})}{\partial\xi^i}\mathbf{e}_j+T^{ij}\Gamma^l_{ji}\mathbf{e}_l\\&=\left(\frac{1}{\mathcal{J}}\frac{\partial(\mathcal{J}T^{ij})}{\partial\xi^i}+T^{il}\Gamma^j_{li}\right)\mathbf{e}_j\end{aligned} \tag{1.2.236}$$

Which is a fairly simple form in the end. One other useful form is given by

$$\begin{aligned}\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}&=\boldsymbol{\nabla}\cdot(T^i_{\cdot j}\mathbf{e}_i\mathbf{e}^j)=\boldsymbol{\nabla}\cdot(T^i_{\cdot j}\mathbf{e}_i)\mathbf{e}^j+T^i_{\cdot j}\mathbf{e}_i\cdot\nabla\mathbf{e}^j\\&=\frac{1}{\mathcal{J}}\frac{\partial(\mathcal{J}T^i_{\cdot j})}{\partial\xi^i}\mathbf{e}^j+T^i_{\cdot j}\mathbf{e}_i\cdot\mathbf{e}^k\frac{\partial\mathbf{e}^j}{\partial\xi^k}=\frac{1}{\mathcal{J}}\frac{\partial(\mathcal{J}T^i_{\cdot j})}{\partial\xi^i}\mathbf{e}^j-T^i_{\cdot j}\delta^k_i\Gamma^l_{jk}\mathbf{e}^l\\&=\frac{1}{\mathcal{J}}\frac{\partial(\mathcal{J}T^i_{\cdot j})}{\partial\xi^i}\mathbf{e}^j-T^i_{\cdot j}\Gamma^l_{ji}\mathbf{e}^l\\&=\left(\frac{1}{\mathcal{J}}\frac{\partial(\mathcal{J}T^i_{\cdot j})}{\partial\xi^i}-T^i_{\cdot l}\Gamma^j_{li}\right)\mathbf{e}^j\end{aligned} \tag{1.2.237}$$

One last cautionary note is in order, again. Online and in some texts you will see $\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}^{\intercal}$ as the "divergence". This has been done so that Gauss's Law can be written simply as

$$\cancel{\iiint\operatorname{div}(\overleftrightarrow{\mathbf{T}})\,\mathrm{d}V=\iiint\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}^{\intercal}\,\mathrm{d}V=\oiint\overleftrightarrow{\mathbf{T}}\cdot\,\mathrm{d}\mathbf{S}} \tag{1.2.238}$$

We don't need to worry about such problems in our text because we have $\iint\mathrm{d}\mathbf{S}\cdot\mathbf{f}$ rather than $\iint\mathbf{f}\cdot\mathrm{d}\mathbf{S}$ so the normal vector applies to the tensor from the left. This will be more fully discussed in the next section.

1.2.10.4 Derivatives on Tensors

Let's actually talk about derivatives when applied to tensors. For this textbook, an operator acting on a tensor acts on the nearest index to its right. Thus, as we saw in the previous Section 1.2.10.3 we have in Cartesian coordinates

$$\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}} = \hat{\mathbf{x}}_k\cdot\frac{\partial(T^{ij}\hat{\mathbf{x}}_i\hat{\mathbf{x}}_j)}{\partial x^k} = \hat{\mathbf{x}}_k\cdot\hat{\mathbf{x}}^i\frac{\partial T_{ij}}{\partial x^k}\hat{\mathbf{x}}_j = \delta_{ik}\frac{\partial T_{ij}}{\partial x^k}\hat{\mathbf{x}}_j = \frac{\partial T_{ij}}{\partial x^i}\hat{\mathbf{x}}_j \tag{1.2.239}$$

and ***not*** on the second index so

$$\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}} \neq \frac{\partial T^{ij}}{\partial x^j}\hat{\mathbf{e}}_i \tag{1.2.240}$$

This latter definition can be convenient in certain situations (when you put the normal associated with a differential surface to the right side in an integral, this definition does not require tensors to be transposed when dotting into it), but becomes a source of confusion because then the del operator is no longer directly operating on the thing to its right. Thus, expressions become more difficult to understand in more complicated expressions. If we wish to act on the second index of a tensor, it makes more sense to use a special symbol such as ∇_R or transposes so that (again with Cartesian vectors only)

$$\boldsymbol{\nabla}_R\cdot\overleftrightarrow{\mathbf{T}} = \boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}^{\intercal} = \frac{\partial T^{ij}\hat{\mathbf{e}}_j}{\partial x^j} \tag{1.2.241}$$

Luckily, such an operation is rarely needed, and so this will be the only time this operator $\boldsymbol{\nabla}_R\cdot$ is seen in this book. With the principle that operators act on the nearest index we see the generalizations for cross products as well

$$(\mathbf{V}\times\overleftrightarrow{\mathbf{T}}) = V_i\mathbf{e}^i\times T_{jk}\mathbf{e}^j\mathbf{e}^k = V^i\mathbf{e}_i\times T^{jk}\mathbf{e}_j\mathbf{e}_k = \cdots \tag{1.2.242}$$

with all the different representations of course possible. However the most useful representations are

$$(\mathbf{V}\times\overleftrightarrow{\mathbf{T}})^{\cdot l}_k = \epsilon_{ijk}JV^iT^{jl} \tag{1.2.243}$$

$$(\mathbf{V}\times\overleftrightarrow{\mathbf{T}})_{kl} = \epsilon_{ijk}JV^iT^j_{\cdot l} \tag{1.2.244}$$

$$(\mathbf{V}\times\overleftrightarrow{\mathbf{T}})^{kl} = \epsilon^{ijk}\mathcal{J}V_iT^{\cdot l}_j \tag{1.2.245}$$

$$(\mathbf{V}\times\overleftrightarrow{\mathbf{T}})^k_{\cdot l} = \epsilon^{ijk}\mathcal{J}V_iT_{jl} \tag{1.2.246}$$

$$(\overleftrightarrow{\mathbf{T}}\times\mathbf{V})_{il} = \epsilon_{jkl}JT^{\cdot j}_iV^k \tag{1.2.247}$$

$$(\overleftrightarrow{\mathbf{T}}\times\mathbf{V})^i_{\cdot l} = \epsilon_{jkl}JT^{ij}V^k \tag{1.2.248}$$

$$(\overleftrightarrow{\mathbf{T}}\times\mathbf{V})^{il} = \epsilon^{jkl}\mathcal{J}T^i_{\cdot l}V_k \tag{1.2.249}$$

$$(\overleftrightarrow{\mathbf{T}}\times\mathbf{V})^{\cdot l}_i = \epsilon^{jkl}\mathcal{J}T^i_{\cdot j}V_k \tag{1.2.250}$$

One can also look at the curl, but it is defined analogously, but not often used. See Section 1.2.10.2 for how the curl acts on a tensor.

1.2.10.5 Vector Differentiation

Now let's focus on vector differentiation and explain what $\partial \mathbf{e}_i/\partial \xi^j$ means. Once again we will see the danger of confusion when we confuse the components of a vector with a geometric vector. When we have a vector $\mathbf{V}$, we can represent it however we wish, covariantly $\mathbf{V} = V_i \mathbf{e}^i$ or contravariantly $\mathbf{V} = V^i \mathbf{e}_i$. We can write the differential as

$$\mathrm{d}\mathbf{V} = \frac{\partial \mathbf{V}}{\partial \xi^i}\,\mathrm{d}\xi^i \tag{1.2.251}$$

and so the partial derivates can thus be formed via substitution for contravariant and covariant representations

$$\frac{\partial \mathbf{V}}{\partial \xi^j} = \frac{\partial}{\partial \xi^j}\left[V^i \mathbf{e}_i\right] = \frac{\partial V^i}{\partial \xi^j}\mathbf{e}_i + V^i \frac{\partial \mathbf{e}_i}{\partial \xi^j} \tag{1.2.252}$$

$$\frac{\partial \mathbf{V}}{\partial \xi^j} = \frac{\partial}{\partial \xi^j}\left[V_i \mathbf{e}^i\right] = \frac{\partial V_i}{\partial \xi^j}\mathbf{e}^i + V_i \frac{\partial \mathbf{e}^i}{\partial \xi^j} \tag{1.2.253}$$

Now, we can then find what these vectors are in contravariant form

$$\begin{aligned}\frac{\partial \mathbf{V}}{\partial \xi^j}\cdot \mathbf{e}^k \mathbf{e}_k &= \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)^k \mathbf{e}_k = \frac{\partial V^i}{\partial \xi^j}\mathbf{e}_i \cdot \mathbf{e}^k \mathbf{e}_k + V^i \frac{\partial \mathbf{e}_i}{\partial \xi^j}\cdot \mathbf{e}^k \mathbf{e}_k \\ &= \frac{\partial V^k}{\partial \xi^j}\mathbf{e}_k + V^i \frac{\partial \mathbf{e}_i}{\partial \xi^j}\cdot \mathbf{e}^k \mathbf{e}_k\end{aligned} \tag{1.2.254}$$

or covariant form

$$\begin{aligned}\frac{\partial \mathbf{V}}{\partial \xi^j}\cdot \mathbf{e}_k \mathbf{e}^k &= \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)_k \mathbf{e}^k = \frac{\partial V_i}{\partial \xi^j}\mathbf{e}^i \cdot \mathbf{e}_k \mathbf{e}^k + V_i \frac{\partial \mathbf{e}^i}{\partial \xi^j}\cdot \mathbf{e}_k \mathbf{e}^k \\ &= \frac{\partial V_k}{\partial \xi^j}\mathbf{e}^k + V_i \frac{\partial \mathbf{e}^i}{\partial \xi^j}\cdot \mathbf{e}_k \mathbf{e}^k\end{aligned} \tag{1.2.255}$$

It is less useful, but also possible to find alternate contravariant and covariant forms

$$\begin{aligned}\frac{\partial \mathbf{V}}{\partial \xi^j}\cdot \mathbf{e}^k \mathbf{e}_k &= \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)^k \mathbf{e}_k = \frac{\partial V_i}{\partial \xi^j}\mathbf{e}^i \cdot \mathbf{e}^k \mathbf{e}_k + V_i \frac{\partial \mathbf{e}^i}{\partial \xi^j}\cdot \mathbf{e}^k \mathbf{e}_k \\ &= \frac{\partial V_i}{\partial \xi^j} g^{ik}\mathbf{e}_k + V_i \frac{\partial \mathbf{e}^i}{\partial \xi^j}\cdot \mathbf{e}^k \mathbf{e}_k\end{aligned} \tag{1.2.256}$$

$$\begin{aligned}\frac{\partial \mathbf{V}}{\partial \xi^j}\cdot \mathbf{e}_k \mathbf{e}^k &= \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)_k \mathbf{e}^k = \frac{\partial V^i}{\partial \xi^j}\mathbf{e}_i \cdot \mathbf{e}_k \mathbf{e}^k + V^i \frac{\partial \mathbf{e}_i}{\partial \xi^j}\cdot \mathbf{e}_k \mathbf{e}^k \\ &= \frac{\partial V^i}{\partial \xi^j} g_{ik}\mathbf{e}^k + V^i \frac{\partial \mathbf{e}_i}{\partial \xi^j}\cdot \mathbf{e}_k \mathbf{e}^k\end{aligned} \tag{1.2.257}$$

which require metric elements from g^{ij} or g_{ij}.

We can then note that if we want to use the contravariant components of $\mathbf{V}$ for computations we should use (1.2.254). If we want to use the covariant components of $\mathbf{V}$ for the computation we use (1.2.255).

The notation and terminology used is that $(\partial \mathbf{V}/\partial \xi^j)^k$ and $(\partial \mathbf{V}/\partial \xi^j)_k$ are the components of the covariant derivative of $\mathbf{V}$. It is called a covariant derivative because if you view the components

of a vector as what a vector is, then the derivative of a vector is simply the derivative of its components. The covariant derivative then takes into account that $\mathbf{e}_i$ and $\mathbf{e}^i$ can vary in space. Thus a covariant derivative of $\mathbf{V}$ is not the same as a derivative of the components of $\mathbf{V}$. Thus we see that the "derivative of $\mathbf{V}$" is confusing because some people mean that $\mathbf{V}$ is its components, in which case the extra terms coming from basis vectors are not easy to see as necessary. In our formulation where $\mathbf{V}$ is a geometric object, then it is clear that the derivative (indeed, the covariant derivative) is the proper object for looking at changes because we explicitly show our coordinates in the representation through the basis vectors.

People then call

$$\begin{aligned}\mathrm{d}\mathbf{V} &= \frac{\partial \mathbf{V}}{\partial \xi^j}\,\mathrm{d}\xi^j \\ &= \frac{\partial V^k}{\partial \xi^j}\mathbf{e}_k\,\mathrm{d}\xi^j + V^i\frac{\partial \mathbf{e}_i}{\partial \xi^j}\cdot\mathbf{e}^k\mathbf{e}_k\,\mathrm{d}\xi^j \\ &= \frac{\partial V_k}{\partial \xi^j}\mathbf{e}^k\,\mathrm{d}\xi^j + V_i\frac{\partial \mathbf{e}^i}{\partial \xi^j}\cdot\mathbf{e}_k\mathbf{e}^k\,\mathrm{d}\xi^j\end{aligned} \tag{1.2.258}$$

the absolute differential or the intrinsic differential.

These "extra" terms are intuitive and transparent with vectors as geometrical objects, but very surprising when thinking of vectors as arrays of numbers. When vectors are components, one must "derive" the extra terms by enforcing that the vector components satisfy the covariant or contravariant transformation rules between coordinate systems. These terms can then seem as rather obscure extra terms added to the natural vector derivative, rather than terms that make perfect sense from a geometric point of view.

In order to make sure that the full, that is covariant derivatives are used, a new notation is often used with indices where

$$V^k_{;j} \equiv \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)^k = \frac{\partial V^k}{\partial \xi^j} + V^i\frac{\partial \mathbf{e}_i}{\partial \xi^j}\cdot\mathbf{e}^k \tag{1.2.259}$$

$$V_{k;j} \equiv \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)_k = \frac{\partial V_k}{\partial \xi^j} + V_i\frac{\partial \mathbf{e}^i}{\partial \xi^j}\cdot\mathbf{e}^k \tag{1.2.260}$$

Let's now explore how to write these in terms of the metric coefficients. We can write

$$\frac{\partial \mathbf{e}_i}{\partial \xi^j} = \frac{\partial \mathbf{e}_i}{\partial \xi^j}\cdot\mathbf{e}^k\mathbf{e}_k = \left(\frac{\partial \mathbf{e}_i}{\partial \xi^j}\right)^k\mathbf{e}_k \tag{1.2.261}$$

$$\frac{\partial \mathbf{e}_i}{\partial \xi^j} = \frac{\partial \mathbf{e}_i}{\partial \xi^j}\cdot\mathbf{e}_k\mathbf{e}^k = \left(\frac{\partial \mathbf{e}_i}{\partial \xi^j}\right)_k\mathbf{e}^k \tag{1.2.262}$$

We note that we can swap the j and k indices without changing things in the following term

$$\frac{\partial \mathbf{e}_k}{\partial \xi^j} = \frac{\partial}{\partial \xi^j}\frac{\partial \mathbf{x}}{\partial \xi^k} = \frac{\partial}{\partial \xi^k}\frac{\partial \mathbf{x}}{\partial \xi^j} = \frac{\partial \mathbf{e}_j}{\partial \xi^k} \tag{1.2.263}$$

We can then use that $\mathbf{e}^k = g^{kl}\mathbf{e}_l$ so that

$$\frac{\partial \mathbf{e}_i}{\partial \xi^j}\cdot\mathbf{e}^k = \frac{\partial \mathbf{e}_i}{\partial \xi^j}\cdot g^{kl}\mathbf{e}_l \tag{1.2.264}$$

We can then split this in half and use (1.2.263) so that

$$\frac{\partial \mathbf{e}_i}{\partial \xi^j} \cdot \mathbf{e}^k = \frac{\partial \mathbf{e}_i}{\partial \xi^j} \cdot g^{kl}\mathbf{e}_l = \frac{g^{kl}}{2}\left[\mathbf{e}_l \cdot \frac{\partial \mathbf{e}_i}{\partial \xi^j} + \mathbf{e}_l \cdot \frac{\partial \mathbf{e}_i}{\partial \xi^j}\right] = \frac{g^{kl}}{2}\left[\mathbf{e}_l \cdot \frac{\partial \mathbf{e}_i}{\partial \xi^j} + \mathbf{e}_l \cdot \frac{\partial \mathbf{e}_j}{\partial \xi^i}\right] \tag{1.2.265}$$

Now we can continue some changes. We'd like to get the $\mathbf{e}_i$ and $\mathbf{e}_l$ into dot products with each other. We can use

$$\frac{\partial \mathbf{e}_i \cdot \mathbf{e}_l}{\partial \xi^j} = \mathbf{e}_l \cdot \frac{\partial \mathbf{e}_i}{\partial \xi^j} + \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_l}{\partial \xi^j} \tag{1.2.266}$$

$$\frac{\partial \mathbf{e}_j \cdot \mathbf{e}_l}{\partial \xi^i} = \mathbf{e}_l \cdot \frac{\partial \mathbf{e}_j}{\partial \xi^i} + \mathbf{e}_j \cdot \frac{\partial \mathbf{e}_l}{\partial \xi^i} \tag{1.2.267}$$

Thus,

$$\frac{\partial \mathbf{e}_i}{\partial \xi^j} \cdot \mathbf{e}^k = \frac{g^{kl}}{2}\left[\mathbf{e}_l \cdot \frac{\partial \mathbf{e}_i}{\partial \xi^j} + \mathbf{e}_l \cdot \frac{\partial \mathbf{e}_j}{\partial \xi^i}\right] = \frac{g^{kl}}{2}\left[\frac{\partial(\mathbf{e}_i \cdot \mathbf{e}_l)}{\partial \xi^j} - \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_l}{\partial \xi^j} + \frac{\partial\left(\mathbf{e}_j \cdot \mathbf{e}_l\right)}{\partial \xi^i} - \mathbf{e}_j \cdot \frac{\partial \mathbf{e}_l}{\partial \xi^i}\right] \tag{1.2.268}$$

We then use

$$\mathbf{e}_i \cdot \frac{\partial \mathbf{e}_l}{\partial \xi^j} + \mathbf{e}_j \cdot \frac{\partial \mathbf{e}_l}{\partial \xi^i} = \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial \xi^l} + \mathbf{e}_j \cdot \frac{\partial \mathbf{e}_i}{\partial \xi^l} = \frac{\partial(\mathbf{e}_i \cdot \mathbf{e}_j)}{\partial \xi^l} \tag{1.2.269}$$

So that

$$\begin{aligned}\frac{\partial \mathbf{e}_i}{\partial \xi^j} \cdot \mathbf{e}^k &= \frac{g^{kl}}{2}\left[\frac{\partial(\mathbf{e}_i \cdot \mathbf{e}_l)}{\partial \xi^j} + \frac{\partial\left(\mathbf{e}_j \cdot \mathbf{e}_l\right)}{\partial \xi^i} - \frac{\partial(\mathbf{e}_i \cdot \mathbf{e}_j)}{\partial \xi^l}\right] \\ &= \frac{g^{kl}}{2}\left[\frac{\partial g_{il}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^l}\right]\end{aligned} \tag{1.2.270}$$

We can then use

$$\frac{\partial}{\partial \xi^j}\left(\mathbf{e}_i \cdot \mathbf{e}^k\right) = \frac{\partial \delta_i^k}{\partial \xi^j} = 0 \tag{1.2.271}$$

So

$$\mathbf{e}_i \cdot \frac{\partial \mathbf{e}^k}{\partial \xi^j} = -\mathbf{e}^k \cdot \frac{\partial \mathbf{e}_i}{\partial \xi^j} \tag{1.2.272}$$

and we can write

$$\frac{\partial \mathbf{e}^i}{\partial \xi^j} \cdot \mathbf{e}_k = -\frac{g^{il}}{2}\left[\frac{\partial g_{kl}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^k} - \frac{\partial g_{kj}}{\partial \xi^l}\right] \tag{1.2.273}$$

If we rewrite our covariant derivatives with metric components, we find

$$V^k_{;j} \equiv \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)^k = \frac{\partial V^k}{\partial \xi^j} + V^i \frac{g^{kl}}{2}\left[\frac{\partial g_{il}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^l}\right] \tag{1.2.274}$$

$$V_{k;j} \equiv \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)_k = \frac{\partial V_k}{\partial \xi^j} - V_i \frac{g^{il}}{2}\left[\frac{\partial g_{kl}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^k} - \frac{\partial g_{kj}}{\partial \xi^l}\right] \tag{1.2.275}$$

Because people don't like writing all of that above, people have defined symbols, called the Christoffel symbols, in order to simplify notation. Sometimes the Christoffel symbols are called connection coefficients or affine connections. These are just symbolic notations for the part in square brackets above. There are multiple notations for them, but it should be remembered that these symbols are not tensors themselves.[35]

The Christoffel symbols of the second kind are defined via

$$\mathbf{e}^k \cdot \frac{\partial \mathbf{e}_i}{\partial \xi^j} \equiv \left\{ \begin{array}{cc} k \\ i & j \end{array} \right\} = \Gamma^k_{ij} = \frac{g^{kl}}{2}\left[\frac{\partial g_{il}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^l}\right] \tag{1.2.276}$$

The Christoffel symbols of the first kind are similar to the previous symbols but dotting the tangent basis vectors on the outside

$$\mathbf{e}_k \cdot \frac{\partial \mathbf{e}_i}{\partial \xi^j} \equiv \left[\begin{array}{cc} k \\ i & j \end{array} \right] = [k, ij] = \Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right] \tag{1.2.277}$$

This can be seen by going back to (1.2.264) and then following the derivation without the factor of g^{kl} included. Because of our identity (1.2.263), it is clear that the indices i and j above can be switched without changing the values, so

$$\Gamma^k_{ij} = \left\{ \begin{array}{cc} k \\ i & j \end{array} \right\} = \left\{ \begin{array}{cc} k \\ j & i \end{array} \right\} = \Gamma^k_{ji} \tag{1.2.278}$$

$$\Gamma_{k,ij} = [k, ij] = \left[\begin{array}{cc} k \\ i & j \end{array} \right] = \left[\begin{array}{cc} k \\ j & i \end{array} \right] = [k, ji] = \Gamma_{k,ji} \tag{1.2.279}$$

It is also clear from this that

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{1.2.280}$$

$$\Gamma_{k,ij} = g_{kl}\Gamma^l_{ij} \tag{1.2.281}$$

The Christoffel symbols of the first kind tend to be less useful, though it can be used for mixed tensor representations. Since ∇ is most easily represented as $\hat{\mathbf{e}}^i \frac{\partial}{\partial \xi^i}$, using $\Gamma_{k,ij}$ is not as easy of a form to work with.

It should be noted that given a coordinate system the above can be written as

$$(\nabla \mathbf{V})^{\cdot k}_{j} \equiv \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)^k \mathbf{e}^j \mathbf{e}_k = \left[\frac{\partial V^k}{\partial \xi^j} + V^i \Gamma^k_{ij}\right] \mathbf{e}^j \mathbf{e}_k \tag{1.2.282}$$

$$(\nabla \mathbf{V})_{jk} \equiv \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)_k \mathbf{e}^j \mathbf{e}^k = \left[\frac{\partial V_k}{\partial \xi^j} - V_i \Gamma^i_{kj}\right] \mathbf{e}^j \mathbf{e}^k \tag{1.2.283}$$

$$(\nabla \mathbf{V})^{jk} \equiv \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)^k g^{jl} \mathbf{e}_l \mathbf{e}_k = \left[\frac{\partial V^k}{\partial \xi^j} g^{jl} + V^i g^{jl} \Gamma^k_{ij}\right] \mathbf{e}_l \mathbf{e}_k \tag{1.2.284}$$

$$(\nabla \mathbf{V})^{j}_{\cdot k} \equiv \left(\frac{\partial \mathbf{V}}{\partial \xi^j}\right)_k g^{jl} \mathbf{e}_l \mathbf{e}^k = \left[\frac{\partial V_k}{\partial \xi^j} g^{jl} - V_i g^{jl} \Gamma^i_{kj}\right] \mathbf{e}_l \mathbf{e}^k \tag{1.2.285}$$

[35] In addition, some people use a different definition for the Christoffel symbols. So always check that any literature you are using is using my definition, which is by far the most common.

as the contravariant, covariant, and mixed representations of the tensor $\nabla\mathbf{V}$.

To summarize the rest of our results

$$\frac{\partial \mathbf{e}_i}{\partial \xi^j} = \frac{\partial \mathbf{e}_i}{\partial \xi^j} \cdot \mathbf{e}^k \mathbf{e}_k = \Gamma^k_{ij} \mathbf{e}_k \tag{1.2.286}$$

$$\frac{\partial \mathbf{e}^i}{\partial \xi^j} = \frac{\partial \mathbf{e}^i}{\partial \xi^j} \cdot \mathbf{e}_k \mathbf{e}^k = -\frac{\partial \mathbf{e}_k}{\partial \xi^j} \cdot \mathbf{e}^i \mathbf{e}^k = -\Gamma^i_{jk} \mathbf{e}^k = -\Gamma^i_{kj} \mathbf{e}^k \tag{1.2.287}$$

$$\frac{\partial \mathbf{e}_i}{\partial \xi^j} = \frac{\partial \mathbf{e}_i}{\partial \xi^j} \cdot \mathbf{e}_k \mathbf{e}^k = \Gamma_{k,ij} \mathbf{e}^k \tag{1.2.288}$$

$$\frac{\partial \mathbf{e}^i}{\partial \xi^j} = \frac{\partial \mathbf{e}^i}{\partial \xi^j} \cdot \mathbf{e}^k \mathbf{e}_k = \frac{\partial \mathbf{e}^i}{\partial \xi^j} \cdot g^{kl} \mathbf{e}_l \mathbf{e}_k = -g^{kl} \frac{\partial \mathbf{e}_l}{\partial \xi^j} \cdot \mathbf{e}^i \mathbf{e}_k = -g^{kl} \Gamma^i_{lj} \mathbf{e}_k = -g^{kl} \Gamma^i_{jl} \mathbf{e}_k \tag{1.2.289}$$

$$\Gamma^k_{ij} = \frac{g^{kl}}{2} \left[\frac{\partial g_{il}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^l} \right] = g^{kl} \Gamma_{l,ij} \tag{1.2.290}$$

$$\Gamma_{k,ij} = \frac{1}{2} \left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k} \right] = g_{kl} \Gamma^l_{ij} \tag{1.2.291}$$

For your convenience, I will also list them with swapped indices because different texts decide on different orders of i, j, k.

$$\frac{\partial \mathbf{e}_j}{\partial \xi^k} = \Gamma^i_{jk} \mathbf{e}_i \tag{1.2.292}$$

$$\frac{\partial \mathbf{e}^j}{\partial \xi^k} = -\Gamma^i_{jk} \mathbf{e}^i \tag{1.2.293}$$

$$\frac{\partial \mathbf{e}_j}{\partial \xi^k} = \Gamma_{i,jk} \mathbf{e}^i \tag{1.2.294}$$

$$\frac{\partial \mathbf{e}^j}{\partial \xi^k} = -g^{il} \Gamma^j_{kl} \mathbf{e}_l \tag{1.2.295}$$

$$\Gamma^i_{jk} = \frac{g^{il}}{2} \left[\frac{\partial g_{jl}}{\partial \xi^k} + \frac{\partial g_{kl}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^l} \right] = g^{il} \Gamma_{l,jk} \tag{1.2.296}$$

$$\Gamma_{i,jk} = \frac{1}{2} \left[\frac{\partial g_{ij}}{\partial \xi^k} + \frac{\partial g_{ik}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^i} \right] = g_{il} \Gamma^l_{jk} \tag{1.2.297}$$

and

$$\frac{\partial \mathbf{e}_i}{\partial \xi^k} = \Gamma^j_{ik} \mathbf{e}_j \tag{1.2.298}$$

$$\frac{\partial \mathbf{e}^i}{\partial \xi^k} = -\Gamma^j_{ik} \mathbf{e}^j \tag{1.2.299}$$

$$\frac{\partial \mathbf{e}_i}{\partial \xi^k} = \Gamma_{j,ik} \mathbf{e}^j \tag{1.2.300}$$

$$\frac{\partial \mathbf{e}^i}{\partial \xi^k} = -g^{jl} \Gamma^i_{kl} \mathbf{e}_l \tag{1.2.301}$$

$$\Gamma^j_{ik} = \frac{g^{jl}}{2} \left[\frac{\partial g_{il}}{\partial \xi^k} + \frac{\partial g_{kl}}{\partial \xi^i} - \frac{\partial g_{ik}}{\partial \xi^l} \right] = g^{jl} \Gamma_{l,ik} \tag{1.2.302}$$

$$\Gamma_{j,ik} = \frac{1}{2} \left[\frac{\partial g_{ij}}{\partial \xi^k} + \frac{\partial g_{jk}}{\partial \xi^j} - \frac{\partial g_{ik}}{\partial \xi^j} \right] = g_{jl} \Gamma^l_{ik} \tag{1.2.303}$$

1.2.11 Useful Relations

Some useful relations used for integration along lines, surfaces, and volumes can be derived from differentials among the coordinates. For arclengths, we care about arclengths along a coordinate curve. Because we are along a specific coordinate curve (with index i') we can write

$$\mathrm{d}l_{i'} \equiv |\,\mathrm{d}\mathbf{x}^{i'}\,| \equiv \sqrt{\frac{\partial \mathbf{x}}{\partial \xi^{i'}}\,\mathrm{d}\xi^{i'} \cdot \frac{\partial \mathbf{x}}{\partial \xi^{i'}}\,\mathrm{d}\xi^{i'}} = \sqrt{\mathbf{e}_{i'}\,\mathrm{d}\xi^{i'} \cdot \mathbf{e}_{i'}\,\mathrm{d}\xi^{'}} \tag{1.2.304}$$

using differentials. That is, we have along our arc $\mathrm{d}l_{i'}$ used that $\xi^{i'}$ is the only varying coordinate and so $\mathrm{d}\xi^{j'} = \mathrm{d}\xi^{k'} = 0$. Thus

$$\mathrm{d}l_{i'} = \sqrt{\mathbf{e}_{i'} \cdot \mathbf{e}_{i'}}\,\mathrm{d}\xi^{i'} = \sqrt{g_{i'i'}}\,\mathrm{d}\xi^{i'} = h_{i'}\,\mathrm{d}\xi^{i'} \tag{1.2.305}$$

We can then use the reciprocal formulas for $\mathbf{e}_i$ from (1.2.64)-(1.2.65) to find

$$\mathbf{e}_{i'} \cdot \mathbf{e}_{i'} = \mathcal{J}^2 |\mathbf{e}^{j'} \times \mathbf{e}^{k'}|^2 \tag{1.2.306}$$

to write

$$\mathrm{d}l_{i'} = \mathcal{J}|\mathbf{e}^{j'} \times \mathbf{e}^{k'}|\,\mathrm{d}\xi^{i'} = \mathcal{J}|\,\nabla\xi^{j'} \times \nabla\xi^{k'}|\,\mathrm{d}\xi^{i'} \tag{1.2.307}$$

Next we can find the differential area along a surface where $\xi^{i'}$ is constant. Thus only $\xi^{j'}$ and $\xi^{k'}$ are allowed to vary and $\mathrm{d}\xi^{i'} = 0$. We can thus use $\mathrm{d}S_{i'}$ as the definition of a surface along constant $\xi^{i'}$.

$$\begin{aligned} \mathrm{d}S_{i'} &\equiv |\,\mathrm{d}\mathbf{X}_{j'} \times \mathrm{d}\mathbf{X}_{k'}| = \left|\frac{\partial \mathbf{X}}{\partial \xi^{j'}} \times \frac{\partial \mathbf{X}}{\partial \xi^{k'}}\right| \mathrm{d}\xi^{j'}\,\mathrm{d}\xi^{k'} \\ &= |\mathbf{e}_{j'} \times \mathbf{e}_{k'}|\,\mathrm{d}\xi^{j'}\,\mathrm{d}\xi^{k'} \end{aligned} \tag{1.2.308}$$

We can then use the reciprocal vector relations (1.2.64)-(1.2.65) to find

$$\mathbf{e}_{j'} \times \mathbf{e}_{k'} = \mathcal{J}\mathbf{e}^{i'} = \mathcal{J}\,\nabla\xi^{i'} \tag{1.2.309}$$

and (remember that this is along a surface where $\xi^{i'}$ is constant)

$$\mathrm{d}\mathbf{S}_{i'} = \pm\mathcal{J}\,\nabla\xi^{i'}\,\mathrm{d}\xi^{j'}\,\mathrm{d}\xi^{k'} \tag{1.2.310}$$

$$\mathrm{d}S_{i'} = \mathcal{J}|\,\nabla\xi^{i'}|\,\mathrm{d}\xi^{j'}\,\mathrm{d}\xi^{k'} \tag{1.2.311}$$

When using a normal surface, one must choose the outward direction appropriately against $\xi^{i'}$. That is, we must choose the outward normal based (and so $\pm$) on what direction $\nabla\xi^{i'}$ points in.

One other format is using

$$|\mathbf{e}_{j'} \times \mathbf{e}_{k'}|^2 = (\mathbf{e}_{j'} \times \mathbf{e}_{k'}) \cdot (\mathbf{e}_{j'} \times \mathbf{e}_{k'}) \tag{1.2.312}$$

Then we can use the vector relation

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{1.2.313}$$

so that

$$\mathrm{d}S_{i'} = \sqrt{g_{j'j'}g_{k'k'} - (g_{j'k'})^2}\,\mathrm{d}\xi^{j'}\,\mathrm{d}\xi^{k'} \tag{1.2.314}$$

Finally, the differential volume is simply given by

$$\mathrm{d}^3x = \mathrm{d}\mathbf{x}_1 \cdot \mathrm{d}\mathbf{x}_2 \times \mathrm{d}\mathbf{x}_3 = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi^3 = \mathcal{J}\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi^3 \tag{1.2.315}$$

1.2.12 Pseudovectors and Pseudoscalars

There is another way of dividing up vectors, scalars, and even tensors that is a purely geometric property. "Proper" vectors, are then often called polar vectors, true vectors, or just "vectors". These polar vectors have the property that when reflected across the origin, the orientation of the vector also reflects [given coordinates $\mathbf{x} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ then a polar vector transforms as $\mathbf{V}(-\mathbf{x}) = -\mathbf{V}(\mathbf{x})$)]. An axial vector or pseudovector is a type of vector that remains invariant so that $\mathbf{v}(-\mathbf{x}) = \mathbf{v}(\mathbf{x})$. A polar vector and pseudovector both transform the same (as geometric vectors) under "proper rotations" which just means under normal rotation with no reflection. If we write out a coordinate transformation as a matrix R for our particular coordinate system, then $\det \mathsf{R} = 1$ is a proper rotation and $\det \mathsf{R} = -1$ is an improper rotation (so like a reflection, perhaps with a rotation, though). Using matrices is the more general definition, but a parity inversion ($\mathbf{x} \to -\mathbf{x}$) is the easiest test for a geometric vector being a polar vector or pseudovector type.[36]

Similarly we can define a pseudoscalar g as a scalar object such that $g(-\mathbf{x}) = -g(\mathbf{x})$ whereas a true scalar or polar scalar[37] f is a scalar type that does not change under this improper rotation $f(-\mathbf{x}) = f(\mathbf{x})$. An example of a pseudoscalar for polar vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ is given by $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$. This means the Jacobian determinant is in fact a pseudoscalar.

We can then define pseudotensors by building up a tensor from vectors and using the properties of the vectors (of either polar or pseudo types). In which case we call a tensor T of order N that transforms such that $\mathsf{T}(-\mathbf{x}) = (-1)^N \mathsf{T}(\mathbf{x})$ a true tensor type or polar tensor type. A pseudotensor or axial tensor S of order N transforms such that $\mathsf{S}(-\mathbf{x}) = (-1)^{N+1} \mathsf{S}(\mathbf{x})$.

Physically, we must have pseudotensors equal pseudotensors and tensors equal tensors or else the equations are not true under improper rotations (so parity inversion, or mirroring). Thus, this can serve as a check that equations are correct, like dimensional analysis. This is simply a statement that we expect the physics of a situation not to change should we watch it in a mirror. In fact, some physical processes may not respect this coordinate transformation, but in plasma physics we do not expect parity inversion to alter the physics of the situation. Unfortunately there is no separate notation for polar and pseudo types of scalars, vectors, and tensors, so that one must either consult a list or test each scalar, vector, or tensor on its own to determine if it is a polar type or pseudo type.

I will show how polar and pseudovectors are related under common operations. Similar relations apply for tensors and pseudotensors in general.

First, when adding pseudovectors, the result is a pseudovector and similarly for vectors. Let vectors be upper case $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and pseudovectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Let $\mathbf{S}$ or $\mathbf{s}$ be a test vector (so it could be a polar vector type or a pseudovector type). Then

$$\mathbf{S}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \mathbf{B}(\mathbf{x}) \tag{1.2.316}$$

$$\mathbf{S}(-\mathbf{x}) = \mathbf{A}(-\mathbf{x}) + \mathbf{B}(-\mathbf{x}) = -[\mathbf{A}(\mathbf{x}) + \mathbf{B}(\mathbf{x})] = -\mathbf{S}(\mathbf{x}) \tag{1.2.317}$$

[36] I may sometimes say pseudovector or polar vector rather than pseudovector type or polar vector type. It is shorter, and in my nomenclature both pseudovectors and polar vectors are types of vectors.

[37] I have never heard someone use "polar" scalar, but I introduce it so I can use it consistently for scalars, vectors, and tensors.

showing $\mathbf{S}$ is a vector. Similarly for pseudovectors we find

$$\mathbf{s}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x}) \tag{1.2.318}$$
$$\mathbf{s}(-\mathbf{x}) = \mathbf{a}(-\mathbf{x}) + \mathbf{b}(-\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) \tag{1.2.319}$$

so $\mathbf{s}$ is a pseudovector in this case. If we do a mix of both types then the resultant quantity is neither a polar vector or a pseudovector. We see this via

$$\mathbf{S}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \mathbf{B}(\mathbf{x}) \tag{1.2.320}$$
$$\mathbf{S}(-\mathbf{x}) = \mathbf{a}(-\mathbf{x}) + \mathbf{B}(-\mathbf{x}) = \mathbf{a}(\mathbf{x}) - \mathbf{B}(\mathbf{x}) \neq \begin{cases} \mathbf{S}(\mathbf{x}) \\ -\mathbf{S}(\mathbf{x}) \end{cases} \tag{1.2.321}$$

Therefore $\mathbf{S}$ in this case is a vector of neither the polar vector type or the pseudovector type.

Then for dot products with polar vectors we find

$$f(\mathbf{x}) = \mathbf{A}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) \tag{1.2.322}$$
$$f(-\mathbf{x}) = \mathbf{A}(-\mathbf{x}) \cdot \mathbf{B}(-\mathbf{x}) = \mathbf{A}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) = f(\mathbf{x}) \tag{1.2.323}$$

so f is a scalar. For dot products with pseudovectors we find

$$f(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{x}) \tag{1.2.324}$$
$$f(-\mathbf{x}) = \mathbf{a}(-\mathbf{x}) \cdot \mathbf{b}(-\mathbf{x}) = -\mathbf{a}(\mathbf{x}) \cdot (-\mathbf{b}(\mathbf{x})) = f(\mathbf{x}) \tag{1.2.325}$$

and f is a scalar again. For a mixed dot product

$$g(\mathbf{x}) = \mathbf{A}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{x}) \tag{1.2.326}$$
$$g(-\mathbf{x}) = \mathbf{A}(-\mathbf{x}) \cdot \mathbf{b}(-\mathbf{x}) = \mathbf{A}(\mathbf{x}) \cdot (-\mathbf{b}(\mathbf{x})) = -g(\mathbf{x}) \tag{1.2.327}$$
$$g(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) \tag{1.2.328}$$
$$g(-\mathbf{x}) = \mathbf{a}(-\mathbf{x}) \cdot \mathbf{B}(-\mathbf{x}) = -\mathbf{a}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) = -g(\mathbf{x}) \tag{1.2.329}$$

so g is a pseudoscalar.

We use that for two polar vectors in a cross product we have

$$\mathbf{s}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) \tag{1.2.330}$$
$$\mathbf{s}(-\mathbf{x}) = \mathbf{A}(-\mathbf{x}) \times \mathbf{B}(-\mathbf{x}) = [-\mathbf{A}(\mathbf{x})] \times [-\mathbf{B}(\mathbf{x})] = \mathbf{A}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) \tag{1.2.331}$$

and so indeed two polar vectors in a cross product produces a pseudovector. For two pseudovectors we find

$$\mathbf{s}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \times \mathbf{b}(\mathbf{x}) \tag{1.2.332}$$
$$\mathbf{s}(-\mathbf{x}) = \mathbf{a}(-\mathbf{x}) \times \mathbf{b}(-\mathbf{x}) = \mathbf{a}(\mathbf{x}) \times \mathbf{b}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) \tag{1.2.333}$$

and so a pseudovector once again. For a mix we have

$$\mathbf{S}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) = -\mathbf{B}(\mathbf{x}) \times \mathbf{a}(\mathbf{x}) \tag{1.2.334}$$
$$\mathbf{S}(-\mathbf{x}) = \mathbf{a}(-\mathbf{x}) \times \mathbf{B}(-\mathbf{x}) = \mathbf{a}(\mathbf{x}) \times [-\mathbf{B}(\mathbf{x})] = -\mathbf{a} \times \mathbf{B}(\mathbf{x}) = -\mathbf{S}(\mathbf{x}) \tag{1.2.335}$$
$$-\mathbf{S}(-\mathbf{x}) = \mathbf{B}(-\mathbf{x}) \times \mathbf{a}(-\mathbf{x}) = [-\mathbf{B}(\mathbf{x})] \times \mathbf{a}(\mathbf{x}) = -\mathbf{B}(\mathbf{x}) \times \mathbf{a}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) \tag{1.2.336}$$

showing that **S** is a polar vector when you cross a pseudovector with a polar vector.

As I have stated, if you expect inversion to not change the physics then you can use these properties to check that equations are consistent. If you think physics does not respect inversion, then you have discovered a breaking of a usual symmetry and should decide if it is physically realistic and explain the new physics.

We can also consider physics under time reversal (a parity inversion for time). We require most classical physical equations to be invariant under time reversal. This implies that some quantities have $q(-t) = -q(t)$. We will call such a quantity odd in time reversal, whereas $q(-t) = q(t)$ is even.

We summarize our results and show time reversal in Table 1.5.

Object	Definition	Type	Time Reversal
polar scalar	$f(-\mathbf{x}) = f(\mathbf{x})$		
pseudoscalar	$g(-\mathbf{x}) = -g(\mathbf{x})$		
polar vector	$\mathbf{A}(-\mathbf{x}) = -\mathbf{A}(\mathbf{x})$		
pseudovector	$\mathbf{a}(-\mathbf{x}) = \mathbf{a}(\mathbf{x})$		
order N polar tensor	$\mathbf{T}(-\mathbf{x}) = (-1)^N\mathbf{T}(\mathbf{x})$		
order N pseudotensor	$\mathbf{t}(-\mathbf{x}) = (-1)^{N+1}\mathbf{t}(\mathbf{x})$		
$f \cdot f$	scalar multiplication	polar scalar	
$f \cdot g$	scalar multiplication	pseudoscalar	
$g \cdot g$	scalar multiplication	polar scalar	
$\mathbf{A} \cdot \mathbf{B}$	dot product	polar scalar	
$\mathbf{A} \cdot \mathbf{b}$	dot product	pseudoscalar	
$\mathbf{a} \cdot \mathbf{b}$	dot product	polar scalar	
$\mathbf{A} \times \mathbf{B}$	cross product	pseudovector	
$\mathbf{A} \cdot \mathbf{b}$	cross product	polar vector	
$\mathbf{a} \cdot \mathbf{b}$	cross product	pseudovector	
$\mathbf{x}$	position	polar vector	even
$\mathbf{v}$	velocity	polar vector	odd
$\mathbf{p}$	momentum	polar vector	odd
$\mathbf{L}$	angular momentum	polar vector	odd
$\mathbf{F}$	Force	polar vector	even
$\boldsymbol{\tau}$	torque	pseudovector	even
$mv^2/2 = p^2/(2m)$	kinetic energy	polar scalar	even
$U(\mathbf{x})$	potential energy	polar scalar	even
ρ_q	electric charge density	polar scalar	even
$\mathbf{J}$	current density	polar vector	odd
$\mathbf{E}$	electric field	polar vector	even
$\mathbf{A}$	magnetic vector potential	polar vector	odd
$\mathbf{B}$	magnetic field	pseudovector	odd
$\mathbf{H}$	H field	pseudovector	odd
$\mathbf{E} \times \mathbf{H}$	Poynting vector	polar vector	odd
$\overleftrightarrow{\boldsymbol{\sigma}}$	Maxwell stress tensor	polar tensor	even

Table 1.5: The vector types are shown for various operations. Here f is a polar scalar, g is a pseudoscalar, and lower case tensors are pseudotensors and upper case tensors are polar tensors. Examples of physical quantities of various types are also shown.

1.3 Fourier Series

What would life be without arithmetic, but a scene of horrors?

— Sydney Smith

In this chapter we review the definition of Fourier series and delve into their use in toroidal coordinate systems. In doing so, we will see how to interpret m and n, poloidal and toroidal Fourier components, respectively, (often called "modes"[38]) for plasma dynamics. We will then consider the Fourier transforms and Laplace transforms. Finally, we will consider the ballooning transform, which is used in specific approximation schemes and is related to the Fourier transform.

1.3.1 Definition

Let's begin with the motivation and definition of a Fourier series. This will lead into the concept of a Fourier transform (and later a Laplace transform), which have a stunning array of uses in mathematical physics. To start, we will start with a function f that is differentiable and square integrable. To a certain degree, some of these assumptions can be relaxed mathematically, but as a physicist, I will assume any physical situation will satisfy the constraints necessary to make the definition sensible. For convenience, we won't consider pathological cases, and so "nice" functions that are differentiable as many times as we need will be considered.

In one dimension, let's consider $f(x)$ for the real variable x. Suppose we wish to approximate $f(x)$. Let's also suppose that we only care about x over a finite domain from $[-L/2, L/2]$ (so of length L). In fact, we will just consider the function to be periodic in length L, so that $[-3L/2, -L/2]$ and $[L/2, 3L/2]$ are identical to $[-L/2, L/2]$, which enforces $f(-L/2) = f(L/2)$.[39] Then how can we approximate such a function? We can always default to a Taylor series, but they become inaccurate as we get far from the point we are approximating. We want something that better approximates the function over the entire domain. Instead, we can look at the function, and pick out the important wavenumbers and use those to approximate the function.[40] What does this mean? It means we write our function as a sum of wavenumbers. For example, if we choose to have $2N + 1$ wavenumber terms, we write

$$f(x) \approx \sum_{j=-N}^{N} f_j \exp(2i\pi jx/L) = \sum_j f_j \exp(ijkx) \tag{1.3.1}$$

where i is the imaginary unit and the wavenumber is often defined as $k = 2\pi/L$. The question then becomes what are these f_j that are now independent of x. We can use that

$$\int_{-\pi}^{\pi} \mathrm{d}x \;\; \exp(ijx)\exp(ij'x) = 2\pi\delta_{j,-j'} \tag{1.3.2}$$

[38] This naming convention is unfortunate since components are not necessarily modes, or eigenstates of a system.

[39] This is not, strictly speaking, necessary to form the series, but if it is not true you would almost certainly be better served reformulating the problem so that it does, like with a Chebyshev series. See especially Boyd[5], who is an excellent resource and excellent writer. He has a great sense of humor and his writing style is not boring. Fourier series are most accurate when dealing with periodic functions.

[40] If you prefer x to be time than space, then we look at the important frequencies rather than wavenumbers.

where $\delta_{j,-j'}$ equals 0 unless $j = -j'$. Thus if we multiply (1.3.1) by $\exp(-2i\pi j'x/L)$ and integrate over our domain we find

$$\int_{-L/2}^{L/2} \mathrm{d}x\ f(x)\exp(-2i\pi j'x/L) = \int_{-L/2}^{L/2} \mathrm{d}x\ \sum_{j=-N}^{N} f_j \exp(2i\pi jx/L)\exp(-2i\pi j'x/L) \tag{1.3.3}$$

We can use that f_j is independent of x and that the sum and integral can be interchanged[41] so that

$$\int_{-L/2}^{L/2} \mathrm{d}x\ f(x)\exp(2i\pi j'x/L) = \sum_{j=-N}^{N} f_j \int_{-L/2}^{L/2} \mathrm{d}x\ \exp(i2\pi jx/L)\exp(-2i\pi j'x/L) \tag{1.3.4}$$

define $y = 2\pi x/L$ and this becomes

$$\int_{-L/2}^{L/2} \mathrm{d}x\ f(x)\exp(-2i\pi j'x/L) = \sum_{j=-N}^{N} \frac{L}{2\pi} f_j \int_{-\pi}^{\pi} \mathrm{d}y\ \exp(ijy)\exp(-ij'y) = \sum_{j=-N}^{N} L\delta_{j,j'} f_j = Lf_{j'} \tag{1.3.5}$$

which can be rewritten as

$$f_j = \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x\ f(x)\exp(-2i\pi jx/L) \tag{1.3.6}$$

This means that [remember $f(x)$ is a real valued function here]

$$\begin{aligned} f_{-j} &= \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x\ f(x)\exp(-2i\pi(-j)x/L) = \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x\ f(x)\exp(2i\pi jx/L) \\ &= \left[\frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x\ f(x)\exp(-2i\pi jx/L)\right]^* = f_j^* \end{aligned} \tag{1.3.7}$$

for * the complex conjugation operation. In general, for multiple dimensions we can treat x, y, and z independently and so we'd find for $\mathbf{k} = 2\pi p/L_x\hat{\mathbf{x}} + 2\pi q/L_y\hat{\mathbf{y}} + 2\pi r/L_z\hat{\mathbf{z}}$ where $x \in [-L_x/2, L_x/2]$, $y \in [-L_y/2, L_y/2]$, and $z \in [-L_z/2, L_z/2]$

$$f(\mathbf{x}) \approx \sum_{p=-N_p}^{N_p}\sum_{q=-N_q}^{N_q}\sum_{r=-N_r}^{N_r} f_{p,q,r}\exp\left(2i\pi\left[px/L_x + qy/L_y + rz/L_z\right]\right) = \sum_{p=-N_p}^{N_p}\sum_{q=-N_q}^{N_q}\sum_{r=-N_r}^{N_r} f_{p,q,r}\exp(i\mathbf{k}\cdot\mathbf{x}) \tag{1.3.8}$$

$$f_{p,q,r} = \frac{1}{L_xL_yL_z}\iiint \mathrm{d}V\ f(\mathbf{x})\exp(-i\mathbf{k}\cdot\mathbf{x}) \tag{1.3.9}$$

Especially in physics, you will see a reduced notation where the summation is over $k_x = 2\pi p/L_x$, $k_y = 2\pi q/L_y$, and $k_z = 2\pi r/L_z$. The sum is then not over integers but the values allowed for k_x,

[41] If it were not possible to exchange the order of the integral and sum, that would mean that we must have some sort of divergent series or integral. Thus we are not working with "nice" functions.

k_y, and k_z. There is often an even further reduction using $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ with the assumed p, q, r dependences. Thus

$$f(\mathbf{x}) \approx \sum_{p=-N_p}^{N_p} \sum_{q=-N_q}^{N_q} \sum_{r=-N_r}^{N_r} f_{p,q,r} \exp(i\mathbf{k}\cdot\mathbf{x}) \equiv \sum_{k_x} \sum_{k_y} \sum_{k_z} f_{k_x,k_y,k_z} \exp(i\mathbf{k}\cdot\mathbf{x}) \equiv \sum_{k} f_{k_x,k_y,k_z} \exp(i\mathbf{k}\cdot\mathbf{x}) \tag{1.3.10}$$

$$f_{k_x,k_y,k_z} = \frac{1}{L_x L_y L_z} \iiint \mathrm{d}V \; f(\mathbf{x}) \exp(-i\mathbf{k}\cdot\mathbf{x}) \tag{1.3.11}$$

One is then left to determine what the permissible k and components of $\mathbf{k}$ are with this notation. This strong implicit component makes the notation less useful in direct calculation, but makes it easier to write and so perform manipulations for Fourier series formulas.

The Fourier series approximation allows us to do especially well with periodic functions. In many contexts people prefer to write the series only with sin and cos rather than an exponential. If we were to convert it to sines and cosines using the famous Euler formula $\exp(i\phi) = \cos\phi + \sin\phi$ we find

$$f(x) \approx \sum_{j=-N}^{N} f_j \exp(ijkx) = \sum_{j=-N}^{N} [f_j \cos(jkx) + i f_j \sin(jkx)] \tag{1.3.12}$$

We can then use $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ so

$$\begin{aligned} f(x) &\approx f_0 + \sum_{j=1}^{N} [(f_j + f_{-j})\cos(jkx) + i f_j \sin(jkx) + i f_{-j} \sin(-jkx)] \\ &\approx f_0 + \sum_{j=1}^{N} [(f_j + f_{-j})\cos(jkx) + i(f_j - f_{-j}) \sin(jkx)] \\ &\approx f_0 + \sum_{j=1}^{N} \left[(f_j + f_j^*)\cos(jkx) + i(f_j - f_j^*) \sin(jkx)\right] \end{aligned} \tag{1.3.13}$$

Where I have rearranged terms because when using sines and cosines we don't use negative values for j. Also remember $f_{-j} = f_j^*$. Perhaps unfortunately, the conventional definition for cosines and sines is given by

$$a_j = \frac{2}{L} \int_{-L/2}^{L/2} \mathrm{d}x \; f(x) \cos(kx) \tag{1.3.14}$$

$$b_j = \frac{2}{L} \int_{-L/2}^{L/2} \mathrm{d}x \; f(x) \sin(kx) \tag{1.3.15}$$

and so there is an extra factor of 2 to be accounted for so that for $j > 0$

$$a_j \equiv \frac{f_j + f_j^*}{2} = \Re f_j \tag{1.3.16}$$

$$b_j \equiv \frac{f_j - f_j^*}{2i} = \Im f_j \tag{1.3.17}$$

with $a_0 = 2f_0$ and $b_0 = 0$. So that the series is written as

$$f(x) \approx \frac{a_0}{2} + \sum_{j=1}^{N} a_j \cos\left(\frac{2\pi jx}{L}\right) + \sum_{j=1}^{N} b_j \sin\left(\frac{2\pi jx}{L}\right) \tag{1.3.18}$$

with the definitions for a_j and b_j given above.

As we let the $N_{p,q,r} \to \infty$ we get a more accurate approximation. If we think of our function's domain as expanding such that $L \to \infty$, as well, we get to the Fourier transform to be explored later in Section 1.3.3. It is worth noting here that $L \to \infty$ implies that $k \to 0$, so that we are considering smaller and smaller wavenumbers as the domain increases.

1.3.2 Series in Toroidal-like Coordinates

In plasma physics, we often use a coordinate system of (r, θ, ζ) where r is a radial-like variable, θ is a poloidal variable, and ζ is a toroidal variable. For a torus, r is the minor radius, θ is the angle that goes around the torus the short way, and ζ is the angle that goes around the torus the long way. Because in many applications r is a flux function, we only care about variation along that surface, and so we wish to Fourier expand only in the angles and not the radial-like flux coordinate variable r. We define $f_{r'}(\theta, \zeta) \equiv g(r = r', \theta, \zeta)$ and then drop the subscript r' as this is done for any r of interest. We desire

$$f(\theta, \zeta) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_{m,n} \exp(i(m\theta - n\zeta)) \tag{1.3.19}$$

where I have used a fairly common definition of the Fourier modes for θ and ζ. Unfortunately there is no convention universally followed in plasma physics for m and n Fourier series. This is for a variety of reasons including that it is not necessarily consistent with the Fourier series defined above, but also having to do with the positive directions of θ and ζ. One could imagine three other reasonable choices

$$f(\theta, \zeta) \stackrel{?}{=} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_{m,n} \exp(i(m\theta + n\zeta)) \tag{1.3.20}$$

$$f(\theta, \zeta) \stackrel{?}{=} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_{m,n} \exp(i(-m\theta + n\zeta)) \tag{1.3.21}$$

$$f(\theta, \zeta) \stackrel{?}{=} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_{m,n} \exp(-i(m\theta + n\zeta)) \tag{1.3.22}$$

It is a hazard of the profession that there is no standard definition and one must then carefully follow negative signs if an author diverges from the definition in (1.3.19). The reason for the choice in (1.3.19) is so that the safety factor for plasmas (we will cover this later in Section 2.2.3.1) is defined as a positive value. The safety factor on rational surfaces then simply becomes $q = m/n$ with this definition. When using (1.3.20) one must realize that $q = -m/n$. With (1.3.21) $q = (-m)/(-n) = m/n$ again,[42] but the helicity is in the opposite direction of the q in (1.3.19).

[42] The cancelling minus signs emphasize the opposite helicity.

Similarly, (1.3.22) yields $q = m/(-n) = -m/n$ the same value as for (1.3.20), but with opposite helicity.[43]

Using our "standard" definition, then we can work out what the function would look like spatially. If certain $f_{m,n}$ are more prominent, then one speaks of the (poloidal) $m = M$ "mode"[44] being dominant or the (toroidal) $n = N$ "mode" being dominant. If a specific combination of Fourier components is dominant then we can speak of the (M, N) "mode" being dominant. Once again, caution must be taken when interpreting anything of the form (M, N). While poloidal "mode" numbers often precede toroidal "mode" numbers in the parentheses, it is by no means universal.[45] In addition, some will write n/m instead of m/n when talking of the "mode" structure. I will now speak of Fourier components instead of "modes" so that there is no misinterpretations that they must represent the solution of a linear eigenmode.[46]

Now that we have explained some of the nomenclature, let's look at some plots and explain how plasma physicists often characterize them. The simple explanation is that if you are shown an (R, Z) plot, that is a plot at constant ζ, then you count the number of peaks in a periodic shape going around the magnetic axis clockwise or counterclockwise at a certain "radius" (not necessarily the circular radius, but a constant for a radial-like variable), and this will equal the poloidal component number M. If someone plots a constant Z surface, then you count the number of peaks around the torus and this will be the toroidal component number N.

This can easily be seen if you are able to divide up the value you are looking at into a steady state or equilibrium value[47] with a perturbation on top. The full images are shown on the left for $m = 0, 1, 2, 3$ in Figure 1.4 and Figure 1.5. One can see that there is a circular structure around the peak value (if the minor radius $r = 0$ is located at the peak value, then the structure is around $r = 0.4$ to $r = 0.5$). As we go around this circular structure we will see the values form peaks and troughs. The number of peaks (or troughs) gives us m. This is much clearer when we can separate out these circular structure perturbations, as seen in the right of the figures with Perturbation for the titles. It is easy to count the peaks (or troughs) and see what $m = M$ structures look like.

1.3.2.1 Perturbations and Fourier Series

There is also one other way that Fourier components are used to describe plasma behavior. This is to explain how the perturbation of the plasma from a perfect torus (or a perfect cylinder) appears. We will consider a cylinder for easier visualization (one "just" needs to connect the ends of the

[43] Generally this is not a problem because $q > 0$ or $q < 0$ completely and so losing the sign does not lose any information about the geometry of the problem, but if q has both negative and positive values, such as in a reverse field pinch, then it is important to know the sign of q.

[44] Technically speaking, unless the function's Fourier components are solutions to the linear eigenmodes, these are not modes. They are simply Fourier components and should be called such; however, it is common to hear of the Fourier components being called modes, and one should be prepared to encounter this abuse of notation. I will put mode in quotation marks to emphasize the fact that these are Fourier components and not necessarily actual modes.

[45] That is, if you are given $(3, 1)$ you have to see what the author is using. It could be (M, N) or (N, M).

[46] There is some nuance between the terms eigenmode and eigenfunction, but it almost never presents a problem to understanding. The important idea is that one gets an eigenvalue that characterizes the solution. In fact, it would be more appropriate in English to call these characteristic values, characteristic modes, and characteristic functions, but we have universally adopted the German *eigen* instead.

[47] These are usually interpreted as the linear eigenfunction. In nonlinear settings, it is less clear what is meant in general. It may mean perturbations upon an MHD equilibrium state.

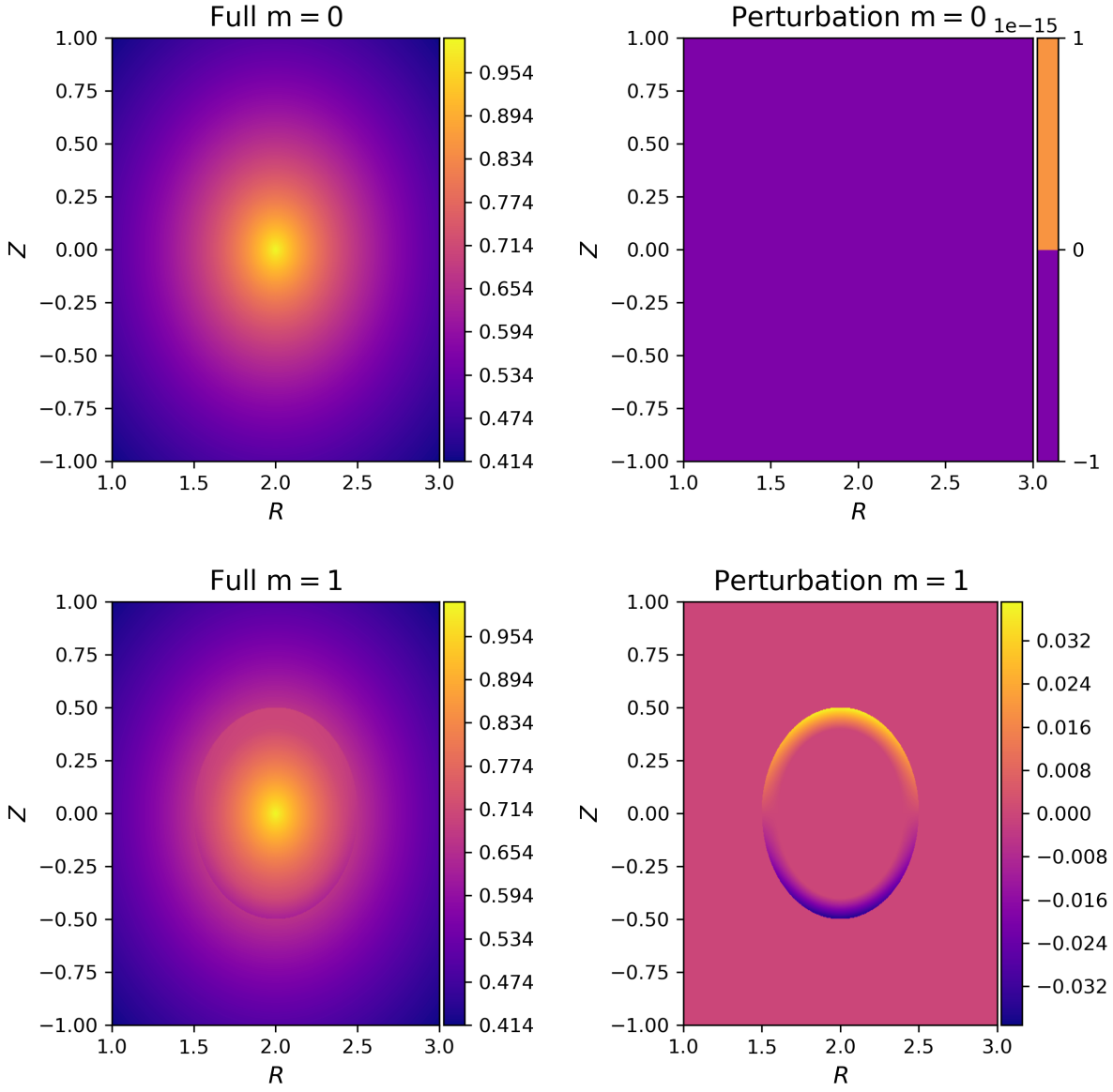

Figure 1.4: These images show how to find Fourier components of structures from contour plots for $m = 0$ and $m = 1$. These will show poloidal structure because they are poloidal cuts of the torus. A simulated value is shown (similar to pressure) with the structure on top on the "Full" image on the left. Just the extra circular structure, considered a "Perturbation", is shown on the right. Note that while I called this a perturbation, there is no actual need for the structure to be smaller than the actual values. The Fourier structure is there regardless of numerical values.

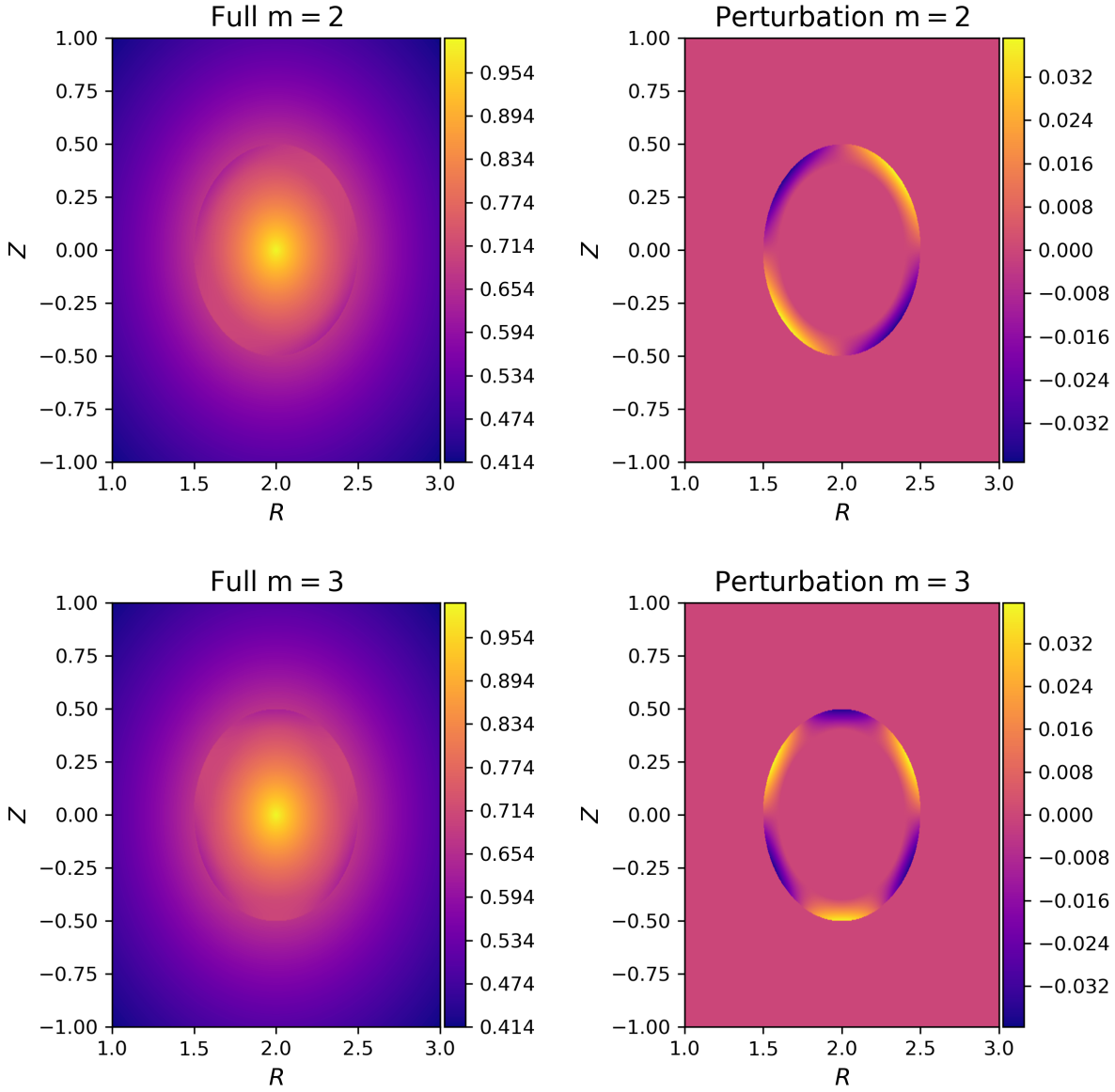

Figure 1.5: These images show how to find Fourier components of structures from contour plots for $m = 2$ and $m = 3$. These will show poloidal structure because they are poloidal cuts of the torus. A simulated value is shown (similar to pressure) with the structure on top on the "Full" image on the left. Just the extra circular structure, considered a "Perturbation", is shown on the right. Note that while I called this a perturbation, there is no actual need for the structure to be smaller than the actual values. The Fourier structure is there regardless of numerical values.

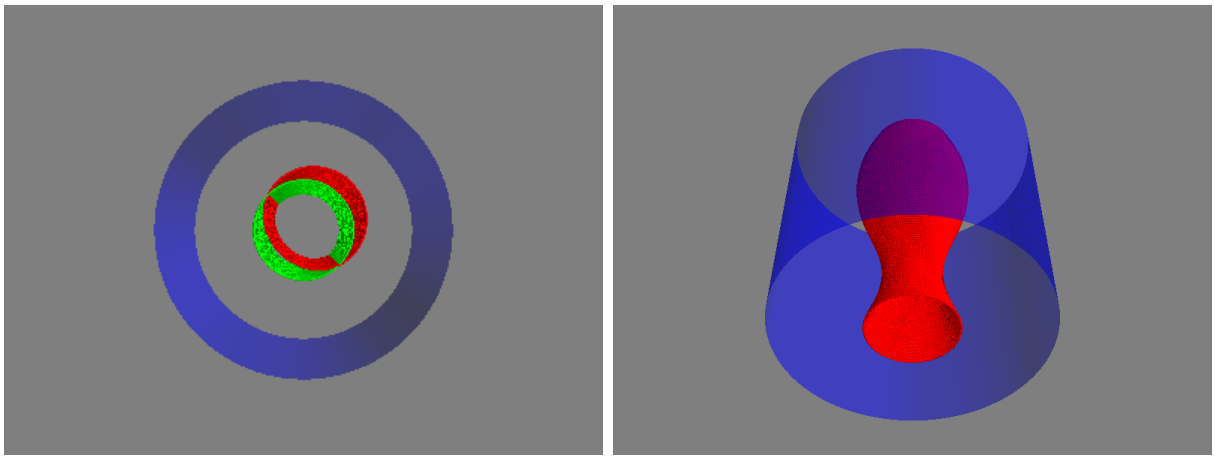

Figure 1.6: The left figure shows an $m = 1$, $n = 0$ kink against a reference $m = 0$, $n = 0$ surface. The rightmost figure shows an $m = 0$, $n = 1$ external kink against a $m = n = 0$ surface.

cylinder together in a circle to get the toroidal version). We then can use the cylindrical coordinate system (R, Z, θ) [note this is right-handed so θ increases in the opposite direction of φ in the usual cylindrical system (r, φ, z) system]. For convenience let $\zeta = Z/L$ with $Z \in [0, L]$. If we write perturbations to the cylindrical surface at radius R' where $R = R'$ is constant, then we may write that in general for a perturbation, the radius is given by

$$R = R' + \delta R' = R' + r' \exp(i(m\theta - n\zeta)) \tag{1.3.23}$$

where $r' \ll R'$ for a perturbation and so then giving m and n tells us what the new shape of the plasma looks like. Because many linear instabilities cause a specific m and n to grow, the shape of the plasma can actually give us a good amount of information about the behavior of the instability (or one could consider it the opposite, the behavior of the instability determines the shape of the plasma). Thus if one speaks of a perturbation to the plasma of some (m, n) form, one is telling you the shape of the toroidal plasma. This is rather difficult to visualize at first, but with a little practice this becomes a quick and useful way of seeing what is going on. We will consider the perturbation to be of the form $\sin(m\theta - n\zeta)$ for convenience (this is simply so that at $\theta = 0, \zeta = 0$ the surface is unperturbed). If one holds $n = 0$ then increasing m makes the cylinder more and more wavy. If one holds $m = 0$ and increases n, the cylinder decreases in diameter and increases as one goes across the cylinder along Z. Combining them then gives grooves in the cylinder that twist around it. Some pictures of these shapes are shown in Figure 1.6 and 1.7.

If you would like to play with this yourself to gain intuition, use your favorite plotting software with the following equations

$$x = \sin(\theta)\left[R' + r' \sin\left(m\theta - \frac{nz}{R'}\right)\right] \tag{1.3.24}$$

$$y = \cos(\theta)\left[R' + r' \sin\left(m\theta - \frac{nz}{R'}\right)\right] \tag{1.3.25}$$

with z varying from $[0, L]$ with $L = 2\pi R'$. Here the major radius R', perturbation r', m, and n are constants that you should choose to make it convenient to see the behavior. Some useful values are $R' = 1$, and $r' = 0.35$ to see m and n make some easily distinguishable shapes.

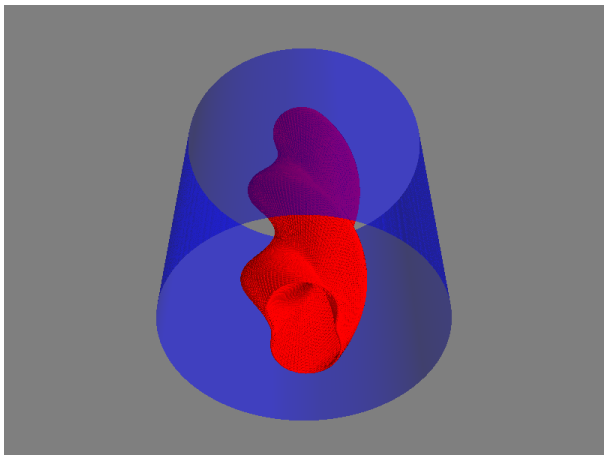

Figure 1.7: The figure shows an $m = 2$, $n = 2$ external kink against a reference $m = 0$, $n = 0$ surface.

1.3.3 Fourier Transforms

It is time to consider Fourier transforms after having looked at Fourier series. As I said before, they can be viewed as a generalization of the Fourier series when $L \to \infty$ and $N \to \infty$. To explain exactly what I mean by that, we let the domain of the function become infinite, and the object we called f_j becomes a function of the wavenumber k often denoted $\widetilde{f}(k)$. We can consider x and k Fourier pairs or Fourier conjugate variables.[48] That is instead of a sum, we go to the continuous series. Thus we define the Fourier transform for variables x and k as

$$\mathcal{F}\left\{f(x)\right\} \equiv \widetilde{f}(k) \equiv \int_{-\infty}^{\infty} \mathrm{d}x \;\; \exp(-ikx) f(x)\,, \tag{1.3.26}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(k)\right\} \equiv f(x) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp(ikx)\widetilde{f}(k)\,. \tag{1.3.27}$$

It should be noted that the factor of $1/(2\pi)$ is often split differently, often with $1/\sqrt{2\pi}$ on the Fourier ($\mathcal{F}$) and inverse Fourier ($\mathcal{F}^{-1}$) terms. I prefer the $1/(2\pi)$ fully on the inverse Fourier transform as it is both common and helps distinguish the inverse transform from the forward transform in my mind.

A physics proof can be constructed in the following manner for one dimension (the generalization to multiple dimensions easily follows). We write our Fourier series with a $\Delta j \;= (j+1) - j = 1$,

$$f(x) \approx \sum_{j=-N}^{N} f_j \exp(2\pi i j x/L) = \sum_{j} f_j \exp(2\pi i j x/L)\Delta j \tag{1.3.28}$$

We then define $\Delta k_{\,j} = \Delta\left(\frac{2\pi j}{L}\right) = \frac{2\pi \Delta j}{L}$ and write

$$f(x) \approx \sum_{j} \frac{L}{2\pi} f_{k_j} \exp(ik_j x)\Delta k_{\,j} \tag{1.3.29}$$

Now if we let $N \to \infty$ while simultaneously letting $L \to \infty$, we see that the we have essentially written the definition of a Riemann integral, where we get smaller and smaller partitions Δk as

[48] In quantum physics position x and momentum p have a similar conjugate relationship.

$L \to \infty$, while summing a larger and larger number of terms from $N \to \infty$. Thus we can go from a discrete sum to a continuous integral as we take both limits.[49] To simplify notation we can write $Lf_{k_j}/2\pi = \widetilde{f}(k)$ and let $k_j \to k$ for simplicity of notation. We then have

$$f(x) = \int_{-\infty}^{\infty} \mathrm{d}k \ \widetilde{f}(\mathbf{k}) \exp(ikx) \tag{1.3.30}$$

where we have used the "identity" (it isn't an identity, but a loose identification)

$$\widetilde{f}(k) = \frac{L}{2\pi} f_{k_j} \tag{1.3.31}$$

where $k_j \to k$ as $N \to \infty$. We can rewrite

$$f_{k_j} = \frac{1}{L} \int_{-L/2}^{L/2} \mathrm{d}x \ f(x) \exp(-ik_j x) \tag{1.3.32}$$

and so we find

$$\widetilde{f}(k) = \lim_{N,L\to\infty} \frac{L}{2\pi} \frac{1}{L} \int_{-L/2}^{L/2} \mathrm{d}x \ f(x) \exp(-ik_j x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \ f(x) \exp(-ikx) \tag{1.3.33}$$

This gives us the definitions we previously stated.

Often, we want to do all three spatial variables as Fourier pairs, $\mathbf{x}$ and $\mathbf{k}$:

$$\mathcal{F}\{f(\mathbf{x})\} \equiv \widetilde{f}(\mathbf{k}) \equiv \iiint_{-\infty}^{\infty} \mathrm{d}^3x \ \exp(-i\mathbf{k}\cdot\mathbf{x}) f(\mathbf{x}) \,, \tag{1.3.34}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\mathbf{k})\right\} \equiv f(\mathbf{x}) \equiv \iiint_{-\infty}^{\infty} \frac{\mathrm{d}^3k}{(2\pi)^3} \ \exp(i\mathbf{k}\cdot\mathbf{x}) \widetilde{f}(\mathbf{k}) \,, \tag{1.3.35}$$

where

$$\iiint_{-\infty}^{\infty} \mathrm{d}^3\xi \ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\xi_x \, \mathrm{d}\xi_y \mathrm{d}\xi_z \ , \tag{1.3.36}$$

and the limits of $-\infty$ to ∞ are often omitted.

Now, to show that the inverse is actually the inverse, let's say $f(x)$ is one of our "nice" functions whose Fourier transform is

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} \mathrm{d}x \ \exp(-ikx) f(x) \,. \tag{1.3.37}$$

Now we apply the inverse Fourier transform (using a dummy variable x' to keep us from making mistakes in labels)

$$\mathcal{F}^{-1}\{\mathcal{F}\{f(x)\}\} = \mathcal{F}^{-1}\left\{\int_{-\infty}^{\infty} \mathrm{d}x \ \exp(-ikx) f(x)\right\} \tag{1.3.38}$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp(ikx) \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp(-ikx') f(x') = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x' \ \exp(ik(x-x')) f(x') \,. \tag{1.3.39}$$

[49] Note that for this to work, it must not matter whether we let $N \to \infty$ occur before $L \to \infty$ or vice versa. The limit only makes sense if we can do it in either order.

because we have well-defined "nice" function, we can switch the order of integration

$$\mathcal{F}^{-1}\left\{\mathcal{F}\left\{f(x)\right\}\right\} = \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp(ik(x-x'))f(x') = \int_{-\infty}^{\infty} \mathrm{d}x'\ f(x') \underbrace{\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \exp(ik[x-x'])}_{\delta(x-x')} \tag{1.3.40}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}x'\ \delta(x-x')\, f(x') = f(x)\,, \tag{1.3.41}$$

as stated. This did require us to use the identity

$$\int_{-\infty}^{\infty} \mathrm{d}k\ \exp(ik[x-x']) = 2\pi\delta(x-x') \tag{1.3.42}$$

where the Dirac delta function is defined by $(A < B)$

$$\int_{A}^{B} \mathrm{d}x' f(x')\delta(x-x') = \begin{cases} f(x) & \text{if } A < x' < B \\ 0 & \text{otherwise} \end{cases} \tag{1.3.43}$$

but a mathematics textbook can justify this. It is required, as we see, so that the Fourier and inverse Fourier series are actually inverses. This of course means that $\exp(ik[x-x'])$ is used as a generalized function, sometimes called a distribution.

See Appendix A for a table of common Fourier transforms.

1.3.4 Laplace Transforms

The Laplace transform is a mathematical transform that usually (assuming that it exists) is defined as (s may be a complex number)

$$\mathfrak{L}\left\{f(t)\right\} \equiv F(s) \equiv \int_{0}^{\infty} \mathrm{d}t\ \exp(-st)f(t)\,. \tag{1.3.44}$$

Then, the inverse Laplace transform (also assuming it exists) is found to be $[\Im\{\sigma\} = 0]$

$$\mathfrak{L}^{-1}\left\{F(s)\right\} \equiv f(t) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathrm{d}s}{2\pi i} \exp(st)F(s)\,. \tag{1.3.45}$$

Often people write this as

$$\mathfrak{L}^{-1}\left\{F(s)\right\} \equiv f(t) = \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} \frac{\mathrm{d}s}{2\pi i} \exp(st)F(s)\,,$$

if they want to make it more rigorous. The use of the Laplace transform is that it brings in a sense of causality, and makes solving initial value problem differential equations easier.

In plasma physics literature, we often find that the Laplace transform is redefined such that $\omega \to is$ $(-i\omega \to s)$ so that $-i\,\mathrm{d}\omega = \mathrm{d}s$ and we then have

$$\mathfrak{L}\left\{f(t)\right\} = F(-i\omega) = \int_{0}^{\infty} \mathrm{d}t\ \exp(i\omega t)f(t)\,, \tag{1.3.46}$$

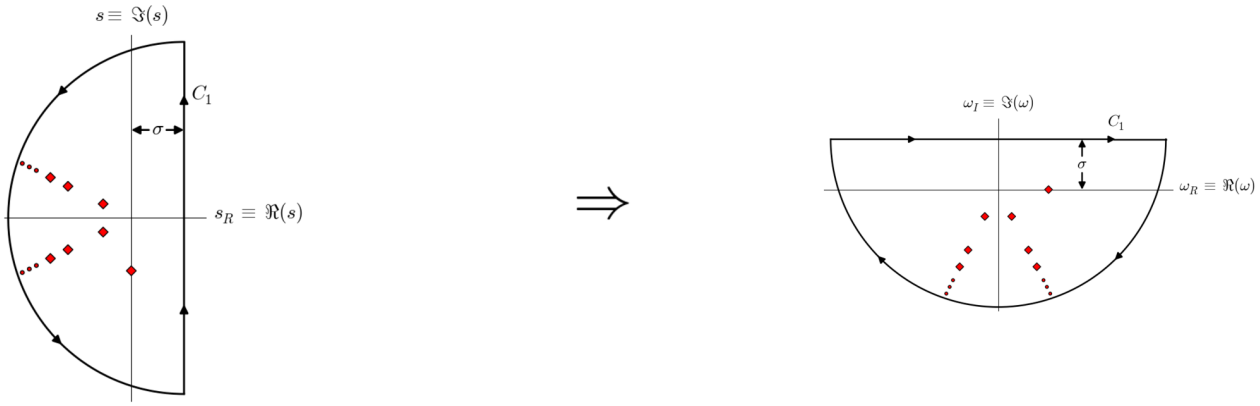

Figure 1.8: Transformation from s complex space to ω complex space. The red diamonds indicate poles.

and the inverse

$$\mathfrak{L}^{-1}\left\{F(-i\omega)\right\} \equiv f(t) = \int_{i(\sigma-i\infty)}^{i(\sigma+i\infty)} \frac{-i\,\mathrm{d}\omega}{2\pi i}\,\exp(-i\omega t)F(-i\omega) \tag{1.3.47}$$

$$= -\int_{i\sigma+\infty}^{i\sigma-\infty} \frac{\mathrm{d}\omega}{2\pi}\,\exp(-i\omega t)F(-i\omega) \tag{1.3.48}$$

$$= \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{\mathrm{d}\omega}{2\pi}\,\exp(-i\omega t)F(-i\omega)\,. \tag{1.3.49}$$

Now instead of keeping the notation $F(-i\omega)$, we denote a new function $\widehat{f}(\omega)$ so that our plasma physics definition of $\mathfrak{L}$ and its inverse are (the n on the $\mathfrak{L}_n$ is to indicate a new definition)

$$\mathfrak{L}_n\left\{f(t)\right\} \equiv \widehat{f}(\omega) \equiv \int_0^\infty \mathrm{d}t\;\exp(i\omega t)f(t)\,, \tag{1.3.50}$$

$$\mathfrak{L}_n^{-1}\left\{\widehat{f}(\omega)\right\} \equiv f(t) = \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{\mathrm{d}\omega}{2\pi}\exp(-i\omega t)\widehat{f}(\omega)\,. \tag{1.3.51}$$

This just shows that we have changed the definition so that the tables for $F(s)$ and $f(t)$ are tables for $\widehat{f}(\omega) = F(-i\omega)$ and $f(t)$.

Geometrically, this redefinition of the Laplace transform just rotates the regular Laplace transform (s) function by 90° counterclockwise (or anticlockwise, if you prefer) and reverses the direction of the contour (because the $-$ in the $-i\omega = s$ definition switches the order of integration) as shown in Figure 1.8. That is, instead of integrating along a vertical line ("imaginary" line) in the complex s plane, we integrate along a horizontal line ("real" line) in the complex ω plane. The σ just offsets from the imaginary in s (real in ω) axis so that all the poles are to the left of the line in s (below the line in ω).

We will prove the inverse Laplace transform is the inverse Laplace transform for $\mathfrak{L}_n\{f(t)\}$ with $f(t)$ being a "nice" function. We begin from

$$\mathfrak{L}_n\left\{f(t)\right\} = \int_0^\infty \mathrm{d}t\;\exp(i\omega t)f(t)\,. \tag{1.3.52}$$

Now we apply the inverse Fourier transform (using a dummy variable t' to keep us from making mistakes)

$$\mathfrak{L}_n^{-1}\left\{\mathfrak{L}_n\left\{f(t)\right\}\right\} = \mathfrak{L}_n^{-1}\left\{\int_0^\infty \mathrm{d}t'\ \exp(i\omega t')f(t')\right\} \tag{1.3.53}$$

$$= \int_{i\sigma-\infty}^{i\sigma+\infty}\frac{\mathrm{d}\omega}{2\pi}\exp(-i\omega t)\int_0^\infty \mathrm{d}t'\ \exp(i\omega t')f(t') = \int_{i\sigma-\infty}^{i\sigma+\infty}\frac{\mathrm{d}\omega}{2\pi}\int_0^\infty \mathrm{d}t'\ \exp(i\omega[t-t'])f(t')\,. \tag{1.3.54}$$

Now with mild assumptions on $f(t)$ we can switch the order of integration (let's also use $\omega' = \omega - i\sigma$ so $\mathrm{d}\omega' = \mathrm{d}\omega$)

$$\mathfrak{L}_n^{-1}\left\{\mathfrak{L}_n\left\{f(t)\right\}\right\} = \int_0^\infty \mathrm{d}t'\ \int_{i\sigma-\infty}^{i\sigma+\infty}\frac{\mathrm{d}\omega}{2\pi}\exp(i\omega[t-t'])f(t') \tag{1.3.55}$$

$$= \int_0^\infty \mathrm{d}t'\ f(t')\underbrace{\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\exp(ii\sigma[t'-t])\exp(i\omega'[t'-t])}_{\exp(\sigma[t-t'])\delta(t'-t)} \tag{1.3.56}$$

$$= \int_0^\infty \mathrm{d}t'\ \delta(t'-t)\ \exp(\sigma[t-t'])f(t') = \exp(\sigma t)\int_0^\infty \delta(t'-t)\ \exp(-\sigma t')f(t') \tag{1.3.57}$$

$$= \exp(\sigma t)\begin{cases}\exp(-\sigma t)f(t) & t>0\\ 0 & t<0\end{cases} = \begin{cases}f(t) & t>0\,,\\ 0 & t<0\,,\end{cases} \tag{1.3.58}$$

which is what we desired (because the Laplace transform only involves $f(t)$ from $t>0$). Of course, one should note that $\sigma \geq 0$ for this to work and, in fact, large enough that the contour encloses all poles in a way such that the poles are below the line $\Im[\omega] = \sigma$ in the ω complex plane.

See Appendix A for a table of common Laplace transforms.

1.3.5 Ballooning Transforms

> *But someone whose knowledge is limited has to use great foresight in his choice of method if he is not to be blocked by meeting calculations beyond his powers.*
>
> — W. W. Sawyer[29, p. 136]

In many situations it is advantageous to go from periodic variables to non-periodic variables. This can be useful when we wish to apply theorems to analytic variables and the periodic variables cause problems because there is no longer a clear ordering that can be imposed. In plasma physics, such things occur with the poloidal θ and toroidal angles ζ. Often it would be wonderful to apply the JWKB approximation so that we can rewrite our equations in a form where there is a phase relationship. However, this causes difficulties with periodic functions and a requirement of varying slowly spatially. For we have three properties that we wish to impose on our solution. First we have a long parallel wavelength because things equalize along magnetic field lines (and so along magnetic flux surfaces) very quickly. Second, the perpendicular wavelengths are short (indeed, it is in this sense that the parallel wavelengths are long). When traversing magnetic surfaces, things can change much more quickly than along them. This can be represented as $k_\perp \gg k_\parallel$. These first two properties are known to be true in some of the most unstable configurations of

toroidal plasmas (ballooning modes). Finally, we desire periodicity in both angles ζ and θ. These three properties prove to be difficult to retain all together within a consistent approximation. The original solution to this problem was given by Connor, Hastie, Taylor[8], and is an excellent resource for understanding the problem.

Suppose we have reduced our problem to radial and poloidal variation only because we have an axisymmetric problem. This can be reduced to solving an eigenvalue problem. Then for operator L we have

$$L(r,\theta)[f(r,\theta)] = \lambda f(r,\theta) \tag{1.3.59}$$

That is, if we desire a solution f to some equation representing a wave-like oscillation, it would be best if we could write

$$f \propto \exp(iS) \tag{1.3.60}$$

for some S that is slowly varying. I have assumed S complex so $S = S_r + iS_i$. Often the above is written $f \propto F\exp(iS_r)$ with $F = \exp(-S_i)$ real and slowly varying. When dealing with axisymmetric systems, ζ's phase poses no problems, but the poloidal angle will not necessarily be simple. If we assume that we can write these as Fourier components in ζ then we can let n be the Fourier component number and our solutions to a f will have $f \propto F\exp(in(\zeta + S_r))$. We simply eliminate any ζ dependence because with axisymmetry it is easy to figure out the solutions and so $f(r,\theta,\zeta) \to f(r,\theta) \propto F\exp(inS_r)$. However, if there is shear in the magnetic field,[50] a problem is imposed. For we want $f(r,\theta)$ to vary slowly for us to justify our approximation with S. But if the magnetic field shears, then this means the magnetic field is twisting differently at every different r so that the magnetic field structure is different as we go out in radius. When we have $f \propto \exp(inS)$, we want an ordering where $k_\parallel \ll 1$ so we can say $k_\parallel = \epsilon$ for an ordering parameter ϵ. Then for the part of f that has

$$\mathbf{B}\cdot\nabla f \propto \mathbf{B}\cdot\nabla S \propto k_\parallel \hat{\mathbf{b}}\cdot\nabla S = 0 + \mathcal{O}(\epsilon) \tag{1.3.61}$$

The above magnetic differential equation implies that the lowest order would yield

$$S(r,\theta) = \zeta - q(r)\theta \tag{1.3.62}$$

The problem now is that $S(r,\theta)$ is no longer periodic in ζ and θ. You might think that we can get around this by restricting ζ and θ to values between 0 and 2π. However, if $q(r)$ varies in r, then ζ and θ will have severe difficulties changing to appropriate values and will no longer be slowly varying as we move in r. Notice that if the shear were zero or very small, then we wouldn't have to worry about this because then the periodic values for each r would not have to change much as we change r.

If we didn't have to have periodicity in ζ and θ, we could make our job easier because we would not have to worry about S changing in just the right way. That is, we'd like not having to try and enforce S being periodic.

The way to do this is to use a different expansion for f. We introduce

$$f(r,\theta) = \sum_m \exp(-im\theta)\int_{-\infty}^{\infty} \mathrm{d}\eta\ \exp(im\eta)\widehat{f}(r,\eta) \tag{1.3.63}$$

[50] That is, we have magnetic shear.

We see that we now have $\widehat{f}(r,\eta)$ with η being a variable that covers $-\infty$ to ∞ without a periodicity constraint. Instead, the periodicity is imposed by the summation over the $\exp(-im\theta)$. To get to this form we use a Fourier-like expansion in θ

$$f(r,\theta) = \sum_m \exp(-im\theta) f_m(r) \tag{1.3.64}$$

We can view $f_m(r)$ as $f(r,m)$ for m an integer. We then analytically continue $f(r,m)$ to non-integer values.[51] Remember that since $f_m(r)$ is a Fourier series coefficient, we have

$$f_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \ \exp(im\theta) f(r,\theta) \tag{1.3.65}$$

for integer m. Assuming that we can analytically continue this function, we can extend to non integer m. Consider its conjugate variable to be η and so

$$\widetilde{f}(r,\eta) = \int_{-\infty}^{\infty} \mathrm{d}m \ \exp(-im\eta) f(r,m) \tag{1.3.66}$$

$$f(r,m) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\eta}{2\pi} \exp(im\eta)\widetilde{f}(r,\eta) \tag{1.3.67}$$

By convention for ballooning transformations, one works with a different variable $\widehat{f}(r,\eta) = 2\pi\widetilde{f}(r,\eta)$ instead so that we then write

$$f(r,\theta) = \sum_m \exp(-im\theta) \int_{-\infty}^{\infty} \mathrm{d}\eta \ \exp(im\eta)\widehat{f}(r,\eta) \tag{1.3.68}$$

We can then substitute this into our original eigenvalue problem

$$L(r,\theta)[f(r,\theta)] = \lambda f(r,\theta) \tag{1.3.69}$$

$$L(r,\theta)[\sum_m \exp(-im\theta)\int_{-\infty}^{\infty} \mathrm{d}\eta \ \exp(im\eta)\widehat{f}(r,\eta)] = \lambda\sum_m \exp(-im\theta)\int_{-\infty}^{\infty} \mathrm{d}\eta \ \exp(im\eta)\widehat{f}(r,\eta) \tag{1.3.70}$$

Now because the $\exp(-im\theta)$ form an orthogonal complete set, it is clear that this must be true for each m. Indeed, we can then change our operator into one that operates with η instead of θ since all θ dependence is through $\exp(-im\theta)$. That is we have

$$\frac{\partial^j}{\partial\theta^j}\left[\sum_m \exp(-im\theta)\int_{-\infty}^{\infty} \mathrm{d}\eta \ \exp(im\eta)\widehat{f}(r,\eta)\right] \tag{1.3.71}$$

$$= \left[\sum_m \exp(-im\theta)\int_{-\infty}^{\infty} \mathrm{d}\eta \ (-im)^j \exp(im\eta)\widehat{f}(r,\eta)\right] \tag{1.3.72}$$

$$= \left[\sum_m \exp(-im\theta)\int_{-\infty}^{\infty} \mathrm{d}\eta \ (-1)^j \frac{\partial^j}{\partial\eta^j}\left\{\exp(im\eta)\right\}\widehat{f}(r,\eta)\right] \tag{1.3.73}$$

[51] If you are worried about this, Carlson's theorem from copmplex analysis allows us to find the unique analytic function that does this continuation if we impose bounds on the growth of the function as it approaches infinity.

$$
\begin{aligned}
&= \Bigg[\sum_m \exp(-im\theta)(-1)^j \int_{-\infty}^{\infty} \mathrm{d}\eta \, \Bigg(\frac{\partial^j}{\partial \eta^j} \left\{ \exp(im\eta) \widehat{f}(r,\eta) \right\} \\
&\qquad + \sum_{\ell<j} C(im,\ell) \frac{\partial^\ell}{\partial \eta^\ell} \left[\exp(im\eta) \widehat{f}(r,\eta) \right] \\
&\qquad + (-1)^j \exp(im\eta) \frac{\partial^j}{\partial \eta^j} \widehat{f}(r,\eta) \Bigg) \Bigg] \qquad (1.3.74) \\
&= \Bigg[\sum_m \exp(-im\theta) \Bigg(\sum_{\ell<j} \left[C(im,\ell) \exp(im\eta) \frac{\partial^\ell \widehat{f}(r,\eta)}{\partial \eta^\ell} \right]_{\eta=-\infty}^{\infty} \\
&\qquad + (-1)^j \left[\exp(im\eta) \frac{\partial^j \widehat{f}(r,\eta)}{\partial \eta^j} \right]_{\eta=-\infty}^{\infty} \qquad (1.3.75) \\
&\qquad + \overbrace{(-1)^{2j}}^{=1} \int_{-\infty}^{\infty} \mathrm{d}\eta \, \frac{\partial^j}{\partial \eta^j} \widehat{f}(r,\eta) \Bigg) \Bigg]
\end{aligned}
$$

For some function $C(im,\ell)$ that outputs some integer times a power of (im) for each ℓ.[52] So long as all $\frac{\partial^\ell \widehat{f}(r,\eta)}{\partial \eta^\ell}$ goes to zero at $\eta = \pm\infty$ for $\ell < j$ then we have

$$
\begin{aligned}
&\frac{\partial^j}{\partial \theta^j} \left[\sum_m \exp(-im\theta) \int_{-\infty}^{\infty} \mathrm{d}\eta \, \exp(im\eta) \widehat{f}(r,\eta) \right] \\
&= \left[\sum_m \exp(-im\theta) \int_{-\infty}^{\infty} \mathrm{d}\eta \, \frac{\partial^j}{\partial \eta^j} \widehat{f}(r,\eta) \right]
\end{aligned} \qquad (1.3.76)
$$

And so we can form a new operator $\mathcal{L}(r,\eta)$ from $L(r,\theta)$ that satisfies

$$\mathcal{L}(r,\eta)[\widehat{f}(r,\eta)] = \lambda \widehat{f}(r,\eta) \qquad (1.3.77)$$

when

$$L(r,\theta)[f(r,\theta)] = \lambda f(r,\theta) \qquad (1.3.78)$$

You might now complain about two things. First, I only showed that the integral over η satisfies the relationship. This can be easily remedied. We first assume that we are dealing with convergent series and well-defined functions so that we can swap the sum over m with the integration and find

$$\int_{-\infty}^{\infty} \mathrm{d}\eta \, \left\{ \sum_m \exp(-im(\theta-\eta)) \left[\frac{\partial^j}{\partial \eta^j} \widehat{f}(r,\eta) - \exp(-im(\theta-\eta)) \lambda \widehat{f}(r,\eta) \right] \right\} = 0 \qquad (1.3.79)$$

We can then use that the $\exp(-im\theta)$ form an orthogonal complete set or expansion. This means that we must have for each m that

$$\exp(im\eta) \frac{\partial^j}{\partial \eta^j} \widehat{f}(r,\eta) - \exp(im\eta) \lambda \widehat{f}(r,\eta) = 0 \qquad (1.3.80)$$

$$\frac{\partial^j}{\partial \eta^j} \widehat{f}(r,\eta) - \lambda \widehat{f}(r,\eta) = 0 \qquad (1.3.81)$$

[52] Faà di Bruno's formula is essentially what we need once again.

This must be true for all m because each exponential is independent of all others. Thus the object multiplying $\exp(-im\theta)$ must be zero to guarantee that we get the full integral to be zero.

The other, and more difficult, problem is whether the steps we have taken are valid. As physicists, this task is much easier. We do the steps without worrying about whether the function we are looking for actually exists. Mathematicians have to show that each step works and prove the existence of a solution. We simply do the steps, plug the solution back in at the end and see if it works. The proof is a constructive one for each situation we encounter.[53] It is like we approach a physical problem and learn it requires that a bicycle fly into the air. A mathematician will prove that such solutions may exist if we define what a "bicycle" is. A physicist can build the bicycle and check if it flies by experiment. If it does, then we're good. If we get a solution through this method and then check it satisfies the original problem, then it does not really matter to us whether the steps were technically valid. They guided us towards a correct answer.

In any case, this new $\widehat{f}(r,\eta)$ can now use an eikonal-like solution without the problems of periodicity since η ranges over all the real numbers, and so the S can now vary slowly in the eikonal as a function of η instead of θ. So once the ballooning transformation has been applied, we can then perform all the usual approximations of an eikonal and get an approximation for when $n \gg 1$. The condition is $n \gg 1$ because we use $n = \frac{1}{\delta}$ in our JWKB approximation.

1.4 Asymptology

> *If there is a problem you can't solve, then there is an easier problem you can solve; find it.*
>
> — George Pólya

This section is all about determining the properties of expressions in limits, typically towards infinity. This is typically called asymptotic approximation, asymptotic analysis, or something similar, but I thought I would go with the somewhat more whimsical term asymptology introduced by Martin Kruskal.[54] He coined the term asymptology as "the art of dealing with applied mathematical systems in limiting cases."[11][55] The essay is well worth reading, and outlines seven principles for asymptology.

1. **The Principle of Simplification.** Asymptological (limiting) analysis tends to simplify the system considered.
2. **The Principle of Recursion.** Treat non-dominant terms as if they were known.
3. **The Principle of Interpretation.** Find variables in which the given problem becomes a perturbation problem.

[53] This reminds me of a joke mathematicians purportedly tell about physicists. A mathematician thinks a physicist is a person who sums the first three terms of a divergent series to get an answer. In fact physicists can get a last laugh in. Asymptotic expansions are often divergent, but offer great approximations in the first few terms as we will see in Section 1.4 on asymptology! Thus summing the first three terms of a divergent series may actually be the right thing to do!

[54] He was one of the great mathematical physicists of the twentieth century. He is the Kruskal of the Kruskal-Schwarzchild problems, Kruskal-Shafranov instabilities, BGK (Bernstein-Greene-Kruskal) modes, and one of the people to help develop the MHD energy principle. He and Norman Zabusky figured out how solitons were solutions to the Korteweg-de-Vries (KdV) equation. This is only a partial list of his accomplishments.

[55] Kruskal's contribution to this reference is available here.

4. **The Principle of Wild Behavior.** Cases of apparent overdeterminism[56]arise because some of the solutions behave wildly in the limit. This means that when the solution we want is not a wildly varying one, we have gained in simplicity by going to a simpler system, by sacrificing[57] one of the solutions we do not care about. When the wildly varying solution is the solution we actually want, we need to reformulate our problem.
5. **The Principle of Annihilation.** Find a complete set of annihilators[58] of the terms which persist in the limit, apply them to the original system, and then go to the limit.
6. **The Principle of Maximum Balance.** Keep terms that are of undetermined magnitude for maximum flexibility and generality.
7. **The Principle of Mathematical Nonsense.** When mathematically nonsensical terms appear, the asymptology has been performed incorrectly or has not been carried out fully.

These principles are best explained with examples, and it is often easiest to explain with very simple examples.

Consider trying to solve a problem of the form

$$\epsilon^2 x^3 + x^2 - \epsilon x - 9 = 0 \tag{1.4.1}$$

when x is large (much greater than 1). Suppose $\epsilon > 0$ is a small parameter. Then there are all sorts of orderings possible, but not all of them are going to be consistent. We can start with $x \gg 1$ so only the highest powers of x will matter. In this case

$$\epsilon^2 x^3 + x^2 = x^2(\epsilon^2 x + 1) \approx 0 \tag{1.4.2}$$

which means $x = 0$ (not a useful solution) or $x = -\epsilon^{-2}$. But then $x < 0$. We can then try $\epsilon \ll 1$ so the only parts of x that will matter are

$$x^2 - 9 = 0 \Rightarrow x_0 = 3 \tag{1.4.3}$$

Note how this is a simpler system to solve and so involves **The Principle of Simplification** as we are solving a quadratic instead of a cubic equation. In addition, we tried one set of variables and found they did not give us a simplification. When we chose to expand based on ϵ we did get an answer illustrating **The Principle of Interpretation**. This is the zeroth order solution when $\epsilon = 0$. We could then write

$$x = \sqrt{9 + \epsilon x^2(1 - \epsilon x)} = 3\sqrt{1 + \epsilon \frac{x^2}{3}(1 - \epsilon x)} \tag{1.4.4}$$

We could view the x on the left hand side as x_1 and plug in x_0 on the right hand side to find a new term for $x = x_0 + x_1 + \cdots$. This is **The Principle of Recursion** where we treat non-dominant unknowns as if they were known. In which case we find

$$x_1 = 3\sqrt{1 + \epsilon \frac{9}{3}(1 - \epsilon 3)} = 3\sqrt{1 + \epsilon(3 - 9\epsilon)} \tag{1.4.5}$$

[56]By overdeterminism, I mean something like solving three linear equations with only two variables. The system is overdetermined because one only needs two linear equations to determine the two variables, and so the overdetermined case may not have a solution (is inconsistent with three equations) or one of the equations is just a restatement of one of the others.

[57]It is sacrificed in the sense that it wildly varies and so cannot be solved for in any meaningful way.

[58]Annihilators are mathematical objects that return zero when applied to a term.

And so on, for recursion. Such series are often non-convergent, but so long as you only write a finite amount, they can be accurate. Such a series is called an asymptotic series since it may decrease in accuracy as you increase the number of terms.[59]

This example, also lets us examine **The Principle of Wild Behavior** to some degree. When we changed our problem into a quadratic, we are implicitly saying x does not get so large that it overwhelms ϵ. In so doing, we ignored the solution that grew enormous as $\epsilon \to 0$, the solution that goes like $-\epsilon^{-2}$. That root is behaving "wildly" by running to infinity. Luckily, in this case we didn't care about the unbounded root. In other cases, we would have to form a new asymptotic representation and see if it works.

I think the other principles are fairly simple to understand, so let's now consider some examples. The JWKB approximation will be covered in Section 1.12, where we assume that we can write the solution as the exponential of a series with a specific ordering.

The notation for asymptotics[60] was noted in the Section 1.1.3, but we shall extend it a bit as well. Given a function $f(x)$, if we have an asymptotic series $g(x)$ that is true as $x \to \infty$ then this is typically written

$$f(x) \sim g(x) \tag{1.4.6}$$

with $\sim$ implicitly indicating that $x \to \infty$. Such an expression could be read as f is asymptotic to g (as $x \to \infty$). I prefer the notation

$$f(x) \overset{x\to\infty}{\longrightarrow} g(x) \tag{1.4.7}$$

and so will use that instead of the $\sim$ notation which in physics often has the meaning of proportional to. In any case, all of the above have the meaning that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1 \tag{1.4.8}$$

This definition could have problems should $g(x) = 0$ as it limits to its value and so the more rigorous description

$$f(x) - g(x) = o(g(x)) \Rightarrow \lim_{x\to a} \frac{f(x) - g(x)}{g(x)} = 0 \tag{1.4.9}$$

is sometimes used as the defining feature of $f(x)$ being asymptotic to $g(x)$ (in whatever limit). See 1.1.3 if you need a reminder on what little O notation means. Here the limit is usually obvious from context if it is not explicitly given.

1.4.1 Integral Asymptology

There are numerous asymptology methods, but I will go over a few of the common ones. The method of steepest descent, the method of stationary phase, or Laplace's method are all various

[59] The loss of accuracy may be surprising at first, but remember that if you have $f(x)$ and $f(x)$ acts like $\sum_i g_i(x)$ as $x \to \infty$, then unless $f(x)$ is finite as $x \to \infty$, for a fixed $x < \infty$, $g_i(x)$ will eventually start adding gigantic terms that overshoot the result for a fixed x. In short, we are interested in how $f(x)$ acts as x is finite but extremely large.

[60] A synonym for asymptology most of the time, though asymptology is meant to be broader in scope.

incarnations of a single idea. They are extremely useful ways to get an asymptotic expansion for integrals. Suppose we have an integral of the form

$$\int_a^b \mathrm{d}t\ g(t)\exp(-Mf(t)) \tag{1.4.10}$$

for M some large, positive constant. We'd like to find a good approximation of the integral in a simpler form. The way to doing this is using that if $\exp(-Mf(t))$ is only large near the minimum of $f(t)$ then only values near that point really count. Suppose $f'(t_0) = 0$.[61] Then if $a \leq t \leq b$ we can integrate only around the region near t_0 and get an answer (we can Taylor expand $f(t)$ around t_0 if it is not a simple expression). The method of steepest descent allows us to consider integrals in complex space [you may have to take into account residues then], and the method of stationary phase is used for $\exp[-iMf(t)]$, but saying you still only care about the critical points[62] of $f(t)$. The method of steepest descent is technically only for saddle points, rather than generic critical points, but the ideas are all similar. The idea is still that critical points of $f(t)$ are in essence, the points that contribute the most to the integral. Examples will be especially useful to understand this.

Consider

$$\int_0^\pi \mathrm{d}t\ t^{-1/2}\cos(t)\exp(-xt) \tag{1.4.11}$$

where $x \to \infty$. In the formulation above we could consider $M = x$. We can recognize that $\exp(-xt)$ is a "controlling factor". That is, it is the term that will make the integral zero nearly everywhere except near $t = 0$. With this, we consider an integral only around this region and so

$$\int_0^\pi \mathrm{d}t\ t^{-1/2}\cos(t)\exp(-xt) \overset{x\to\infty}{\longrightarrow} \int_0^\epsilon \mathrm{d}t\ t^{-1/2}\cos(t)\exp(-xt) \tag{1.4.12}$$

We can then simplify via $xt = u$ so $\mathrm{d}u = x\,\mathrm{d}t$ and

$$\int_0^\pi \mathrm{d}t\ t^{-1/2}\cos(t)\exp(-xt) \overset{x\to\infty}{\longrightarrow} \int_0^\epsilon \mathrm{d}u\ \frac{1}{x}\left(\frac{u}{x}\right)^{-1/2}\cos\left(\frac{u}{x}\right)\exp(-u) \tag{1.4.13}$$

Now we can use that in this small region we have the approximation

$$\cos(u/x) \overset{x\to\infty}{\longrightarrow} 1 - \frac{1}{2}\frac{u^2}{x^2} + \frac{1}{24}\frac{u^4}{x^4} \tag{1.4.14}$$

Next, we recognize that $\exp(-u)$ will overwhelm any power of u and so we can extend our limits from $(0,\epsilon)$ to $(0,\infty)$ because this will not add too much thanks to our controlling factor. Thus

$$\begin{aligned}\int_0^\pi \mathrm{d}t\ t^{-1/2}\cos(t)\exp(-xt) &\overset{x\to\infty}{\longrightarrow} \frac{1}{x}\int_0^\epsilon \mathrm{d}u\ \left(\frac{u}{x}\right)^{-1/2}\left[1 - \frac{1}{2}\frac{u^2}{x^2} + \frac{1}{24}\frac{u^4}{x^4}\right]\exp(-u)\\ &\overset{x\to\infty}{\longrightarrow} \frac{1}{x}\int_0^\infty \mathrm{d}u\ \left[\frac{u^{-1/2}}{x^{-1/2}} - \frac{1}{2}\frac{u^{3/2}}{x^{3/2}} + \frac{1}{24}\frac{u^{7/2}}{x^{7/2}}\right]\exp(-u)\end{aligned} \tag{1.4.15}$$

[61] Make sure it is a minimum and not a maximum. Otherwise the term will not be a dominant one in the integral.
[62] You probably only care about the stationary points where the derivative is defined.

We can then use

$$\int_0^\infty \mathrm{d}u\ u^n \exp(-u) = \Gamma(n+1) = n! \tag{1.4.16}$$

where I use $\Gamma(1/2) = (-1/2)! = \sqrt{\pi}$. Then

$$\int_0^\pi \mathrm{d}t\ t^{-1/2}\cos(t)\exp(-xt) \overset{x\to\infty}{\longrightarrow} \frac{1}{x}\left[\sqrt{\pi}\sqrt{x} - \frac{\sqrt{\pi}}{2}\frac{\frac{1}{2}\frac{3}{2}}{x^{3/2}} + \frac{\sqrt{\pi}}{24}\frac{\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}}{x^{7/2}}\right] \tag{1.4.17}$$

Which we simplify to

$$\begin{aligned}\int_0^\pi \mathrm{d}t\ t^{-1/2}\cos(t)\exp(-xt) &\overset{x\to\infty}{\longrightarrow} \frac{\sqrt{\pi}}{\sqrt{x}}\left[1 - \frac{3}{8x^{4/2}} + \frac{35}{128x^{8/2}}\right] \\ &\overset{x\to\infty}{\longrightarrow} \frac{\sqrt{\pi}}{\sqrt{x}}\left[1 - \frac{3}{8x^2} + \frac{35}{128x^4}\right]\end{aligned} \tag{1.4.18}$$

In general, we have for $f'(t_0) = 0$ and $a < t_0 < b$ and $u = t - t_0$ that

$$\begin{aligned}&\int_a^b \mathrm{d}t\ g(t)\exp(-Mf(t)) \\ &\overset{M\to\infty}{\longrightarrow} \int_{t_0-\epsilon}^{t_0+\epsilon} \mathrm{d}t\ \left[g(t_0) + (t-t_0)g'(t_0) + \cdots\right]\exp[-Mf(t_0) - M(t-t_0)\cancel{f'(t_0)} - M(t-t_0)^2 f''(t_0)] \\ &\overset{M\to\infty}{\longrightarrow} \exp[-Mf(t_0)]\int_{t_0-\epsilon}^{t_0+\epsilon} \mathrm{d}t\ \left[g(t_0) + (t-t_0)g'(t_0) + \cdots\right]\exp[-M(t-t_0)^2 f''(t_0)] \\ &\overset{M\to\infty}{\longrightarrow} \exp[-Mf(t_0)]\int_{-\infty}^{\infty} \mathrm{d}t\ \left[g(t_0) + (t-t_0)g'(t_0) + \cdots\right]\exp[-M(t-t_0)^2 f''(t_0)] \\ &\overset{M\to\infty}{\longrightarrow} \exp[-Mf(t_0)]\int_{-\infty}^{\infty} \mathrm{d}u\ \left[g(t_0) + ug'(t_0) + u^2 g''(t_0)\cdots\right]\exp[-Mu^2 f''(t_0)]\end{aligned} \tag{1.4.19}$$

where we can use that these are now Gaussian integrals on the right hand side with the identity

$$\int_{-\infty}^\infty \mathrm{d}u\ u^{2n}\exp(-\alpha u^2) = \frac{\Gamma(\frac{2n+1}{2})}{\alpha^{(2n+1)/2}} \tag{1.4.20}$$

and so

$$\begin{aligned}&\int_a^b \mathrm{d}t\ g(t)\exp(-Mf(t)) \\ &\overset{M\to\infty}{\longrightarrow} \exp[-Mf(t_0)]\left[g(t_0)\sqrt{\frac{\pi}{Mf''(t_0)}} + 0 + g''(t_0)\sqrt{\pi}\frac{3}{2\left(Mf''(t_0)\right)^{3/2}} + \cdots\right]\end{aligned} \tag{1.4.21}$$

The method of steepest descent simply generalizes this method for complex values where you deform your contour integral so that you go through zeros of $f'(t)$ but where the imaginary part of $f(t)$ is constant. In this case, then we are dealing with a saddle point in the complex plane.[63]

[63] That is a critical point where the gradient along orthogonal directions are zero, but that is not a local extremum (since it can be a relative minimum along one direction and a relative maximum along a different, orthogonal direction).

To find out if a point is a saddle point, you form the Hessian for $f(\mathtt{x})$ for variables $\mathtt{x}$ given by $\frac{\partial^2 f}{\partial \mathtt{x}\partial \mathtt{x}}$. We then determine if something is a saddle point by evaluating the Hessian at the critical points $\mathtt{x}_i$ where $\frac{\partial f}{\partial \mathtt{x}} = \mathbf{0}$. This means that we use the Hessian definition

$$\mathbf{H} \equiv \frac{\partial^2 f}{\partial \mathtt{x}\partial \mathtt{x}} = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial^2 x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial^2 x_n} \end{bmatrix} \tag{1.4.22}$$

evaluated at the $\mathtt{x}_i$ and find the eigenvalues of this matrix. If the eigenvalues are all positive or all negative then you have a relative extremum, and if the eigenvalues are both positive and negative, then you have a saddle point. Note that for complex $\mathtt{z}$, the Hessian generalization uses

$$\mathbf{H} \equiv \frac{\partial^2 f}{\partial \mathtt{z}\partial \bar{\mathtt{z}}} \tag{1.4.23}$$

and this is $\mathbf{0}$ if we have an analytic function f. Thus analytic functions' Hessians are not useful tests.[64] Note also that if the determinant of the Hessian is exactly zero at a critical point, then it is considered a degenerate critical point and we cannot use the above test to determine if the critical point is a saddle point.

Let's now do a famous example, Stirling's approximation. We start with

$$N! = \Gamma(N+1) = \int_0^\infty \mathrm{d}t\ t^N \exp(-t) \tag{1.4.24}$$

We'd like to find $N!$ when $N \to \infty$. The t^N factor is inconvenient since it will grow large as $N \to \infty$, but $\exp(-t)$ pulls it smaller as $t \to \infty$. We'd prefer to have our point where the integral most contributes be near a finite value, not at $t \to \infty$. So we rewrite $t^N = \exp(N \ln t)$ and have

$$N! = \int_0^\infty \mathrm{d}t\ \exp(-t + N \ln t) \tag{1.4.25}$$

Now we have a minimum of $-t + N \ln t$ at $-1 + \frac{N}{t} = 0 \Rightarrow t = N$. Since N is the thing that is growing, it would be convenient to rescale so that we put the maximum at $s = 0$. This can be most simply achieved by using $t = N(1+s)$ Then at $s = 0$ we have $t = N$. This means $\mathrm{d}t = N\,\mathrm{d}s$ and we have

$$\begin{aligned} N! &= \frac{1}{N} \int_{-1}^\infty \mathrm{d}s\ \exp[-N(1+s) + N \ln N + N \ln(1+s)] \\ &= N \exp(-N + N \ln N) \int_{-1}^\infty \mathrm{d}s\ \exp[-Ns + N \ln(1+s)] \end{aligned} \tag{1.4.26}$$

We finally have a suitable form for Laplace's method.

$$(1+s)^{-1} \xrightarrow{s\to 0} 1 - s + s^2 - s^3 + s^4 + \cdots \tag{1.4.27}$$

$$\ln(1+s) = \int \mathrm{d}x\ (1+s)^{-1} \xrightarrow{s\to 0} s - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} + \frac{s^5}{5} + \cdots \tag{1.4.28}$$

[64] If this confuses you, ask yourself what it means for a complex function to have an extremum. The complex numbers are not ordered, so what does a critical point even signify?

Then we can write (make these normalized by $Ns^2/2 = u^2$ or $u = \sqrt{N/2}s$ so $\mathrm{d}u = \sqrt{N/2}\,\mathrm{d}s$)

$$\begin{aligned} N! &\overset{N\to\infty}{\longrightarrow} N\exp(-N+N\ln N)\int_{-1}^{\infty}\mathrm{d}s\ \exp\left[-N\left(\sum_{j=2}^{\infty}\frac{(-1)^j s^j}{j}\right)\right] \\ &\overset{N\to\infty}{\longrightarrow} N\exp(-N+N\ln N)\int_{-\epsilon}^{\epsilon}\mathrm{d}s\ \exp\left[-N\left(\sum_{j=2}^{\infty}\frac{(-1)^j s^j}{j}\right)\right] \\ &\overset{N\to\infty}{\longrightarrow} N\sqrt{\frac{2}{N}}\exp(-N+N\ln N)\int_{-\epsilon\sqrt{N/2}}^{\epsilon\sqrt{N/2}}\mathrm{d}u\ \exp\left[-\left(u^2+\sum_{j=3}^{\infty}N\sqrt{\frac{2}{N}}^{j}\frac{(-1)^j u^j}{j}\right)\right] \end{aligned} \tag{1.4.29}$$

Now we'd like to have up to $\mathcal{O}(N^{-1})$ terms. This means we need to think about expanding the $j=3$ and $j=4$ terms which can contribute because our controlling factor is a Gaussian and so when we expand the other terms, any odd functions (that is odd polynomial expansion terms) will vanish via symmetry. Thus, we have to consider the $j=4$ term. We use

$$\sum_{j=3}^{\infty}\left(\frac{2}{N}\right)^{j/2}\frac{(-1)^j u^j}{j} \overset{u,s\to 0}{\longrightarrow} \frac{2^{3/2}}{N^{3/2}}\frac{u^3}{3}+\frac{2^2}{N^2}\frac{u^4}{4} \tag{1.4.30}$$

$$-N\sum_{j=3}^{\infty}\left(\frac{2}{N}\right)^{j/2}\frac{(-1)^j u^j}{j} \overset{u,s\to 0}{\longrightarrow} \frac{2^{3/2}}{N^{1/2}}\frac{u^3}{3}-\frac{2^2}{N}\frac{u^4}{4} \tag{1.4.31}$$

where, as we will see, there can be no other terms that contribute to $\mathcal{O}(N^{-1})$. We then expand these exponentials in a Taylor series expansion around $u, s = 0$ and find

$$\begin{aligned} \exp\left(\frac{2^{3/2}}{N^{1/2}}\frac{u^3}{3}\right) &\overset{u,s\to 0}{\longrightarrow} 1+\frac{2^{3/2}}{N^{1/2}}\frac{u^3}{3}+\left(\frac{2^{3/2}}{N^{1/2}}\frac{u^3}{3}\right)^2/2 \\ &\overset{u,s\to 0}{\longrightarrow} 1+\frac{\sqrt{8}}{N^{1/2}}\frac{u^3}{3}+\frac{8}{N}\frac{u^6}{18} \end{aligned} \tag{1.4.32}$$

$$\begin{aligned} \exp\left(-\frac{2^2}{N}\frac{u^4}{4}\right) &\overset{u,s\to 0}{\longrightarrow} 1-\frac{2^2}{N}\frac{u^4}{4}+\left(\frac{2^2}{N}\frac{u^4}{4}\right)^2/2 \\ &\overset{u,s\to 0}{\longrightarrow} 1-\frac{u^4}{N}+\frac{1}{N^2}\frac{u^8}{2} \end{aligned} \tag{1.4.33}$$

and we see that we can throw away the $\mathcal{O}(N^{-2})$ term from the $j=4$ term. And

$$\begin{aligned} -N\sum_{j=3}^{4}\left(\frac{2}{N}\right)^{j/2}\frac{u^j}{j} &\overset{u,s\to 0}{\longrightarrow} \left(1+\frac{\sqrt{8}}{N^{1/2}}\frac{u^3}{3}+\frac{8}{N}\frac{u^6}{18}\right)\left(1-\frac{u^4}{N}\right)+\mathcal{O}(N^{-2}) \\ &\overset{u,s\to 0}{\longrightarrow} \left(1+\frac{\sqrt{8}}{N^{1/2}}\frac{u^3}{3}+\frac{8}{N}\frac{u^6}{18}-\frac{u^4}{N}\right)+\mathcal{O}(N^{-2}) \end{aligned} \tag{1.4.34}$$

Finally, we put this into our integral and expand the limits out to infinity and see

$$\begin{aligned}
N! &\overset{N\to\infty}{\longrightarrow} \sqrt{2N}\exp(-N+N\ln N)\int_{-\epsilon\sqrt{N/2}}^{\epsilon\sqrt{N/2}}\mathrm{d}u\ \left(1+\frac{\sqrt{8}}{N^{1/2}}\frac{u^3}{3}+\frac{8}{N}\frac{u^6}{18}-\frac{u^4}{N}\right)\exp(-u^2)\\
&\overset{N\to\infty}{\longrightarrow} \sqrt{2N}\exp(-N+N\ln N)\int_{-\infty}^{\infty}\mathrm{d}u\ \left(1+\frac{\sqrt{8}}{N^{1/2}}\frac{u^3}{3}+\frac{8}{N}\frac{u^6}{18}-\frac{u^4}{N}\right)\exp(-u^2)\\
&\overset{N\to\infty}{\longrightarrow} \sqrt{2N}\exp(-N+N\ln N)\int_{-\infty}^{\infty}\mathrm{d}u\ \left(1+\frac{\sqrt{8}}{\cancel{N^{1/2}}}\frac{\cancel{u^3}}{3}+\frac{8}{N}\frac{u^6}{18}-\frac{u^4}{N}\right)\exp(-u^2)\\
&\overset{N\to\infty}{\longrightarrow} \sqrt{2N}\exp(-N+N\ln N)\int_{-\infty}^{\infty}\mathrm{d}u\ \left(1+\frac{8}{N}\frac{u^6}{18}-\frac{u^4}{N}\right)\exp(-u^2)\\
&\overset{N\to\infty}{\longrightarrow} \sqrt{2N}\exp(-N+N\ln N)\left(\sqrt{\pi}+\frac{8}{N}\frac{\sqrt{\pi}5\cdot 3\cdot 1}{2^3(18)}-\frac{\sqrt{\pi}3\cdot 1}{2^2(N)}\right)\\
&\overset{N\to\infty}{\longrightarrow} \sqrt{2\pi N}\exp(-N+N\ln N)\left(1+\frac{1}{N}\left[\frac{5}{6}-\frac{3}{4}\right]\right)\\
&\overset{N\to\infty}{\longrightarrow} \sqrt{2\pi N}\exp(-N+N\ln N)\left(1+\frac{1}{12N}\right)
\end{aligned} \tag{1.4.35}$$

This can be rewritten as

$$N! \overset{N\to\infty}{\longrightarrow} \sqrt{2\pi N}\frac{N^N}{\exp(N)}\left(1+\frac{1}{12N}\right) \tag{1.4.36}$$

which is Stirling's approximation. Because people typically don't care about the prefactor (the $\sqrt{2\pi N}$) which is small compared to N^N, this is often written as

$$\ln N! \overset{N\to\infty}{\longrightarrow} N\ln N - N + \mathcal{O}(\ln N) \tag{1.4.37}$$

One can also derive Watson's lemma which says for $\lambda > -1$ and any $0 < T \leq \infty$ with appropriately nice (infinitely differentiable at $t = 0$) functions $t^\lambda g(t)$ that we have

$$\int_0^T \mathrm{d}t\ t^\lambda g(t)\exp(-xt) \overset{N\to\infty}{\longrightarrow} \sum_{n=0}^{\infty}\frac{g^{(n)}(0)\Gamma(\lambda+n+1)}{n!x^{\lambda+n+1}} \tag{1.4.38}$$

You should think about why $\lambda > -1$ is necessary in general, and about situations where our Laplace's method works when $\lambda \leq -1$.

Finally, we can use integration by parts to find some asymptotic expressions. For example, consider $\mathrm{erfc}(x)$ defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{d}t\ \exp(-t^2) \tag{1.4.39}$$

Then we could integrate by parts with $v = \exp(-t^2)$ so $\mathrm{d}v = -2t\exp(-t^2)$ meaning $u = \frac{1}{-2t}$ and $\mathrm{d}u = \frac{1}{2t^2}\,\mathrm{d}t$ so that

$$\begin{aligned}
\mathrm{erfc}(x) &= \frac{2}{\sqrt{\pi}}\left[\left\{\frac{\exp(-t^2)}{-2t}\right\}_x^\infty - \int_x^\infty \mathrm{d}t\ \frac{\exp(-t^2)}{2t^2}\right]\\
&= \frac{1}{\sqrt{\pi}}\left[\frac{\exp(-x^2)}{x} - \int_x^\infty \mathrm{d}t\ \frac{\exp(-t^2)}{t^2}\right]
\end{aligned} \tag{1.4.40}$$

One then notes that for $x \to \infty$ that

$$\int_x^\infty \mathrm{d}t\ \exp(-t^2) = o\left(\int_x^\infty \mathrm{d}t\ \frac{\exp(-t^2)}{t^2}\right) \tag{1.4.41}$$

because $t^{-2} = o(1)$ as $t \to \infty$. Thus the last integral can be ignored. If we desire extra terms, we can show that for

$$I_n(x) = \int_x^\infty \mathrm{d}t\ \frac{\exp(-t^2)}{t^{2n}} \tag{1.4.42}$$

with $I_0 = \frac{2}{\sqrt{\pi}}\operatorname{erfc}(x)$ we find by integration by parts with $v = \exp(-t^2)$, $\mathrm{d}v = -2t\exp(-t^2)\,\mathrm{d}t$, $u = \frac{-1}{2t^{2n+1}}$ and $\mathrm{d}u = \frac{2n+1}{2t^{2n+2}}$ so

$$\begin{aligned} I_n &= \left[\frac{-\exp(-t^2)}{2t^{2n+1}}\right]_x^\infty - \int_x^\infty \mathrm{d}t\ \frac{(2n+1)\exp(-t^2)}{2t^{2n+2}} \\ &= \frac{\exp(-x^2)}{2x^{2n+1}} - \frac{2n+1}{2} I_{n+1} \end{aligned} \tag{1.4.43}$$

Then we have

$$I_0 = \frac{\exp(-x^2)}{2x} - \frac{1}{2} I_1 \tag{1.4.44}$$

$$I_0 = \frac{\exp(-x^2)}{2x} - \frac{1}{2}\frac{\exp(-x^2)}{2x^3} + \frac{1}{2}\frac{3}{2} I_2 \tag{1.4.45}$$

where we can then use that $I_n = o(I_0)$ and so we have an asymptotic series. Thus

$$\operatorname{erfc}(x) \overset{x\to\infty}{\longrightarrow} \frac{2}{\sqrt{\pi}} I_0 = \frac{2}{\sqrt{\pi}}\left[\frac{\exp(-x^2)}{2x} - \frac{1}{2}\frac{\exp(-x^2)}{2x^3} + \cdots\right] \tag{1.4.46}$$

$$\operatorname{erfc}(x) \overset{x\to\infty}{\longrightarrow} \frac{\exp(-x^2)}{x\sqrt{\pi}}\left[1 - \frac{1}{2x^2} + \cdots\right] \tag{1.4.47}$$

$$\operatorname{erfc}(x) \overset{x\to\infty}{\longrightarrow} \frac{\exp(-x^2)}{x\sqrt{\pi}}\left[1 \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2x^2)^n}\right] \tag{1.4.48}$$

One should note that these are all asymptotic series, and so one should realize that the sums will diverge for a fixed x (or N with Stirling's approximation). This is because the functions themselves often diverge when approaching the limit. The important thing to realize is that though both sides may diverge, they diverge in a similar way so that their ratio approaches one, which makes our values accurate. That is the relative error is very small, while the absolute error can grow arbitrarily large. For example, for large x, one finds that

$$\operatorname{erfc}(x) \approx \frac{\exp(-x^2)}{x\sqrt{\pi}} \tag{1.4.49}$$

is generally good enough. If we analyzed this in more detail we could derive that the error in taking only N terms of the approximation is $\mathcal{O}(x^{1-2N}\exp(-x^2))$. So for $x \approx 10$ with $N = 1$ we see we have error $\mathcal{O}(\exp(-10^2)/10)$ or $\mathcal{O}(3.72 \times 10^{-45})$ which is sufficient for most applications... That is

$$\operatorname{erfc}(10) \approx 2.088 \times 10^{-45} \tag{1.4.50}$$

$$\frac{\exp(-10^2)}{10\sqrt{\pi}} \approx 2.099 \times 10^{-45} \tag{1.4.51}$$

which is a fairly impressive accuracy for just one term.

1.4.2 Other Asymptotic Series

> *When iterating, three things may happen and two of them are bad. An iteration may converge, may converge too slowly to be useful, or diverge.*
>
> — J. P. Boyd[5, p. 514]

It is often useful to find an asymptotic series form for a function. As we saw for integrals, recursion is often useful, but so is using Taylor series by using that if something is going to ∞ then its inverse is going to zero. Thus, if we want to approximate $\ln(\exp(x)+1)$ as $x \to \infty$ we can use that

$$\begin{aligned}\ln(\exp(x)+1) &= \ln(\exp(x)[1+\exp(-x)]) = \ln(\exp(x)) + \ln(1+\exp(-x)) \\ &= x + \ln[1+\exp(-x)]\end{aligned} \tag{1.4.52}$$

Then $\exp(-x) = 1/\exp(x) \equiv \epsilon \overset{x\to\infty}{\longrightarrow} 0$ so we have

$$\ln(1+\epsilon) \overset{x\to\infty}{\longrightarrow} \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} + \cdots \tag{1.4.53}$$

and so

$$\ln(\exp(x)+1) \overset{x\to\infty}{\longrightarrow} x + \exp(-x) - \frac{\exp(-2x)}{2} + \frac{\exp(-3x)}{3} + \cdots \tag{1.4.54}$$

Such cases usually involve factoring out terms that we cannot deal with in such a way that they leave terms that we can approximate.

Of special interest are cases where we want to know how the inverse function acts. It is often quite difficult to get a closed form solution of an inverse function for a given function. However, asymptology allows us to find inverses in a limit that can help us understand how such functions can be inverted (this can help in finding annihilators, as well). Suppose you are given the function $f(x) = x^3 + x^2$. Try and find the inverse of such a function. Remember that the inverse $f^{-1}(x)$ is such that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$. If you can find a simple way of expressing the inverse of $f(x) = x^3 + x^2$, hats off to you. I do not know of any. One could try to find the inverse by saying $f(y) = x$ and solving for y in terms of x. Then $y = f^{-1}(x)$ is the inverse. Looking at our case you have

$$x = y^3 + y^2 = y^2(y+1) \tag{1.4.55}$$

which means that we need to use a cubic formula to solve for y in terms of x. It is not pretty. However, if we are only interested in the case where $x \to \infty$, then we can use that

$$f(x) \overset{x\to\infty}{\longrightarrow} x^3 \tag{1.4.56}$$

$$f^{-1}(x) \overset{x\to\infty}{\longrightarrow} x^{1/3} \tag{1.4.57}$$

We can surmise that extra corrections are likely to be powers of x of the form ax^α with $\alpha < 1/3$ so that they are subdominant to $x^{1/3}$. If we simply try

$$f^{-1}(x) \overset{x\to\infty}{\longrightarrow} x^{1/3} + ax^\alpha \tag{1.4.58}$$

and plug this into $f(f^{-1}(x)) \overset{x\to\infty}{\longrightarrow} x$ we see we get

$$(x^{1/3} + ax^\alpha)^3 + (x^{1/3} + ax^\alpha)^2 \overset{x\to\infty}{\longrightarrow} x \tag{1.4.59}$$

We see the left-hand side can be written as

$$x + 3x^{2/3}ax^{\alpha} + 3x^{1/3}a^2x^{2\alpha} + a^3x^{3\alpha} + x^{2/3} + 2ax^{1/3+\alpha} + a^2x^{2\alpha} \tag{1.4.60}$$

Now we need $\alpha < 1/3$ which means that $2\alpha < 2/3$ and $3\alpha < 1$. Thus, among the terms that are left, the dominant ones should be $x^{2/3+\alpha}$ or possibly $x^{3\alpha}$ because $2/3 + \alpha \leq 1$ [so that x is the dominant term in (1.4.60)]. It must be these because clearly $1/3 + \alpha < 2/3 + \alpha$ always and for $\alpha > 0$, $3\alpha > 2\alpha > \alpha$. In addition the unadorned by a or α term $x^{2/3}$ is clearly the term among all of these that must be matched. First let's try $x^{3\alpha}$ is dominant.[65] Then we must have

$$x^{2/3} + a^3x^{3\alpha} \overset{x\to\infty}{\longrightarrow} 0 \tag{1.4.61}$$

which would mean $\alpha = 2/9$ and $a^3 = -1$ or $a = -1$. Then $2/3 + \alpha = 8/9$ and so we have a contradiction that $x^{3\alpha} = x^{6/9}$ is dominant over $x^{2/3+\alpha} = x^{8/9}$, which is clearly false as $x \to \infty$.

So let's try $x^{2/3+\alpha}$ as the dominant term. Then we have

$$x^{2/3} + 3ax^{2/3+\alpha} \overset{x\to\infty}{\longrightarrow} 0 \tag{1.4.62}$$

which would mean $\alpha = 0$ and $a = -1/3$. Then $x^{3\alpha} = x^0$ is subdominant to $x^{2/3+\alpha} = x^{2/3}$ for $x \to \infty$ and this is a possible solution.

Then we have as our approximation that

$$f^{-1}(x) = x^{1/3} - \frac{1}{3} \tag{1.4.63}$$

We can repeat the process by guessing a form

$$f^{-1}(x) = x^{1/3} - \frac{1}{3} + bx^{\beta} \tag{1.4.64}$$

with $\beta < 0$. Then we try $f(f^{-1}(x)) = x$ again and find

$$\left(x^{1/3} - \frac{1}{3} + bx^{\beta}\right)^3 + \left(x^{1/3} - \frac{1}{3} + bx^{\beta}\right)^2 \overset{x\to\infty}{\longrightarrow} x \tag{1.4.65}$$

with the left hand side yielding (we will just immediately use for any $a > 0$ that $a + \beta > 2\beta$ or $a > \beta$ which must be true since $\beta < 0$ is a solution. In addition, we can now use that $x^{2\beta}$ is small in comparison to x^{β} since we know $\beta < 0$)

$$\left(x^{1/3} - \frac{1}{3}\right)^3 + 3\left(x^{1/3} - \frac{1}{3}\right)^2 bx^{\beta} + \mathcal{O}(x^{2\beta}) + \left(x^{1/3} - \frac{1}{3}\right)^2 + 2\left(x^{1/3} - \frac{1}{3}\right) bx^{\beta} + \mathcal{O}(x^{2\beta}) \tag{1.4.66}$$

$$\begin{aligned} x - \cancel{x^{2/3}} + \frac{x^{1/3}}{3} - \frac{1}{27} + 3\left(x^{2/3} - \frac{2}{3}x^{1/3} + \frac{1}{9}\right) bx^{\beta} + \mathcal{O}(x^{2\beta}) \\ + \left(\cancel{x^{2/3}} - \frac{2}{3}x^{1/3} + \frac{1}{9}\right) + 2\left(x^{1/3} - \frac{1}{3}\right) bx^{\beta} + \mathcal{O}(x^{2\beta}) \end{aligned} \tag{1.4.67}$$

[65] Experience, skill, and luck will let you in the future choose more wisely.

Let's assume that $x^{2/3+\beta}$ is the next dominant term. Then the dominant terms unadorned by β or b are the $x^{1/3}$ terms so

$$\cancel{x} + 3bx^{2/3+\beta} + x^{1/3}\left(\frac{1}{3} - \frac{2}{3}\right) \overset{x\to\infty}{\longrightarrow} \cancel{x} \tag{1.4.68}$$

or

$$3bx^{2/3+\beta} \overset{x\to\infty}{\longrightarrow} \frac{1}{3}x^{1/3} \tag{1.4.69}$$

yielding $b = \frac{1}{9}$ and $\beta = -1/3$ consistent with $\beta < 0$. We have

$$f^{-1}(x) \overset{x\to\infty}{\longrightarrow} x^{1/3} - \frac{1}{3} + \frac{1}{9}x^{-1/3} \tag{1.4.70}$$

We could clearly keep going further if we so desired, but this is the kernel of the idea.

One other quick example is finding the inverse of $f(x) = x\exp(x)$ as $x \to \infty$. Our initial guess is that $\exp(x)$ dominates and so $f^{-1}(x) = \ln(x)$. We could guess that a further term will be of the form $g(x)$ so

$$f^{-1}(x) \overset{x\to\infty}{\longrightarrow} \ln x + g(x) \tag{1.4.71}$$

and then use that $f^{-1}(f(x)) \overset{x\to\infty}{\longrightarrow} x$ (note how the order of this is different than before)

$$\ln(x\exp(x)) + g(x\exp(x)) \overset{x\to\infty}{\longrightarrow} x \tag{1.4.72}$$

$$\ln x + \cancel{x} + g(x\exp(x)) \overset{x\to\infty}{\longrightarrow} \cancel{x} \tag{1.4.73}$$

$$\ln x \overset{x\to\infty}{\longrightarrow} -g(x\exp(x)) \tag{1.4.74}$$

Now as $x \to \infty$ we expect $g(x\exp(x)) \overset{x\to\infty}{\longrightarrow} g(\exp(x))$ and so we have

$$g(\exp(x)) \overset{x\to\infty}{\longrightarrow} -\ln x \tag{1.4.75}$$

Thus if we put in $y = \ln(x)$ we see

$$g(\exp(y)) = g(\exp(\ln(x))) = g(x) \tag{1.4.76}$$

$$\ln(y) = \ln(\ln(x)) \tag{1.4.77}$$

and so using x instead of $\exp(x)$ means we have[66]

$$g(x) \overset{x\to\infty}{\longrightarrow} -\ln[\ln(x)] \tag{1.4.78}$$

And we could try the same thing again with another new additional function $h(x)$.

$$f^{-1}(x) \overset{x\to\infty}{\longrightarrow} \ln(x) - \ln[\ln(x)] + h(x) \tag{1.4.79}$$

$$f^{-1}(f(x)) \overset{x\to\infty}{\longrightarrow} x \tag{1.4.80}$$

[66]Since $\exp(x) > 0$ we restrict $x > 0$ but we have $x \to \infty$ so this imposes no new penalty.

which implies

$$\begin{aligned}
&\ln(x\exp(x)) - \ln[\ln(x\exp(x))] + h(x\exp(x)) = \ln(x) + x - \ln[\ln x + x] + h(x\exp(x)) \\
&= \ln(x) + x - \ln\left[x\left(\frac{\ln x}{x} + 1\right)\right] + h(x\exp(x)) \\
&= \cancel{\ln(x)} + x - \cancel{\ln(x)} - \ln\left(1 + \frac{\ln x}{x}\right) + h(x\exp(x))
\end{aligned} \tag{1.4.81}$$

We can use that $\ln(x)/x \equiv \epsilon \overset{x\to\infty}{\longrightarrow} 0$ and $h(x\exp(x)) \overset{x\to\infty}{\longrightarrow} h(\exp(x))$ and so Taylor expand $\ln(1+\epsilon)$ to find

$$\cancel{x} + h(\exp(x)) - \frac{\ln x}{x} + \frac{[\ln x]^2}{x^2} + \cdots \overset{x\to\infty}{\longrightarrow} \cancel{x} \tag{1.4.82}$$

$$h(\exp(x)) \overset{x\to\infty}{\longrightarrow} \frac{\ln x}{x} \tag{1.4.83}$$

$$h(x) \overset{x\to\infty}{\longrightarrow} \frac{\ln[\ln x]}{\ln x} \tag{1.4.84}$$

and so

$$f^{-1}(x) \overset{x\to\infty}{\longrightarrow} \ln(x) - \ln[\ln(x)] + \frac{\ln[\ln x]}{\ln x} \tag{1.4.85}$$

We could of course continue the process.

One is led to consider the Lambert W function given as $W(x)$ which solves $W\exp(W) = x$. In other words, the Lambert W function is implicitly defined as the inverse of the function $f(w) = w\exp(w)$. In fact we could consider z a complex number instead of x a real number for the full Lambert W functions with various branch cuts W_k. We focus on the principal branch W_0 which I will denote just as W. We might ask what does W look like as $x \to \infty$? Well, with our previous methods we have a simple way of figuring this out. We have $f(x) = f(w)$ and we desire an expression for $W(x) = f^{-1}(x)$ when $x \to \infty$. Lo and behold, my previous example seems a bit more relevant! For we then have

$$W(x) \overset{x\to\infty}{\longrightarrow} \ln(x) - \ln[\ln(x)] + \frac{\ln[\ln x]}{\ln x} \tag{1.4.86}$$

Finally, we can deal with multiple variable equations with similar techniques. Suppose we had an equation like

$$y^3 - 2xy^2 + x^2y - x = 0 \tag{1.4.87}$$

and wanted to know how the expression behaved as $x \to \infty$. Now we have all sorts of choices for y. If $y \to \infty$ as well, then we need to think about how quickly it goes in comparison to x. One simple way to deal with this is to introduce $\xi = \epsilon x$ and $\psi = \delta y$ with $\epsilon \ll 1$ and $\delta \ll 1$. This makes ξ and ψ the same "order" terms, and is essentially just stretching out x and y. Then we can think of $x, y \gg 1$ as $\epsilon, \delta \ll 1$. The equation becomes

$$\delta^{-3}\psi^3 - 2\delta^{-2}\epsilon^{-1}\xi\psi^2 + \epsilon^{-2}\delta^{-1}\xi^2\psi - \epsilon^{-1}\xi = 0 \tag{1.4.88}$$

Or

$$\psi^3 - 2\delta\epsilon^{-1}\xi\psi^2 + \epsilon^{-2}\delta^2\xi^2\psi - \delta^3\epsilon^{-1}\xi = 0 \tag{1.4.89}$$

We see the above can then be rewritten as

$$\psi\left(\psi^2 - 2\delta\epsilon^{-1}\xi\psi + \epsilon^{-2}\delta^2\xi^2\right) = \delta^3\epsilon^{-1}\xi \tag{1.4.90}$$

$$\psi\left(\psi - \delta\epsilon^{-1}\xi\right)^2 = \delta^3\epsilon^{-1}\xi \tag{1.4.91}$$

Now, I think it is not a stretch to try $\delta/\epsilon = 1$ so that ψ and ξ balance on the left hand side and the δ^3/ϵ term on the right is non-dominant. That is $\psi = \xi + \mathcal{O}(\delta)$. In fact we can even do better. Let's assume $\psi = \xi + a\delta\xi^\alpha$ for $\alpha < 1$. Then

$$(\xi + a\delta\xi^\alpha)(a\delta\xi^\alpha)^2 = \delta^2\xi \tag{1.4.92}$$

$$a^2\delta^2\xi^{1+2\alpha} + a^3\delta^3\xi^{3\alpha} = \delta^2\xi \tag{1.4.93}$$

and clearly the δ^2 terms must match with the δ^3 term being subdominant. Thus $1 + 2\alpha = 1$ and $a^2 = 1$ so $\alpha = 0$ and $a = 1$. Thus

$$\psi = \xi + \delta \tag{1.4.94}$$

We can try one more term via $\psi = \xi + \delta + a\delta^2\xi^\alpha$ again. Then

$$\left(\xi + \delta + a\delta^2\xi^\alpha\right)(\delta + a\delta^2\xi^\alpha)^2 = \delta^2\xi \tag{1.4.95}$$

$$\delta^2\left(\xi + \delta + \delta^2\xi^\alpha\right)(1 + 2a\delta\xi^\alpha + a^2\delta^2\xi^{2\alpha}) = \delta^2\xi \tag{1.4.96}$$

$$\cancel{\delta^2\xi} + \delta^3\left(1 + 2a\xi^{1+\alpha}\right) + \mathcal{O}(\delta^4) = \cancel{\delta^2\xi} \tag{1.4.97}$$

$$\delta^3\left(1 + 2a\xi^{1+\alpha}\right) = \mathcal{O}(\delta^4) \tag{1.4.98}$$

so $a = -\frac{1}{2}$ and $1 + \alpha = 0$ so $\alpha = -1$, and our final solution is

$$\psi = \xi + \delta - \delta^2\frac{\xi^{-1}}{2} + \mathcal{O}(\delta^3) \tag{1.4.99}$$

Then we can use $\psi = \delta y$ and $\xi = \delta x$ in our ordering so that

$$\delta y \overset{x,y\to\infty}{\longrightarrow} \delta x + \delta - \delta\frac{x^{-1}}{2} \tag{1.4.100}$$

$$y \overset{x,y\to\infty}{\longrightarrow} x + 1 - \frac{x^{-1}}{2} \tag{1.4.101}$$

This approach is flexible, and could have been used in our previous cases. Sometimes it is simpler to deal with small parameters rather than things going to infinity.

1.4.3 Asymptotic Matching

A common problem is that we have an equation with an ordering parameter that we can solve when the ordering parameter is either very large or very small and this corresponds to a specific region in the equation. When the parameter is intermediate, we have no such solution, and so we

desire a way to form some sort of intermediate solution. It seems like we should be able to do this by matching the two solution regions together.

To make this concrete, we will consider the time-independent Schrödinger equation. We will use an eikonal form to find a solution when terms are not singular which is near the region where the energy is the potential energy $E = V(x)$, the classical turning points. We will just consider one dimension. We have the equation

$$\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = (U - E)\psi \tag{1.4.102}$$

We use $\epsilon = \hbar^2/(2m)$ as a small parameter. Then with $\widehat{U} = U - E$ we write

$$\epsilon\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \widehat{U}\psi \tag{1.4.103}$$

We have the solutions in the region outside classical turning points in Section 1.12.2. Thus we only need to be near a region where the classical turning points are. Let's assume initially that there is only one classical turning point located at $x = x_0$. Then we can Taylor expand near this point and find

$$\widehat{U} = \overbrace{\widehat{U}_0}^{\widehat{U}(x_0)=0} + (x - x_0)\overbrace{\left(\frac{\mathrm{d}\widehat{U}}{\mathrm{d}x}\right)_{x=x_0}}^{\widehat{U}_1(x_0)} + (x - x_0)^2\overbrace{\left(\frac{\mathrm{d}^2\widehat{U}}{\mathrm{d}x^2}\right)_{x=x_0}}^{\widehat{U}_2(x_0)} + \cdots \tag{1.4.104}$$

At the turning point we must have $\widehat{U}_0 = 0$ by definition. If we consider a region $x - x_0 \propto \epsilon$ we can write

$$\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = (x - x_0)\widehat{U}_1\psi \tag{1.4.105}$$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = (x - x_0)\overbrace{\widehat{u}_1}^{2m\widehat{U}_1/\hbar^2}\psi \tag{1.4.106}$$

One must then recognize that this is of the form

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - x\psi = 0 \tag{1.4.107}$$

which is the Airy equation with solutions of Airy functions. That is for (1.4.107) the solutions are

$$\psi(x) = C_1\,\mathrm{Ai}(x) + C_2\,\mathrm{Bi}(x) \tag{1.4.108}$$

with $\mathrm{Ai}(x)$ satisfying $\psi \to 0$ as $x \to \infty$ and $\mathrm{Bi}(x)$ has the same amplitude of oscillation as $\mathrm{Ai}(x)$ as $x \to -\infty$ but differing in phase by $\pi/2$. It turns out we can write these as integrals of the form

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \mathrm{d}t\ \cos\left(\frac{t^3}{3} + xt\right) \tag{1.4.109}$$

$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \mathrm{d}t\ \left[\exp\left(\frac{-t^3}{3} + xt\right) + \sin\left(\frac{t^3}{3} + xt\right)\right] \tag{1.4.110}$$

In our case we see that we have

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \overbrace{(x-x_0)\widehat{u}_1}^{\text{like } x}\psi \tag{1.4.111}$$

and so our x is not in the right form for the Airy function solutions. First, we take $y = (x-x_0)(\widehat{u}_1)^\alpha$ and define $\Psi(y) = \psi(x)$ so

$$(\widehat{u}_1^{2\alpha})\frac{\mathrm{d}^2\Psi(y)}{\mathrm{d}y^2} = y(\widehat{u}_1)^{1-\alpha}\Psi(y) \tag{1.4.112}$$

$$\frac{\mathrm{d}^2\Psi(y)}{\mathrm{d}y^2} = y(\widehat{u}_1)^{1-\alpha-2\alpha}\Psi(y) \tag{1.4.113}$$

if we choose $\alpha = \frac{1}{3}$ then $\widehat{u}_1$ disappears on both sides and we get the Airy equation for $\Psi(y)$. Thus

$$\Psi(y) = C_1\,\mathrm{Ai}(y) + C_2\,\mathrm{Bi}(y) \tag{1.4.114}$$

And we have $\Psi(y) = \psi(x)$ with $y = (x-x_0)(\widehat{u}_1)^{1/3}$ and so

$$\psi(x) = C_1\,\mathrm{Ai}([x-x_0][\widehat{u}_1]^{1/3}) + C_2\,\mathrm{Bi}([x-x_0][\widehat{u}_1]^{1/3}) \tag{1.4.115}$$

Now for the asymptotic analysis. This is the solution near $x = x_0$ which we will call the inner region. As we go to the right we think of this as going to $x \to \infty$ and when we go to the left, as $x \to -\infty$. Asymptotically we have

$$\mathrm{Ai}(x) \overset{x\to\infty}{\longrightarrow} \frac{\exp\left(-\frac{2}{3}x^{3/2}\right)}{2\sqrt{\pi}x^{1/4}} \tag{1.4.116}$$

$$\mathrm{Bi}(x) \overset{x\to\infty}{\longrightarrow} \frac{\exp\left(\frac{2}{3}x^{3/2}\right)}{2\sqrt{\pi}x^{1/4}} \tag{1.4.117}$$

$$\mathrm{Ai}(x) \overset{x\to-\infty}{\longrightarrow} \frac{\sin\left(\frac{2}{3}x^{3/2}+\frac{\pi}{4}\right)}{\sqrt{\pi}x^{1/4}} \tag{1.4.118}$$

$$\mathrm{Bi}(x) \overset{x\to-\infty}{\longrightarrow} \frac{\cos\left(\frac{2}{3}x^{3/2}+\frac{\pi}{4}\right)}{\sqrt{\pi}x^{1/4}} \tag{1.4.119}$$

Our solution in the outer region is given by [see (1.12.67)]

$$\psi^o(x) \approx \frac{C_+\exp\left[\int_{x_0}^x \mathrm{d}x'\,\sqrt{\frac{2m}{\hbar^2}(U-E)}\right] + C_-\exp\left[-\int_{x_0}^x \mathrm{d}x'\,\sqrt{\frac{2m}{\hbar^2}(U-E)}\right]}{(\frac{2m}{\hbar^2}|U-E|)^{1/4}} \tag{1.4.120}$$

Without loss of generality Consider $U-E<0$ for $x>x_0$ (that is, $E>U$, so a classically allowed region and I will denote as $\to$ since x is to the right of x_0). Then in this region we get $\sqrt{(U-E)} = i\sqrt{|U-E|}$ and so we get waves. We can then consider $U-E>0$ for $x<x_0$ (the $\leftarrow$ region). In this case we have exponentials with $C_+^{\leftarrow}$ a negatively decaying exponential and $C_-^{\leftarrow}$ a growing exponential. Clearly, only $C_+^{\rightarrow}$ is a physical solution, as we can't have something exponentially growing and have a normalized ψ.

In the case we chose with $U-E<0$ for $x>x_0$ we have both coefficients which I will just keep as C_+ and C_- without $\to$ to keep them separate from the inner region values.

For matching onto the inner region values, we note that our choices enforce that $\widehat{u}_1 < 0$ because $U - E$ is decreasing as x crosses x_0. If we use the same Taylor series approximation for $\widehat{U}$ on the left outer region we see that for $x < x_0$ so $(x-x_0)\widehat{u}_1 > 0$ and we replace $(x-x_0)\widehat{u}_1$ with $(x_0-x)|\widehat{u}_1|$ for convenience and $U - E > 0$ we get

$$\psi^{o\leftarrow}(x) \approx \frac{C_+^{\leftarrow} \exp\left[-\int_x^{x_0} \mathrm{d}x' \ \sqrt{(x_0-x)|\widehat{u}_1|}\right]}{[(x_0-x)|\widehat{u}_1|]^{1/4}} \tag{1.4.121}$$

using $t' = x_0 - x'$ and $t = x_0 - x > 0$ we write this as

$$\psi^{o\leftarrow}(x) \approx \frac{C_+^{\leftarrow} \exp\left[-\int_0^{t} \mathrm{d}t' \ \sqrt{t'|\widehat{u}_1|}\right]}{[t|\widehat{u}_1|]^{1/4}} = \frac{C_+^{\leftarrow}}{(|\widehat{u}_1|t)^{1/4}} \exp\left(-\frac{(\widehat{u}_1 t)^{3/2}}{\frac{3}{2}}\right) \tag{1.4.122}$$

$$\approx \frac{C_+^{\leftarrow}}{(|\widehat{u}_1|t)^{1/4}} \exp\left(-\frac{2(|\widehat{u}_1|t)^{3/2}}{3}\right) \tag{1.4.123}$$

Matching with (1.4.116) [remember that although $x - x_0 < 0$ that $(\widehat{u}_1)^{1/3} < 0$ as well so we are in the $x \to +\infty$ limit] we see that

$$\frac{C_+^{\leftarrow}}{(|\widehat{u}_1|t)^{1/4}} \exp\left(-\frac{2(|\widehat{u}_1|t)^{3/2}}{3}\right) = \frac{\exp\left(-\frac{2}{3}(|\widehat{u}_1|t)^{3/2}\right)}{2\sqrt{\pi}(|\widehat{u}_1|t)^{1/4}} \tag{1.4.124}$$

which means $C_+^{\leftarrow} = \frac{1}{2\sqrt{\pi}}$ is a requirement. This means we don't care about the $\mathrm{Bi}(t)$ solution.

Next we look at $x > x_0$ and use $t' = x' - x_0$ with $t = x - x_0 > 0$ to find

$$\psi^{o\rightarrow}(x) \approx \frac{C_+ \exp\left[i\int_0^t \mathrm{d}t' \ \sqrt{t|\widehat{u}_1|}\right] + C_- \exp\left[-i\int_0^t \mathrm{d}t' \ \sqrt{t|\widehat{u}_1|}\right]}{[t|\widehat{u}_1|]^{1/4}} \tag{1.4.125}$$

which simplifies to

$$\psi^{o\rightarrow}(x) \approx \frac{C_+ \exp\left[i\sqrt{|\widehat{u}_1|}\frac{2t^{3/2}}{3}\right] + C_- \exp\left[-i\sqrt{|\widehat{u}_1|}\frac{2t^{3/2}}{3}\right]}{[t|\widehat{u}_1|]^{1/4}} \tag{1.4.126}$$

We rewrite (1.4.118) (define $\zeta = \frac{2}{3}(|\widehat{u}_1|^{1/3}t)^{3/2}$ for convenience, remembering that $(\widehat{u}_1)^{1/3}$ is negative so we are in the correct limit)

$$\mathrm{Ai}(t) \overset{x\to-\infty}{\longrightarrow} \frac{\exp\left(i\zeta + i\frac{\pi}{4}\right) - \exp\left(-i\zeta - i\frac{\pi}{4}\right)}{2i\sqrt{\pi}(|\widehat{u}_1|t)^{1/4}} \tag{1.4.127}$$

Then

$$C_+ = \frac{-i\exp(i\pi/4)}{2\sqrt{\pi}} \tag{1.4.128}$$

$$C_- = \frac{i\exp(-i\pi/4)}{2\sqrt{\pi}} \tag{1.4.129}$$

Then we must have using $C_+^{\rightarrow} = \frac{1}{2\sqrt{\pi}}$ that

$$C_+ = -i\exp(i\pi/4)C_+^{\leftarrow} = \exp\left(-i\pi/4\right)C_+^{\leftarrow} \tag{1.4.130}$$

$$C_- = i\exp(-i\pi/4)C_+^{\leftarrow} = \exp(i\pi/4)C_+^{\leftarrow} \tag{1.4.131}$$

Now assume there is one other turning point at x_1 to the right of x_0 and $x > x_1$ is a classically forbidden region. Then the form of the solution near x_1 will be the same but there can now be a phase difference from the new coefficients there. So in order to get a single valued, square integrable solution we need to think about the argument in the complex exponentials. We will soon see that we must have

$$\int_{x_0}^{x_1} \mathrm{d}x' \, \sqrt{\frac{2m}{\hbar^2}|U-E|} = \left(n + \frac{1}{2}\right)\pi \tag{1.4.132}$$

for n a non-negative integer so that the forms of C_+ and C_- either exchange places or go back to the same values when connecting across turning point regions. That is, we recover the Bohr-Sommerfeld quantization rule (with the "Maslov correction" of 1/2 allowing "swapping"). Had we chosen different boundary conditions at x_0 and x_1 [for example, we could force $\psi = 0$ there] we could get different Maslov correction factors, which just come from asymptotic matching.

All other types of asymptotic matching problems are similar. We deal with an inner region and "extend" its solution to infinity with an asymptotic expression, and then use the freedom in the outer solution to match the inner solution to the outer solution, consistent with any physical requirements we may have.

Thus, our full solution is for the inner region ψ^i and outer region ψ^o given by

$$\psi^i \approx C_+^{\rightarrow} \, \mathrm{Ai}(|\widehat{u}_1|^{1/3}(x - x_0)) \quad x < x_0 \tag{1.4.133}$$

$$\psi^o \approx C_+^{\rightarrow} \frac{\exp\left(-\frac{i\pi}{4} + \frac{i}{\hbar}\int_{x_0}^{x} \mathrm{d}x' \, \sqrt{\frac{2m}{\hbar^2}|U-E|}\right) + \exp\left(\frac{i\pi}{4} - \frac{i}{\hbar}\int_{x_0}^{x} \mathrm{d}x' \, \sqrt{\frac{2m}{\hbar^2}|U-E|}\right)}{\left(\frac{2m}{\hbar^2}|U-E|\right)^{1/4}} \quad x_0 < x < x_1 \tag{1.4.134}$$

where x_1 is ∞ if there are no other turning points.

If we extend to another region with turning point x_1 [with $U - E < 0$ to the left of x_1 to be consistent with x_0 and $U - E > 0$ to the right of x_1][67] to the right of x_0 so $x_1 > x_0$, then we must have

$$\psi^{i_1} \approx C_{1-}^{\rightarrow} \, \mathrm{Ai}(|\widehat{u}_{1,2}|^{1/3}(x - x_1)) \quad x > x_1 \tag{1.4.135}$$

$$\psi^{o_1 \leftarrow} \approx C_{1+}^{\rightarrow} \frac{\exp\left(\frac{i\pi}{4} + \frac{i}{\hbar}\int_{x_1}^{x} \mathrm{d}x' \, \sqrt{\frac{2m}{\hbar^2}|U-E|}\right) + \exp\left(-\frac{i\pi}{4} - \frac{i}{\hbar}\int_{x_1}^{x} \mathrm{d}x' \, \sqrt{\frac{2m}{\hbar^2}|U-E|}\right)}{\left(\frac{2m}{\hbar^2}|U-E|\right)^{1/4}} \quad x_0 < x < x_1 \tag{1.4.136}$$

$$\psi^{o_1 \rightarrow} \approx C_{1+}^{\rightarrow} \frac{\exp\left(\frac{-1}{\hbar}\int_{x_1}^{x} \mathrm{d}x' \, \sqrt{\frac{2m}{\hbar^2}|U-E|}\right)}{\left(\frac{2m}{\hbar^2}|U-E|\right)^{1/4}} \quad x > x_1 \tag{1.4.137}$$

with $\widehat{u}_{1,2}$ the analog of $\widehat{u}_1$ but now positive and at $x = x_1$ instead of x_0. $C_{1+}^{\rightarrow}$ is the coefficient of the function around x_1 that must be consistent with exponential decay to the right of x_1. Note how the $\pm i\pi/4$ switch because when we match this time we have to remember that $x < x_1$ [whereas before $x > x_0$ in this same region] and so the integrals as written are negative. Because ψ^{i_1} must also

[67] This means that the classically allowed region is surrounded by classically forbidden regions.

match onto the "outer" solution $\psi^{o\rightarrow}$ we see that we must get the matching quantization condition mentioned before.

We see that we need matching completely within $x_0 < x < x_1$. Thus we require for this region

$$\begin{aligned} &C_+^{\leftarrow}\left[\exp\left(-\frac{i\pi}{4}+i\int_{x_0}^{x}\mathrm{d}x'\ \alpha\right)+\exp\left(\frac{i\pi}{4}-\int_{x_0}^{x_1}\mathrm{d}x'\ \alpha\right)\right] \\ &=C_{1-}^{\rightarrow}\left[\exp\left(\frac{i\pi}{4}+i\int_{x_1}^{x}\mathrm{d}x'\ \alpha\right)+\exp\left(-\frac{i\pi}{4}-i\int_{x_1}^{x}\mathrm{d}x'\ \alpha\right)\right] \end{aligned} \tag{1.4.138}$$

$$2C_+^{\leftarrow}\cos\left(\int_{x_0}^{x}\mathrm{d}x'\ \alpha-\frac{\pi}{4}\right)=2C_{1-}^{\rightarrow}\cos\left(\int_{x_1}^{x}\mathrm{d}x'\ \alpha+\frac{\pi}{4}\right) \tag{1.4.139}$$

$$C_+^{\leftarrow}\cos\left(\int_{x_0}^{x}\mathrm{d}x'\ \alpha-\frac{\pi}{4}\right)=C_{1-}^{\rightarrow}\cos\left(-\int_{x}^{x_1}\mathrm{d}x'\ \alpha+\frac{\pi}{4}\right) \tag{1.4.140}$$

The only posible way this is true is when $C_+^{\leftarrow}=C_{1-}^{\rightarrow}$ the phases are only off by a certain amount (since they are periodic in 2π) or if $C_+^{\leftarrow}=-C_{1-}^{\rightarrow}$ and the phases are off by a certain amount. We can't really know *a priori* which way it is, but we can choose them consistently after we decide on a condition for the integral. Thus we can alter the phase by $n\pi$, since we can choose the constants to be negatives of each other rather than simply equal to each other. This means

$$\int_{x_0}^{x}\mathrm{d}x'\ \alpha-\frac{\pi}{4}=-\int_{x}^{x_1}\mathrm{d}x'\ \alpha+\frac{\pi}{4}+n\pi \tag{1.4.141}$$

so

$$\int_{x_0}^{x_1}\mathrm{d}x'\ \alpha=\frac{\pi}{2}+n\pi=\left(n+\frac{1}{2}\right)\pi \tag{1.4.142}$$

which is the quantization condition we had before.

If we had enforced the coefficients to be the same, however, we'd find

$$\int_{x_0}^{x}\mathrm{d}x'\ \alpha-\frac{\pi}{4}=-\int_{x}^{x_1}\mathrm{d}x'\ \alpha+\frac{\pi}{4}+2n\pi \tag{1.4.143}$$

which would then enforce

$$\int_{x_0}^{x_1}\mathrm{d}x'\ \alpha=\left(2n+\frac{1}{2}\right)\pi \tag{1.4.144}$$

an uglier looking relation.

1.4.4 Other Ideas

Of course, asymptology as envisioned by Kruskal involves many other ideas, including perturbation series methods that we will consider in other sections. I could include everything under asymptology, but I think that they are important enough to merit their own sections. So JWKB in Section 1.12, eikonals in Section 1.12.1 , linearization in Section 1.6, Taylor series in Section 1.5, and so on are all elements of asymptology, but will be covered in their respective sections.

1.5 Taylor Series

Truth . . . is much too complicated to allow anything but approximations.

— John von Neumann

A physicist's best friend is a Taylor series. You may think I am joking, but the ubiquity of this approximation in all of physics cannot be overstated. A Taylor series is a way of approximating a function around a value. The advantage is that it is quite accurate near that value. The disadvantage is that the convergence rate of that function is determined by the function's properties in complex space, meaning that functions that look perfect on the real line may have surprising convergence properties.[68] So let's get to them. A Taylor approximation uses information at a single point about a function to approximate it elsewhere. The simple idea is that if we know the value at a point and then the value of the derivative at that point, we can simply use that to extrapolate. And, if we know the second derivative at that point, we can use that to extrapolate even better, etc. As one would expect, the accuracy decreases as one goes away from the chosen point, and so only certain functions will work for this. Luckily, they are the functions most used in physics and we often are interested at places where only a couple terms are sufficient for the accuracy we desire. For a point z_0 (complex or real) the Taylor approximation of a function $f(z)$ with (z complex or real) is given by (define $\Delta z \equiv z - z_0$)

$$f(z) \approx f(z_0)(z-z_0) + f'(z_0)(z-z_0)^2 + \frac{f''(z_0)}{2}(z-z_0)^3 + \cdots + \frac{f^{(n)}(z_0)(z-z_0)^n}{n!} + \cdots \tag{1.5.1}$$

$$f(z) \approx f(z_0)\Delta z + f'(z_0)(\Delta z)^2 + \frac{f''(z_0)}{2}(\Delta z)^3 + \cdots + \frac{f^{(n)}(z_0)(\Delta z)^n}{n!} + \cdots \tag{1.5.2}$$

with $f^n(z)$ being the nth derivative of $f(z)$ and $f^{(0)}(z) \equiv f(z)$. Or more compactly

$$f(z) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(\Delta z)^n}{n!} \tag{1.5.3}$$

Now if we are talking about z complex, then there is a generalization that is often useful to take which is to use the Laurent series, where

$$f(z) \approx \sum_{n=-\infty}^{\infty} a_n (\Delta z)^n \tag{1.5.4}$$

$$a_n = \frac{1}{2\pi i} \oint_\gamma \mathrm{d}z \, \frac{f(z)}{(z-z_0)^{n+1}} \tag{1.5.5}$$

where the integral is a contour integral and γ is a closed curve that goes counterclockwise round the point z_0. The function $f(z)$ must be holomorphic, which is just to say it is a special "nice" complex function, in order for the above to make sense.

Assuming that the functions are "nice", then the above expressions can even use an equals sign instead of an approximate sign, and we can treat the Taylor series as definitions of the functions.

[68] A surprising convergence property could, in principle, be good, but usually it means a pole in the complex plane makes the approximation worse than one would expect.

This is all basic stuff, but what about multiple variables? The answer is easier in index notation first. We then write $f(z_j)$ where j is some positive integer. Thus f is a function of J variables $z_1, \ldots, z_J$. Then let the approximation point be $z_0 = (z_{0,0}, z_{0,1}, \ldots, z_{0,j}, \ldots, z_{0,J})$,Then we have

$$f(z_j) \approx \sum_{n_1=0}^{\infty} \cdots \sum_{n_j=0}^{\infty} \cdots \sum_{n_J=0}^{\infty} \frac{(z_1 - z_{0,1})^{n_0} \cdots (z_j - z_{0,j})^{n_j} \cdots (z_J - z_{0,J})^{n_J}}{n_0! \cdots n_j! \cdots n_J!} \left. \frac{\partial^{n_1+\cdots+n_j+\cdots+n_J} f}{\partial z_1^{n_1} \cdots \partial z_j^{n_j} \cdots \partial z_J^{n_J}} \right|_{z=z_0} \tag{1.5.6}$$

For z complex, the above is only well defined if each complex variable satisfies the Cauchy-Riemann relations. This is most easily expressed using Wirtinger derivatives. First for a single variable $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ the Cauchy-Riemann relations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1.5.7}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.5.8}$$

The Wirtinger derivatives are defined via ($z = x + iy$)

$$\frac{\partial f(z)}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f(x+iy) \tag{1.5.9}$$

$$\frac{\partial f(z)}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(x+iy) \tag{1.5.10}$$

This is fairly reasonable for if we viewed $z = x + iy$ with $x = (z + z^*)/2$ and $y = (z - z^*)/(2i)$ with the complex conjugate of z given by $\bar{z} = z^*$ considered independent of z, then

$$\frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial x}\right)_y \frac{\partial x}{\partial z} + \left(\frac{\partial f}{\partial y}\right)_x \frac{\partial y}{\partial z} = \frac{\partial f}{\partial x}\frac{1}{2} + \frac{\partial f}{\partial y}\frac{1}{2i} \tag{1.5.11}$$

$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial f}{\partial x}\right)_y \frac{\partial x}{\partial \bar{z}} + \left(\frac{\partial f}{\partial y}\right)_x \frac{\partial y}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{1}{2} + \frac{\partial f}{\partial y}\frac{-1}{2i} \tag{1.5.12}$$

which gives the correct Wirtinger derivative results. If we consider a full derivative with respect to $z = x + iy$ and consider it as a function of two real variables x and y we will find (use $2\Delta x = \Delta z + \Delta\bar{z}$ and $2i\Delta y = \Delta z - \Delta\bar{z}$)

$$\left.\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right|_{z=z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \mathcal{O}([\Delta x, \Delta y]^2) - f(z_0)}{\Delta z} \tag{1.5.13}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{1}{2}\frac{\partial f}{\partial x}(\Delta z + \Delta\bar{z}) - \frac{i}{2}\frac{\partial f}{\partial y}(\Delta z - \Delta\bar{z})}{\Delta z} + \mathcal{O}([\Delta z]^2) \tag{1.5.14}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}}{2}\Delta z + \frac{\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}}{2}\Delta\bar{z}}{\Delta z} + \mathcal{O}([\Delta z]^2) \tag{1.5.15}$$

$$= \lim_{\Delta z \to 0} \left[\frac{\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}}{2} + \frac{\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}}{2}\frac{\Delta\bar{z}}{\Delta z} + \mathcal{O}([\Delta z]^2)\right] \tag{1.5.16}$$

$$= \left.\frac{\partial f}{\partial z}\right|_{z=z_0} + \left.\frac{\partial f}{\partial \bar{z}}\right|_{z=z_0} \left.\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}\right|_{z=z_0} \tag{1.5.17}$$

Because $\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}$ is not differentiable anywhere, we must require $\frac{\partial f}{\partial \bar{z}}|_{z=z_0} = 0$. Using the previous notation this means

$$\frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} + i^2\frac{\partial v}{\partial y}\right) = 0 \tag{1.5.18}$$

separating real and imaginary parts this then indeed says

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.5.19}$$

Thus with this generalization, for `z` an array of complex variables with components z_j we simply must have $\partial f/\partial \bar{z}_j = 0$ for all the z_j. Note that this is very restrictive for functions. There is no such restriction for z_j real, the more common case in physics. Note also, that the Wirtinger derivative makes our job much easier. We do not have to separate out z_j into two real variables. We simply check if the function has the complex conjugate in it. If it does not then we have an analytic function. If it does, we have to see if there is cancellation. Thus we can actually just interpret the Wirtinger derivative as a derivative with respect to the variable $\bar{z}_j$ holding z_j constant or vice versa. Let's now show that the Wirtinger derivatives satisfy properties similar to regular differentiation rules. First, let's show some amusing properties

$$\overline{\frac{\partial f}{\partial z_j}} = \frac{\partial \bar{f}}{\partial \bar{z}_j} \tag{1.5.20}$$

$$\overline{\frac{\partial f}{\partial \bar{z}_j}} = \frac{\partial \bar{f}}{\partial z_j} \tag{1.5.21}$$

This is shown by the definitions

$$\overline{\frac{\partial f}{\partial z_j}} = \frac{1}{2}\overline{\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}} = \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x_j} + i\frac{\partial \bar{f}}{\partial y_j}\right) = \frac{\partial \bar{f}}{\partial \bar{z}_j} \tag{1.5.22}$$

$$\overline{\frac{\partial f}{\partial \bar{z}_j}} = \frac{1}{2}\overline{\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}} = \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x_j} - i\frac{\partial \bar{f}}{\partial y_j}\right) = \frac{\partial \bar{f}}{\partial z_j} \tag{1.5.23}$$

Next, that a product of $f(z_j)$ and $g(z_j)$ with (complex) constants α and β yields the usual rule

$$\frac{\partial(\alpha f + \beta g)}{\partial z_j} = \alpha\frac{\partial f}{\partial z_j} + \beta\frac{\partial g}{\partial z_j} \tag{1.5.24}$$

$$\frac{\partial(\alpha f + \beta g)}{\partial \bar{z}_j} = \alpha\frac{\partial f}{\partial \bar{z}_j} + \beta\frac{\partial g}{\partial \bar{z}_j} \tag{1.5.25}$$

This follows from the definition once again.

$$\begin{aligned}\frac{\partial\,(\alpha f + \beta g)}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial(\alpha f + \beta g)}{\partial x_j} - i\frac{\partial(\alpha f + \beta g)}{\partial y_j}\right) = \frac{1}{2}\left(\alpha\frac{\partial f}{\partial x_j} + \beta\frac{\partial g}{\partial x_j} - i\alpha\frac{\partial f}{\partial y_j} - i\beta\frac{\partial g}{\partial y_j}\right) \\ &= \frac{\alpha}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right) + \frac{\beta}{2}\left(\frac{\partial g}{\partial x_j} - i\frac{\partial g}{\partial y_j}\right) \\ &= \alpha\frac{\partial f}{\partial z} + \beta\frac{\partial g}{\partial z}\end{aligned} \tag{1.5.26}$$

$$\begin{aligned}\frac{\partial\left(\alpha f+\beta g\right)}{\partial\bar{z}_j} &= \frac{1}{2}\left(\frac{\partial(\alpha f+\beta g)}{\partial x_j}+i\frac{\partial(\alpha f+\beta g)}{\partial y_j}\right)=\frac{1}{2}\left(\alpha\frac{\partial f}{\partial x_j}+\beta\frac{\partial g}{\partial x_j}+i\alpha\frac{\partial f}{\partial y_j}+i\beta\frac{\partial g}{\partial y_j}\right)\\ &= \frac{\alpha}{2}\left(\frac{\partial f}{\partial x_j}+i\frac{\partial f}{\partial y_j}\right)+\frac{\beta}{2}\left(\frac{\partial g}{\partial x_j}+i\frac{\partial g}{\partial y_j}\right)\\ &= \alpha\frac{\partial f}{\partial\bar{z}}+\beta\frac{\partial g}{\partial\bar{z}}\end{aligned} \tag{1.5.27}$$

The product rule is messier, but we have

$$\frac{\partial(fg)}{\partial z_j}=\frac{\partial f}{\partial z_j}g+f\frac{\partial g}{\partial z_j} \tag{1.5.28}$$

$$\frac{\partial(fg)}{\partial\bar{z}_j}=\frac{\partial f}{\partial\bar{z}_j}g+f\frac{\partial g}{\partial\bar{z}_j} \tag{1.5.29}$$

Once again, we can use the definition to prove this result.

$$\begin{aligned}\frac{\partial(fg)}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial(fg)}{\partial x_j}-i\frac{\partial(fg)}{\partial y_j}\right)=\frac{1}{2}\left(\frac{\partial f}{\partial x_j}g+f\frac{\partial g}{\partial x_j}-i\frac{\partial f}{\partial y_j}g-if\frac{\partial g}{\partial y_j}\right)\\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x_j}-i\frac{\partial f}{\partial y_j}\right)g+\frac{1}{2}f\left(\frac{\partial g}{\partial x_j}-i\frac{\partial g}{\partial y_j}\right)\\ &= \frac{\partial f}{\partial z_j}g+f\frac{\partial g}{\partial z_j}\end{aligned} \tag{1.5.30}$$

$$\begin{aligned}\frac{\partial(fg)}{\partial\bar{z}_j} &= \frac{1}{2}\left(\frac{\partial(fg)}{\partial x_j}+i\frac{\partial(fg)}{\partial y_j}\right)=\frac{1}{2}\left(\frac{\partial f}{\partial x_j}g+f\frac{\partial g}{\partial x_j}+i\frac{\partial f}{\partial y_j}g+if\frac{\partial g}{\partial y_j}\right)\\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x_j}+i\frac{\partial f}{\partial y_j}\right)g+\frac{1}{2}f\left(\frac{\partial g}{\partial x_j}+i\frac{\partial g}{\partial y_j}\right)\\ &= \frac{\partial f}{\partial\bar{z}_j}g+f\frac{\partial g}{\partial\bar{z}_j}\end{aligned} \tag{1.5.31}$$

Finally, we have the dreaded chain rule.

$$\frac{\partial f(\mathsf{g}(\mathsf{z}))}{\partial z_j}=\frac{\partial f}{\partial\mathsf{g}}\cdot\frac{\partial\mathsf{g}}{\partial z_j}+\frac{\partial f}{\partial\bar{\mathsf{g}}}\cdot\frac{\partial\bar{\mathsf{g}}}{\partial z_j}=\sum_k\frac{\partial f}{\partial z_k}\bigg|_{\mathsf{z}=\mathsf{g}(\mathsf{z})}\frac{\partial\mathsf{g}_k}{\partial z_j}+\sum_k\frac{\partial f}{\partial\bar{z}_k}\bigg|_{\mathsf{z}=\mathsf{g}(\mathsf{z})}\frac{\partial\bar{\mathsf{g}}_k}{\partial z_j} \tag{1.5.32}$$

$$\frac{\partial f(\mathsf{g}(\mathsf{z}))}{\partial\bar{z}_j}=\frac{\partial f}{\partial\mathsf{g}}\cdot\frac{\partial\mathsf{g}}{\partial\bar{z}_j}+\frac{\partial f}{\partial\bar{\mathsf{g}}}\cdot\frac{\partial\bar{\mathsf{g}}}{\partial\bar{z}_j}=\sum_k\frac{\partial f}{\partial z_k}\bigg|_{\mathsf{z}=\mathsf{g}(\mathsf{z})}\frac{\partial\mathsf{g}_k}{\partial\bar{z}_j}+\sum_k\frac{\partial f}{\partial\bar{z}_k}\bigg|_{\mathsf{z}=\mathsf{g}(\mathsf{z})}\frac{\partial\bar{\mathsf{g}}_k}{\partial\bar{z}_j} \tag{1.5.33}$$

The second equality is simply using what $\frac{\partial f}{\partial\mathsf{g}}$ actually means. For the chain rule to make sense, g outputs an array of complex variables for f. Then we can separate g into components g_k where g_k corresponds to the same functional input as z_k for the function f. It depends on multiple z_k, so then we can take the derivatives with respect to the z_k and put the value for $\mathsf{z}=\mathsf{g}(\mathsf{z})$ in instead of z. If you do not see why this is equivalent, then you should investigate it yourself. For us, it is simpler to just deal with g as a whole. To prove the chain rule we apply the definitions on the right hand side (let the function g be broken into two real functions $\mathsf{g}=\mathsf{g}_x+i\mathsf{g}_y$, note that the

dot product is necessary to give the right answer)

$$\frac{\partial f}{\partial \mathsf{g}} \cdot \frac{\partial \mathsf{g}}{\partial z_j} = \frac{1}{4}\left(\frac{\partial f}{\partial \mathsf{g}_x} - i\frac{\partial f}{\partial \mathsf{g}_y}\right) \cdot \left(\frac{\partial(\mathsf{g}_x + i\mathsf{g}_y)}{\partial x_j} - i\frac{\partial(\mathsf{g}_x + i\mathsf{g}_y)}{\partial y_j}\right)$$
$$= \frac{1}{4}\left(\frac{\partial f}{\partial \mathsf{g}_x} - i\frac{\partial f}{\partial \mathsf{g}_y}\right) \cdot \overbrace{\left(\frac{\partial \mathsf{g}_x}{\partial x_j} + \frac{\partial \mathsf{g}_y}{\partial y_j} + i\left[\frac{\partial \mathsf{g}_y}{\partial x_j} - \frac{\partial \mathsf{g}_x}{\partial y_j}\right]\right)}^{\gamma_{\mathsf{g}}} \tag{1.5.34}$$

$$\frac{\partial f}{\partial \bar{\mathsf{g}}} \cdot \frac{\partial \bar{\mathsf{g}}}{\partial z_j} = \frac{1}{4}\left(\frac{\partial f}{\partial \mathsf{g}_x} + i\frac{\partial f}{\partial \mathsf{g}_y}\right) \cdot \left(\frac{\partial(\mathsf{g}_x - i\mathsf{g}_y)}{\partial x_j} - i\frac{\partial(\mathsf{g}_x - i\mathsf{g}_y)}{\partial y_j}\right)$$
$$= \frac{1}{4}\left(\frac{\partial f}{\partial \mathsf{g}_x} + i\frac{\partial f}{\partial \mathsf{g}_y}\right) \cdot \overbrace{\left(\frac{\partial \mathsf{g}_x}{\partial x_j} - \frac{\partial \mathsf{g}_y}{\partial y_j} + i\left[-\frac{\partial \mathsf{g}_y}{\partial x_j} - \frac{\partial \mathsf{g}_x}{\partial y_j}\right]\right)}^{\Gamma_{\bar{\mathsf{g}}}} \tag{1.5.35}$$

(1.5.36)

and so if we add them we find

$$\frac{\partial f}{\partial \mathsf{g}} \cdot \frac{\partial \mathsf{g}}{\partial z_j} + \frac{\partial f}{\partial \bar{\mathsf{g}}} \cdot \frac{\partial \bar{\mathsf{g}}}{\partial z_j} = \frac{1}{4}\left[\frac{\partial f}{\partial \mathsf{g}_x} \cdot (\gamma_{\mathsf{g}} + \Gamma_{\bar{\mathsf{g}}}) + i\frac{\partial f}{\partial \mathsf{g}_y} \cdot (\Gamma_{\bar{\mathsf{g}}} - \gamma_{\mathsf{g}})\right] \tag{1.5.37}$$

$$\gamma_{\mathsf{g}} + \Gamma_{\bar{\mathsf{g}}} = 2\frac{\partial \mathsf{g}_x}{\partial x_j} - 2i\frac{\partial \mathsf{g}_x}{\partial y_j} \tag{1.5.38}$$

$$\Gamma_{\bar{\mathsf{g}}} - \gamma_{\mathsf{g}} = -2\frac{\partial \mathsf{g}_y}{\partial y_j} - 2i\frac{\partial \mathsf{g}_y}{\partial x_j} \tag{1.5.39}$$

which means

$$\frac{\partial f}{\partial \mathsf{g}} \cdot \frac{\partial \mathsf{g}}{\partial z_j} + \frac{\partial f}{\partial \bar{\mathsf{g}}} \cdot \frac{\partial \bar{\mathsf{g}}}{\partial z_j} = \frac{1}{2}\left[\frac{\partial f}{\partial \mathsf{g}_x} \cdot \left(\frac{\partial \mathsf{g}_x}{\partial x_j} - i\frac{\partial \mathsf{g}_x}{\partial y_j}\right) - i\frac{\partial f}{\partial \mathsf{g}_y} \cdot \left(\frac{\partial \mathsf{g}_y}{\partial y_j} + i\frac{\partial \mathsf{g}_y}{\partial x_j}\right)\right] \tag{1.5.40}$$

$$= \frac{\partial f}{\partial \mathsf{g}_x} \cdot \frac{\partial \mathsf{g}_x}{\partial z_j} + \frac{1}{2}\frac{\partial f}{\partial \mathsf{g}_y} \cdot \left(\frac{\partial \mathsf{g}_y}{\partial x_j} - i\frac{\partial \mathsf{g}_y}{\partial y_j}\right) \tag{1.5.41}$$

$$= \frac{\partial f}{\partial \mathsf{g}_x} \cdot \frac{\partial g_x}{\partial z_j} + \frac{\partial f}{\partial \mathsf{g}_y} \cdot \frac{\partial \mathsf{g}_y}{\partial z_j} = \frac{\partial f}{\partial z_j} \tag{1.5.42}$$

In the same way, when we take $\frac{\partial f(\mathsf{g}(\mathsf{z}))}{\partial \bar{z}_j}$ we find

$$\frac{\partial f}{\partial \mathsf{g}} \cdot \frac{\partial \mathsf{g}}{\partial \bar{z}_j} = \frac{1}{4}\left(\frac{\partial f}{\partial \mathsf{g}_x} - i\frac{\partial f}{\partial \mathsf{g}_y}\right) \cdot \left(\frac{\partial(\mathsf{g}_x + i\mathsf{g}_y)}{\partial x_j} + i\frac{\partial(\mathsf{g}_x + i\mathsf{g}_y)}{\partial y_j}\right)$$
$$= \frac{1}{4}\left(\frac{\partial f}{\partial \mathsf{g}_x} - i\frac{\partial f}{\partial \mathsf{g}_y}\right) \cdot \overbrace{\left(\frac{\partial \mathsf{g}_x}{\partial x_j} - \frac{\partial \mathsf{g}_y}{\partial y_j} + i\left[\frac{\partial \mathsf{g}_x}{\partial y_j} + \frac{\partial \mathsf{g}_y}{\partial x_j}\right]\right)}^{\bar{\gamma}_{\mathsf{g}}} \tag{1.5.43}$$

$$\frac{\partial f}{\partial \bar{\mathsf{g}}} \cdot \frac{\partial \bar{\mathsf{g}}}{\partial \bar{z}_j} = \frac{1}{4}\left(\frac{\partial f}{\partial \mathsf{g}_x} + i\frac{\partial f}{\partial \mathsf{g}_y}\right) \cdot \left(\frac{\partial(\mathsf{g}_x - i\mathsf{g}_y)}{\partial x_j} + i\frac{\partial(\mathsf{g}_x - i\mathsf{g}_y)}{\partial y_j}\right)$$
$$= \frac{1}{4}\left(\frac{\partial f}{\partial \mathsf{g}_x} + i\frac{\partial f}{\partial \mathsf{g}_y}\right) \cdot \overbrace{\left(\frac{\partial \mathsf{g}_x}{\partial x_j} + \frac{\partial \mathsf{g}_y}{\partial y_j} + i\left[\frac{\partial \mathsf{g}_x}{\partial y_j} - \frac{\partial \mathsf{g}_y}{\partial x_j}\right]\right)}^{\bar{\Gamma}_{\bar{\mathsf{g}}}} \tag{1.5.44}$$

(1.5.45)

and so if we add them we find

$$\frac{\partial f}{\partial \mathtt{g}}\cdot\frac{\partial \mathtt{g}}{\partial \bar{z}_j}+\frac{\partial f}{\partial \bar{\mathtt{g}}}\cdot\frac{\partial \bar{\mathtt{g}}}{\partial \bar{z}_j}=\frac{1}{4}\left[\frac{\partial f}{\partial \mathtt{g}_x}\cdot\left(\bar{\gamma}_{\mathtt{g}}+\bar{\Gamma}_{\bar{\mathtt{g}}}\right)+i\frac{\partial f}{\partial \mathtt{g}_y}\cdot\left(\bar{\Gamma}_{\bar{\mathtt{g}}}-\bar{\gamma}_{\mathtt{g}}\right)\right] \tag{1.5.46}$$

$$\bar{\gamma}_{\mathtt{g}}+\bar{\Gamma}_{\bar{\mathtt{g}}}=2\frac{\partial \mathtt{g}_x}{\partial x_j}+2i\frac{\partial \mathtt{g}_x}{\partial y_j} \tag{1.5.47}$$

$$\bar{\gamma}_{\bar{\mathtt{g}}}-\bar{\gamma}_{\mathtt{g}}=2\frac{\partial \mathtt{g}_y}{\partial y_j}-2i\frac{\partial \mathtt{g}_y}{\partial x_j} \tag{1.5.48}$$

which means

$$\frac{\partial f}{\partial \mathtt{g}}\cdot\frac{\partial \mathtt{g}}{\partial \bar{z}_j}+\frac{\partial f}{\partial \bar{\mathtt{g}}}\cdot\frac{\partial \bar{\mathtt{g}}}{\partial \bar{z}_j}=\frac{1}{2}\left[\frac{\partial f}{\partial \mathtt{g}_x}\cdot\left(\frac{\partial \mathtt{g}_x}{\partial x_j}+i\frac{\partial \mathtt{g}_x}{\partial y_j}\right)+i\frac{\partial f}{\partial \mathtt{g}_y}\cdot\left(\frac{\partial \mathtt{g}_y}{\partial y_j}-i\frac{\partial \mathtt{g}_y}{\partial x_j}\right)\right] \tag{1.5.49}$$

$$=\frac{\partial f}{\partial \mathtt{g}_x}\cdot\frac{\partial \mathtt{g}_x}{\partial \bar{z}_j}+\frac{1}{2}\frac{\partial f}{\partial \mathtt{g}_y}\cdot\left(\frac{\partial \mathtt{g}_y}{\partial x_j}+i\frac{\partial \mathtt{g}_y}{\partial y_j}\right) \tag{1.5.50}$$

$$=\frac{\partial f}{\partial \mathtt{g}_x}\cdot\frac{\partial \mathtt{g}_x}{\partial \bar{z}_j}+\frac{\partial f}{\partial \mathtt{g}_y}\cdot\frac{\partial \mathtt{g}_y}{\partial \bar{z}_j}=\frac{\partial f}{\partial \bar{z}_j} \tag{1.5.51}$$

Note that we could generalize to complex vectors and tensors all of the previous results so long as we are careful about order. For complex vector arrays, for example, we'd have

$$\frac{\partial \mathtt{f}(\mathtt{g}(\mathtt{z}))}{\partial \mathtt{z}}=\frac{\partial \mathtt{f}}{\partial \mathtt{g}}\cdot\frac{\partial \mathtt{g}}{\partial \mathtt{z}}+\frac{\partial \mathtt{f}}{\partial \bar{\mathtt{g}}}\cdot\frac{\partial \bar{\mathtt{g}}}{\partial \mathtt{z}} \tag{1.5.52}$$

$$\frac{\partial \mathtt{f}(\mathtt{g}(\mathtt{z}))}{\partial \bar{\mathtt{z}}}=\frac{\partial \mathtt{f}}{\partial \mathtt{g}}\cdot\frac{\partial \mathtt{g}}{\partial \bar{\mathtt{z}}}+\frac{\partial \mathtt{f}}{\partial \bar{\mathtt{g}}}\cdot\frac{\partial \bar{\mathtt{g}}}{\partial \bar{\mathtt{z}}} \tag{1.5.53}$$

To summaraize, we have the Wirtinger derivatives satisfying for functions f and g with complex constants α and β and $\mathtt{g}$ a complex vector array such that

$$\frac{\partial f}{\partial z_j}=\frac{1}{2}\left(\frac{\partial f}{\partial x_j}-i\frac{\partial f}{\partial y_j}\right) \tag{1.5.54}$$

$$\frac{\partial f}{\partial \bar{z}_j}=\frac{1}{2}\left(\frac{\partial f}{\partial x_j}+i\frac{\partial f}{\partial y_j}\right) \tag{1.5.55}$$

$$\overline{\frac{\partial f}{\partial z}}=\frac{1}{2}\overline{\frac{\partial f}{\partial x_j}-i\frac{\partial f}{\partial y_j}}=\frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x_j}+i\frac{\partial \bar{f}}{\partial y_j}\right)=\frac{\partial \bar{f}}{\partial \bar{z}_j} \tag{1.5.56}$$

$$\overline{\frac{\partial f}{\partial \bar{z}}}=\frac{1}{2}\overline{\frac{\partial f}{\partial x_j}+i\frac{\partial f}{\partial y_j}}=\frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x_j}-i\frac{\partial \bar{f}}{\partial y_j}\right)=\frac{\partial \bar{f}}{\partial z_j} \tag{1.5.57}$$

$$\frac{\partial(\alpha f+\beta g)}{\partial z_j}=\alpha\frac{\partial f}{\partial z_j}+\beta\frac{\partial f}{\partial z_j} \tag{1.5.58}$$

$$\frac{\partial(\alpha f+\beta g)}{\partial \bar{z}_j}=\alpha\frac{\partial f}{\partial \bar{z}_j}+\beta\frac{\partial f}{\partial \bar{z}_j} \tag{1.5.59}$$

$$\frac{\partial(fg)}{\partial \bar{z}_j}=f\frac{\partial g}{\partial \bar{z}_j}+\frac{\partial f}{\partial \bar{z}_j}g \tag{1.5.60}$$

$$\frac{\partial f(\mathtt{g}(\mathtt{z}))}{\partial z_j}=\frac{\partial f}{\partial \mathtt{g}}\cdot\frac{\partial \mathtt{g}}{\partial z_j}+\frac{\partial f}{\partial \bar{\mathtt{g}}}\cdot\frac{\partial \bar{\mathtt{g}}}{\partial z_j}=\sum_k\left.\frac{\partial f}{\partial z_k}\right|_{\mathtt{z}=\mathtt{g}(\mathtt{z})}\frac{\partial \mathtt{g}_k}{\partial z_j}+\sum_k\left.\frac{\partial f}{\partial \bar{z}_k}\right|_{\mathtt{z}=\mathtt{g}(\mathtt{z})}\frac{\partial \bar{\mathtt{g}}_k}{\partial z_j} \tag{1.5.61}$$

$$\frac{\partial f(\mathtt{g}(\mathtt{z}))}{\partial \bar{z}_j}=\frac{\partial f}{\partial \mathtt{g}}\cdot\frac{\partial \mathtt{g}}{\partial \bar{z}_j}+\frac{\partial f}{\partial \bar{\mathtt{g}}}\cdot\frac{\partial \bar{\mathtt{g}}}{\partial \bar{z}_j}=\sum_k\left.\frac{\partial f}{\partial z_k}\right|_{\mathtt{z}=\mathtt{g}(\mathtt{z})}\frac{\partial \mathtt{g}_k}{\partial \bar{z}_j}+\sum_k\left.\frac{\partial f}{\partial \bar{z}_k}\right|_{\mathtt{z}=\mathtt{g}(\mathtt{z})}\frac{\partial \bar{\mathtt{g}}_k}{\partial \bar{z}_j} \tag{1.5.62}$$

Now if we have all real variables x_j we can write this in vector notation[69] by writing $f(\mathbf{x})$ and thus at point $\mathbf{x}_0$ with $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$ we have

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \Delta\mathbf{x} \cdot \frac{\partial f}{\partial \mathbf{x}}\Big|_{\mathbf{x}_0} + \Delta\mathbf{x}\,\Delta\mathbf{x} : \frac{1}{2}\frac{\partial^2 f}{\partial \mathbf{x}^2}\Big|_{\mathbf{x}_0} + \cdots [\Delta\mathbf{x}]^n \overset{\boldsymbol{n}}{\cdot} \frac{1}{n!}\frac{\partial f^n}{\partial \mathbf{x}^n}\Big|_{\mathbf{x}_0} + \cdots \tag{1.5.63}$$

where the $|_{\mathbf{x}_0}$ indicate evaluate at $\mathbf{x} = \mathbf{x}_0$ after taking the derivative and

$$\mathbf{x}^n = \overbrace{\mathbf{x}\mathbf{x}\cdots\mathbf{x}}^{n \text{ instances}}$$

is interpreted as a nth order polyad or polyadic (or tensor of order n) and not as a dot product and if $\Delta\mathbf{x}$ has J components we have

$$[\Delta\mathbf{x}]^n \overset{\boldsymbol{n}}{\cdot} \frac{\partial^n f}{\partial \mathbf{x}^n}\Big|_{\mathbf{x}_0} = \sum_{j_1=1}^{J} \cdots \sum_{j_n=1}^{J} \frac{\partial^n f}{\partial x_{j_1} \cdots \partial x_{j_n}}\bigg|_{\mathbf{x}_0} (\Delta x_{j_1}) \cdots (\Delta x_{j_n}) \tag{1.5.64}$$

Here $\overset{\boldsymbol{n}}{\cdot}$ can be thought of as a generalization of the dot and double dot products. A more "conventional" representation using the nabla operator with

$$\nabla^{(n)} \equiv \overbrace{\nabla\nabla\cdots\nabla}^{n \text{ instances}}$$

would be

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \Delta\mathbf{x} \cdot \nabla f|_{\mathbf{x}_0} + \Delta\mathbf{x}\,\Delta\mathbf{x} : \frac{1}{2}\nabla\nabla f|_{\mathbf{x}_0} + \cdots [\Delta\mathbf{x}]^n \overset{\boldsymbol{n}}{\cdot} \frac{1}{n!}\nabla^{(n)} f|_{\mathbf{x}_0} + \cdots \tag{1.5.65}$$

Finally, it is often useful to approximate vectors and tensors. We can use vector $\mathbf{f}(\mathbf{x})$ and find

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_0) + \Delta\mathbf{x} \cdot \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\bigg|_{\mathbf{x}_0} + \cdots + [\Delta\mathbf{x}]^n \overset{\boldsymbol{n}}{\cdot} \frac{1}{n!}\frac{\partial^n \mathbf{f}}{\partial \mathbf{x}^n}\bigg|_{\mathbf{x}_0} + \cdots \tag{1.5.66}$$

where the notation does all of the work for us. Here the definition is given by (assuming three dimensions, $J = 3$)

$$\nabla\mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \sum_{j_1=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial f^j}{\partial x^{j_1}} + f^i \Gamma^j_{ij_1}\right)\mathbf{e}^{j_1}\mathbf{e}_j \tag{1.5.67}$$

$$\nabla\mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \sum_{j_1=1}^{3}\left(\frac{\partial f_j}{\partial x^{j_1}} - f_i \Gamma^i_{jj_1}\right)\mathbf{e}^{j_1}\mathbf{e}^j \tag{1.5.68}$$

Further derivatives become even more of a headache to calculate. Using Cartesian coordinates makes the calculations much easier, as we can ignore all the metric component parts. In those cases, we can write

$$\nabla^{(n)}\mathbf{f} = \frac{\partial^n \mathbf{f}}{\partial \mathbf{x}^n} = \sum_{j_1=0}^{3} \cdots \sum_{j_n=0}^{3}\left(\frac{\partial^n f^j}{\partial x^{j_1} \cdots \partial x^{j_n}}\right)\hat{\mathbf{x}}_{j_1} \cdots \hat{\mathbf{x}}_{j_n}\hat{\mathbf{x}}_j \tag{1.5.69}$$

with $\hat{\mathbf{x}}_j$ the usual Cartesian unit vectors $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}$, $\hat{\mathbf{x}}_2 = \hat{\mathbf{y}}$, $\hat{\mathbf{x}}_3 = \hat{\mathbf{z}}$.

[69] This can even be generalized to vector arrays $\mathbf{x}$ if we do not enforce the connection coefficients below.

1.6 Why Linearization Works

All linear problems are alike, but each nonlinear problem is nonlinear in its own way.

— Paraphrase of *Anna Karenina*

We can also comment on the concept of linearization and its generalizations (though the generalizations are not used nearly as often in plasma physics). First, let us explain what nonlinear equations and linear equations are. A linear equation has the unknown appear in each term one or zero times. A nonlinear equation is any equation that is not linear.[70] Suppose we have some nonlinear differential equations

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{V}) = 0 \tag{1.6.1}$$

$$\frac{1}{\gamma - 1}\left[\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T\right] = -nT\boldsymbol{\nabla} \cdot \mathbf{V} \tag{1.6.2}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} \tag{1.6.3}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.6.4}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{V} \times \mathbf{B}) \tag{1.6.5}$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla\mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla p \tag{1.6.6}$$

with $p = nT$. Note that these are very similar to the Ideal MHD equations.

So ∇p or $\frac{\partial \mathbf{V}}{\partial t}$ are linear while terms like $\mathbf{V} \cdot \nabla\mathbf{V}$ or $\mathbf{J} \times \mathbf{B}$ are nonlinear.

Now, we assume that there is a solution to these equations and consider a perturbation away from this steady state solution, sometimes called the equilibrium solution.[71] A major assumption is in deciding the ordering of terms. But for each q we will assume that it can be ordered such that

$$q = q_0 + \delta q_1 + \delta^2 q_2 + \cdots = \sum_{i=0}^{\infty} \delta^i q_i \tag{1.6.7}$$

with δ a parameter of some sort such that $\delta \ll 1$ and the $q_i = \mathcal{O}(1)$, that is, are all of a similar order so that δ actually does useful work. This assumption that all different unknowns have this sort of dependence is not something actually required of any solution. For linearization (only using $i = 0, 1$), this is more justified in that we will only keep linear terms and so small changes in one term must distribute to small changes in other quantities that are coupled. But it is important to note that each term could have a different δ (perhaps call it δ_{q_i} for each quantity q_i) and then different orders would need to be balanced based on the relative ordering of the δ_{q_i}'s with each other (that is we need relations $\delta_{q_i} = \mathcal{O}(\delta_{q_0}^{n_{q_i}})$ for some n_{q_i} so that we can relate each δ_{q_i} to all the other δ_{q_j}). In fact, different types of MHD equations are derived by having different relative values for the various ordering parameters. All that really matters is that one is consistent. Remember

[70] I have heard that this is like dividing the world into bananas and non-bananas. This is a bit unfair. If your world contained a variety of bananas, all edible, and then isolated cases of objects that were clearly not bananas but shared little else in common, it would make sense to have banana and non-banana categories. For us, linear equations are solvable while nonlinear equations generally are not analytically solvable.

[71] It is not usually a thermodynamic equilibrium, and so the use of equilibrium should instead be interpreted more like a mechanical equilibrium where things are unchanging in time.

that by making such assumptions, it is always possible to miss important physical solutions that do not match the ordering you are imposing.

I feel it needs to be belabored that we have assumed a form of solution given by (1.6.7) and then investigate what that implies for q_0, q_1, q_2, ..., etc. If we find that it implies something nonsensical or that the terms are all zero, then our approximation method has failed to give us any insight. There is nothing in a general solution of partial differential equations to guarantee us that linearization will work. In practice, we know that physical situations will provide us with examples where linearization is useful. The further we'd like to extend solutions around these situations, the more difficulty and more calculation we will have to endure.

With this in mind, the reasoning behind linearization is a type of balance between terms. What we will use is a principle called dominant balance in which we say only terms of the same order need to be considered at the same time because the larger terms are "balancing" each other (with the smaller terms small enough they couldn't possibly affect the larger terms to the accuracy we care about). Here is one (admittedly, strange) way to think about it. Suppose there were two owners each with one employee. Suppose they were splitting the money from giant crates containing giant gold blocks of 10 000 kg. There can be really large number of these 10 000 kg gold blocks in a crate. But inside each crate there is one treasure chest containing at most 100 kg of gold. The bosses don't care about any mass of gold less than 100 kg and so as long as no treasure chest is more than that, they leave it as payment for their employees.

There is an even number of gold blocks contained in all of the crates, but they are not split evenly between them. The bosses only care about the gold blocks and the employees are paid with whatever treasure chest they get. First the bosses get their say. They only care about the big gold blocks being equal, and so measure them out equally. They don't care about the treasure chests. This is like a zeroth approximation. The employees, however, may or may not be happy with this, but their voice doesn't count as much.

Suppose the bosses do care about keeping their employees happy, though. They don't want to deal with it themselves, though, and so tell the employees, they can split the treasure chests however they want. Then the employees will balance the treasure chest gold, and we have a first order approximation.

If both the bosses and employees split equally, then we have enforced a balance among the dominant terms, and then the next dominant terms, and so we have equality overall. If the employees had subemployees, and there were some gold rings in each treasure chest (say a couple of grams of gold, and the employees only care about the treasure chest being equal up to a few grams), the employees won't care about splitting the gold rings evenly, but the subemployees would, and so on. Each order only cares about it's share, and if we want overall balance, we must first balance the dominant terms.

The important thing is that at the outset we chose to split the gold rations into sizes that are widely separated in mass so that we can deal with each mass size one at a time (as the various levels of bosses and employees deal only with their gold mass size) so that we can stop the process without worrying about the leftovers being large enough to upset the balance (this would be like the treasure chest containing more gold than one of the gold blocks; the boss would then care about it). Our mathematical method of dividing contributions with an ordering parameter works in the same way.

We consider only the largest contributions first, then we refine to more accurate answers. What is crucial is that we split the quantities on each side of the equality into pieces that are essentially independent of each other because of the vast difference in scale at each step.[72] The ordering parameter tells us how accurately we want our two sides of the equation to approximate each other and ensures that we can split the equation up as we would like. Note the method can easily fail to be accurate if quantities can affect each other because equality of the sides is not actually guaranteed. This would be like in our example allowing a box in the crate to contain 1000 treasure chests. Then the treasure chests may not be a negligible contribution and so if we didn't fairly distribute the treasure chests it would not be an equal split (and the bosses would be unhappy).

Going back to our example of ideal MHD-like equations, we can use the assumption that each parameter can be separated into different contributions with an ordering parameter. The next step is to expand our equations and collect terms of the same order in δ. For linearization, we only care about terms up to order δ^1. For this example, let's take up to δ^2 so we can see how linearization can easily be generalized later. We have for (1.6.1)

$$\frac{\partial}{\partial t}(n_0+\delta n_1+\delta^2 n_2)+\boldsymbol{\nabla}\cdot((n_0+\delta n_1+\delta^2 n_2)\left(\mathbf{V}_0+\delta\mathbf{V}_1+\delta^2\mathbf{V}_2\right))=\mathcal{O}(\delta^3) \tag{1.6.8}$$

$$\begin{aligned}&\left[\frac{\partial n_0}{\partial t}+\boldsymbol{\nabla}\cdot(n_0\mathbf{V}_0)\right]+\delta\left(\frac{\partial n_1}{\partial t}+\boldsymbol{\nabla}\cdot(n_0\mathbf{V}_1+n_1\mathbf{V}_0)\right)\\&\qquad+\delta^2\left[\frac{\partial n_2}{\partial t}+\boldsymbol{\nabla}\cdot(n_1\mathbf{V}_2+n_2\mathbf{V}_1)\right]=\mathcal{O}(\delta^3)\end{aligned} \tag{1.6.9}$$

and then for (1.6.2)

$$\begin{aligned}&\frac{1}{\gamma-1}\left[\frac{\partial}{\partial t}+(\mathbf{V}_0+\delta\mathbf{V}_1+\delta^2\mathbf{V}_2)\cdot\nabla\right]\left[T_0+\delta T_1+\delta^2 T_2\right]\\&=-(n_0+\delta n_1+\delta n_2)(T_0+\delta T_1+\delta^2 T_2)\boldsymbol{\nabla}\cdot\left(\mathbf{V}_0+\delta\mathbf{V}_1+\delta^2\mathbf{V}_2\right)\end{aligned} \tag{1.6.10}$$

leading to

$$\begin{aligned}&\left[\frac{1}{\gamma-1}\left\{\frac{\partial T_0}{\partial t}+\mathbf{V}_0\cdot\nabla T_0\right\}+n_0T_0\boldsymbol{\nabla}\cdot\mathbf{V}_0\right]\\&+\delta\left[\frac{1}{\gamma-1}\left\{\frac{\partial T_1}{\partial t}+\mathbf{V}_0\cdot\nabla T_1+\mathbf{V}_1\cdot\nabla T_0\right\}+(n_1T_0+n_0T_1)\boldsymbol{\nabla}\cdot\mathbf{V}_0+n_0T_0\boldsymbol{\nabla}\cdot\mathbf{V}_1\right]\\&+\delta^2\left[\frac{1}{\gamma-1}\left\{\frac{\partial T_2}{\partial t}+\mathbf{V}_0\cdot\nabla T_2+\mathbf{V}_1\cdot\nabla T_1+\mathbf{V}_2\cdot\nabla T_0\right\}+n_0T_0\boldsymbol{\nabla}\cdot\mathbf{V}_2\right.\\&\qquad\left.+(n_0T_1+n_1T_0)\boldsymbol{\nabla}\cdot\mathbf{V}_1+n_1T_1\boldsymbol{\nabla}\cdot\mathbf{V}_0\right]=\mathcal{O}(\delta^3)\end{aligned} \tag{1.6.11}$$

For the next equation (1.6.3)

$$\frac{\partial}{\partial t}\left[\mathbf{B}_0+\delta\mathbf{B}_1+\delta^2\mathbf{B}_2\right]=\boldsymbol{\nabla}\times(\left[\mathbf{V}_0+\delta\mathbf{V}_1+\delta^2\mathbf{V}_2\right]\times\left[\mathbf{B}_0+\delta\mathbf{B}_1+\delta^2\mathbf{B}_2\right]) \tag{1.6.12}$$

[72] If the treasure chests contained 1000 kg of gold each, you could follow the procedure, but if you stopped at just equalizing the giant gold blocks, the bosses would get upset if they learned they didn't get the share they were promised.

$$\begin{aligned}
&\left[\frac{\partial \mathbf{B}_0}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{V}_0 \times \mathbf{B}_0)\right] \\
&+ \delta \left[\frac{\partial \mathbf{B}_1}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{V}_0 \times \mathbf{B}_1 + \mathbf{V}_1 \times \mathbf{B}_0)\right] \\
&+ \delta^2 \left[\frac{\partial \mathbf{B}_2}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{V}_0 \times \mathbf{B}_2 + \mathbf{V}_1 \times \mathbf{B}_1 + \mathbf{V}_2 \times \mathbf{B}_0)\right] = \mathcal{O}(\delta^3)
\end{aligned} \tag{1.6.13}$$

and then for (1.6.5)

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \left[\mathbf{V}_0 + \delta\mathbf{V}_1 + \delta^2\mathbf{V}_2\right] \cdot \nabla\right\} \left[\mathbf{V}_0 + \delta\mathbf{V}_1 + \delta^2\mathbf{V}_2\right] \\
&= (\boldsymbol{\nabla} \times \left[\mathbf{B}_0 + \delta\mathbf{B}_1 + \delta^2\mathbf{B}_2\right]) \times \left[\mathbf{B}_0 + \delta\mathbf{B}_1 + \delta^2\mathbf{B}_2\right] \\
&- \nabla\left([n_0 + \delta n_1 + \delta^2 n_2][T_0 + \delta T_1 + \delta^2 T_2]\right)
\end{aligned} \tag{1.6.14}$$

$$\begin{aligned}
&\left[\frac{\partial \mathbf{V}_0}{\partial t} + \mathbf{V}_0 \cdot \nabla\mathbf{V}_0 - (\boldsymbol{\nabla} \times \mathbf{B}_0) \times \mathbf{B}_0 + \nabla(n_0 T_0)\right] \\
&+ \delta\left[\frac{\partial \mathbf{V}_1}{\partial t} + \mathbf{V}_1 \cdot \nabla\mathbf{V}_0 + \mathbf{V}_0 \cdot \nabla\mathbf{V}_1\right. \\
&\qquad \left. - (\boldsymbol{\nabla} \times \mathbf{B}_1) \times \mathbf{B}_0 + (\boldsymbol{\nabla} \times \mathbf{B}_0) \times \mathbf{B}_1 + \nabla(n_0 T_1 + n_1 T_0)\right] \\
&+ \delta^2\left[\frac{\partial \mathbf{V}_2}{\partial t} + \mathbf{V}_0 \cdot \nabla\mathbf{V}_2 + \mathbf{V}_1 \cdot \nabla\mathbf{V}_1 + \mathbf{V}_2 \cdot \nabla\mathbf{V}_0\right. \\
&\qquad - \boldsymbol{\nabla} \times (\mathbf{B}_2) \times \mathbf{B}_0 - (\boldsymbol{\nabla} \times \mathbf{B}_1) \times \mathbf{B}_1 - (\boldsymbol{\nabla} \times \mathbf{B}_0) \times \mathbf{B}_2 \\
&\qquad \left. + \nabla(n_0 T_2 + n_1 T_1 + n_2 T_0)\right] = \mathcal{O}(\delta^3)
\end{aligned} \tag{1.6.15}$$

You might worry that I skipped (1.6.4), but it simply states that

$$\boldsymbol{\nabla} \cdot \mathbf{B}_0 + \delta\boldsymbol{\nabla} \cdot \mathbf{B}_1 + \delta^2\boldsymbol{\nabla} \cdot \mathbf{B}_2 = \mathcal{O}(\delta^3) \tag{1.6.16}$$

Because δ changes the order it is then clear for any $\mathbf{B}_j$ that we have $\boldsymbol{\nabla} \cdot \mathbf{B}_j = 0$.

Now we assumed we had a steady state so $\frac{\partial q_0}{\partial t} = 0$ for all quantities stating

$$\boldsymbol{\nabla} \cdot (n_0 \mathbf{V}_0) = 0 \tag{1.6.17}$$

$$\frac{1}{\gamma - 1}\mathbf{V}_0 \cdot \nabla T_0 = -n_0 T_0 \boldsymbol{\nabla} \cdot \mathbf{V}_0 \tag{1.6.18}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{V}_0 \times \mathbf{B}_0) = \mathbf{0} \tag{1.6.19}$$

$$\cancelto{0}{\frac{\partial \mathbf{V}_0}{\partial t}} + \mathbf{V}_0 \cdot \nabla\mathbf{V}_0 = (\boldsymbol{\nabla} \times \mathbf{B}_0) \times \mathbf{B}_0 - \nabla(n_0 T_0) \tag{1.6.20}$$

We have eight unknowns and eight equations.

The next order $\mathcal{O}(\delta)$ equations are

$$\frac{\partial n_1}{\partial t} = -\boldsymbol{\nabla}\cdot(n_0\mathbf{V}_1 + n_1\mathbf{V}_0) \tag{1.6.21}$$

$$\frac{1}{\gamma-1}\left\{\frac{\partial T_1}{\partial t} + \mathbf{V}_0\cdot\nabla T_1 + \mathbf{V}_1\cdot\nabla T_0\right\} = -(n_1T_0 + n_0T_1)\boldsymbol{\nabla}\cdot\mathbf{V}_0 + n_0T_0\boldsymbol{\nabla}\cdot\mathbf{V}_1 \tag{1.6.22}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \boldsymbol{\nabla}\cdot(\mathbf{V}_0\times\mathbf{B}_1 + \mathbf{V}_1\times\mathbf{B}_0) \tag{1.6.23}$$

$$\frac{\partial \mathbf{V}_1}{\partial t} + \mathbf{V}_1\cdot\nabla\mathbf{V}_0 + \mathbf{V}_0\cdot\nabla\mathbf{V}_1 = (\boldsymbol{\nabla}\times\mathbf{B}_1)\times\mathbf{B}_0 + (\boldsymbol{\nabla}\times\mathbf{B}_0)\times\mathbf{B}_1 + \nabla(n_0T_1 + n_1T_0) \tag{1.6.24}$$

Once again, 8 equations and 8 unknowns since we have all the subscript 0 terms. In principle, we can solve for all of the first order (subscript 1) terms which are the linearized terms. For example, one can use Fourier transforms if the steady-state terms have no spatial dependences. Note that unless our previous solutions for $\mathcal{O}(\delta^0 = 1)$ terms are especially simple, it might be as difficult to solve this system of equations as the original system without the ordering parameter. Thus, we see that linearization may always work, but it may not always be useful. Context again decides whether linearization will provide us physical insight.

After solving for these we could then go for the next order $\mathcal{O}(\delta^2)$ which would entail solving

$$\frac{\partial n_2}{\partial t} + \boldsymbol{\nabla}\cdot(n_1\mathbf{V}_2 + n_2\mathbf{V}_1) \tag{1.6.25}$$

$$\begin{aligned}\left\{\frac{\partial T_2}{\partial t} + \mathbf{V}_0\cdot\nabla T_2 + \mathbf{V}_1\cdot\nabla T_1 + \mathbf{V}_2\cdot\nabla T_0\right\} + n_0T_0\boldsymbol{\nabla}\cdot\mathbf{V}_2 \\ + (n_0T_1 + n_1T_0)\boldsymbol{\nabla}\cdot\mathbf{V}_1 + n_1T_1\boldsymbol{\nabla}\cdot\mathbf{V}_0\end{aligned} \tag{1.6.26}$$

$$\frac{\partial \mathbf{B}_2}{\partial t} = \boldsymbol{\nabla}\cdot(\mathbf{V}_0\times\mathbf{B}_2 + \mathbf{V}_1\times\mathbf{B}_1 + \mathbf{V}_2\times\mathbf{B}_0) \tag{1.6.27}$$

$$\begin{aligned}\frac{\partial \mathbf{V}_2}{\partial t} + \mathbf{V}_0\cdot\nabla\mathbf{V}_2 + \mathbf{V}_1\cdot\nabla\mathbf{V}_1 + \mathbf{V}_2\cdot\nabla\mathbf{V}_0 \\ = \boldsymbol{\nabla}\times(\mathbf{B}_2)\times\mathbf{B}_0 + (\boldsymbol{\nabla}\times\mathbf{B}_1)\times\mathbf{B}_1 + (\boldsymbol{\nabla}\times\mathbf{B}_0)\times\mathbf{B}_2 \\ - \nabla(n_0T_2 + n_1T_1 + n_2T_0)\end{aligned} \tag{1.6.28}$$

which can in principle be solved once we have all zeroth and first order terms solved for above. We can then write out our solutions using the series we developed above for each quantity q as

$$q \approx q_0 + \delta q_1 + \delta^2 q_2 \tag{1.6.29}$$

Once we have gone to $\mathcal{O}(\delta^2)$ however, it is pretty clear that general solutions become more and more difficult to work out as the equations become longer and longer (not necessarily much more intrinsically difficult, however, but often they actually are more difficult). Usually the second order expansion is considered too much work for a limited gain. For if $\delta \ll 1$ then we are only making a correction of $\delta^2 \ll \delta$. So you need to always weigh the difficulties versus the fruits of the labor. It may be better to find a different approximation than to work out the second order results.

When doing such calculations, if we assume q_0 are constant in time and space, then it is common to assume that the linear (first order) terms are in the form of plane waves with spatial and temporal dependence $\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$ which is equivalent to a Fourier-Laplace transform as long as there are no terms with spatial dependence outside of the first-order terms. When this is the case, note

that the second order $\mathcal{O}(\delta^2)$ equations cannot easily be calculated by the same method, and so finding a solution for the second order equations is actually a fairly difficult task. This is why it is rarely done in practice.

1.7 Multiple Scale Analysis

> *Robert [Oppenheimer] replied bluntly, "I'm in difficulties." Bohr asked, "Are the difficulties mathematical or physical?" When Robert replied, "I don't know," Bohr said, "That's bad."*
>
> — From *American Prometheus*[3, p. 54]

Multiple scale analysis is, essentially, a generalized perturbation method. But the multiple scale analysis works with only a few terms in a series, whereas regular (that is usual) perturbation series (what I covered in the linearization Section 1.6) may require an infinite number of terms to recover useful information. Generally in physics, we think of the multiple scales as being in time, though space is always a possibility as well. I will talk in the language of multiple *time* scale analysis, but one can easily generalize to other situations.

The key idea is assuming that our problem has multiple time scales that we care about and are relevant. In addition these time scales are separated by an ordering parameter ϵ, and we think of things changing ever more slowly on the longer and longer time scales so that the time separation makes sense. Then we have time t written for each of these scales as

$$t = \epsilon^{-j}\tau_j \tag{1.7.1}$$

with each τ_j being a time variable associated with the scale ϵ^j. With these j independent time variables we will then analyze the problem.[73] If we have an equation with an operator $\mathbf{G}$ (that can involve differentiation, integration, etc., all the things we normally see) and given driving function $s(\mathbf{x}, t)$ we can form an equation

$$\mathbf{G}(f(\mathbf{x}, t)) = s(\mathbf{x}, t) \tag{1.7.2}$$

for $f(\mathbf{x}, t)$ with appropriate boundary conditions.

Calling all the τ_j the vector array $\mathtt{T}$, we can write

$$f(\mathbf{x}, t) \to f(\mathbf{x}, \mathtt{T}) \tag{1.7.3}$$
$$s(\mathbf{x}, t) \to s(\mathbf{x}, \mathtt{T}) \tag{1.7.4}$$

We can consider each time scale as independent of the others, but dependent on t itself so that for any generic function $g(t)$ we have

$$\frac{\partial g}{\partial t} = \sum_{j=0}^{\infty} \frac{\partial g}{\partial \tau_j}\frac{\mathrm{d}\tau_j}{\mathrm{d}t} = \sum_{j=0}^{\infty} \epsilon^j \frac{\partial g}{\partial \tau_j} \tag{1.7.5}$$

[73] That is we solve a more complicated problem from a variable perspective and use the extra freedoms inherent in the extra complexity to match onto our original problem. This is like finding the answer to the moments of a Gaussian by noticing that for $F = \int_{-\infty}^{\infty} \mathrm{d}t\ \exp(-\alpha t^2) = \sqrt{\pi/\alpha}$ and so we can take $\frac{\mathrm{d}}{\mathrm{d}\alpha}$ to find the answer instead of redoing each integral. We usually only deal with two scales in multiple scale analysis as it usually becomes more difficult to solve the equations as you go on.

That is, we artifically treat τ_j as independent of each other but dependent on t. Later we use the acutal relationships between the time scales to form our solution. Finally, we use a perturbation series

$$f(\mathbf{x}, \mathtt{T}) = \sum_{j=0}^{\infty} \epsilon^j f_j(\mathbf{x}, \mathtt{T}) \tag{1.7.6}$$

In doing so, we have added $j-1$ extra independent variables which gives too much freedom to our solutions. If we are lucky, as we usually are, then we will enforce $j-1$ new conditions that rid us of secular[74] terms with so-called solvability conditions. These usually turn out to be something like the integral of something or some expression is zero in cases of physical interest. These solvability conditions are not too complicated and so we get a physical solution by finding a more general solution and reducing it to a case consistent with our actual case.

An example will make this clearer. Following the usual approach, we choose the Duffing equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y + \epsilon y^3 = 0 \tag{1.7.7}$$

with $\epsilon \ll 1$ with $y(0) = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t}|_{y=0} = 0$. If we used a typical perturbation series we find

$$y = \sum_{j=0}^{\infty} \epsilon^j y_j \tag{1.7.8}$$

and so find

$$\frac{\mathrm{d}^2 y_0}{\mathrm{d}t^2} + \epsilon \frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + y_0 + \epsilon y_1 + \epsilon y_0^3 = \mathcal{O}(\epsilon^2) \tag{1.7.9}$$

Thus order by order we find

$$\frac{\mathrm{d}^2 y_0}{\mathrm{d}t^2} + y_0 = 0 \tag{1.7.10}$$

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + y_1 + y_0^3 = 0 \tag{1.7.11}$$

If we enforce the boundary conditions, then the solution to the zeroth order equation is clearly $a_0 \cos(t)$. The second equation then states

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + y_1 = -a_0^3 \cos^3 t \tag{1.7.12}$$

We can solve this ODE by any method you choose. The homogenous solution is clearly going to be $b_0 \cos(t)$. The particular solution is

$$\frac{3a_0^3}{8} t \sin(t) - \frac{a_0^3}{32} \cos(3t) \tag{1.7.13}$$

Thus the solution is

$$y \approx a_0 \cos(t) + \epsilon \left[b_0 \cos(t) - \frac{a_0^3}{32} \cos(3t) + \frac{3a_0^3}{8} t \sin(t) \right] \tag{1.7.14}$$

[74] That is terms that grow unphysically, and so for a physical solution should be set to zero somehow.

However, when $t \sim 1/\epsilon$, then our perturbation series fails because then $\epsilon t \sin(t)$ grows to be on the y_0 scale rather than the y_1 scale. This is a secular term.

If we had instead started with a multiple time scale analysis we introduce ($\mathtt{T} = [\tau_0, \tau_1]$)

$$t = \tau_0 + \epsilon\tau_1 \tag{1.7.15}$$

$$y = y_0 + \epsilon y_1 \tag{1.7.16}$$

and use

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial y_0}{\partial \tau_0} + \epsilon\frac{\partial y_0}{\partial \tau_1} + \epsilon\frac{\partial y_1}{\partial \tau_0} + \mathcal{O}(\epsilon^2) \tag{1.7.17}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial \tau_0^2} + 2\epsilon\frac{\partial^2 y}{\partial \tau_1 \partial \tau_0} + \epsilon\frac{\partial^2 y_1}{\partial \tau_0^2} \tag{1.7.18}$$

and so our equation becomes

$$\frac{\partial^2 y_0}{\partial \tau_0^2} + 2\epsilon\frac{\partial^2 y_0}{\partial \tau_1 \partial \tau_0} + \epsilon\frac{\partial^2 y_1}{\partial \tau_0^2} + y_0 + \epsilon y_1 + \epsilon y_0^3 = \mathcal{O}(\epsilon^2) \tag{1.7.19}$$

and so order by order we get

$$\frac{\partial^2 y_0}{\partial \tau_0^2} + y_0 = 0 \tag{1.7.20}$$

$$\frac{\partial^2 y_1}{\partial \tau_0^2} + 2\frac{\partial^2 y_0}{\partial \tau_1 \partial \tau_0} + y_0^3 = 0 \tag{1.7.21}$$

Now the first equation clearly has a solution of the form

$$y_0 = a(\tau_1)\cos(\tau_0) = A(\tau_1)\exp(i\tau_0) + A^*(\tau_1)\exp(-i\tau_0) \tag{1.7.22}$$

We then have

$$\begin{aligned} y_0^3 &= A(\tau_1)^3\exp(3i\tau_0) + 3A(\tau_1)^2A^*(\tau_1)\exp(2i\tau_0 - 2i\tau_0) \\ &\quad + 3A(\tau_1)A^*(\tau_1)^2\exp(i\tau_0 - 2i\tau_0) + A(\tau_1)^3\exp(-3i\tau_0) \end{aligned} \tag{1.7.23}$$

$$= A(\tau_1)^3\exp(3i\tau_0) + 3A(\tau_1)^2A^*(\tau_1)\exp(i\tau_0) + 3A(\tau_1)A^*(\tau_1)^2\exp(-i\tau_0) + A(\tau_1)^3\exp(-3i\tau_0) \tag{1.7.24}$$

$$= a(\tau_1)^3\cos^3(\tau_0) \tag{1.7.25}$$

$$\begin{aligned} \frac{\partial^2 y_0}{\partial \tau_1 \partial \tau_0} &= \frac{\partial}{\partial \tau_0}\left[\frac{\partial A(\tau_1)}{\partial \tau_1}\exp(i\tau_0)\right] + \frac{\partial}{\partial \tau_0}\left[\frac{\partial A^*(\tau_1)}{\partial \tau_1}\exp(-i\tau_0)\right] \\ &= i\frac{\partial A(\tau_1)}{\partial \tau_1}\exp(i\tau_0) + -i\frac{\partial A^*(\tau_1)}{\partial \tau_1}\exp(i\tau_0) \end{aligned} \tag{1.7.26}$$

$$= -\frac{\partial a(\tau_1)}{\partial \tau_1}\sin\tau_0 \tag{1.7.27}$$

So that the first order equation reads

$$\frac{\partial^2 y_1}{\partial \tau_1^2} + y_1 = -a^3\cos^3(\tau_0) + \frac{\partial a(\tau_1)}{\partial \tau_1}\sin\tau_0 \tag{1.7.28}$$

We'd pretty clearly like to have the $\sin\tau_0$ term on the right disappear. The $\cos^3(\tau_0)$ term also hides a $\cos\tau_0$ term that will cause trouble. That is we have

$$\cos^3\tau_0 = \frac{3\cos\tau_0 + \cos(3\tau_0)}{4} \tag{1.7.29}$$

and so we want

$$-\frac{3a^3\cos\tau_0}{4} + \frac{\partial a}{\partial\tau_1}\sin\tau_0 = 0 \tag{1.7.30}$$

Now we have a problem. It seems like a will depend on τ_0 which would mean our perturbation series assumptions about a are wrong. So we revert to the A equations, because it is clear that it is not simple to remove the τ_0 dependence with cosines and sines (though, in principal it is possible, as we will see by adding in phases to the cos and sin terms). We then see that we want the coefficients of $\exp(\pm i\tau_0)$ to be zero or else we will have secular terms. These seem like two equations, but they are simply complex conjugates of each other so we only need to solve

$$-3A^2A^* - 2i\frac{\mathrm{d}A}{\mathrm{d}\tau_1} = 0 \tag{1.7.31}$$

which has no difficult to eliminate τ_0 dependence. We write $A = R(\tau_1)\exp(i\theta(\tau_1))$ and see the above equation then says

$$-3R^3\exp(i\theta) - 2i\left[\frac{\mathrm{d}R}{\mathrm{d}\tau_1} + iR\frac{\mathrm{d}\theta}{\mathrm{d}\tau_1}\right]\exp(i\theta) = 0 \tag{1.7.32}$$

Then the real and imaginary parts [after dividing by $\exp(i\theta)$] say

$$-3R^3 + 2R\frac{\mathrm{d}\theta}{\mathrm{d}\tau_1} = 0 \tag{1.7.33}$$

$$\frac{\mathrm{d}R}{\mathrm{d}\tau_1} = 0 \tag{1.7.34}$$

so R is a constant with respect to τ_1, say $R_0 = R(0)$ with $\theta_0 = \theta(0)$ and we find

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau_1} = \frac{3R^2}{2} \tag{1.7.35}$$

$$\theta = \frac{3R^2}{2}\tau_1 + \theta_0 \tag{1.7.36}$$

This then yields

$$A = R_0\exp\left(i\frac{3R_0^2}{2}\tau_1 + i\theta_0\right) \tag{1.7.37}$$

This is the solvability condition that we were looking for before, as it will remove the secular term. Thus we find

$$Y_0 = A\exp(i\tau_0) + A^*\exp(-i\tau_0) = R_0\exp\left(i\tau_0 + i\frac{3R_0^2}{2}\tau_1 + i\theta_0\right) - R_0\exp\left(-i\tau_0 - i\frac{3R_0^2}{2}\tau_1 - i\theta_0\right) \tag{1.7.38}$$

$$= 2R_0\cos\left(\tau_0 + \frac{3R_0^2}{2}\tau_1 + \theta_0\right) \tag{1.7.39}$$

We require $Y_0(0) = 1$ and $Y_0'(0) = 0$. Thus $R_0 = \frac{1}{2}$ and the solution then becomes

$$Y_0 = \cos\left(t + \frac{3}{8}\epsilon t + \theta_0\right) \tag{1.7.40}$$

and we have succeeded at "zeroth" order. Note that this solution does not present the same problem as $t \sim 1/\epsilon$. We now simply require that ϵt be $\mathcal{O}(1)$ for our solution to be all right.

However, failure is often more instructive than success. What if we tried this method on an oscillator equation where the oscillation frequency changes on the slow time scale. That is, we want to solve

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \omega^2(\epsilon t)y(t) = 0 \tag{1.7.41}$$

If we introduced $\epsilon t = \tau$ then the equation above becomes

$$\epsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + \omega^2(\tau)y(\tau) = 0 \tag{1.7.42}$$

which is the one dimensional Schrödinger equation that we have solved elsewhere in this book. So for multiple scale analysis we introduce

$$t = \tau_0 + \epsilon^{-1}\tau_1 \tag{1.7.43}$$

$$y = y_0 + \epsilon y_1 \tag{1.7.44}$$

Remember that ω is slowly varying, and we see that we have

$$\frac{\partial^2 y_0}{\partial \tau_0^2} + 2\epsilon \frac{\partial^2 y_0}{\partial \tau_0 \partial \tau_1} + \epsilon \frac{\partial^2 y_1}{\partial \tau_0^2} + \omega^2(\tau_1)y_0 + \epsilon\omega^2(\tau_1)y_1 = 0 \tag{1.7.45}$$

So zeroth order yields

$$\frac{\partial^2 y_0}{\partial \tau_0^2} + \omega^2 y_0 = 0 \tag{1.7.46}$$

and first order yields

$$2\frac{\partial^2 y_0}{\partial \tau_0 \partial \tau_1} + \frac{\partial^2 y_1}{\partial \tau_0^2} + \omega^2(\tau_1)y_1 = 0 \tag{1.7.47}$$

The zeroth order equation should yield

$$y_0 = A(\tau_1)\exp(i\omega(\tau_1)\tau_0) + A^*(\tau_1)\exp(-i\omega(\tau_1)\tau_0) \tag{1.7.48}$$

The first order equation states

$$\frac{\partial^2 y_1}{\partial \tau_0} + \omega^2(\tau_1)y_1 = -2\frac{\partial}{\partial \tau_1}\left[i\omega A\exp(i\omega\tau_0) - i\omega A^*\exp(-i\omega\tau_0)\right] \tag{1.7.49}$$

The right hand side can be expanded out to

$$\begin{aligned}
&-2\frac{\partial\omega}{\partial\tau_1}\left[iA\exp(i\omega\tau_0)-iA^*\exp(-i\omega\tau_0)\right] \\
&\quad -2\left[i\frac{\partial A}{\partial\tau_1}\omega\exp(i\omega\tau_0)-i\frac{\partial A^*}{\partial\tau_1}\omega\exp(-i\omega\tau_0)\right] \\
&\quad -2\left[i^2\tau_0\frac{\partial\omega}{\partial\tau_1}A\omega\exp(i\omega\tau_0)+(-i)^2\tau_0\frac{\partial\omega}{\partial\tau_1}A^*\omega\exp(-i\omega\tau_0)\right]
\end{aligned} \tag{1.7.50}$$

$$\begin{aligned}
&= -2i\exp(i\omega\tau_0)\left[\frac{\partial\omega}{\partial\tau_1}A+\omega\frac{\partial A}{\partial\tau_1}+i\tau_0\frac{\partial\omega}{\partial\tau_1}A\right] \\
&\quad +2i\exp(-i\omega\tau_0)\left[\frac{\partial\omega}{\partial\tau_1}A^*+\omega\frac{\partial A^*}{\partial\tau_1}-i\tau_0\frac{\partial\omega}{\partial\tau_1}A^*\right]
\end{aligned} \tag{1.7.51}$$

Now in order to get the right-hand side secular terms to be zero we have to deal with τ_0 terms which cannot be removed. We cannot get rid of the terms in brackets by a clever use of A (try assuming a nonzero A and see what you find). We must have $A = 0$ to ensure they are zero, which means our approximation was of zero. That's not very useful. In fact, this shows that even if ω, the oscillation, changes on the $\epsilon\tau_0 = \tau_1$ time scale, then multiple scale analysis will fail. That means if we have something that changes on the long time scale, we need to change it into a form where ϵ is times the function of $y(t)$, without an ϵt dependence. For us this would involve using a new time variable so that the term $\omega^2(\epsilon\tau)y \to \epsilon G(y)$ where G is some function of y. One can show that a time variable such as

$$T = \int^{\tau_0} \mathrm{d}t'\ \omega(\epsilon t') = \frac{1}{\epsilon}\int^{\tau_1} \mathrm{d}s\ \omega(s) \tag{1.7.52}$$

will get us into the correct form.[75]

1.7.1 A More Complicated Variant

Finally, let's consider a more difficult example with a twist. Consider a relationship

$$\frac{\mathrm{d}\mathbf{x}'(t')}{\mathrm{d}t'} = \mathbf{f}(\mathbf{x}', t') \tag{1.7.53}$$

First let's introduce a characteristic time scale. Suppose we are interested in solutions that are approximately constant over $\Delta t' = t_0$, and then have a characteristic length scale L_0 so that $|\delta\mathbf{x}'| \sim L_0$. Then we introduce $t = t'/t_0$ and $\mathbf{x} = \mathbf{x}'/L_0$ and can write[76]

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \frac{t_0}{L_0}\mathbf{f}(\mathbf{x}, t) \tag{1.7.54}$$

In previous analyses we used $\tau_0 = t$ and created a long time scale $\epsilon\tau_1 = t$. However, there is nothing wrong with choosing $\epsilon = 1/\delta_t$ with $\delta_t \ll 1$. Then we have $\tau_1 = \delta_t t$ is a short time scale instead of a long time scale. We will still require that the oscillation is constant on the long time

[75] For more on multiple scale analysis, see Bender and Orzsag[2], which has illuminating discussions of the limits and benefits of this method.

[76] Usually we set $t_0 = 1$ as a natural frequency and $L_0 = 1$ as a natural length when we solve this analytically, but I'll retain the t_0 and L_0 to help remind us that we are using normalized equations.

scale, however. We can then perform the multiple scale analysis with $t = \tau_0$ and $\tau_1 = \delta_t t$ as normal, though we now wish to solve over τ_0 time scales with small corrections due to τ_1, the short scale. To do so we can introduce a periodicity requirement on τ_1 so that it retains oscillatory behavior. First we write

$$\frac{\mathrm{d}\mathbf{x}(\tau_0,\tau_1)}{\mathrm{d}t} = \frac{t_0}{L_0}\mathbf{f}(\mathbf{x},\tau_0,\tau_1) \tag{1.7.55}$$

$$\frac{\mathrm{d}\mathbf{x}(\tau_0,\tau_1)}{\mathrm{d}t} = \frac{\partial \mathbf{x}}{\partial \tau_0}\frac{\partial \tau_0}{\partial t} + \frac{\partial \mathbf{x}}{\partial \tau_1}\frac{\partial \tau_1}{\partial t} = \frac{\partial \mathbf{x}}{\partial \tau_0} + \frac{1}{\delta_t}\frac{\partial \mathbf{x}}{\partial \tau_1} \tag{1.7.56}$$

For periodicity, we can introduce a period T such that $\frac{\Delta\tau_1}{T} = \delta_t$ for a characteristic time in τ_1 (which is $\delta_t \Delta t$ for characteristic times in t which we set to $\Delta t \sim 1$ via using a time scale t_0). We can then introduce the series for $\mathbf{x}$

$$\mathbf{x}(\tau_0,\tau_1) = \mathbf{x}_0(\tau_0,\tau_1) + \delta_t \mathbf{x}_1(\tau_0,\tau_1) + \cdots \tag{1.7.57}$$

Note that if we average over τ_1 we use

$$\langle q(\mathbf{X},\tau_0,\tau_1)\rangle_{\tau_1} \equiv \frac{\int_0^T \mathrm{d}s\; q(\mathbf{X},\tau_0,s)}{\int_0^T \mathrm{d}s} = \frac{1}{T}\int_0^{2\pi} \mathrm{d}s\; q(\mathbf{X},\tau_0,s) \tag{1.7.58}$$

It would then be nice to filter out $\mathbf{x}$ into components that are independent of τ_1. That is

$$\langle \mathbf{x}(\tau_0,\tau_1)\rangle_{\tau_1} = \mathbf{X}(\tau_0) \tag{1.7.59}$$

$$\langle \mathbf{x}(\tau_0,\tau_1)\rangle_{\tau_1} = \mathbf{X}_0(\tau_1) + \delta_t \mathbf{X}_1(\tau_1) + \cdots \tag{1.7.60}$$

Now we can define

$$\mathbf{x}(\tau_0,\tau_1) - \mathbf{X}(\tau_0) \equiv \kappa\boldsymbol{\xi}(\mathbf{X}(\tau_0),\tau_0,\tau_1) \tag{1.7.61}$$

However, there is now a question of how close $\mathbf{x}$ and $\mathbf{X}$ are, which is why there is an ordering parameter κ. We will see if we can get some new information by imposing a new ordering. For example, if $\mathbf{X}(\tau_0)$ is essentially the same as $\mathbf{x}$ at the spatial scales we are interested in, then $\boldsymbol{\xi}$ can be considered small and the ordering parameter κ will be small.

So now we can expand $\mathbf{X}$ and $\boldsymbol{\xi}$ in perturbation series

$$\mathbf{X} = \mathbf{X}_0(\tau_0) + \delta_t \mathbf{X}_1(\tau_0) + \cdots \tag{1.7.62}$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}_0(\mathbf{X},\tau_0,\tau_1) + \delta_t \boldsymbol{\xi}_1(\mathbf{X},\tau_0,\tau_1) + \cdots \tag{1.7.63}$$

However, now $\boldsymbol{\xi}_i$ has a dependence on $\mathbf{X}$, which has a dependence on τ_0. We want the full τ_0 variation assuming τ_1 is independent of τ_0. Thus

$$\frac{\partial \mathbf{x}}{\partial \tau_0} = \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau_0} + \overbrace{\kappa\frac{\partial \boldsymbol{\xi}}{\partial \tau_0} + \kappa\frac{\partial \mathbf{X}}{\partial \tau_0}\cdot\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{X}}}^{\kappa\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}\tau_0}} \tag{1.7.64}$$

We then have for $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$ that

$$\frac{\partial \mathbf{X}_0}{\partial \tau_0} + \kappa\frac{\mathrm{d}\boldsymbol{\xi}_0}{\mathrm{d}\tau_0} + \frac{\kappa}{\delta_t}\frac{\partial \boldsymbol{\xi}_0}{\partial \tau_1} + \delta_t\frac{\partial \mathbf{X}_1}{\partial \tau_0} + \kappa\delta_t\frac{\mathrm{d}\boldsymbol{\xi}_1}{\mathrm{d}\tau_0} + \kappa\frac{\partial \boldsymbol{\xi}_1}{\partial \tau_1} + \mathcal{O}(\delta_t^2 + \kappa^2) \tag{1.7.65}$$

Finally, we can rewrite $\mathbf{f}$ so

$$\mathbf{f}(\mathbf{x}, \tau_0, \tau_1) = \mathbf{f}(\mathbf{X} + \kappa\boldsymbol{\xi}, \tau_0, \tau_1) \tag{1.7.66}$$

which makes it easier to Taylor expand

$$\mathbf{f}(\mathbf{X} + \kappa\boldsymbol{\xi}, \tau_0, \tau_1) = \mathbf{f}(\mathbf{X}, \tau_0, \tau_1) + \kappa\boldsymbol{\xi} \cdot \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{X}} + \mathcal{O}(\kappa^2) \tag{1.7.67}$$

Clearly, the above could be rewritten as

$$\mathbf{f}(\mathbf{X}_0 + \delta_t\mathbf{X}_1 + \kappa\boldsymbol{\xi}_0 + \cdots, \tau_0, \tau_1) = \mathbf{f}(\mathbf{X}_0, \tau_0, \tau_1) + [\delta_t\mathbf{X}_1 + \kappa\boldsymbol{\xi}_0] \cdot \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{X}_0} + \mathcal{O}(\kappa^2, \delta_t^2) \tag{1.7.68}$$

but this form is actually in some sense a worse equation, because it is less accurate once we know $\mathbf{X}$. Either form is admissible.

Now, we'd like our equations to be consistent, which means that when we average them over τ_1 we'd like there to be terms balancing on all sides. We are now confronted with the κ conundrum. What should it be in terms of δ_t. Let's consider a couple of different options. First suppose $\kappa = 1$. Then the dominant term will be $\frac{\partial \boldsymbol{\xi}_0}{\partial \tau_1} = \mathbf{0}$. This means $\boldsymbol{\xi}_0(\mathbf{X}, \tau_0, \tau_1) = \boldsymbol{\xi}_0(\mathbf{X}, \tau_0)$ which means $\boldsymbol{\xi}_0 = \mathbf{0}$. We then require $\langle \mathbf{x}_0 \rangle_{\tau_1} = \mathbf{X}_0$ by definition which means that if $\mathbf{x}$ has τ_1 dependence then $\langle \boldsymbol{\xi}_0 \rangle_{\tau_1} \neq \mathbf{0}$. Thus $\kappa = 1$ is a contradiction. This tells us that $\kappa = \delta_t^\alpha$ for $\alpha \leq 0$ is impossible.[77] However, we are left with freedom for α a positive integer. For $\alpha > 1$ we would have

$$\frac{\partial \mathbf{X}_0}{\partial \tau_0} = \frac{t_0}{L_0}\mathbf{f}(\mathbf{X}, \tau_0, \tau_1) \tag{1.7.69}$$

This is problematic since $\mathbf{f}$ is a function of τ_1 while the $\frac{\partial \mathbf{X}_0}{\partial \tau_0}$ is not. This means that we would require f not be a function of the small time scale, a requirement we cannot impose since we assumed it has such a dependence! This is the crucial clue. What term has τ_1 dependence that with a proper α could rise to $\mathcal{O}(1)$. Clearly $\delta_t^{\alpha-1}\frac{\partial \boldsymbol{\xi}_0}{\partial \tau_1}$ is the term, as it is the only one that could do the job. Then $\alpha = 1$ and we see

$$\frac{\partial \mathbf{X}_0}{\partial \tau_0} + \frac{\partial \boldsymbol{\xi}_0}{\partial \tau_1} = \frac{t_0}{L_0}\mathbf{f}(\mathbf{X}, \tau_0, \tau_1) \tag{1.7.70}$$

If we average over τ_1 we find

$$\left\langle \frac{\partial \mathbf{X}_0}{\partial \tau_0} + \frac{\partial \boldsymbol{\xi}_0}{\partial \tau_1} \right\rangle_{\tau_1} = \frac{t_0}{L_0} \langle \mathbf{f}(\mathbf{X}, \tau_0, \tau_1) \rangle_{\tau_1} \tag{1.7.71}$$

$$\frac{\partial \mathbf{X}_0}{\partial \tau_0} = \frac{t_0}{L_0} \langle \mathbf{f}(\mathbf{X}, \tau_0, \tau_1) \rangle_{\tau_1} \tag{1.7.72}$$

which is now fine as a solvability condition. And we see we recover that $\boldsymbol{\xi}_0$ is the part minus the average, as we desired

$$\frac{\partial \boldsymbol{\xi}_0}{\partial \tau_0} = \frac{t_0}{L_0}\mathbf{f} - \langle \mathbf{g} \rangle_{\tau_1} \tag{1.7.73}$$

$$\boldsymbol{\xi}_0 - \boldsymbol{\xi}_0(\tau_1 = 0) = \frac{t_0}{L_0}\int_0^{\tau_0} \mathrm{d}\tau' \ \left[\mathbf{f} - \langle \mathbf{f} \rangle_{\tau_1}\right]_{\tau_1 = \tau'} \tag{1.7.74}$$

[77] If $\alpha < 0$ we would still have a leading term of $\frac{\partial \boldsymbol{\xi}_0}{\partial \tau_1} = \mathbf{0}$.

And to first order we have

$$\frac{\partial \mathbf{X}_1}{\partial \tau_0} + \frac{\mathrm{d}\boldsymbol{\xi}_0}{\mathrm{d}\tau_1} + \frac{\partial \boldsymbol{\xi}_1}{\partial \tau_1} = \frac{t_0}{L_0}\left[\mathbf{X}_1 + \boldsymbol{\xi}_0\right] \cdot \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{X}} \tag{1.7.75}$$

Remember that

$$\frac{\mathrm{d}\boldsymbol{\xi}_i}{\mathrm{d}\tau_0} = \frac{\partial \boldsymbol{\xi}_i}{\partial \tau_0} + \frac{\partial \mathbf{X}}{\partial \boldsymbol{\tau}_0} \cdot \frac{\partial \boldsymbol{\xi}_i}{\partial \mathbf{X}} = \frac{\partial \boldsymbol{\xi}_i}{\partial \tau_0} + \frac{\partial \mathbf{X}_0}{\partial \tau_0} \cdot \frac{\partial \boldsymbol{\xi}_i}{\partial \mathbf{X}_0} + \mathcal{O}(\delta_t) \tag{1.7.76}$$

Thus we find

$$\frac{\partial \mathbf{X}_1}{\partial \tau_0} + \frac{\partial \boldsymbol{\xi}_0}{\partial \tau_0} + \frac{\partial \mathbf{X}_0}{\partial \tau_0} \cdot \frac{\partial \boldsymbol{\xi}_0}{\partial \mathbf{X}_0} + \frac{\partial \boldsymbol{\xi}_1}{\partial \tau_1} = \frac{t_0}{L_0}\left[\boldsymbol{\xi}\right] \cdot \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{X}} = \frac{t_0}{L_0}\left[\mathbf{X}_1 + \boldsymbol{\xi}_0\right] \cdot \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{X}_0} \tag{1.7.77}$$

Then the solubility condition is going to come from the τ_1 average again, and we find

$$\left\langle \frac{\partial \mathbf{X}_1}{\partial \tau_0} + \cancel{\frac{\partial \boldsymbol{\xi}_0}{\partial \tau_0}} + \cancel{\frac{\partial \mathbf{X}_0}{\partial \tau_0} \cdot \frac{\partial \boldsymbol{\xi}_0}{\partial \mathbf{X}_0}} + \cancel{\frac{\partial \boldsymbol{\xi}_1}{\partial \tau_1}} \right\rangle_{\tau_1} = \frac{t_0}{L_0}\left\langle \left[\boldsymbol{\xi}\right] \cdot \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{X}} \right\rangle_{\tau_1} \tag{1.7.78}$$

The right-hand side $\boldsymbol{\xi}\cdot$ doesn't cancel out of the τ_1 average because it is multiplying $\partial \mathbf{f}/\partial \mathbf{x}$, and so it no longer necessarily averages out to zero. And so our solvability criterion is

$$\frac{\partial \mathbf{X}_1}{\partial \tau_0} = \frac{t_0}{L_0}\left\langle \left[\boldsymbol{\xi}\right] \cdot \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{X}} \right\rangle_{\tau_1} = \frac{t_0}{L_0}\left\langle \left[\mathbf{X}_1 + \boldsymbol{\xi}_0\right] \cdot \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{X}_0} \right\rangle_{\tau_1} \tag{1.7.79}$$

Clearly the $\mathbf{X}_1 + \boldsymbol{\xi}_0$ form is difficult to work with, and since we have $\mathbf{f}(\mathbf{X}, \tau_0, \tau_1)$ just as easily as $\mathbf{f}(\mathbf{x}, \tau_0, \tau_1)$ at this order, that form is to be preferred. Indeed, we can even replace $\boldsymbol{\xi}$ with $\boldsymbol{\xi}_0$ and retain that the expression is accurate to $\mathcal{O}(\delta_t^2)$. This means we can write with $\tau_0 \to t$

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau_0} = \frac{\mathrm{d}\mathbf{X}_0}{\mathrm{d}\tau_0} + \delta_t \frac{\mathrm{d}\mathbf{X}_1}{\mathrm{d}\tau_0} + \mathcal{O}(\delta_t^2) \tag{1.7.80}$$

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{X}_0}{\mathrm{d}t} + \delta_t \frac{\mathrm{d}\mathbf{X}_1}{\mathrm{d}t} + \mathcal{O}(\delta_t^2) \tag{1.7.81}$$

And so plugging in what we have found from the solvability conditions we find

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \frac{t_0}{L_0}\left\langle \mathbf{f}(\mathbf{X}, t, \tau_1) \right\rangle_{\tau_1} + \frac{t_0}{L_0}\epsilon \left\langle \boldsymbol{\xi}_0 \cdot \left[\frac{\partial \mathbf{f}(\mathbf{X}, t, \tau_1)}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{X}} \right\rangle_{\tau_1} + \mathcal{O}(\delta_t^2) \tag{1.7.82}$$

We can translate this into a plasma relevant situation by considering $\delta_t \sim t_0\Omega$. Then what we have found for $\boldsymbol{\xi}$ is called the gyromotion, and we find the gyrocenter $\mathbf{X}$ moves based on a combination of the gyroaverage of the force plus a gyroaverage of the gradient of the force dotted into the gyromotion. Finally, there is one more thing to think about. The question of how to actually do the τ_1 average when we return to just the single t time variable. The answer is that we simply have to consider any ϵt terms as if they are a separate variable when performing the averages of $\mathbf{f}(\mathbf{x}, t)$. If you have formulated the problem correctly, then ϵt terms should naturally occur in the form of $\mathbf{f}$. Indeed, sometimes we start off with two variables that have such a scale separation for $\mathbf{f}$ so that $\mathbf{f}(\mathbf{x}, \tau_0, \tau_1)$ is not an ansatz for simplification, but a truth (such as gyrophase and time when thinking about a phase space).

1.8 The Calculus of Variations

> *And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?*
>
> — George Berkeley

The calculus of variations, sometimes called variational calculus, is a way of applying calculus to "generalized" functions. You can see that George Berkeley may not have been impressed in extending calculus beyond functions.[78] We will explore the basics of what functionals are, how we can take something like a derivative, look at a couple of examples, and consider the use of variational methods in computational algorithms.

1.8.1 Basics of Variational Calculus

Variational calculus really is calculus with functionals. A functional is essentially a function of functions. So for a functional S written as $S[f]$ with f being a function, the input is a function in the same way that for $f(x)$ the x is an input. It is customary to use square brackets to indicate a functional rather than a function. Just like functions, we want our functionals to spit out real or complex numbers, which we then associate with the function. What sort of magic is a functional, then? Well, typically we think of a functional as an integral. Technically, they don't have to be, but the vast majority of the time when we [physicists] talk of functionals we are thinking of integrals.[79] Thus

$$S[f] = \int_{t_0}^{t_1} \mathrm{d}t\ f(t) \tag{1.8.1}$$

where t_0 and t_1 are fixed is a possible form for a functional $S[f]$. We could also write

$$S[f] = \int_{t_0}^{t_1} \mathrm{d}t\ L(t, f(t), f'(t), \ldots) \tag{1.8.2}$$

where L is a function that takes t, f, and its derivatives. In fact, if we choose $t_1 = t$ where t is an unknown complex or real value, then we can use the same ideas, so long as we understand that t is fixed when doing the integration. In addition, it is typical to consider functionals that spit out real numbers so that we do not have to deal with complex calculus theory. So we will then consider real functionals.

A question you should ask yourself is what it will mean for $f(t)$ and $f'(t)$ to be "independent" of each other. An example may be a useful crutch to understanding here. Suppose we had $f(x) = 1 + x + x^2$. We could write this as $f(x, x^2) = 1 + x + x^2$. Are x and x^2 independent of each other? It clearly depends on what is meant by independent. If we mean $\frac{\partial x^2}{\partial x} = 0$ when considering

[78] In full fairness to Berkeley, we now have the limit formalism and nonstandard analysis to make infinitesimals rigorous. Back in his time, his critiques rang rather true.

[79] What an "integral" is, is actually not always so simple. Should we allow $\int_{t_0}^{t_1} \mathrm{d}t\ \delta(t)f(t)$ to count? Even though $\delta(t)$, the Dirac delta function, is not really a function (but a generalized distribution)? Further, one could have functionals of the form $S[f] = [\int_{t_0}^{t_1} \mathrm{d}t\ f(t)]/[1 + \int_{t_0}^{t_1} g(t)f(t)]$ which is not "just" an integral.

x^2 a function of x, then they are not independent. But suppose we ask a different question. Suppose we ask how f varies for a given x and x^2 (with x^2 not necessarily $x * x$ under our initial consideration). One way of answering such a question is to consider x and x^2 independent of each other (so $\partial x^2/\partial x = \frac{\partial x}{\partial x^2} = 0$ because we think of x^2 as y and so not necessarily the same as x^2) and use a chain rule. Clearly, this is equivalent to considering a function $f(x, y)$ and later setting $y = x^2$, but we can use the x^2 notation so long as we realize it is actually independent of x at this point in our calculation. We then have

$$\begin{aligned} \mathrm{d}f &= \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial x^2}\,\mathrm{d}(x^2) = \frac{\partial}{\partial x}\left(1 + x + x^2\right)\,\mathrm{d}x + \frac{\partial}{\partial x^2}\left(1 + x + x^2\right)\,\mathrm{d}(x^2) \\ &= \left[\cancel{\frac{\partial 1}{\partial x}} + \frac{\partial x}{\partial x} + \cancel{\frac{\partial x^2}{\partial x}}\right]\,\mathrm{d}x + \left[\cancel{\frac{\partial 1}{\partial x^2}} + \cancel{\frac{\partial x}{\partial x^2}} + \frac{\partial x^2}{\partial x^2}\right]\,\mathrm{d}(x^2) = \mathrm{d}x + \mathrm{d}(x^2) \end{aligned} \tag{1.8.3}$$

$$\begin{aligned} \mathrm{d}f &= \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y = \frac{\partial}{\partial x}\left(1 + x + y\right)\,\mathrm{d}x + \frac{\partial}{\partial y}\left(1 + x + y\right)\,\mathrm{d}y \\ &= \left[\cancel{\frac{\partial 1}{\partial x}} + \frac{\partial x}{\partial x} + \cancel{\frac{\partial y}{\partial x}}\right]\,\mathrm{d}x + \left[\cancel{\frac{\partial 1}{\partial y}} + \cancel{\frac{\partial x}{\partial y}} + \frac{\partial y}{\partial y}\right]\,\mathrm{d}y = \mathrm{d}x + \mathrm{d}y \end{aligned} \tag{1.8.4}$$

where the second makes it clear how this actually works. We can then use that $\mathrm{d}y = \mathrm{d}(x^2) = 2x\,\mathrm{d}x$

$$\mathrm{d}f = (\frac{\partial f}{\partial x} + 2x\left[\frac{\partial f}{\partial y}\right]_{y=x^2})\,\mathrm{d}x = (\frac{\partial f}{\partial x} + 2x\frac{\partial f}{\partial x^2})\,\mathrm{d}x = (1 + 2x)\,\mathrm{d}x \tag{1.8.5}$$

and so x and x^2 are not independent when we finally put values in for x, but were before we enforced the relationship between them. So the $\frac{\partial f}{\partial x} + 2x\frac{\partial f}{\partial x^2}$ part acts as if x and x^2 are independent for taking derivatives, but that is because we were not originally enforcing the connection between x and x^2, but were using y which later happens to equal x^2. It is in this sense that it always makes sense to consider a function as varying on its dependent variables, regardless of the dependencies of those dependent variables on each other. If those dependent variables have dependencies on each other, we can incorporate that information later. For functionals, the situation is completely analogous. When we consider variations in $S[f, f']$, we are actually considering variations in $S[f, g]$ and then using that $g = f'$ after considering the variations. When we are considering independent variations (keeping f and g separate), we can consider $\frac{\partial f}{\partial g} = 0$ since these are completely arbitrary functions. It is only when we impose the condition $g = f'$ that this condition no longer formally holds [this happens, for example, when we perform integration by parts or assign boundary conditions].

We can now ask what is an extremal value of S over some function space for f. This just means, out of the permissible f we allow, we want ours to gives a critical point (usually a minimum or maximum). It is important to consider what functions f we allow, since otherwise we could have pathological functions that are not continuous or differentiable. For us in physics world, we will only consider continuous functions, and I will mostly only consider differentiable functions (up to the required order we need). Now, to find the extremal value for f given a path (we fix t_0 and t_1), then we need to find the variation in $S[f]$. This is denoted δS. We can define a new thing called a functional derivative for many functionals. We use the definition for $S[f]$ that for $\delta f = \epsilon\phi$ an arbitrary function[80] that

$$\delta S\left[f, \delta f\right] \equiv \lim_{\epsilon\to 0}\frac{S[f + \epsilon\phi] - S[f]}{\epsilon} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(S[f + \epsilon\phi]\right)_{\epsilon=0} \tag{1.8.6}$$

[80] Arbitrary up to satisfying our admissibility constraints.

Because δf is an arbitrary function, this is often abbreviated to $\delta S\left[f\right]=\delta S$, omitting the brackets. When this is proportional to $\epsilon\phi=\delta f$ we can write the derivative as $\frac{\delta S}{\delta f}$ to denote the variation of S with respect to f (when it exists). That is using $\epsilon\phi=\delta f$ we define the part in front of δf as the variational/functional derivative via

$$\frac{\delta S}{\delta f}\delta f\ =\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[S[f+\epsilon\phi]\right]_{\epsilon=0,\phi\to\delta f} \tag{1.8.7}$$

We can then write either (in the cases where our functionals are integrals)

$$\delta S\ =\int_{t_0}^{t_1}\mathrm{d}t\ \frac{\delta S}{\delta f}\delta f \tag{1.8.8}$$

$$\delta S\ =\int_{t_0}^{t_1}\mathrm{d}t\ \frac{\delta S}{\delta f}\phi \tag{1.8.9}$$

depending upon whether we wish to actually write out the ϕ portion of δf. The ϕ is in some sense visually more sound as it connects to the mathematically rigorous method in a simple and obvious way, but with the δf it is also clear what is meant and it is visually more understandable as a notation by itself. I will prefer the δf notation, but I'll go through one example with the $\epsilon\phi$ so that you can understand the connection.

In fact, our functional δS as a variation is the same as what is called a Gâteaux derivative. This is a generalization of a directional derivative. The $\frac{\delta S}{\delta f}$, when it exists, is then a Frechét derivative. Suppose you have function F and want the "derivative" of F at a position x in the direction g in whatever space[81] we are dealing with. Then the Gâteaux derivative is defined to be

$$\mathrm{d}F(u;g)=\lim_{\epsilon\to0}\frac{F(u+\epsilon g)-F(u)}{\epsilon}=\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[F(u+\epsilon g)\right]_{\epsilon=0} \tag{1.8.10}$$

when it exists. The Frechét derivative requires that the variation be linear in g.

As an example, consider

$$S[f]=\begin{cases}\int_{t_0}^{t_1}\mathrm{d}t\ \frac{[f(t)]^3}{[f'(t)]^2+[f(t)]^2} & f\neq0\\ 0 & f=0\end{cases} \tag{1.8.11}$$

which can be viewed through the integrand L as

$$L(t,f,f')=\begin{cases}\frac{[f]^3}{[f']^2+[f]^2} & f\neq0\\ 0 & f=f'=0\end{cases} \tag{1.8.12}$$

Then we can consider the variation when f [and also f'] is the zero function.

$$\delta S\left[0,\epsilon g\right]=\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[S[0+\epsilon g]\right]_{\epsilon=0} \tag{1.8.13}$$

$$=\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\int_{t_0}^{t_1}\mathrm{d}t\ \frac{\epsilon^3g^3}{\epsilon^2g'^2+\epsilon^2g^2}\right]_{\epsilon=0}=\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\int_{t_0}^{t_1}\mathrm{d}t\ \epsilon\frac{g^3}{g'^2+g^2}\right]_{\epsilon=0} \tag{1.8.14}$$

$$=\int_{t_0}^{t_1}\mathrm{d}t\ \frac{g^3}{g'^2+g^2} \tag{1.8.15}$$

[81] These functions are no longer functions of real and complex variables, but of all sorts of spaces (such as Hilbert spaces).

which cannot depend on g (that is, g, g') linearly. Thus $\frac{\delta S}{\delta f}$ does not exist at $f = f' = 0$. However, at any location besides $f = f' = 0$ we find

$$\delta S\,[f, \epsilon g] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,[S[f + \epsilon g]]_{\epsilon=0} \tag{1.8.16}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\int_{t_0}^{t_1} \mathrm{d}t\ \frac{[f + \epsilon g]^3}{[f + \epsilon g]^2[f' + \epsilon g']^2 + \epsilon^2 g^2}\right]_{\epsilon=0} \tag{1.8.17}$$

$$= \int_{t_0}^{t_1} \mathrm{d}t\ \frac{f^2(3f'^2 g - 2ff'g' + f^2 g)}{(f^2 + f'^2)^2} \tag{1.8.18}$$

which is linear in g [if we integrate by parts] (that is, it is linear in g, g', etc.). So, when L has no problematic points, we can see that we have $\frac{\delta S}{\delta f}$ existing everywhere. We mostly deal with such functionals in physics.

To make this truly rigorous, you can consult a mathematics textbook, preferably explaining the subtleties of Gâteaux derivatives but the idea is fairly simple. We now can just apply the normal rules of calculus and differentials, treating the functions as if they were variables, with the idea that we can expand the functions in Taylor-like series retaining only first order in δf terms for the first variation. For $L(t, f, f', \ldots, f^{(n)})$ with all of the $f^{(j)}$ being independent variables for L, we can apply the same rule, but we can simply realize that this is just the usual differentials we had been using before so that

$$\delta L = \frac{\delta L}{\delta t}\delta t + \frac{\delta L}{\delta f}\delta f + \sum_{j=1}^{n} \frac{\delta L}{\delta f^{(j)}}\delta f^{(j)} = \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial f}\delta f + \sum_{j=1}^{n} \frac{\partial L}{\partial f^{(j)}}\delta f^{(j)} \tag{1.8.19}$$

Here you simply have to recognize that I have defined $\frac{\delta L}{\delta f} = \frac{\partial L}{\partial f}$ when L is a function rather than a functional. Also, δt is typically zero, (and pretty much always is in any actual application of the calculus of variations).

Now, given S and $L = L(t, f, f')$, we can find the first variation via[82]

$$\begin{aligned}
\delta S &= [S[f + \delta f\,] - S[f]] + \mathcal{O}(\delta^2) \\
&= \int_{t_0}^{t_1} \mathrm{d}t\ \left[\frac{L(t, f + \delta f\,, f' + \delta f\,) - L(t, f, f')}{\epsilon}\right] + \mathcal{O}(\delta^2) \\
&= \int_{t_0}^{t_1} \mathrm{d}t\ \left[\frac{\cancel{L(t, f, f')} + \frac{\delta L}{\delta f}\epsilon\phi + \frac{\delta L}{\delta f'}\epsilon\phi' + \mathcal{O}(\delta^2) - \cancel{L(t, f, f')}}{\epsilon}\right] \\
&= \int_{t_0}^{t_1} \mathrm{d}t\ \left[\frac{\partial L}{\partial f}\phi + \frac{\partial L}{\partial f'}\phi'\right]
\end{aligned} \tag{1.8.20}$$

where we have used $\delta f' = \epsilon\phi'(t)$. Then we can integrate by parts and use that $\phi(t_1) = \phi(t_0) = 0$

[82] I abuse notation by adding $\mathcal{O}(\delta^2)$ to mean take only terms with a δf (or $\delta f^{(j)}$ for any jth derivative), but ignore any terms $(\delta f^{(j)})(\delta f^{(k)})$ or higher where we act as if the δ's are an ordering parameter like the ϵ's in $\epsilon\phi$.

by definition.

$$\begin{aligned}\delta S &= \int_{t_0}^{t_1} \mathrm{d}t \ \left[\frac{\partial L}{\partial f}\phi + \frac{\partial L}{\partial f'}\phi'\right] = \int_{t_0}^{t_1} \mathrm{d}t \ \left[\frac{\partial L}{\partial f}\phi + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial f'}\phi\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial f'}\right)\phi\right] \\ &= \cancel{\left[\frac{\partial L}{\partial f'}\phi\right]_{t_0}^{t_1}} + \int_{t_0}^{t_1} \mathrm{d}t \ \left[\frac{\partial L}{\partial f}\phi - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial f'}\right)\phi\right] \\ &= \int_{t_0}^{t_1} \mathrm{d}t \ \left[\frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial f'}\right)\right]\phi = \int_{t_0}^{t_1} \mathrm{d}t \ \frac{\delta S}{\delta f}\phi\end{aligned} \tag{1.8.21}$$

We have then found the Euler-Lagrange equation is the variational derivative of S.

You might worry that our expansion does not make sense because f and f' are not independent of each other. What you have to remember is that for $L(x, f, f')$, is that they are. For $L(x, f, f') = L(x, f, g)$, L doesn't care that $g = f'$. L thinks of it as a separate variable. We enforce a relationship through $\delta f\,' = \frac{\mathrm{d}\delta f}{\mathrm{d}t}$, or by using the same ϕ (or derivative of ϕ) when perturbing f and g. This is why it is fine to consider f and f' as separate variables for looking at variations. Only along the actual path/trajectory do we actually have a further relationship.

In practice, the pseudorigorous[83] way of doing the above is via $\delta f\,(t_0) = \delta f\,(t_1) = 0$. We use the δf formulation assuming that S has no problematic points so that we can write

$$\delta S = S[f + \delta f\,] - S[f] + \mathcal{O}(\delta^2) \tag{1.8.22}$$

by which I mean that δS is only the part of $S[f + \delta f\,] - S[f]$ that contains a single δ of f or some derivative of f in it. That is we ignore all contributions that have a δ times another δ. Remember, once again, that this assumes that we have nice enough functions that we get things linear in δf (and its derivatives). Then we find with the δf rather than $\epsilon\phi$ formulation that

$$\begin{aligned}\delta S &= S[f + \delta f\,] - S[f] + \mathcal{O}(\delta^2) \\ &= \int_{t_0}^{t_1} \mathrm{d}t \ \left[L(t, f + \delta f\,, f' + \delta f'\,) - L(t, f, f')\right] + \mathcal{O}(\delta^2) \\ &= \int_{t_0}^{t_1} \mathrm{d}t \ \left[\cancel{L(t, f, f')} + \frac{\delta L}{\delta f}\delta f \ + \frac{\delta L}{\delta f'}\delta f' \ - \cancel{L(t, f, f')}\right] + \mathcal{O}(\delta^2) \\ &= \int_{t_0}^{t_1} \mathrm{d}t \ \left[\frac{\partial L}{\partial f}\delta f \ + \frac{\partial L}{\partial f'}\delta f'\right] + \mathcal{O}(\delta^2) \\ &= \int_{t_0}^{t_1} \mathrm{d}t \ \left[\frac{\partial L}{\partial f}\delta f \ + \frac{\partial L}{\partial f'}\frac{\mathrm{d}\delta f}{\mathrm{d}t}\right] + \mathcal{O}(\delta^2) \\ &= \int_{t_0}^{t_1} \mathrm{d}t \ \left[\frac{\partial L}{\partial f}\delta f \ + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial f'}\delta f\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial f'}\right)\delta f\right] + \mathcal{O}(\delta^2) \\ &= \cancel{\left[\frac{\partial L}{\partial f'}\delta f\right]_{t_0}^{t_1}} + \int_{t_0}^{t_1} \mathrm{d}t \ \left[\frac{\partial L}{\partial f}\delta f \ - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial f'}\right)\delta f\right] + \mathcal{O}(\delta^2) \\ &= \int_{t_0}^{t_1} \mathrm{d}t \ \left[\frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial f'}\right)\right]\delta f \ + \mathcal{O}(\delta^2) \\ &= \int_{t_0}^{t_1} \mathrm{d}t \ \frac{\delta S}{\delta f}\delta f \ + \mathcal{O}(\delta^2)\end{aligned} \tag{1.8.23}$$

[83] Pseudorigorous because it is in essence a shorthand for the rigorous way, and I have never actually proven that functional derivatives have a sensible definition.

Leading again to

$$\frac{\delta S}{\delta f} = \frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial f'}\right) \tag{1.8.24}$$

Then we can find the extremal function by setting this to zero, yielding

$$\frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial f'}\right) = 0 \tag{1.8.25}$$

This can also be thought of as saying that the coefficient of δf must be zero so that δS is zero.

Because the above process is somewhat laborious, people often write

$$\delta S = \delta \int_{t_0}^{t_1} \mathrm{d}t\ L(t, f, f') = \int_{t_0}^{t_1} \mathrm{d}t\ \left[\cancel{\frac{\delta L}{\delta t}}\delta t + \frac{\delta L}{\delta f}\delta f + \frac{\delta L}{\delta f'}\delta f'\right] \tag{1.8.26}$$

and then proceed, but you should realize that this is a shorthand that works because of the laborious procedure outlined above. It is correct, and as we will see, the generalization is correct because we are essentially just writing out a Taylor series, but the above is not the definition. Also, remember that $\delta t = 0$ because we don't allow variation in t for this process.

Now, we can extend to higher variations. This can involve subtleties, but in general it is a straightforward procedure. In this case you go to $\mathcal{O}(\epsilon^3)$ or $\mathcal{O}(\delta^3)$ and only consider the terms proportional to ϵ^2 or $(\delta f)^2$ using that

$$\delta^2 S[f, \epsilon\phi] = \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\left[S[f + \epsilon\phi]\right]_{\epsilon=0} \tag{1.8.27}$$

$$\delta^2 S = \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\left[S[f + \epsilon\phi]\right]_{\epsilon=0,\phi\to\delta f} \tag{1.8.28}$$

We can use a shorthand form

$$\delta^2 S = S[f + \delta f] - S[f] - \delta S + \mathcal{O}(\delta^3) \tag{1.8.29}$$

Again, the $\mathcal{O}(\delta^3)$ means that $\delta^2 S$ contains only contributions where two δ's (or less, though these are eliminated by δS) appear in each term. One of the subtleties possible in higher derivatives is whether you consider the same direction when doing the variation. For our functional derivatives, it does not make much sense to use different directions, but one can consider this as a variation of the Gâteaux derivatives. That is one could consider the first variation with respect to $\epsilon\phi$ but the second variation with respect to $\kappa\psi$ and have ϵ and κ go to zero. In any case, I will adopt as a definition of the second variation (and higher) the above definitions.

For the example we looked at previously, we find the second variation to be given by

$$\begin{aligned}\delta^2 S &= \int_{t_0}^{t_1} \mathrm{d}t\ \left[L(t, f + \delta f, f' + \delta f') - L(t, f, f') - \delta S\right] \\ &= \int_{t_0}^{t_1} \mathrm{d}t\ \left[\frac{\partial^2 L}{\partial f^2}(\delta f)^2 + 2\frac{\partial^2 L}{\partial f \partial f'}\delta f\,\delta f' + \frac{\partial^2 L}{\partial \delta f' \partial \delta f'}(\delta f')^2\right]\end{aligned} \tag{1.8.30}$$

This simply is the second variation. Note how there is nothing else that can be said and we don't have a simple form $\frac{\delta^2 S}{\delta f^2}(\delta f)^2$ in general. [Let $\partial^2 L/\partial\delta f'\partial\delta f' \equiv \partial^2 L/\partial\delta f'^2$]

If we have a $\delta S = 0$ then we can make some simplifications. Note $2\delta f\,(\delta f\,') = \frac{\mathrm{d}(\delta f)^2}{\mathrm{d}t}$ and we see

$$\delta^2 S = \int_{t_0}^{t_1} \mathrm{d}t \left[\frac{\partial^2 L}{\partial f^2}(\delta f)^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^2 L}{\partial f \partial f'}(\delta f)^2\right] - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^2 L}{\partial f \partial f'}(\delta f)^2 + \frac{\partial^2 L}{\partial \delta f'^2}(\delta f')^2\right] \tag{1.8.31}$$

$$\delta^2 S = \int_{t_0}^{t_1} \mathrm{d}t \left[\left(\frac{\partial^2 L}{\partial f^2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^2 L}{\partial f \partial f'}\right)(\delta f)^2 + \frac{\partial^2 L}{\partial \delta f'^2}(\delta f')^2\right] \tag{1.8.32}$$

$$\delta^2 S = \int_{t_0}^{t_1} \mathrm{d}t \left[\frac{\partial}{\partial f}\cancel{\left(\frac{\partial L}{\partial f} - \frac{\partial}{\partial f}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial f'}\right)}(\delta f)^2 + \frac{\partial^2 L}{\partial \delta f'^2}(\delta f')^2\right] \tag{1.8.33}$$

$$\delta^2 S = \int_{t_0}^{t_1} \mathrm{d}t \left[\frac{\partial^2 L}{\partial \delta f'^2}(\delta f')^2\right] \tag{1.8.34}$$

This implies that $\frac{\partial^2 L}{\partial \delta f'^2}$ determines the sign of $\delta^2 S$ at $\delta S = 0$. This is important because if $\delta^2 S > 0$ then we have a minimum and if $\delta^2 S < 0$ then we have a maximum, just as for the second derivative test. In our case, we see that this test simplifies to $\frac{\partial^2 L}{\partial \delta f'^2} \gtrless 0 \Rightarrow \delta^2 S \gtrless 0$ and $\delta^2 S = 0$ is inconclusive. Note that $\frac{\partial^2 L}{\partial \delta f'^2}$ may not be wholly positive or negative, in which case we simply have to calculate the integral.

Note that in general we can define nth variations, but they are hardly every used in practice. In addition, for a function $S[f, g]$, we define $\frac{\delta S}{\delta g}$ similar to a partial derivative: we vary g but not f. So given $S[f, g]$, and $T[f]$ we define

$$\delta S\,[f, g; 0, \epsilon\phi] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(S[f, g + \epsilon\phi]\right)_{\epsilon=0} \tag{1.8.35}$$

$$\delta S\,[f, g; \epsilon\phi, 0] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(S[f + \epsilon\phi, g]\right)_{\epsilon=0} \tag{1.8.36}$$

$$\delta S\,[f, g; 0, \delta g\,] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(S[f, g + \epsilon\phi]\right)_{\epsilon=0,\phi\to\delta g} \tag{1.8.37}$$

$$\delta S\,[f, g; \delta f\,, 0] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(S[f + \epsilon\phi, g]\right)_{\epsilon=0,\phi\to\delta f} \tag{1.8.38}$$

$$\delta^n T\,[f; \epsilon\phi] = \frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\left(T[f + \epsilon\phi]\right)_{\epsilon=0} \tag{1.8.39}$$

$$\delta^n T\,[f; \delta f\,] = \frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\left(T[f + \epsilon\phi]\right)_{\epsilon=0,\phi\to\delta f} \tag{1.8.40}$$

To finish the thought on second (and higher order) variations, we could also consider

$$\delta^2 T\,[f; \epsilon\phi, \tau\psi] = \frac{\partial^2}{\partial\tau\partial\epsilon}\left(T[f + \epsilon\phi + \tau\psi]\right)_{\tau=\epsilon=0} \tag{1.8.41}$$

Generally, the second variation is only interesting when $\epsilon\phi$ and $\tau\psi$ are the same and so we get the second variation formula from above with 2ϵ instead of ϵ, which can be absorbed into the arbitrary ϕ.

Last, mathematically we have the "usual" properties of derivatives. For $S[f]$ and $T[f]$ we find

$$\delta(T + S) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(T[f + \epsilon\phi] + S[f + \epsilon\phi]\right)_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(T[f + \epsilon\phi]\right)_{\epsilon=0} + \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(S[f + \epsilon\phi]\right)_{\epsilon=0} = \delta T + \delta S \tag{1.8.42}$$

$$\delta(TS) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(T[f + \epsilon\phi]S[f + \epsilon\phi]\right)_{\epsilon=0} = T[f]\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(S[f + \epsilon\phi]\right)_{\epsilon=0} + \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(T[f + \epsilon\phi]\right)_{\epsilon=0} S[f] = T\delta S + \delta S\,T \tag{1.8.43}$$

There is a generalized chain rule for Gâteaux derivatives, but for us functionals cannot take another functional as an argument since a functional outputs a scalar and takes in a function.

Similarly, there is nothing that prevents us from considering a functional involving functions of multiple dimensions, such as an L with $L(\mathtt{x}, f(\mathtt{x}), \frac{\partial f}{\partial \mathtt{x}}, \ldots)$ so long as our functional still gives us a real number as an output. Thus

$$S[f] = \int_{V_0} \mathrm{d}^n x \; L(\mathtt{x}, f(\mathtt{x}), \frac{\partial f}{\partial \mathtt{x}}, \ldots) \tag{1.8.44}$$

is perfectly fine (assume that L includes any Jacobian factor). Here the volume V_0 is fixed, and we require that on the boundary of V_0, given by $\partial V_0 = S_0$, that our function f is not allowed to vary. For example, choose $L = \frac{\partial f}{\partial \mathtt{x}} \cdot \frac{\partial f}{\partial \mathtt{x}}$, a fairly common example in reality, and consider a boundary value problem where we want to find the set of functions that can solve our problem for certain given conditions on the boundary (consider Dirichlet conditions where we force f to be something on the boundary; clearly we can also consider any set, though we must incorporate them into δf and $\frac{\partial \delta f}{\partial \mathtt{x}}$, etc.). For simplicity, we will call call $\mathtt{p} = \frac{\partial f}{\partial \mathtt{x}}$ and $\delta \mathtt{p} = \frac{\partial \delta f}{\partial \mathtt{x}}$ so $\delta \mathtt{p}$ is $\mathcal{O}(\delta f\,)$. Then the variation of L is given by

$$L(f + \delta f\,, \mathtt{p} + \delta \mathtt{p}\,) - L(f, \mathtt{p}) = \frac{\partial L}{\partial \mathtt{p}} \cdot \delta \mathtt{p} \; + \mathcal{O}([\delta f\,]^2) \tag{1.8.45}$$

and so

$$\delta S \; = \int_{V_0} \mathrm{d}^3 x \; \frac{\partial L}{\partial \mathtt{p}} \cdot \delta \mathtt{p} \tag{1.8.46}$$

With Dirichlet boundary conditions we know that if we can get δf into surface terms, it vanishes (or is a constant, at the least, but we will consider the vanishing case). Thus we write the above as (and then use the divergence theorem)

$$\begin{aligned} \delta S \; &= \int_{V_0} \mathrm{d}^3 x \; \frac{\partial L}{\partial \mathtt{p}} \cdot \frac{\partial}{\partial \mathtt{x}} \delta f \; = \int_{V_0} \mathrm{d}^3 x \; \left[\frac{\partial}{\partial \mathtt{x}} \cdot \left(\frac{\partial L}{\partial \mathtt{p}} \delta f \right) - \frac{\partial}{\partial \mathtt{x}} \cdot \left(\frac{\partial L}{\partial \mathtt{p}} \right) \delta f \right] \\ &= \cancel{\int_{S_0} \mathrm{d}^2 x \; \hat{\mathbf{n}} \cdot \frac{\partial L}{\partial \mathtt{p}} \delta f} - \int_{V_0} \mathrm{d}^3 x \; \frac{\partial}{\partial \mathtt{x}} \cdot \left(\frac{\partial L}{\partial \mathtt{p}} \right) \delta f \\ &= - \int_{V_0} \mathrm{d}^3 x \; \frac{\partial}{\partial \mathtt{x}} \cdot \left(\frac{\partial L}{\partial \mathtt{p}} \right) \delta f \end{aligned} \tag{1.8.47}$$

Thus, in this case, we require $\frac{\partial}{\partial \mathtt{x}} \cdot \frac{\partial L}{\partial \mathtt{p}} = 0$ since δf can be arbitary. If this feels like a strange requirement we can consider a more concrete interpretation. If we took the case that f is an electricstatic potential with $\mathtt{x} = \mathbf{x}$ the physical position in space, then $\nabla f = \mathtt{p} = -\mathbf{E}$ is the negative of the electric field (in a purely electrostatic situation). Thus we are saying that we need with $L = \nabla f \cdot \nabla f = \mathtt{p} \cdot \mathtt{p}$ that

$$\frac{\partial L}{\partial \mathtt{p}} = \frac{\partial \mathtt{p} \cdot \mathtt{p}}{\partial \mathtt{p}} = \overbrace{\frac{\partial \mathtt{p}}{\partial \mathtt{p}}}^{=\mathbb{1}} \cdot \mathtt{p} + \mathtt{p} \cdot \frac{\partial \mathtt{p}}{\partial \mathtt{p}} = 2\mathtt{p} \tag{1.8.48}$$

$$\frac{\partial}{\partial \mathtt{x}} \cdot \frac{\partial L}{\partial \mathtt{p}} = 2 \frac{\partial}{\partial \mathtt{x}} \cdot \mathtt{p} = 2 \frac{\partial}{\partial \mathtt{x}} \cdot \frac{\partial f}{\partial \mathtt{x}} = -2 \boldsymbol{\nabla} \cdot \mathbf{E} \tag{1.8.49}$$

So the divergence of **E** being zero (i.e., no charge in the region) will minimize the above integral over a given volume given some Dirichlet boundary conditions. Since the above is related to the electrostatic potential energy, we see that this gives us a lowest potential energy state. That is, Laplace's equation fo rthe potential f, given by $\nabla^2 f = 0$ is what we require to minimize f in the physical region.

There is also another use of variational principles, especially with computations. In this case we may be interested in finding a good enough approximation for some problem. Usually we then say let's restrict the permissible functions to something simple I can deal with, such as a sum of basic analytic fucntions. Because we are using simpler functions, this means that we won't get perfect accuracy, but we may ask what set of functions provides only a small amount of error. Here the error is a functional. You may ask, how do we know the error? For if we had the solution, we could simply use that. If we have the function we are trying to find f being approximated by something like $\psi = \sum_i a_i \psi_i$ then the error

$$E[\psi_i] = \int_{x_0}^{x_1} \mathrm{d}x \ [f - a_i \psi_i] \tag{1.8.50}$$

is not very useful. But remember, f must be satisfying some equation. Suppose it is a differential equation. Say

$$\frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} + p^2 f = 0 \tag{1.8.51}$$

subject to $f = 0$ on the boundary of the volume V_0. Then clearly the error in ψ can be written as

$$E[\psi] = \int_{V_0} \mathrm{d}^3 x \ \left(\frac{\partial \psi}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} + p^2 \psi \right) \tag{1.8.52}$$

We would like the extremal position for ψ (we can then check if it makes the error small, or if it is the largest possible error), and so we can use the calculus of variations. Clearly, if ψ can represent f perfectly, then we will simply find $\psi = f$ as our solution. If ψ cannot represent f then we will get the best approximation available. In reality, most people recast the problem immediately by having $E(\psi) \to E(a_i)$ because we choose ψ_i linearly independent of each other. Then it is simply a usual minimization problem because $E(a_i)$ is a function.

1.8.2 Variation with Constraints

> *There is also the satisfaction of "making friends with the function", to use Tai-Tsun Wu's poetical phrase: brute force numerical computation is often no better at giving insight into the physics of the structure of the solution than television is at giving insight into real human behavior.*
>
> — J. P. Boyd[5, p. 542]

In general, for a $L(t, \mathtt{f}, \mathtt{f}', \ldots)$, we may have some condition $G(t, \mathtt{f}) = 0$. This G may be an integral relation as above, or it may be something we want true of the function at every part in the domain. The important thing is it can be written as an equation equalling zero and involves only $\mathtt{f}$.[84] For then we can enforce it by adding the constraint into our variational problem. This means we must

[84] Such constraints are called holonomic from Greek for "whole law". A holonomic constraint corresponds to an

impose that the constraints disappear with a displacement $\delta \mathtt{f}$ over all of the variations, and not just over our final path. The failure of this to be true for non-holonomic constraints is the essential reason why we cannot do this for $G(t, \mathtt{f}, \mathtt{f}', \ldots) = 0$. Remember

$$S = \int_{t_0}^{t_1} \mathrm{d}t \; L \tag{1.8.53}$$

and we find

$$\delta G \; = \frac{\partial G}{\partial t}\delta t \; + \frac{\partial G}{\partial \mathtt{f}} \cdot \delta \mathtt{f} \tag{1.8.54}$$

Thus, when we write

$$\delta S \; = 0 \tag{1.8.55}$$

we add in the variation of $\delta \lambda G$ (with $\lambda = \lambda(t)$ a new coordinate parameter) and consider $\widetilde{S} = S + \lambda G$ via

$$\delta \widetilde{S} \; = \delta S \; + \delta(\, \lambda G) = \int_{t_0}^{t_1} \mathrm{d}t \;\; [\delta L \; + \lambda \delta G \; + G \delta \lambda \,] \tag{1.8.56}$$

$$\delta \widetilde{S} \; = \int_{t_0}^{t_1} \mathrm{d}t \;\; \left[\left(\frac{\partial L}{\partial \mathtt{f}} + \frac{\partial G}{\partial \mathtt{f}} \right) \cdot \delta \mathtt{f} \; + \sum_{j>0} \left\{ \frac{\partial L}{\partial \mathtt{f}^{(j)}} \right\} \cdot \delta \mathtt{f}^{(j)} \; + G \delta \lambda \right] \tag{1.8.57}$$

It is worth considering multiple constraints $G_i(t, \mathtt{f}) = 0$ for some number of i less than the degrees of freedom.[85]

$$\delta \widetilde{S} \; = \int_{t_0}^{t_1} \mathrm{d}t \;\; \left[\left(\frac{\partial L}{\partial \mathtt{f}} + \sum_i \lambda_i \frac{\partial G_i}{\partial \mathtt{f}} \right) \delta \mathtt{f} \; + \sum_{j>0} \left\{ \frac{\partial L}{\partial \mathtt{f}^{(j)}} \right\} \widetilde{\mathtt{f}}^{(j)} + \sum_{i=1} G_i \delta \lambda_{\,i} \right] \tag{1.8.58}$$

This gives us (assume $\delta \mathtt{f}^{\,(j)} = 0$ at the endpoints) [86]

$$\frac{\partial L}{\partial \mathtt{f}} + \sum_i \lambda_i \frac{\partial G_i}{\partial \mathtt{f}} + \sum_{j>0} \left[(-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \frac{\partial L}{\partial \mathtt{f}^{(j)}} \right] = 0 \tag{1.8.59}$$

If we solved the original S without the λ_i we could clearly use some new set of functions that incorporate the G_i constraints. That is when we have $G_i(\mathtt{f}, t) = 0$ this means some of the $\mathtt{f}$ are not independent of each other. We form the $\widetilde{\mathtt{f}}$ from the $\mathtt{f}$ that are independent of each other. Then we must have a relationship $\mathtt{F}(\widetilde{\mathtt{f}}, t) = \mathtt{f}$ satisfied by the $\widetilde{\mathtt{f}}$, and so $G_i(t, \mathtt{f}) = G_i(t, \mathtt{F}) = 0$ is

integrable system. Thus a non-holonomic constraint depends on the path in configuration space. Other than a special case of semiholonomic constraints (G depends only linearly on the $\mathtt{f}'$ in such a way that it actually is a holonomic constraint in disguise), the Lagrange multiplier method for constraints will not work. See Flynn[12] and Flygare[13] for approaches to non-holonomic constraints using D'Alembert's principle along with another way of showing the Lagrange multiplier method fails for non-holonomic constraints in general.

[85] So that this is not an overconstrained problem. This just says the number of G_i is less than the number of elements in $\mathtt{f}$.

[86] Note that $(-1)^j$ are necessary because if the number of integration by parts is j, then we introduce j (-1)'s for that term.

satisfied. Clearly if we use this new set $\widetilde{\mathtt{f}}$ we must have, through the Euler-Lagrange equations, that

$$\frac{\partial L}{\partial \widetilde{\mathtt{f}}} + \sum_j (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \frac{\partial L}{\partial \widetilde{\mathtt{f}}^{(j)}} = 0 \tag{1.8.60}$$

Thus, if we wish to switch between $\widetilde{\mathtt{f}}$ and $\mathtt{f}$ we can easily see

$$\frac{\partial L}{\partial \widetilde{\mathtt{f}}} = \frac{\partial L}{\partial \mathtt{f}} \cdot \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \tag{1.8.61}$$

For the next order we see

$$\frac{\partial L}{\partial \widetilde{\mathtt{f}}'} = \frac{\partial L}{\partial \mathtt{f}'} \cdot \frac{\partial \mathtt{f}'}{\partial \widetilde{\mathtt{f}}'} \tag{1.8.62}$$

Now if we view $\mathtt{f} = \mathtt{F}(\widetilde{\mathtt{f}}, t)$ then we can more simply write $\mathtt{f} = \mathtt{f}(\widetilde{\mathtt{f}}, t)$ and we have

$$\frac{\partial \mathtt{f}'}{\partial \widetilde{\mathtt{f}}'} = \frac{\partial}{\partial \widetilde{\mathtt{f}}'} \frac{\mathrm{d}\mathtt{f}}{\mathrm{d}t} = \frac{\partial}{\partial \widetilde{\mathtt{f}}'} \left[\frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \widetilde{\mathtt{f}}' + \frac{\partial \mathtt{f}}{\partial t} \right] \tag{1.8.63}$$

$$= \cancel{\frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}} \partial \widetilde{\mathtt{f}}'} \widetilde{\mathtt{f}}'} + \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} + \cancel{\frac{\partial}{\partial t} \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}'}} \tag{1.8.64}$$

where the last relation uses $\mathtt{f} = \mathtt{F}(\widetilde{\mathtt{f}}, t)$ has no dependence on $\widetilde{\mathtt{f}}'$ and so $\partial \mathtt{f} / \partial \widetilde{\mathtt{f}}' = \mathbf{0}$. Thus

$$\frac{\partial \mathtt{f}'}{\partial \widetilde{\mathtt{f}}'} = \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \tag{1.8.65}$$

The next order has

$$\begin{aligned}
\frac{\partial \mathtt{f}''}{\partial \widetilde{\mathtt{f}}''} &= \frac{\partial}{\partial \widetilde{\mathtt{f}}''} \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \widetilde{\mathtt{f}}' + \frac{\partial \mathtt{f}}{\partial t} \right] \\
&= \frac{\partial}{\partial \widetilde{\mathtt{f}}''} \left[\frac{\partial \mathtt{f}'}{\partial \widetilde{\mathtt{f}}} \widetilde{\mathtt{f}}' + \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \widetilde{\mathtt{f}}'' + \frac{\partial^2 f}{\partial \widetilde{\mathtt{f}} \partial t} + \frac{\partial^2 \mathtt{f}}{\partial t^2} \right] \\
&= \frac{\partial}{\partial \widetilde{\mathtt{f}}''} \left[\left(\frac{\partial^2 \mathtt{f}}{\partial \widetilde{\mathtt{f}}^2} + \frac{\partial^2 \mathtt{f}}{\partial t \partial \widetilde{\mathtt{f}}} \right) \widetilde{\mathtt{f}}' + \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \widetilde{\mathtt{f}}'' + \frac{\partial^2 \mathtt{f}}{\partial \widetilde{\mathtt{f}} \partial t} + \frac{\partial^2 \mathtt{f}}{\partial t^2} \right] \\
&= \frac{\partial}{\partial \widetilde{\mathtt{f}}''} \left(\cancel{\frac{\partial^2 \mathtt{f}}{\partial^2 \widetilde{\mathtt{f}}} \widetilde{\mathtt{f}}' + \frac{\partial \mathtt{f}}{\partial t \partial \widetilde{\mathtt{f}}} \widetilde{\mathtt{f}}'} \right) + \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \frac{\partial \widetilde{\mathtt{f}}''}{\partial \widetilde{\mathtt{f}}''} + \frac{\partial}{\partial \widetilde{\mathtt{f}}''} \cancel{\frac{\partial^2 \mathtt{f}}{\partial t^2}} \\
&= \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}}
\end{aligned} \tag{1.8.66}$$

where we use that $\partial \mathtt{f} / \partial \widetilde{\mathtt{f}}^{(j)} = \mathbf{0}$ for all j except $j = 0$ and that $\partial \widetilde{\mathtt{f}}^{(j)} / \partial \widetilde{\mathtt{f}}^{(k)} = \mathbf{0}$ except for $j = k$. Any other order can be written as

$$\begin{aligned}
\frac{\partial \mathtt{f}^{(j)}}{\partial \widetilde{\mathtt{f}}^{(j)}} &= \frac{\partial}{\partial \widetilde{\mathtt{f}}^{(j)}} \frac{\mathrm{d}^j}{\mathrm{d}t^j} [\mathtt{f}] = \frac{\partial}{\partial \widetilde{\mathtt{f}}^{(j)}} \left[\frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \cdot \widetilde{\mathtt{f}}^{(j)} + \sum_{i=1}^{j-1} H_i(\mathtt{f}, \partial \mathtt{f}) \cdot \widetilde{\mathtt{f}}^i \right] \\
&= \cancel{\frac{\partial}{\partial \widetilde{\mathtt{f}}^{(j)}} \left(\frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \right) \widetilde{\mathtt{f}}^{(j)}} + \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \cdot \frac{\partial \widetilde{\mathtt{f}}^{(j)}}{\partial \widetilde{\mathtt{f}}^{(j)}} + \cancel{\frac{\partial}{\partial \widetilde{\mathtt{f}}^{(j)}} \sum_{i=1}^{j-1} H_i(\mathtt{f}, \partial \mathtt{f}) \cdot \widetilde{\mathtt{f}}^{(i)}}
\end{aligned} \tag{1.8.67}$$

where $H_i(\mathtt{f}, \partial\mathtt{f})$ means any number of partial derivative of $\mathtt{f}$ with respect to t or $\widetilde{\mathtt{f}}$. Clearly $\partial H_i/\partial\widetilde{\mathtt{f}}^{(j)} = 0$ for $j \geq 1$ because f has no dependence on any derivatives of $\widetilde{\mathtt{f}}$. In addition $\partial\widetilde{\mathtt{f}}^{(i)}/\partial\widetilde{\mathtt{f}}^{(j)} = 0$ for $i < j$ because they are assumed to be independent as we discussed above. Thus we get the given cancellations and we find in general

$$\frac{\partial \mathtt{f}^{(j)}}{\partial\widetilde{\mathtt{f}}^{(j)}} = \frac{\partial \mathtt{f}}{\partial\widetilde{\mathtt{f}}} \tag{1.8.68}$$

When using Newton dot notation, this is usually called canceling the dots because we have

$$\frac{\partial \mathtt{q}}{\partial\widetilde{\mathtt{q}}} = \frac{\partial \dot{\mathtt{q}}}{\partial\dot{\widetilde{\mathtt{q}}}} = \frac{\partial \ddot{\mathtt{q}}}{\partial\ddot{\widetilde{\mathtt{q}}}} = \cdots \tag{1.8.69}$$

and it looks like you just cancel the dots from a fraction. This is another abuse of notation, but it is a convenient one. Note that if $\mathtt{f}$ depended on the $\widetilde{\mathtt{f}}^{(j)}$ cancelling the dots would not work! The fact that we can cancel the dots is the key reason that we can use Lagrange multipliers, as we will see. Armed with this, we can continue on our path of justifying Lagrange multipliers. Remember we have

$$\frac{\partial L}{\partial\widetilde{\mathtt{f}}} + \sum_j (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial\widetilde{\mathtt{f}}^{(j)}} = 0 \tag{1.8.70}$$

and so

$$\frac{\partial L}{\partial\widetilde{\mathtt{f}}^{(j)}} = \frac{\partial \mathtt{f}^{(j)}}{\partial\widetilde{\mathtt{f}}^{(j)}} \cdot \frac{\partial L}{\partial \mathtt{f}^{(j)}} = \frac{\partial \mathtt{f}}{\partial\widetilde{\mathtt{f}}} \cdot \frac{\partial L}{\partial \mathtt{f}^{(j)}} \tag{1.8.71}$$

Thus we can write

$$\frac{\partial \mathtt{f}}{\partial\widetilde{\mathtt{f}}} \cdot \left[\frac{\partial L}{\partial \mathtt{f}} + \sum_j (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial \mathtt{f}}\right] = 0 \tag{1.8.72}$$

Thus we see $\frac{\partial \mathtt{f}}{\partial\widetilde{\mathtt{f}}}$ is perpendicular to the Euler-Lagrange equations without the constraint. This means the projection of the original Euler-Lagrange equations along the $G_i = 0$ defining surface are zero. In other words, when we insert the $\mathtt{f}$ and also impose $G_i = 0$, we find that the Euler-Lagrange equations must be zero when dotted by $\frac{\partial \mathtt{f}}{\partial\widetilde{\mathtt{f}}}$.

We remember that our $\delta\widetilde{S}$ automatically enforces $G_i = 0$ and has extra force terms

$$\sum_i \lambda_i \frac{\partial G_i}{\partial \mathtt{f}} \tag{1.8.73}$$

Note that for each G_i we have (from $G_i = 0$)

$$\mathbf{0} = \frac{\partial G_i}{\partial\widetilde{\mathtt{f}}} = \frac{\partial \mathtt{f}}{\partial\widetilde{\mathtt{f}}} \cdot \frac{\partial G_i}{\partial \mathtt{f}} \tag{1.8.74}$$

Thus the projection of $\partial G_i/\partial\mathtt{f}$ by dotting $\frac{\partial \mathtt{f}}{\partial\widetilde{\mathtt{f}}}$ into it is by our very construction equal to zero.

Our Lagrange multiplier equation would state if we assumed all $\delta\mathtt{f}$ independent that we would have

$$\frac{\partial L}{\partial \mathtt{f}} + \sum_{j>0} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial \mathtt{f}} = -\sum_i \lambda_i \frac{\partial G_i}{\partial \mathtt{f}} \tag{1.8.75}$$

So we see that with $\delta\widetilde{S}$ we get a solution of the form

$$\mathbf{0} = \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \cdot \left[-\sum_i \lambda_i \frac{\partial G_i}{\partial \mathtt{f}} \right] = \frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \cdot \left[\frac{\partial L}{\partial \mathtt{f}} + \sum_{j>0} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \frac{\partial L}{\partial \mathtt{f}} \right] \tag{1.8.76}$$

This means by treating the $\delta\mathtt{f}$ independent with the Lagrange multipliers, we actually get back that

$$\frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}} \cdot \left[\frac{\partial L}{\partial \mathtt{f}} + \sum_{j} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \frac{\partial L}{\partial \mathtt{f}} \right] = \mathbf{0} \tag{1.8.77}$$

along with $G_i = 0$ enforced by the $\delta\lambda_i$ terms vanishing. Thus we have the same set of equations as we did when we solved it the "proper way" without extra degrees of freedom, where we saw the solution had to be zero when dotted by $\frac{\partial \mathtt{f}}{\partial \widetilde{\mathtt{f}}}$. Then this solution with Lagrange multipliers is the same as the method using the $\widetilde{\mathtt{f}}$ that has the constraints intrinsically, and so is an equivalent solution method. The extra terms are the constraint forces. We can view this in geometric language. We made sure that our solution is along the surface created by the constraints $G_i = 0$, and saw that the solution is the point where the contour surfaces of G_i and L coincide. That is where the gradients of G_i and L point parallel or antiparallel.

Thus, when we find the minimal L as if the $\mathtt{f}$ has no constraints on it, we get the same form of solution as when we solve with $\widetilde{\mathtt{f}}$ with the proper constraints. The λ_i are there to allow us the freedom necessary to put the constraints on G_i and get the $\mathtt{f}$ as if they had no constraints. Just remember that Lagrange multipliers allow us to solve a new problem that will automatically satisfy the constraints we required of the original problem. The geometric idea is that a point that satisfies the $\frac{\partial L}{\partial \mathtt{f}} - \lambda \frac{\partial G}{\partial \mathtt{f}} = 0$ will have $\frac{\partial L}{\partial \mathtt{f}}$ and $\frac{\partial G}{\partial \mathtt{f}}$ parallel. Consider a given function and a constraint (the black line) in Figure 1.9. That is if we consider the contours of constant L and the contours of constant G, then a maximum or minimum value satisfying the constraint must have the gradients of L and G pointing in the same direction. That is, the perpendicular to the contours of L and G at some point must be parallel or antiparallel. Otherwise we could move along the G contour to some other point and have L get greater or smaller (because the gradient of L is then along G), contradicting our original assumption that we were at a minimum or maximum.

Suppose we want to enforce a constraint on our problem with ψ above. For example, if we want to ensure we find a minimization, we might desire

$$\int_{V_0} \mathrm{d}^3x \; r(\mathbf{x})\psi^2 = 1 \tag{1.8.78}$$

for some weighting function $r(\mathbf{x})$ so that ψ can't get arbitrarily large.

$$\begin{aligned} \delta E \; &= \delta E \; + \delta G \; = \int_{V_0} \mathrm{d}^3x \; \left[2\frac{\partial \delta\psi}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} + 2p^2\psi\delta\psi \; + 2\lambda r\psi\delta\psi \right] \\ &= 2\cancel{\left[\frac{\partial}{\partial \mathbf{x}} \cdot \left(\frac{\partial \psi}{\partial \mathbf{x}} \delta\psi \right) \right]_{\partial V_0}} + \int_{V_0} \mathrm{d}^3x \; \left[-2\frac{\partial \psi}{\partial \mathbf{x}} \delta\psi \; + 2(p^2 + \lambda r)\psi\delta\psi \right] \\ &= 2\int_{V_0} \mathrm{d}^3x \; \left[-\frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} + (p^2 + \lambda r)\psi \right] \delta\psi \end{aligned} \tag{1.8.79}$$

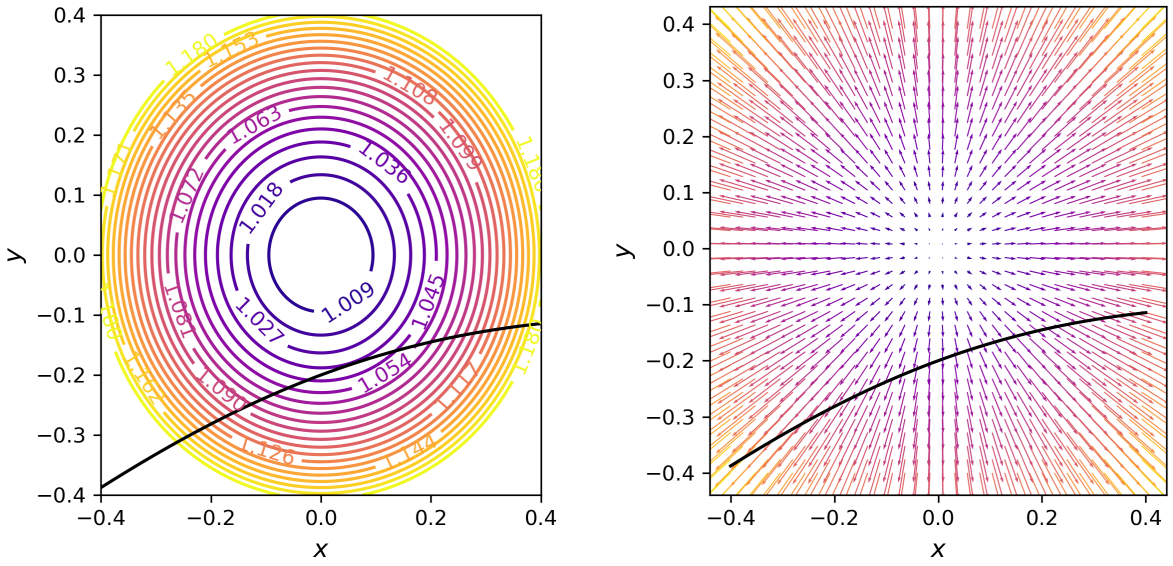

Figure 1.9: We can see how the maximum value of the contours is where the constraint line (black) is exactly tangential to the contours. This means that the gradient of the function points exactly perpendicular to the constraint line as seen above.

and so the ψ we require must satisfy

$$\frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} = (p^2 + \lambda r)\psi \tag{1.8.80}$$

with λ a constant. You might think this is not much better than our original problem, but, in fact, this is fairly incredible. We reduced minimization over an integral with a constraint down to a partial differential equation. In addition, if we multiply by ψ and integrate we find that our λ is given by

$$\int_{V_0} \mathrm{d}^3x \; \frac{\partial}{\partial \mathbf{x}} \cdot \psi \frac{\partial \psi}{\partial \mathbf{x}} = \cancel{\int_{V_0} \frac{\partial}{\partial \mathbf{x}} \cdot \left(\frac{\partial \psi}{\partial \mathbf{x}} \psi \right)} - \int_{V_0} \mathrm{d}^3x \; \frac{\partial \psi}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} \tag{1.8.81}$$

$$\int_{V_0} \mathrm{d}^3x \; (p^2 + \lambda r)\psi^2 = \lambda + \int_{V_0} \mathrm{d}^3x \; p^2\psi^2 \tag{1.8.82}$$

$$Q[\psi] \equiv \int_{V_0} \mathrm{d}^3x \; \left[\frac{\partial \psi}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} + p^2\psi^2 \right] \tag{1.8.83}$$

$$\lambda = -Q[\psi] \tag{1.8.84}$$

In fact, this problem is usually presented as minimize ψ for functional[87]

$$S[\psi] \equiv \frac{Q[\psi]}{R[\psi]} \tag{1.8.85}$$

$$Q[\psi] \equiv \int_{V_0} \mathrm{d}^3x \; \left[\frac{\partial \psi}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} + p^2\psi^2 \right] \tag{1.8.86}$$

$$R[\psi] \equiv \int_{V_0} \mathrm{d}^3x \; r(\mathbf{x})\psi^2 \tag{1.8.87}$$

[87] This method uses the Rayleigh quotient.

since $\delta Q = \delta E$ from before without the Lagrange multiplier constraint, we realize

$$\delta S[\psi] = -\frac{Q[\psi]}{R[\psi]^2}\delta R + \frac{\delta Q}{R[\psi]} \tag{1.8.88}$$

We can multiply through by $R[\psi]$ since $\delta S = 0$ will still be zero in this case

$$0 = R[\psi]\delta S = -S[\psi]\delta R + \delta Q \tag{1.8.89}$$

$$0 = -S[\psi]\int_{V_0} \mathrm{d}^3x\ 2r\psi\delta\psi + 2\int_{V_0}\mathrm{d}^3x\ \left[-\frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial\psi}{\partial\mathbf{x}} + p^2\psi\right]\delta\psi \tag{1.8.90}$$

$$0 = -S[\psi]r\psi - \frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial\psi}{\partial\mathbf{x}} + p^2\psi \tag{1.8.91}$$

$$\frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial\psi}{\partial\mathbf{x}} = \left(p^2 - S[\psi]r\right)\psi \tag{1.8.92}$$

which we recognize as the same problem as we found with the original method with a constraint in (1.8.80), but now with $\lambda = -S[\psi] = -Q[\psi]/R[\psi]$. Back then we had the restriction imposed that $R[\psi] = 1$, and so we see we do get the same solution.

1.9 Hamiltonians and Hamilton-Jacobi Equations

> *In almost all textbooks, even the best, this principle is presented so that it is impossible to understand (K. Jacobi Lectures on Dynamics, 1842-1843). I do not choose to break with tradition.*
>
> — V. I. Arnol'd, on the principle of least action[31, p. 246]

This will be a whirlwind tour of classical mechanics. First, we start with the Lagrangian. I will denote it $\mathcal{L}$.[88] It can be motivated by considering virtual forces, but I will simply posit that it is given by the kinetic energy minus the potential energy as $\mathcal{L} = T - V$. We will form a set of generalized coordinates $\mathtt{x}_i$ with i indexing each particle and the set of velocities associated with them $\mathtt{v}_i = \frac{\mathrm{d}\mathtt{x}_i}{\mathrm{d}t} = \dot{\mathtt{x}}_i$ where time is t. We have for Cartesian position $\mathbf{X}_i = \mathbf{X}(\mathtt{x}_i|_{\text{all } i})$ and velocity $\mathbf{V}_i = \dot{\mathbf{X}}_i = \dot{\mathbf{X}}_i(\dot{\mathtt{x}}_i|_{\text{all } i})$ so that

$$T(\dot{\mathtt{x}}_1, \ldots, \dot{\mathtt{x}}_n, t) \equiv \frac{1}{2}\sum_{i=1}^{n} m_i\mathbf{V}_i\cdot\mathbf{V}_i \tag{1.9.1}$$

where we use the expressions of $\mathbf{V}_i$ in terms of $\mathtt{v}_i$ and t so that $\mathbf{V}_i\cdot\mathbf{V}_i$ is fully in terms of $\mathtt{v}_i$ and t. Then we use a potential V given by

$$V = V(\mathtt{x}_1, \ldots, \mathtt{x}_n, \dot{\mathtt{x}}_1, \ldots, \dot{\mathtt{x}}_n, t) \tag{1.9.2}$$

For conservative potentials then $V = V(\mathtt{x}_1, \ldots, \mathtt{x}_n, t)$. I will then write $\mathtt{x}$ and $\mathtt{v} = \dot{\mathtt{x}}$ to indicate all of the $\mathtt{x}_i$ and $\mathtt{v}_i$, respectively. To elaborate the calculation, we have that $\mathbf{X}_i = \mathbf{X}_i(\mathtt{x}, t)$ so that

[88] Sometimes L is used for a Lagrangian and $\mathcal{L}$ is used for a Lagrangian density in some fields of physics. The Lagrangian density satisfies $L = \int \mathrm{d}^n q\ \mathcal{L}$ where n is the number of generalized positions, so an integral over the volume of generalized positions. Similarly, some write the Hamiltonian H and Hamiltonian density $\mathcal{H}$, but I will just use $\mathcal{H}$ for the Hamiltonian in this section.

each $\mathbf{X}_i$ depends on all of the generalized coordinates $\mathtt{x}$. This means through the chain rule that

$$\frac{\mathrm{d}\mathbf{X}_i}{\mathrm{d}t} = \sum_{j=1}^{n} \frac{\partial \mathbf{X}_i}{\partial \mathtt{x}_j}\dot{\mathtt{x}}_j + \frac{\partial \mathbf{X}_i}{\partial t} \tag{1.9.3}$$

which allows us to write the expression for T quite easily.

Note that if we switch to a relativistic case, you may expect we use a new kinetic energy given by

$$T = \sum_{i=1}^{n} [\gamma_i(\dot{\mathtt{x}}_i) - 1] m_{0i} c^2 \tag{1.9.4}$$

but this is wrong. In fact, the correct formulation is

$$\mathcal{L} = -\sum_{i=1}^{n} \frac{m_{0i}c^2}{\gamma_i(\dot{\mathtt{x}}_i)} - V(\mathtt{x}, \dot{\mathtt{x}}, t) \tag{1.9.5}$$

If we have holonomic constraints this means by definition that they can be expressed via

$$f_j(\mathtt{x}, t) = 0 \tag{1.9.6}$$

for each holonomic constraint j (assume there are m of them). A holonomic constraint just means that the state of the system is completely determined by its location, and is not path dependent. A non-holonomic constraint is path dependent and not treatable by standard Lagrangian or Hamiltonian mechanics. Note that if the Lagrangian $\mathcal{L}$ is explicitly time dependent, this means t appears in $\mathcal{L}$ outside of any $\mathtt{x}_i$ and $\dot{\mathtt{x}}_i$. The Lagrangian is always a function of t implicitly through the generalized positions and velocities. Then Lagrangian mechanics states that the Lagrangian satisfies a form of the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \mathtt{x}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathtt{x}}_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial \mathtt{x}_i} = 0 \tag{1.9.7}$$

where λ_j is a Lagrange multiplier giving a generalized force associated with each holonomic constraint. Note that this is true for each $\mathtt{x}_i$. Clearly each holonomic constraint removes one degree of freedom because we can use a holonomic constraint to write an $\mathtt{x}_j$ in terms of all the other $\mathtt{x}_i$ with $i \neq j$. Thus this will have $3n - m$ degrees of freedom.

If we instead use no holonomic constraints, but use only the necessary generalized coordinates, then the equation simplifies down to what is typically called the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \mathtt{x}_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathtt{x}}_i} \tag{1.9.8}$$

which is the traditionally used form.

One can then easily show that this yields Newton's laws under regular Cartesian coordinates, and so it is equivalent.

We can then form the action from[89]

$$S = \int_{t_0}^{t} \mathrm{d}t \; \mathcal{L}(\mathtt{x}(t), \dot{\mathtt{x}}(t), t) \tag{1.9.9}$$

If we use a variational principle, called Hamilton's principle, then the action should be "minimized".[90] Then using variational calculus, we write this as

$$\delta S \; = 0 = \int_{t_0}^{t} \mathrm{d}t \; \delta\mathcal{L} \tag{1.9.10}$$

where variational calculus uses the same rules as differentials, so (using the summation convention)

$$\delta\mathcal{L} \; = \frac{\delta\mathcal{L}}{\delta \mathtt{x}_i}\delta\mathtt{x}_i \; + \frac{\delta\mathcal{L}}{\delta \dot{\mathtt{x}}_i}\delta\dot{\mathtt{x}}_i \; + \frac{\delta\mathcal{L}}{\delta t}\delta t \tag{1.9.11}$$

Now, we have to fix the endpoints in time and so $\delta t \; = 0$ by definition. We must also then have $\delta\mathtt{x}_i\left(t_0\right) \; = 0 \; = \delta\mathtt{x}_i\left(t\right)$ so that there is no variation at the ends of the integral. We can then use that we can reverse the order of variation and time derivatives for well behaved functions so that $\delta\dot{\mathtt{x}}_i \; = \frac{\mathrm{d}}{\mathrm{d}t}\delta\mathtt{x}_i$. We can then write this as a total differential for $\frac{\mathrm{d}}{\mathrm{d}t}$ writing

$$\frac{\delta\mathcal{L}}{\delta \dot{\mathtt{x}}_i}\delta\dot{\mathtt{x}}_i \; = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\delta\mathcal{L}}{\delta \dot{\mathtt{x}}_i}\delta\mathtt{x}_i\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\mathcal{L}}{\delta \dot{\mathtt{x}}_i}\right)\delta\mathtt{x}_i \tag{1.9.12}$$

which then means the integral becomes

$$\begin{aligned} \delta S \; = 0 &= \int_{t_0}^{t} \mathrm{d}t \; \left[\frac{\delta\mathcal{L}}{\delta \mathtt{x}_i}\delta\mathtt{x}_i \; + \frac{\mathrm{d}}{\mathrm{d}t}\cancel{\left[\frac{\delta\mathcal{L}}{\delta \dot{\mathtt{x}}_i}\delta\mathtt{x}_i\right]} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\mathcal{L}}{\delta \dot{\mathtt{x}}_i}\right)\delta\mathtt{x}_i\right] \\ &= \int_{t_0}^{t} \mathrm{d}t \; \left\{\left[\frac{\delta\mathcal{L}}{\delta \mathtt{x}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\mathcal{L}}{\delta \dot{\mathtt{x}}_i}\right)\right]\delta\mathtt{x}_i\right\} \end{aligned} \tag{1.9.13}$$

In order for this to vanish then the coefficient of $\delta\mathtt{x}_i$ must be zero identically. But we see that this is just the Euler-Lagrange equation.

At this point, you may be asking yourself, where are the Hamiltonians and what is the Hamilton-Jacobi equation? Do not worry, I will now explain.

First, Hamiltonian mechanics is yet another formulation of classical mechanics. If we impose a Legendre transform on the Lagrangian, we will get a Hamiltonian. The Legendre transform only works on convex functions. A convex function is a function such that when you draw a straight line between any two points, the function values are below that line. For differentiable functions, this means that the second derivative is non-negative everywhere. If we fix $\mathtt{x}$ then the $\mathcal{L}(\dot{\mathtt{x}}, \mathtt{x}, t)$ is such a convex function[91] because $\mathbf{V}\cdot\mathbf{V}\;\frac{\partial}{\partial\mathbf{V}}\frac{\partial(\mathbf{V}\cdot\mathbf{V})}{\partial\mathbf{V}} = 2\mathbb{1}$. For any sensible set $\mathtt{x}$, then the $\dot{\mathtt{x}}$ will also follow a similar form so that $\frac{\partial}{\partial \mathtt{v}}\frac{\partial\mathcal{L}}{\partial \mathtt{v}} \geq \mathbf{0}$ and so the Legendre transform is well-defined.[92] The

[89]For this integral, I will use the often looked down upon notation of $\int_{t_0}^{t} \mathrm{d}t \; f(t)$ rather than introducing a dummy variable t' and writing $\int_{t_0}^{t} \mathrm{d}t' \; f(t')$ because I think it's pretty obvious that this won't cause us any confusion.

[90]This is always how it is phrased even though what is always shown is simply that we find a stationary point, i.e., a minimum, maximum, or some sort of saddle point.

[91]For multiple dimensions, one can look at the whether the shape formed "above" the function's values is convex.

[92]Note that the generalization to complex valued functions is not often used, so that one should not count on this working with complex valued functions.

Legendre transform has the Lagrangian $\mathcal{L}$ change to the Hamiltonian $\mathcal{H}$ via defining conjugate momenta $\mathtt{p}_i(\mathtt{x}_i, \dot{\mathtt{x}}_i, t) = \frac{\partial \mathcal{L}}{\partial \dot{\mathtt{x}}_i}$ so that[93]

$$\mathcal{H} = \sum_i \dot{\mathtt{x}}_i \frac{\partial \mathcal{L}}{\partial \dot{\mathtt{x}}_i} - \mathcal{L} = \sum_i \dot{\mathtt{x}}_i \mathtt{p}_i - \mathcal{L} \tag{1.9.14}$$

where one puts in $\dot{\mathtt{x}}_i(\mathtt{p})$ so that we know the $\dot{\mathtt{x}}_i$ in terms of the $\mathtt{p}_j$. We then would like to know what equations the Hamiltonian $\mathcal{H}$ satisfies (just as the Lagrangian satisfies the Euler-Lagrange equations). Typically generalized coordinates use the variable q rather than x, so I will now switch to this notation for Hamiltonian form. Thus we define $\mathtt{q}_i = \mathtt{x}_i$ and $\dot{\mathtt{q}}_i = \dot{\mathtt{x}}_i$. We write the differential of $\mathcal{L}$ to determine this via

$$\mathrm{d}\mathcal{L} = \sum_i \left[\frac{\partial \mathcal{L}}{\partial \mathtt{q}_i} \mathrm{d}\mathtt{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{\mathtt{q}}_i} \mathrm{d}\dot{\mathtt{q}}_i \right] + \frac{\partial \mathcal{L}}{\partial t} \mathrm{d}t \tag{1.9.15}$$

and we use the definition $\mathtt{p}_i = \frac{\partial \mathcal{L}}{\partial \dot{\mathtt{q}}_i}$ so this becomes

$$\begin{aligned} \mathrm{d}\mathcal{L} &= \sum_i \left[\frac{\partial \mathcal{L}}{\partial \mathtt{q}_i} \mathrm{d}\mathtt{q}_i + \mathtt{p}_i \, \mathrm{d}\dot{\mathtt{q}}_i \right] + \frac{\partial \mathcal{L}}{\partial t} \mathrm{d}t \\ &= \sum_i \left[\frac{\partial \mathcal{L}}{\partial \mathtt{q}_i} \mathrm{d}\mathtt{q}_i + \mathrm{d}\left(\mathtt{p}_i \mathtt{q}_i\right) - \dot{\mathtt{q}}_i \, \mathrm{d}\mathtt{p}_i \right] + \frac{\partial \mathcal{L}}{\partial t} \mathrm{d}t \end{aligned} \tag{1.9.16}$$

where if we take the full differentials to the left hand side we see

$$\mathrm{d}\left(\mathcal{L} - \sum_i \mathtt{p}_i \mathtt{q}_i \right) = \sum_i \left[\frac{\partial \mathcal{L}}{\partial \mathtt{q}_i} \mathrm{d}\mathtt{q}_i - \dot{\mathtt{q}}_i \, \mathrm{d}\mathtt{p}_i \right] + \frac{\partial \mathcal{L}}{\partial t} \mathrm{d}t \tag{1.9.17}$$

$$-\mathrm{d}\mathcal{H} = \sum_i \left[\frac{\partial \mathcal{L}}{\partial \mathtt{q}_i} \mathrm{d}\mathtt{q}_i - \dot{\mathtt{q}}_i \, \mathrm{d}\mathtt{p}_i \right] + \frac{\partial \mathcal{L}}{\partial t} \mathrm{d}t \tag{1.9.18}$$

$$\mathrm{d}\mathcal{H} = \sum_i \left[\dot{\mathtt{q}}_i \, \mathrm{d}\mathtt{p}_i - \frac{\partial \mathcal{L}}{\partial \mathtt{q}_i} \mathrm{d}\mathtt{q}_i \right] - \frac{\partial \mathcal{L}}{\partial t} \mathrm{d}t \tag{1.9.19}$$

And we can find the differential of $\mathcal{H}$ directly as

$$\mathrm{d}\mathcal{H} = \sum_i \left[\frac{\partial \mathcal{H}}{\partial \mathtt{q}_i} \mathrm{d}\mathtt{q}_i + \frac{\partial \mathcal{H}}{\partial \mathtt{p}_i} \mathrm{d}\mathtt{p}_i + \frac{\partial \mathcal{H}}{\partial t} \mathrm{d}t \right] \tag{1.9.20}$$

we can subtract these two expressions and since we are assuming the independence of $\mathtt{q}_i$, $\dot{\mathtt{q}}_i$ and t, their differential coefficients must vanish and we have

$$0 = \left[\sum_i \left(-\frac{\partial \mathcal{L}}{\partial \mathtt{q}_i} - \frac{\partial \mathcal{H}}{\partial \mathtt{q}_i} \right) \right] \mathrm{d}\mathtt{q}_i + \left[\sum_i \left(\dot{\mathtt{q}}_i - \frac{\partial \mathcal{H}}{\partial \dot{\mathtt{q}}_i} \right) \right] \mathrm{d}\mathtt{p}_i + \left(-\frac{\partial \mathcal{L}}{\partial t} - \frac{\partial \mathcal{H}}{\partial t} \right) \mathrm{d}t \tag{1.9.21}$$

And so

$$\frac{\partial \mathcal{H}}{\partial \mathtt{q}_i} = -\frac{\partial \mathcal{L}}{\partial \mathtt{q}_i} \tag{1.9.22}$$

$$\frac{\partial \mathcal{H}}{\partial \mathtt{p}_i} = \dot{\mathtt{q}}_i \tag{1.9.23}$$

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{1.9.24}$$

[93] We use the same notation $\mathtt{p}$ means all of the $\mathtt{p}_i$.

If we have a solution to the Euler-Lagrange equations then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathtt{q}}_i} = \frac{\partial \mathcal{L}}{\partial \mathtt{q}_i} \tag{1.9.25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathtt{p}_i = \dot{\mathtt{p}}_i = \frac{\partial \mathcal{L}}{\partial \mathtt{q}_i} \tag{1.9.26}$$

and so we have

$$\frac{\partial \mathcal{H}}{\partial \mathtt{q}_i} = -\dot{\mathtt{p}}_i \tag{1.9.27}$$

$$\frac{\partial \mathcal{H}}{\partial \mathtt{p}_i} = \dot{\mathtt{q}}_i \tag{1.9.28}$$

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{1.9.29}$$

Which are Hamilton's equations. If we have n generalized coordinates then this is a $2n+1$ system (if time is included).

Here $\mathtt{p}_i$ is the canonically conjugate momentum to the generalized coordinate $\mathtt{q}_i$. It is possible to transform both $\mathtt{q}_i$ and $\mathtt{p}_i$ such that the new coordinates $\mathtt{Q}_i$ and $\mathtt{P}_i$ are also canonically conjugate and satisfy Hamilton's equations.

There are four generating functions for this. We start by writing the Lagrangian action in two separate ways and making their variation zero. Thus we have have $(H, \mathtt{q}_i, \mathtt{p}_i)$ as one Hamiltonian system and $(K, \mathtt{Q}_i, \mathtt{P}_i)$ as the second Hamiltonian system corresponding to the same Lagrangian and physical situation.

We have

$$\delta \int_{t_0}^{t} \mathrm{d}t \ \left[\mathtt{p}_i\dot{\mathtt{q}}_i - H(\mathtt{q},\mathtt{p},t)\right] = \delta \int_{t_0}^{t} \mathrm{d}t \ \left[\mathtt{P}_i\dot{\mathtt{Q}}_i - K(\mathtt{Q},\mathtt{P},t)\right] = 0 \tag{1.9.30}$$

We can clearly add a total time derivative and could have a scale factor difference between the two integrands and still have zero. That is

$$\delta \int_{t_0}^{t} \mathrm{d}t \ \left[\lambda_{pq}\left(\mathtt{p}_i\dot{\mathtt{q}}_i - H(\mathtt{q},\mathtt{p},t)\right) + \frac{\mathrm{d}g_{pq}}{\mathrm{d}t}\right] - \delta \int_{t_0}^{t} \mathrm{d}t \ \left[\lambda_{PQ}\left(\mathtt{P}_i\dot{\mathtt{Q}}_i - K(\mathtt{Q},\mathtt{P},t)\right) + \frac{\mathrm{d}g_{PQ}}{\mathrm{d}t}\right] = 0 \tag{1.9.31}$$

$$\lambda_{pq}\left(\mathtt{p}_i\dot{\mathtt{q}}_i - H(\mathtt{q},\mathtt{p},t)\right) + \frac{\mathrm{d}g_{pq}}{\mathrm{d}t} - \left\{\lambda_{PQ}\left(\mathtt{P}_i\dot{\mathtt{Q}}_i - K(\mathtt{Q},\mathtt{P},t)\right) + \frac{\mathrm{d}g_{PQ}}{\mathrm{d}t}\right\} = 0 \tag{1.9.32}$$

which can be rewritten as

$$\lambda\left(\mathtt{p}_i\dot{\mathtt{q}}_i - H(\mathtt{q},\mathtt{p},t)\right) = \mathtt{P}_i\dot{\mathtt{Q}}_i - K(\mathtt{Q},\mathtt{P},t) + \frac{\mathrm{d}G}{\mathrm{d}t} \tag{1.9.33}$$

where

$$\lambda = \lambda_{pq}/\lambda_{PQ} \tag{1.9.34}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{1}{\lambda_{PQ}}\left(\frac{\mathrm{d}g_{PQ}}{\mathrm{d}t} - \frac{\mathrm{d}g_{pq}}{\mathrm{d}t}\right) \tag{1.9.35}$$

Generally people choose $\lambda = 1$ and put all of the transformations into $\frac{\mathrm{d}G}{\mathrm{d}t}$. We can then choose G such that it is a variable of two of the $\mathtt{q}$, $\mathtt{Q}$, $\mathtt{p}$, or $\mathtt{P}$ so that we can transform one of the two sides into the other. Thus there are four generating functions. The type 1 generating function $G = G_1(\mathtt{q}, \mathtt{Q}, t)$ so that our above transformation states

$$\mathtt{p}_i\dot{\mathtt{q}}_i - H(\mathtt{q}, \mathtt{p}, t) = \mathtt{P}_i\dot{\mathtt{Q}}_i - K(\mathtt{Q}, \mathtt{P}, t) + \frac{\partial G_1}{\partial t} + \frac{\partial G_1}{\partial \mathtt{q}_i}\dot{\mathtt{q}}_i + \frac{\partial G_1}{\partial \mathtt{Q}_i}\dot{\mathtt{Q}}_i \tag{1.9.36}$$

which we rewrite as

$$\left(\mathtt{p}_i - \frac{\partial G_1}{\partial \mathtt{q}_i}\right)\dot{\mathtt{q}}_i - H(\mathtt{q}, \mathtt{p}, t) + K(\mathtt{Q}, \mathtt{P}, t) - \frac{\partial G_1}{\partial t} = \left(\mathtt{P}_i + \frac{\partial G_1}{\partial \mathtt{Q}_i}\right)\dot{\mathtt{Q}}_i \tag{1.9.37}$$

Since all the canonical coordinates, their conjugate momenta, and the Hamiltonians are independent of each other, then each term in front of a time derivative of a conjugate momenta or coordinate or for the Hamiltonians themselves must vanish separately. Thus

$$\frac{\partial G_1}{\partial \mathtt{q}_i} = \mathtt{p}_i \tag{1.9.38}$$

$$\frac{\partial G_1}{\partial \mathtt{Q}_i} = -\mathtt{P}_i \tag{1.9.39}$$

$$K = H + \frac{\partial G_1}{\partial t} \tag{1.9.40}$$

This allows us to write H and $\frac{\partial G_1}{\partial t}$ in terms of $\mathtt{Q}$ and $\mathtt{P}$.

The type 2 generating function is $G = -\mathtt{Q}_i\mathtt{P}_i + G_2(\mathtt{q}, \mathtt{P}, t)$ and so we get

$$\mathtt{p}_i\dot{\mathtt{q}}_i - H(\mathtt{q}, \mathtt{p}, t) = \mathtt{P}_i\dot{\mathtt{Q}}_i - K(\mathtt{Q}, \mathtt{P}, t) + \frac{\partial G_2}{\partial t} + \frac{\partial G_2}{\partial \mathtt{q}_i}\dot{\mathtt{q}}_i + \frac{\partial G_2}{\partial \mathtt{P}_i}\dot{\mathtt{P}}_i - \mathtt{P}_i\frac{\mathrm{d}\mathtt{Q}_i}{\mathrm{d}t} - \mathtt{Q}_i\frac{\mathrm{d}\mathtt{P}_i}{\mathrm{d}t} \tag{1.9.41}$$

which we rewrite as

$$\left(\mathtt{p}_i - \frac{\partial G_2}{\partial \mathtt{q}_i}\right)\dot{\mathtt{q}}_i - H + K - \frac{\partial G_2}{\partial t} = \left(\frac{\partial G_2}{\partial \mathtt{P}_i} - \mathtt{Q}_i\right)\dot{\mathtt{P}}_i \tag{1.9.42}$$

and so by independence of the full time derivative terms we get

$$\frac{\partial G_2}{\partial \mathtt{q}_i} = \mathtt{p}_i \tag{1.9.43}$$

$$\frac{\partial G_2}{\partial \mathtt{P}_i} = \mathtt{Q}_i \tag{1.9.44}$$

$$K = H + \frac{\partial G_2}{\partial t} \tag{1.9.45}$$

The type 3 generating function is $G = \mathtt{q}_i\mathtt{p}_i + G_3(\mathtt{Q}, \mathtt{P}, t)$ yielding

$$\mathtt{p}_i\dot{\mathtt{q}}_i - H(\mathtt{q}, \mathtt{p}, t) = \mathtt{P}_i\dot{\mathtt{Q}}_i - K(\mathtt{Q}, \mathtt{P}, t) + \frac{\partial G_3}{\partial t} + \frac{\partial G_3}{\partial \mathtt{Q}_i}\dot{\mathtt{Q}}_i + \frac{\partial G_3}{\partial \mathtt{p}_i}\dot{\mathtt{p}}_i + \mathtt{p}_i\frac{\mathrm{d}\mathtt{q}_i}{\mathrm{d}t} + \mathtt{q}_i\frac{\mathrm{d}\mathtt{p}_i}{\mathrm{d}t} \tag{1.9.46}$$

which we rewrite as

$$-H(\mathtt{q}, \mathtt{p}, t) + K - \frac{\partial G_3}{\partial t} = \left(\mathtt{P}_i + \frac{\partial G}{\partial \mathtt{Q}_i}\right)\dot{\mathtt{Q}}_i + \left(\frac{\partial G_3}{\partial \mathtt{p}_i} + \mathtt{q}_i\right)\dot{\mathtt{p}}_i \tag{1.9.47}$$

and so by independence of the full time derivative terms we get

$$\frac{\partial G_3}{\partial \mathtt{Q}_i} = -\mathtt{P}_i \tag{1.9.48}$$

$$\frac{\partial G_3}{\partial \mathtt{p}_i} = -\mathtt{q}_i \tag{1.9.49}$$

$$K = H + \frac{\partial G_3}{\partial t} \tag{1.9.50}$$

The type 4 generating function is $G = \mathtt{q}_i\mathtt{p}_i - \mathtt{Q}_i\mathtt{P}_i + G_4(\mathtt{p}, \mathtt{P}, t)$ yielding

$$\mathtt{p}_i\dot{\mathtt{q}}_i - H(\mathtt{q}, \mathtt{p}, t) = \mathtt{P}_i\dot{\mathtt{Q}}_i - K(\mathtt{Q}, \mathtt{P}, t) + \frac{\partial G_4}{\partial t} + \frac{\partial G_4}{\partial \mathtt{p}_i}\dot{\mathtt{p}}_i + \frac{\partial G_4}{\partial \mathtt{P}_i}\dot{\mathtt{P}}_i + \mathtt{p}_i\frac{\mathrm{d}\mathtt{q}_i}{\mathrm{d}t} + \mathtt{q}_i\frac{\mathrm{d}\mathtt{p}_i}{\mathrm{d}t} - \mathtt{P}_i\frac{\mathrm{d}\mathtt{Q}_i}{\mathrm{d}t} - \mathtt{Q}_i\frac{\mathrm{d}\mathtt{P}_i}{\mathrm{d}t} \tag{1.9.51}$$

which we rewrite as

$$-H(\mathtt{q}, \mathtt{p}, t) + K - \frac{\partial G_4}{\partial t} = \left(\frac{\partial G_4}{\partial \mathtt{p}_i} + \mathtt{q}_i\right)\dot{\mathtt{p}}_i + \left(\frac{\partial G_4}{\partial \mathtt{P}_i} - \mathtt{Q}_i\right)\dot{\mathtt{P}}_i \tag{1.9.52}$$

and so by independence of the full time derivative terms we get

$$\frac{\partial G_4}{\partial \mathtt{P}_i} = \mathtt{Q}_i \tag{1.9.53}$$

$$\frac{\partial G_4}{\partial \mathtt{p}_i} = -\mathtt{q}_i \tag{1.9.54}$$

$$K = H + \frac{\partial G_4}{\partial t} \tag{1.9.55}$$

Finally, we can form the Hamilton-Jacobi equation via a type 2 generating function given by choosing a $G_2(\mathtt{q}, \mathtt{P}, t)$ so that

$$\mathtt{p} = \frac{\partial G_2}{\partial \mathtt{q}} \tag{1.9.56}$$

$$\mathtt{P} = \frac{\partial G_2}{\partial \mathtt{Q}} \tag{1.9.57}$$

$$K(\mathtt{Q}, \mathtt{P}, t) = H(\mathtt{q}, \mathtt{p}, t) + \frac{\partial G_2}{\partial t} \tag{1.9.58}$$

We want to choose a K such that $K = 0$. Thus $\frac{\partial G_2}{\partial t} = -H$ and so

$$\frac{\partial K}{\partial \mathtt{Q}} = \dot{\mathtt{P}} = \mathbf{0} \tag{1.9.59}$$

$$\frac{\partial K}{\partial \mathtt{P}} = -\dot{\mathtt{Q}} = \mathbf{0} \tag{1.9.60}$$

because $K = 0$. This means $\mathtt{P}$ and $\mathtt{Q}$ are constants and so traditionally it is written $\mathtt{P} = \boldsymbol{\alpha}$ and $\mathtt{Q} = \boldsymbol{\beta}$.[94] We can then write

$$G_2(\mathtt{q}, \boldsymbol{\alpha}, t) = S(\mathtt{q}, t) + C \tag{1.9.61}$$

[94]Here $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are not geometric vectors, but vector arrays.

for some constant C. This implies

$$\frac{\partial G_2}{\partial \mathtt{q}} = \frac{\partial S}{\partial \mathtt{q}} = \mathtt{p} \tag{1.9.62}$$

and we then have

$$H(\mathtt{q},\mathtt{p},t) + \frac{\partial G_2}{\partial t} = 0 \tag{1.9.63}$$

$$H(\mathtt{q},\frac{\partial S}{\partial \mathtt{q}},t) + \frac{\partial S}{\partial t} = 0 \tag{1.9.64}$$

We can then find the

$$\boldsymbol{\beta} = \mathtt{Q} = \frac{\partial S(\mathtt{q},\boldsymbol{\alpha},t)}{\partial \boldsymbol{\alpha}} \tag{1.9.65}$$

$S(\mathtt{q},t)$ (remember $\boldsymbol{\alpha}$ are constant) is called Hamilton's principal function. It satisfies

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial S}{\partial \mathtt{q}} \cdot \frac{\mathrm{d}\mathtt{q}}{\mathrm{d}t} + \frac{\partial S}{\partial t} = \mathtt{p} \cdot \dot{\mathtt{q}} - H(\mathtt{q},\mathtt{p}) = \mathcal{L}(\mathtt{q},\mathtt{p}) \tag{1.9.66}$$

and so

$$S = \int \mathrm{d}t\ \mathcal{L} \tag{1.9.67}$$

is indeed the action (at least up to a constant, which is the best we can do with an action).

We can also find S in a more direct manner. Most of the "proofs" I have seen with this method usually involve a surprisingly non-rigorous (and, I daresay, incomprehensible) step. Here we start with a definition of the action, and perform the usual variational calculus on it where we can deform the path/trajectory but don't allow the endpoints to change (so the total time cannot change, but the path can)

$$S = \int_{t_0}^{t} \mathrm{d}t\ \mathcal{L} \tag{1.9.68}$$

$$\delta S\,|_{\delta t=0} = \int_{t_0}^{t} \mathrm{d}t\ \delta\mathcal{L} = \int \mathrm{d}t\ \left(\frac{\delta \mathcal{L}}{\delta \mathtt{q}_i}\delta\, q_i + \frac{\delta \mathcal{L}}{\delta \dot{\mathtt{q}}_i}\delta\dot{\mathtt{q}}_i\right) \tag{1.9.69}$$

We put in the Euler-Lagrange equations $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathtt{q}}_i} = \frac{\partial \mathcal{L}}{\partial \mathtt{q}_i}$ with the variational derivatives being interpreted as partial derivatives, as usual, and so

$$\delta S = \int \mathrm{d}t\ \left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\mathtt{q}}_i}\right]\mathtt{q}_i + \frac{\delta \mathcal{L}}{\delta \dot{\mathtt{q}}_i}\frac{\mathrm{d}}{\mathrm{d}t}\left[\delta \mathtt{q}_i\right]\right) = \int \mathrm{d}t\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\mathtt{q}}_i}\delta \mathtt{q}_i\right] = \frac{\partial \mathcal{L}}{\partial \dot{\mathtt{q}}_i}\delta \mathtt{q}_i = p\delta \mathtt{q}_i \tag{1.9.70}$$

which we then rewrite as

$$\frac{\delta S}{\delta \mathtt{q}_i} = \frac{\partial S}{\partial \mathtt{q}_i} = \mathtt{p}_i \tag{1.9.71}$$

Remember we defined

$$S(\mathtt{q},t) = \int_{t_0}^{t} \mathrm{d}t\ \mathcal{L} \tag{1.9.72}$$

We can note that if we take $\frac{\mathrm{d}}{\mathrm{d}t}$ of both sides, the right side is the antiderivative and so we must get

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial S}{\partial \mathtt{q}} \cdot \frac{\mathrm{d}\mathtt{q}}{\mathrm{d}t} + \frac{\partial S}{\partial t} = \mathcal{L} \tag{1.9.73}$$

if we use $\frac{\partial S}{\partial \mathtt{q}} = \mathtt{p}$ then we have found

$$\frac{\partial S}{\partial t} = \mathcal{L} - \mathtt{p} \cdot \dot{\mathtt{q}} = -\mathcal{H} \tag{1.9.74}$$

Thus, we have shown that we indeed recover the Hamilton-Jacobi equation using S as the action.

I must now explain what I think is a mistake in many of the proofs I have seen using this more direct method. Instead of using the definition of S and a full time derivative, they try to use the calculus of variations to calculate $\frac{\partial S}{\partial t}$. They say something like "we now do a variation but allowing the time endpoint to change while not allowing the trajectory endpoint to change"; this should immediately sound some alarms for you. The entire calculus of variations is predicated on not moving the endpoints in time (or whatever we are integrating over). How can we use variational calculus in a case where one of its assumptions is false? The reasoning then given is usually along these lines. If we allow only t to vary, this means $\mathtt{q}_i$ will vary implicitly with t. However, if we still desire an extremal or "minimal" solution then if we want the same spatial trajectory, we need to move the endpoint q backwards a bit to compensate for the extra time δt that we are allowing and so $\delta \mathtt{q}_i = -\dot{\mathtt{q}}_i \delta t$. In addition we must just get an extra factor of $\mathcal{L}\delta t$ from the extra time we are allowing and so

$$\delta S = \mathcal{L}\delta t - \mathtt{p}_i \dot{\mathtt{q}}_i \delta t = -H\delta t \tag{1.9.75}$$

leading to the claim

$$\frac{\partial S}{\partial t} = -\mathcal{H} \tag{1.9.76}$$

The process that leads to this reasoning has never been properly explained to me. We seem to be working with an idea like

$$\delta S = \int_{t_0,\mathtt{q}_0}^{t,\mathtt{q}-\dot{\mathtt{q}}\delta t} \mathrm{d}t\ \mathcal{L} + \int_{t,\mathtt{q}-\dot{\mathtt{q}}_i\delta t}^{t+\delta t,\mathtt{q}} \mathrm{d}t\ \mathcal{L} - \int_{t_0,\mathtt{q}_0}^{t,\mathtt{q}} \mathrm{d}t\ \mathcal{L} \tag{1.9.77}$$

but getting the desired result from this is not only not at all obvious, it is not even clear it makes sense to call this a variation. If you come across such a proof, I would recommend ignoring it. It probably is not true, and we have the simpler method available that is outlined above using a full time derivative.

1.10 Hamiltonian Form of Magnetic Field Lines

> *To discover to the world something which deeply concerns it, and of which it was previously ignorant; to prove to it that it had been mistaken on some vital point of temporal or spiritual interest, is as important a service as a human being can render to his fellow-creatures.*
>
> — John Stuart Mill[24]

The quotation by Mills may be a bit dramatic for this section, but the discovery that magnetic field lines are a Hamiltonian system does not seem to have penetrated into the physics community outside of plasma physics. I think it is one of those things which deeply concerns the world, and think the story should be better known. Let's now explore how this connection is made.

There is an unfortunate tendency in electromagnetic textbooks to emphasize that the divergence of $\mathbf{B}$ constraint, namely $\boldsymbol{\nabla}\cdot\mathbf{B}=0$, implies that magnetic field lines are closed or that the lines end at "infinity".[95] This is often explained with the evocative wording "a magnetic field line bites its own tail". This statement is quite false. Magnetic field lines often bite their tails, but it is not required. In magnetic confinement devices magnetic field lines often form irrational surfaces where the magnetic field lines never connect back on themselves. In addition, a magnetic field line can ergodically fill a volume such that the magnetic field line, again, does not meet back up with itself. One could actually say that most forms of magnetic field lines do not bite their own tails, as we will show that magnetic field lines are directly analogous to classical trajectories. We will show this by showing that magnetic field lines can be calculated via a Hamiltonian. The magnetic field lines then bite their own tails when there are extra symmetries in the problem[96] (and so in the Hamiltonian) and so the textbook cases can present an unbalanced picture. These extra symmetries allow the Hamiltonian of the magnetic fields to have ignorable coordinates which simplify the magnetic field line structure.

If you have studied Hamiltonian dynamics, you can simply remember that trajectories do not necessarily close on themselves or extend out to infinity. So after we show that magnetic field lines satisfy a Hamiltonian relation, we have proved that they do not necessarily close. Indeed, since most Hamiltonian systems exhibit chaotic trajectories that wander without ever closing on themselves, one could say that the majority of magnetic field lines physically possible do not close on themselves.

We begin with the magnetic vector potential $\mathbf{A}$. We use that

$$\mathbf{B}=\boldsymbol{\nabla}\times(\mathbf{A}+\nabla f)=\boldsymbol{\nabla}\times\mathbf{A} \tag{1.10.1}$$

for some arbitrary function f. Let's choose curvilinear coordinates with a radial-like r, poloidal-like θ, and toroidal-like ζ. Then $f=f(r,\theta,\zeta)$ is the gauge and so we can choose it to be defined such that $f(0,\theta,\zeta)=0$ and

$$\frac{\partial f}{\partial r}=A_r \tag{1.10.2}$$

$$\frac{\partial f}{\partial \theta}=A_\theta-\psi_t \tag{1.10.3}$$

$$\frac{\partial f}{\partial \zeta}=A_\zeta-\psi_p \tag{1.10.4}$$

We will later show that $\psi_t(r,\theta,\zeta)$ is directly related to the toroidal flux and $\psi_p(r,\theta,\zeta)$ is directly related to the poloidal flux, properly defined and single-valued.[97] We can write $\mathbf{A}$ and ∇f in a

[95] A simple google search with "divergence B closed field lines" will present you with many sources saying this. Indeed, even Griffiths' Introduction to Electrodynamics[17, p. 279] says "magnetic field lines do not begin or end anywhere—to do so would require nonzero divergence. They either form closed loops or extend out to infinity". The existence of irrational magnetic flux surfaces disproves this.

[96] The class of problems generally covered in textbooks.

[97] When r is a flux label, then $\psi_t=\psi_t(r)$ and $\psi_p=\psi_p(r)$ and so the toroidal and poloidal fluxes are related and can be written $\psi_t=\psi_t(\psi_p)$ or $\psi_p=\psi_p(\psi_t)$.

covariant representation so that

$$\mathbf{A} = A_r\,\nabla r + A_\theta\,\nabla\theta + A_\zeta\,\nabla\zeta \tag{1.10.5}$$

$$\nabla f = \frac{\partial f}{\partial r}\,\nabla r + \frac{\partial f}{\partial \theta}\,\nabla\theta + \frac{\partial f}{\partial \zeta}\,\nabla\zeta = A_r\,\nabla r + (A_\theta - \psi_t)\,\nabla\theta + (A_\zeta + \psi_p)\,\nabla\zeta \tag{1.10.6}$$

Thus we can write

$$\mathbf{A} = \psi_t\,\nabla\theta + \psi_p\,\nabla\zeta + \nabla f \tag{1.10.7}$$

and so the curl is simply given by

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \nabla\psi_t\times\nabla\theta + \nabla\psi_p\times\nabla\zeta \tag{1.10.8}$$

This form of $\mathbf{B}$ is called the canonical form. This is because it makes us remember that Hamiltonian dynamics uses canonical variables, and this form of $\mathbf{B}$ has the canonical variables for our magnetic field Hamiltonian. It is important to note that this form is entirely general. It does not require the existence of magnetic flux surfaces. In fact it does not require anything of the magnetic field beyond it being divergence free (by construction) even though it has a superficially similar appearance to the conventional forms given to magnetic fields that have nested flux surfaces. This form assumes much less, and so tells us much less about the structure of $\mathbf{B}$.

We now prove that ψ_t is indeed a toroidal flux. We thus do an integral over a constant ζ surface.[98] This implies a relationship between r,θ and ζ over the surface. We can replace r with ψ_t since ψ_t has an r dependence and so it can be used to measure distance in a direction independent of θ and ζ. This means our coordinate system becomes (ψ_t,θ,ζ). The only real worries would be whether r increases when ψ_t increases and that ψ_t monotonically increases or decreases. The answer is to simply choose ψ_t such that it increases as r increases (if a function $f(r)$ decreases as r increases define $g(r) = f(-r)$ so that $g(r)$ increases when r increases) and realize that if ψ_t does not simply increase or decrease, then we will have to solve in separate regions or have some separate way of associating ψ_t with distinct r locations. This can be difficult, but the problem is not insurmountable if we divide ψ_t into regions with unique r and make sure they connect continuously onto each other. We will choose a constant ζ surface such that it is a "disk"[99] centered around the magnetic axis. We then use $\mathcal{J} = 1/(\nabla\psi_t\cdot\nabla\theta\times\nabla\zeta)$ and the form of the surface integral via (1.2.310)

$$\begin{aligned}\iint \mathrm{d}\mathbf{S}_\zeta\cdot\mathbf{B} &= \int \mathrm{d}\psi_t\int \mathrm{d}\theta\;\mathcal{J}\,\nabla\zeta\cdot[\,\nabla\psi_t\times\nabla\theta + \cancel{\nabla\psi_p\times\nabla\zeta}\,]\\ &= \int \mathrm{d}\psi_t\int \mathrm{d}\theta\;\mathcal{J}\,\nabla\zeta\cdot\nabla\psi_t\times\nabla\theta = \int \mathrm{d}\psi_t\int \mathrm{d}\theta\;\mathcal{J}\,\nabla\psi_t\cdot\nabla\theta\times\nabla\zeta\\ &= \int \mathrm{d}\psi_t\int \mathrm{d}\theta\;\frac{\mathcal{J}}{\mathcal{J}} = 2\pi\psi_t\end{aligned} \tag{1.10.9}$$

Thus we see that ψ_t is in fact the normalized toroidal flux. It measures the flux through a constant toroidal angle θ, with the actual toroidal flux through that surface given by $2\pi\psi_t$.

[98] This is essentially saying that a toroidal flux is the flux through a constant toroidal angle. This should be regarded as a definition.

[99] This is simply to say it follows a flux surface if it exists, or to some prescribed radius from the magnetic axis. It need not be an actual disk.

Similarly, we could consider $\psi_p(r,\theta,\zeta)$ to be the radial coordinate instead of ψ_t. We again will have to consider the sign of ψ_p. The sign turns out to be a little more complicated than for the toroidal case, but otherwise the exact same manipulations can be used.

There are two traditional definitions of poloidal flux, the ribbon flux and the disk flux. A disk flux[100] counts the flux along a constant θ surface from the geometric axis outward to some other radial position. The ribbon flux instead privileges the magnetic axis and so is along a constant θ surface from the magnetic axis out to some radial distance. We can then call P_R^d the disk flux from the geometric axis out to radial distance R, and P_r^r the ribbon flux. Indeed, we can write P_r^d if we understand that r just represents some radial variable. We can then note that for typical geometric coordinates we choose θ and r so that $\mathrm{d}P^d = -\,\mathrm{d}P^r$ or $\nabla P^d = -\,\nabla P^r$ and so depending on whether we use the ribbon or the disk flux we can take either sign. This follows from the fact that if $r = r'$ and $R = R'$ are referring to the same position then we choose $P_R^d(R') + P_r^r(r') = P^d(R_0)$ where R_0 is major radius pointing from the geometric axis to the magnetic axis. This implies that the $\theta = \pi$ surface is used for the ribbon flux. We will consider the disk flux for our calculation below. Then

$$\begin{aligned}\iint \mathrm{d}\mathbf{S}_\theta \cdot \mathbf{B} &= \int \mathrm{d}\psi_p \int \mathrm{d}\zeta\ \mathcal{J}\,\nabla\theta\cdot\left[\cancel{\nabla\psi_t\times\nabla\theta} + \nabla\psi_p\times\nabla\zeta\right] \\ &= \int \mathrm{d}\psi_p \int \mathrm{d}\zeta\ \mathcal{J}\,\nabla\theta\cdot\nabla\psi_p\times\nabla\zeta = \int \mathrm{d}\psi_p \int \mathrm{d}\zeta\ -\mathcal{J}\,\nabla\psi_p\cdot\nabla\theta\times\nabla\zeta \\ &= -\int \mathrm{d}\psi_p \int \mathrm{d}\theta\ \frac{\mathcal{J}}{\mathcal{J}} = -2\pi\psi_p\end{aligned} \tag{1.10.10}$$

Thus we have a normalized poloidal flux again with ψ_p corresponding to a poloidal disk flux of $-2\pi\psi_p$ for a content θ surface. We can make this explicit via using ψ_p^d. If instead we designate ψ_p^r the ribbon flux then we would have found the above integral to be $+2\pi\psi_p^r$ because $\nabla\psi_p^r \propto -\nabla r$. This can be summarized as

$$\mathbf{B} = \nabla\psi_t\times\nabla\theta + \nabla\psi_p^d\times\nabla\zeta \tag{1.10.11}$$

$$\mathbf{B} = \nabla\psi_t\times\nabla\theta + \nabla\zeta\times\nabla\psi_p^r = \nabla\psi_t\times\nabla\theta - \nabla\psi_p^r\times\nabla\zeta \tag{1.10.12}$$

Now that we have proven how to choose our gauge explicitly we can return to showing that there is a Hamiltonian form from the canonical form of the magnetic field (1.10.8). We will choose to use the coordinate system (ψ_t,θ,ζ) with $\psi_p^d(r,\theta,\zeta) = \psi_p^d(\psi_t,\theta,\zeta)$. We find the magnetic field lines via the equation

$$B\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\ell} = \mathbf{B} \tag{1.10.13}$$

where ℓ is a parameter that measures the distance along the magnetic field line and so points to a location along the magnetic field line, the position $\mathbf{x}(\ell)$.[101] We could also give positions via $\mathbf{x}(\psi_t,\theta,\zeta)$. Then the chain rule implies

$$\frac{\mathbf{B}}{B} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\ell} = \frac{\partial\mathbf{x}}{\partial\psi_t}\frac{\partial\psi_t}{\partial\ell} + \frac{\partial\mathbf{x}}{\partial\theta}\frac{\partial\theta}{\partial\ell} + \frac{\partial\mathbf{x}}{\partial\zeta}\frac{\partial\zeta}{\partial\ell} = \mathbf{e}_{\psi_t}\frac{\partial\psi_t}{\partial\ell} + \mathbf{e}_\theta\frac{\partial\theta}{\partial\ell} + \mathbf{e}_\zeta\frac{\partial\zeta}{\partial\ell} \tag{1.10.14}$$

[100] Again, not necessarily an actual disk.

[101] Note that one could use $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\ell} = \mathbf{B}$. This way $\frac{\mathrm{d}}{\mathrm{d}\ell}$ can be identified with $\mathbf{B}\cdot\nabla$, and ℓ has units of 'length per magnetic field' rather than just 'length'. Were we to adopt this convention here we would not need to keep putting in factors of B that eventually just cancel.

If we then take $\nabla\psi_t = \mathbf{e}^{\psi_t}$, $\nabla\theta = \mathbf{e}^{\theta}$ or $\nabla\zeta = e^{\zeta}$ we can use the reciprocal relation (1.2.61) to find

$$\mathbf{B}\cdot\nabla\psi_t = B\mathbf{e}^{\psi_t}\cdot\left(\mathbf{e}_{\psi_t}\frac{\partial\psi_t}{\partial\ell} + \cancel{\mathbf{e}_{\theta}\frac{\partial\theta}{\partial\ell}} + \cancel{\mathbf{e}_{\zeta}\frac{\partial\zeta}{\partial\ell}}\right) = B\frac{\partial\psi_t}{\partial\ell} \tag{1.10.15}$$

$$\mathbf{B}\cdot\nabla\theta = B\mathbf{e}^{\theta}\cdot\left(\cancel{\mathbf{e}_{\psi_t}\frac{\partial\psi_t}{\partial\ell}} + \mathbf{e}_{\theta}\frac{\partial\theta}{\partial\ell} + \cancel{\mathbf{e}_{\zeta}\frac{\partial\zeta}{\partial\ell}}\right) = B\frac{\partial\theta}{\partial\ell} \tag{1.10.16}$$

$$\mathbf{B}\cdot\nabla\zeta = B\mathbf{e}^{\zeta}\cdot\left(\cancel{\mathbf{e}_{\psi_t}\frac{\partial\psi_t}{\partial\ell}} + \cancel{\mathbf{e}_{\theta}\frac{\partial\theta}{\partial\ell}} + \mathbf{e}_{\zeta}\frac{\partial\zeta}{\partial\ell}\right) = B\frac{\partial\zeta}{\partial\ell} \tag{1.10.17}$$

So long as these are non-singular (there are no magnetic nulls) we can then use the chain rule again to write

$$\frac{\mathrm{d}f}{\mathrm{d}g} = \frac{\frac{\mathrm{d}f}{\mathrm{d}\ell}}{\frac{\mathrm{d}g}{\mathrm{d}\ell}} = \frac{B\frac{\mathrm{d}f}{\mathrm{d}\ell}}{B\frac{\mathrm{d}g}{\mathrm{d}\ell}} \tag{1.10.18}$$

Finally we can use

$$\begin{aligned}
\nabla\psi_t\cdot\mathbf{B} &= \cancel{B\,\nabla\psi_t\cdot\nabla\psi_t\times\nabla\theta} + B\,\nabla\psi_t\cdot\nabla\psi_p^d\times\nabla\zeta \\
&= \nabla\psi_t\cdot\left[\left(\cancel{\frac{\partial\psi_p^d}{\partial\psi_t}\nabla\psi_t} + \frac{\partial\psi_p^d}{\partial\theta}\nabla\theta + \cancel{\frac{\partial\psi_p^d}{\partial\zeta}\nabla\zeta}\right)\times\nabla\zeta\right] \\
&= B\frac{\partial\psi_p^d}{\partial\theta}\nabla\psi_t\cdot\nabla\theta\times\nabla\zeta = \frac{\partial\psi_p^d}{\partial\theta}\frac{B}{\mathcal{J}}
\end{aligned} \tag{1.10.19}$$

$$\begin{aligned}
\nabla\theta\cdot\mathbf{B} &= \cancel{B\,\nabla\theta\cdot\nabla\psi_t\times\nabla\theta} + B\,\nabla\theta\cdot\nabla\psi_p^d\times\nabla\zeta \\
&= B\,\nabla\theta\cdot\left[\left(\frac{\partial\psi_p^d}{\partial\psi_t}\nabla\psi_t + \cancel{\frac{\partial\psi_p^d}{\partial\theta}\nabla\theta} + \cancel{\frac{\partial\psi_p^d}{\partial\zeta}\nabla\zeta}\right)\times\nabla\zeta\right] \\
&= B\frac{\partial\psi_p^d}{\partial\psi_t}\nabla\theta\cdot\nabla\psi_t\times\nabla\zeta = -\frac{\partial\psi_p^d}{\partial\psi_t}\frac{B}{\mathcal{J}}
\end{aligned} \tag{1.10.20}$$

$$\begin{aligned}
\nabla\zeta\cdot\mathbf{B} &= B\,\nabla\zeta\cdot\nabla\psi_t\times\nabla\theta + B\,\nabla\zeta\cdot\cancel{\nabla\psi_p^d\times\nabla\zeta} \\
&= B\,\nabla\psi_t\cdot\nabla\theta\times\nabla\zeta = \frac{B}{\mathcal{J}}
\end{aligned} \tag{1.10.21}$$

Note that this then also gives the magnetic field line equations

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}\ell} = \frac{\mathbf{B}\cdot\nabla\psi_t}{B} = \frac{B^{\psi_t}}{B} \tag{1.10.22}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\ell} = \frac{\mathbf{B}\cdot\nabla\theta}{B} = \frac{B^{\theta}}{B} \tag{1.10.23}$$

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\ell} = \frac{\mathbf{B}\cdot\nabla\zeta}{B} = \frac{B^{\zeta}}{B} \tag{1.10.24}$$

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}\theta} = \frac{\mathbf{B}\cdot\nabla\psi_t}{\mathbf{B}\cdot\nabla\theta} = \frac{B_t^{\psi}}{B^{\theta}} \tag{1.10.25}$$

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}\zeta} = \frac{\mathbf{B}\cdot\nabla\psi_t}{\mathbf{B}\cdot\nabla\zeta} = \frac{B_t^{\psi}}{B^{\zeta}} \tag{1.10.26}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \frac{\mathbf{B}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\zeta} = \frac{B^{\zeta}}{B^{\theta}} \tag{1.10.27}$$

and all other possible forms (assuming no singular components) which can be summarized by

$$\frac{\mathrm{d}\ell}{B} = \frac{\mathrm{d}\psi_t}{B^{\psi_t}} = \frac{\mathrm{d}\theta}{B^\theta} = \frac{\mathrm{d}\zeta}{B^\zeta} \tag{1.10.28}$$

Make sure to note that $B^i = \nabla\xi^i \cdot \mathbf{B}$, the contravariant components of $\mathbf{B}$.

In any case, we can then write

$$\frac{B\frac{\mathrm{d}\psi_t}{\mathrm{d}\ell}}{B\frac{\mathrm{d}\zeta}{\mathrm{d}\ell}} = \frac{\mathrm{d}\psi_t}{\mathrm{d}\zeta} = \frac{\mathbf{B}\cdot\nabla\psi_t}{\mathbf{B}\cdot\nabla\zeta} = \frac{\frac{\partial\psi_p^d}{\partial\theta}\frac{B}{\mathcal{J}}}{\frac{B}{\mathcal{J}}} = \frac{\partial\psi_p^d}{\partial\theta} \tag{1.10.29}$$

$$\frac{B\frac{\mathrm{d}\theta}{\mathrm{d}\ell}}{B\frac{\mathrm{d}\zeta}{\mathrm{d}\ell}} = \frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \frac{\mathbf{B}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\zeta} = \frac{-\frac{\partial\psi_p^d}{\partial\psi_t}\frac{B}{\mathcal{J}}}{\frac{B}{\mathcal{J}}} = -\frac{\partial\psi_p^d}{\partial\psi_t} \tag{1.10.30}$$

or more cleanly in a more suggestive form

$$\frac{\partial\psi_p^d}{\partial\theta} = \frac{\mathrm{d}\psi_t}{\mathrm{d}\zeta} \tag{1.10.31}$$

$$\frac{\partial\psi_p^d}{\partial\psi_t} = -\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} \tag{1.10.32}$$

If we then identify $\psi_p^d \to -H$, $\theta \to q$, $\psi_t \to p$ and $\zeta \to t$ we see that the (1.10.31)-(1.10.32) would be written as

$$\frac{\partial H}{\partial q} = -\frac{\mathrm{d}p}{\mathrm{d}t} \tag{1.10.33}$$

$$\frac{\partial H}{\partial p} = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{1.10.34}$$

the conventional form of the Hamiltonian for canonical coordinates (q, p) with q generalized positions and p generalized momentum.

If we had used ψ_p^r instead then the identifications would be simpler as $\psi_p^r \to H$, $\theta \to q$, $\psi_t \to p$ and $\zeta \to t$ and

$$\frac{\partial\psi_p^r}{\partial\theta} = -\frac{\mathrm{d}\psi_t}{\mathrm{d}\zeta} \tag{1.10.35}$$

$$\frac{\partial\psi_p^r}{\partial\psi_t} = \frac{\mathrm{d}\theta}{\mathrm{d}\zeta} \tag{1.10.36}$$

which is often used as well.

In any case, we recognize that ψ_t and θ are canonical variables with ψ_t canonically conjugate to θ (and vice versa). One final comment is that we have a 1D "time dependent" (because ζ is playing the role of time) Hamiltonian above, and we could "eliminate" the time dependence by considering ψ_p^d and ζ as coordinates with ψ_p^d the canonically conjugate momentum to canonical coordinate ζ. We then have the identification $\mathtt{Q} = [\theta, \zeta]$ and $\mathtt{P} = [\psi_t, \psi_p^d]$ (for convenience we can write $\mathtt{q} = [\theta, 0]$ and $\mathtt{p} = [\psi_t, 0]$). We can say we are now parameterized by a new time τ with ψ_p independent of this τ. We can use the generating function

$$G_2(\mathtt{q}, \mathtt{P}, \zeta) = \mathtt{P}\cdot\mathtt{q} + \psi_p^d\zeta = \psi_t\theta + \psi_p^d\zeta \tag{1.10.37}$$

and so the new Hamiltonian H (the old one is $-\psi_p$) is given by

$$H(\mathtt{Q},\mathtt{P}) = H(\theta,\zeta,\psi_t,\psi_p^d) = -\psi_p^d(\psi_t,\theta,\zeta) + \left(\frac{\partial G_2}{\partial \zeta}\right)_{\theta,\psi_t,\psi_p^d} = -\psi_p^d(\psi_t,\theta,\zeta) + \psi_p^d \tag{1.10.38}$$

This equation should be read as $-\psi_p^d(\psi_t,\theta,\zeta)$ is the written out form of the previous Hamiltonian $-\psi_p^d$ in terms of the variables ψ_t,θ,ζ, with the last ψ_p^d not written out in those variables because it now stands as a separate coordinate. The new equations are then

$$\frac{\partial H}{\partial \mathtt{Q}} = -\frac{\mathrm{d}\mathtt{P}}{\mathrm{d}\tau} \tag{1.10.39}$$

$$\frac{\partial H}{\partial \mathtt{P}} = \frac{\mathrm{d}\mathtt{Q}}{\mathrm{d}\tau} \tag{1.10.40}$$

with H independent of the time τ. Written out component wise, this says

$$\frac{\partial H}{\partial \theta} = -\frac{\mathrm{d}\psi_t}{\mathrm{d}\tau} \tag{1.10.41}$$

$$\frac{\partial H}{\partial \psi_p^d} = -\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} \tag{1.10.42}$$

$$\frac{\partial H}{\partial \psi_t} = \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \tag{1.10.43}$$

$$\frac{\partial H}{\partial \zeta} = \frac{\mathrm{d}\psi_p^d}{\mathrm{d}\tau} \tag{1.10.44}$$

and we have a 2D τ-independent ("time" independent) Hamiltonian system. Such systems are known to produce stochastic behavior.

We have thus shown how to construct a "time" (really ζ) independent Hamiltonian ψ_p^d (or in ψ_p^r) given toroidal and poloidal fluxes defined by toroidal ζ and poloidal θ angles for generic magnetic fields (without nulls).

The final consideration is the actual time evolution of $\mathbf{B}$. All the above analysis is for actual time held constant. In fact, it is still valid for time dependence. Instead of $\psi(r,\theta,\zeta)$ use $\psi(r,\theta,\zeta,t)$ and $\mathbf{x}(r,\theta,\zeta)$ is now $\mathbf{x}(r,\theta,\zeta,t)$. So long as r,θ,ζ are time independent coordinates, then essentially nothing changes other than $\mathbf{x}$ will now point at different places at different times t.

1.11 Frenet-Serret Formulas

In the previous Hamiltonian section, Section 1.10, we saw that magnetic field lines are analogues of trajectories. We can then apply a similar analysis to directions along magnetic field lines as that done for particle trajectories. This leads to a powerful description of the geometry of the magnetic field lines.

1.11.1 Particle Trajectories

The Frenet-Serret formulas are a complete description of a particle trajectory. They are traditionally given by three unit vectors: the tangent vector $\hat{\mathbf{T}}$, normal vector $\hat{\mathbf{N}}$, and binormal vector $\hat{\mathbf{B}}$,

along with two scalars the curvature κ and torsion τ. Given a specific trajectory, we can parameterize that specific trajectory by time t. That is we write the trajectory as $\mathbf{x}(t)$. We need the trajectory to be nondegenerate, so that the path doesn't cross itself unambiguously (this is easily fixed below, though, through the arclength). We also need the curvature κ to be nonzero. This restriction on κ means given $\mathbf{x}(t)$, that $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}$ are not proportional to each other. To make sure we have a monotonically increasing parameterization we define a new parameter based on time given by s. This s will be the arclength along the trajectory from a specified initial point. If it is nondegenerate, then we do not need to worry about a non-deterministic trajectory. We can write s in terms of time via

$$s = \int_0^t \mathrm{d}\sigma \left|\frac{\mathrm{d}\mathbf{x}(\sigma)}{\mathrm{d}\sigma}\right| \tag{1.11.1}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \left|\frac{\mathrm{d}\mathbf{x}(\sigma)}{\mathrm{d}\sigma}\right|_{\sigma=t} = \left|\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}\right| \tag{1.11.2}$$

With this arclength s as the parameterization, we have $\mathbf{x}(s) = \mathbf{x}(t(s))$. We can easily write the tangent vector for the curve. It is simply $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}$, because s goes along the curve, then this points in that direction. While it is not immediately obvious, this is also a unit vector. This is because

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\frac{1}{|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}|} \tag{1.11.3}$$

which is clearly a unit vector. We call it $\hat{\mathbf{T}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}$. We then take another s derivative of this vector normalize it via the curvature and call this the normal vector $\hat{\mathbf{N}}$

$$\hat{\mathbf{N}} = \frac{\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}}{\left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}\right|} = \frac{1}{\kappa}\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} \tag{1.11.4}$$

Finally, we define the binormal vector via $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$.

Because $\hat{\mathbf{T}}$ is a unit vector, then $\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}$ never points along $\hat{\mathbf{T}}$, and so it is perpendicular to $\hat{\mathbf{T}}$. If you find this verbal argument insufficient, we write

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} \cdot \hat{\mathbf{T}} = \frac{1}{2}\frac{\mathrm{d}\hat{\mathbf{T}}\cdot\hat{\mathbf{T}}}{\mathrm{d}s} = \frac{1}{2}\frac{\mathrm{d}1}{\mathrm{d}s} = 0 \tag{1.11.5}$$

Thus $\hat{\mathbf{N}}$ is indeed orthogonal to $\hat{\mathbf{T}}$. Finally, $\hat{\mathbf{B}}$ is by construction orthogonal to $\hat{\mathbf{N}}$ and $\hat{\mathbf{T}}$. All that remains is to find relations between all of these vectors.

We have a right-handed coordinate system $(\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}})$. Because $\hat{\mathbf{N}}$ is a unit vector, we again have that its derivative is perpendicular to it

$$\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} \cdot \hat{\mathbf{N}} = \frac{1}{2}\frac{\mathrm{d}\hat{\mathbf{N}}\cdot\hat{\mathbf{N}}}{\mathrm{d}s} = \frac{1}{2}\frac{\mathrm{d}1}{\mathrm{d}s} = 0 \tag{1.11.6}$$

This means in our coordinate system we must have

$$\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} = a\hat{\mathbf{T}} + b\hat{\mathbf{B}} \tag{1.11.7}$$

We then have for $\hat{\mathbf{B}}$ that

$$\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} = \frac{\mathrm{d}\hat{\mathbf{T}} \times \hat{\mathbf{N}}}{\mathrm{d}s} = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} \times \hat{\mathbf{N}} + \hat{\mathbf{T}} \times \frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} = \cancel{\frac{\hat{\mathbf{N}}}{\kappa}} \times \hat{\mathbf{N}} = \hat{\mathbf{T}} \times \left[\cancel{a\hat{\mathbf{T}}} + b\hat{\mathbf{B}}\right] = -b\hat{\mathbf{N}} \tag{1.11.8}$$

We define $b = \tau$ to be the torsion (which we will investigate in a moment). We can then use $\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}}$ to rewrite

$$\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} = \frac{\mathrm{d}\hat{\mathbf{B}} \times \hat{\mathbf{T}}}{\mathrm{d}s} = \frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} \times \hat{\mathbf{T}} + \hat{\mathbf{B}} \times \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = -\tau\hat{\mathbf{N}} \times \hat{\mathbf{T}} + \hat{\mathbf{B}} \times \kappa\hat{\mathbf{N}} = \tau\hat{\mathbf{B}} - \kappa\hat{\mathbf{T}} \tag{1.11.9}$$

Thus, we can write out all of our equations as

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = \kappa\hat{\mathbf{N}} \tag{1.11.10}$$

$$\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} = -\kappa\hat{\mathbf{T}} + \tau\hat{\mathbf{B}} \tag{1.11.11}$$

$$\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} = -\tau\hat{\mathbf{N}} \tag{1.11.12}$$

which defines the Frenet-Serret frame. Remember that we could use the time directly as a parameterization so long as we do not have a degenerate trajectory. In that case we have

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t} = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| \kappa\hat{\mathbf{N}} \tag{1.11.13}$$

$$\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}t} = -\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| \kappa\hat{\mathbf{T}} + \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| \tau\hat{\mathbf{B}} \tag{1.11.14}$$

$$\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}t} = -\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| \tau\hat{\mathbf{N}} \tag{1.11.15}$$

Now we can investigate curvature κ and torsion τ in terms of s and t. For s it is especially simple with

$$\kappa = \left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}\right| \tag{1.11.16}$$

$$\tau = \left|\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s}\right| \tag{1.11.17}$$

We would like to remove our parameterization s explicitly from the picture. We first note

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \hat{\mathbf{T}}\frac{\mathrm{d}s}{\mathrm{d}t} \tag{1.11.18}$$

Thus

$$\begin{aligned}\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} &= \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}\frac{\mathrm{d}s}{\mathrm{d}t} + \hat{\mathbf{T}}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + \hat{\mathbf{T}}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \\ &= \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \kappa\hat{\mathbf{N}} + \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\hat{\mathbf{T}}\end{aligned} \tag{1.11.19}$$

Finally, we can find

$$\begin{aligned}
\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3} &= 2\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\kappa\hat{\mathbf{N}} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\frac{\mathrm{d}\kappa}{\mathrm{d}t}\hat{\mathbf{N}} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\kappa\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}t} + \frac{\mathrm{d}^3s}{\mathrm{d}t^3}\hat{\mathbf{T}} + \frac{\mathrm{d}^2s}{\mathrm{d}t^2}\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t} \\
&= \left[2\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\kappa + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\frac{\mathrm{d}\kappa}{\mathrm{d}t}\right]\hat{\mathbf{N}} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\kappa\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} + \frac{\mathrm{d}^3s}{\mathrm{d}t^3}\hat{\mathbf{T}} + \frac{\mathrm{d}^2s}{\mathrm{d}t^2}\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} \\
&= \left[2\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\kappa + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\frac{\mathrm{d}\kappa}{\mathrm{d}t}\right]\hat{\mathbf{N}} + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\kappa\left[-\kappa\hat{\mathbf{T}} + \tau\hat{\mathbf{B}}\right] + \frac{\mathrm{d}^3s}{\mathrm{d}t^3}\hat{\mathbf{T}} + \frac{\mathrm{d}^2s}{\mathrm{d}t^2}\frac{\mathrm{d}s}{\mathrm{d}t}\kappa\hat{\mathbf{N}} \qquad (1.11.20) \\
&= \left[-\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\kappa^2 + \frac{\mathrm{d}^3s}{\mathrm{d}t^3}\right]\hat{\mathbf{T}} + \left[2\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\kappa + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\frac{\mathrm{d}\kappa}{\mathrm{d}t} + \frac{\mathrm{d}^2s}{\mathrm{d}t^2}\frac{\mathrm{d}s}{\mathrm{d}t}\right]\hat{\mathbf{N}} + \tau\kappa\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\hat{\mathbf{B}} \\
&= \left[\frac{\mathrm{d}^3s}{\mathrm{d}t^3} - \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\kappa^2\right]\hat{\mathbf{T}} + \left[3\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\kappa + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\frac{\mathrm{d}\kappa}{\mathrm{d}t}\right]\hat{\mathbf{N}} + \tau\kappa\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\hat{\mathbf{B}}
\end{aligned}$$

Then this means

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \hat{\mathbf{T}}\frac{\mathrm{d}s}{\mathrm{d}t}\times\left[\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\kappa\hat{\mathbf{N}} + \frac{\mathrm{d}^2s}{\mathrm{d}t^2}\hat{\mathbf{T}}\right] \qquad (1.11.21)$$

$$= \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\kappa\hat{\mathbf{T}}\times\hat{\mathbf{N}} = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\kappa\hat{\mathbf{B}} \qquad (1.11.22)$$

So with $\frac{\mathrm{d}s}{\mathrm{d}t} = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|$ and $|\hat{\mathbf{B}}| = 1$ we find

$$\kappa = \frac{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|}{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3|\hat{\mathbf{B}}|} = \frac{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^3} \qquad (1.11.23)$$

We can then find

$$\begin{aligned}
&\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \\
&= \cancel{\left[\left[\frac{\mathrm{d}^3s}{\mathrm{d}t^3} - \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\kappa^2\right]\hat{\mathbf{T}} + \left[3\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\kappa + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\frac{\mathrm{d}\kappa}{\mathrm{d}t}\right]\hat{\mathbf{N}} + \tau\kappa\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\hat{\mathbf{B}}\right]\cdot\left[\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3\kappa\hat{\mathbf{B}}\right]} \\
&= \tau\kappa^3\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^6
\end{aligned} \qquad (1.11.24)$$

Thus we will find that τ is

$$\tau = \frac{\left|\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^6\kappa^2} = \frac{\left|\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^6\frac{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|^2}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^6}} = \frac{\left|\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|^2} \qquad (1.11.25)$$

1.11.1.1 Particle Trajectories Summary

To summarize, when using arclength parameter s we have

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = \kappa\hat{\mathbf{N}} \tag{1.11.26}$$

$$\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} = -\kappa\hat{\mathbf{T}} + \tau\hat{\mathbf{B}} \tag{1.11.27}$$

$$\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} = -\tau\hat{\mathbf{N}} \tag{1.11.28}$$

with

$$\kappa = \left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}\right| \tag{1.11.29}$$

$$\tau = \left|\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s}\right| \tag{1.11.30}$$

and then using time as the parameter

$$\frac{1}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|}\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t} = \kappa\hat{\mathbf{N}} \tag{1.11.31}$$

$$\frac{1}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|}\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}t} = -\kappa\hat{\mathbf{T}} + \tau\hat{\mathbf{B}} \tag{1.11.32}$$

$$\frac{1}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|}\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}t} = -\tau\hat{\mathbf{N}} \tag{1.11.33}$$

with

$$\tau = \frac{\left|\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^6\kappa^2} = \frac{\left|\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^6\frac{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|^2}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^6}} = \frac{\left|\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|}{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|^2} \tag{1.11.34}$$

1.11.2 Darboux Frames

There is yet another twist to this story. In the above, we assumed that the curve can wander wherever it would like in 3D space. This is completely general, but there are situations where we might prefer to enforce something more on our curve. Suppose we have a surface in space that our trajectory is constrained to lie on. That is, the curve is completely on a surface in 3D space (for example, one could consider a trajectory lying only on the surface of a sphere). There is a sense in which we would then like to say that the curvature is zero if we are simply following the surface of the sphere on our trajectory. That is, if we are told to go from one location on the equator to another location on the equator and we take the great circle (that is, we stay on the equator) between these locations, we'd like to say that there is no curvature. We took the geodesic, the shortest path given our constraints.

This is as if we were near a geyser and wished to go across to the other side. The straightest distance is simply to go over the geyser to the other side. But, in reality, there is a fence and a

sidewalk that restricts our choices to going around the geyser. Thus, the fastest way is to simply follow the sidewalk around to the other side (staying on the edge of the sidewalk closest to the geyser). The curvature on this path is nonzero, but this is because of our constraint. We will define the geodesic curvature to be zero because we are taking the fastest path given our constraints.

So, mathematically, how do we deal with this? Well, consider a surface given by the surface unit normal $\hat{\mathbf{n}}$. Note that this $\hat{\mathbf{n}}$ need not be in the same direction as the normal vector $\hat{\mathbf{N}}$ defined above because $\hat{\mathbf{n}}$ is completely determined by the surface itself, rather than the path we are parametrizing. We still have the tangent vector $\hat{\mathbf{T}}$ as above, because this still goes along the path. So, we can construct the right-handed system $(\hat{\mathbf{T}}, \hat{\mathbf{n}}, \hat{\mathbf{T}} \times \hat{\mathbf{n}})$ as our Frenet-Serret-like coordinates, with $\hat{\mathbf{D}} \equiv \hat{\mathbf{T}} \times \hat{\mathbf{n}}$ the binormal vector. There does not appear to be any actual standard basis set, as I have seen the order $(\hat{\mathbf{T}}, \hat{\mathbf{n}} \times \hat{\mathbf{T}}, \hat{\mathbf{n}})$ as well. This formulation is then called the Darboux frame instead of a Frenet-Serret frame. Let's use the arclength parameterization. Our vectors are all unit vectors so we know that the arclength derivative of a unit vector will be zero along that same unit vector. We can then write

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = a_{Tn}\hat{\mathbf{n}} + a_{TD}\hat{\mathbf{D}} \tag{1.11.35}$$

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = a_{nT}\hat{\mathbf{T}} + a_{nD}\hat{\mathbf{D}} \tag{1.11.36}$$

$$\frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s} = a_{DT}\hat{\mathbf{T}} + a_{Dn}\hat{\mathbf{n}} \tag{1.11.37}$$

Now, there are relationships between these coefficients given by

$$\frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s} = \frac{\mathrm{d}(\hat{\mathbf{T}} \times \hat{\mathbf{n}})}{\mathrm{d}s} = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} \times \hat{\mathbf{n}} + \hat{\mathbf{T}} \times \frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} \tag{1.11.38}$$

$$= \left(a_{Tn}\hat{\mathbf{n}} + a_{TD}\hat{\mathbf{D}}\right) \times \hat{\mathbf{n}} + \hat{\mathbf{T}} \times \left(a_{nT}\hat{\mathbf{T}} + a_{nD}\hat{\mathbf{D}}\right) \tag{1.11.39}$$

$$= a_{TD}\hat{\mathbf{D}} \times \hat{\mathbf{n}} + \hat{\mathbf{T}} \times \hat{\mathbf{D}}a_{nD} \tag{1.11.40}$$

$$= -a_{TD}\hat{\mathbf{T}} - a_{nD}\hat{\mathbf{n}} \overset{(1.11.37)}{=} a_{DT}\hat{\mathbf{T}} + a_{Dn}\hat{\mathbf{n}} \tag{1.11.41}$$

The last equality comes from our definition (1.11.37) above. Then we see that the actual equations are

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = 0\hat{\mathbf{T}} + a_{Tn}\hat{\mathbf{n}} + a_{TD}\hat{\mathbf{D}} \tag{1.11.42}$$

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = a_{nT}\hat{\mathbf{T}} + 0\hat{\mathbf{n}} + a_{nD}\hat{\mathbf{D}} \tag{1.11.43}$$

$$\frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s} = -a_{TD}\hat{\mathbf{T}} - a_{nD}\hat{\mathbf{n}} + 0\hat{\mathbf{D}} \tag{1.11.44}$$

We can also use

$$\frac{\mathrm{d}(\hat{\mathbf{T}} \cdot \hat{\mathbf{n}})}{\mathrm{d}s} = 0 = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} \cdot \hat{\mathbf{n}} + \frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} \cdot \hat{\mathbf{T}} = a_{Tn} + a_{nT} \tag{1.11.45}$$

and so finally

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = 0\hat{\mathbf{T}} + a_{Tn}\hat{\mathbf{n}} + a_{TD}\hat{\mathbf{D}} \tag{1.11.46}$$

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = -a_{Tn}\hat{\mathbf{T}} + 0\hat{\mathbf{n}} + a_{nD}\hat{\mathbf{D}} \tag{1.11.47}$$

$$\frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s} = -a_{TD}\hat{\mathbf{T}} - a_{nD}\hat{\mathbf{n}} + 0\hat{\mathbf{D}} \tag{1.11.48}$$

Now, the curvature component in the direction of the surface normal is usually called normal curvature, $\kappa_n = a_{Tn}$ while the curvature not along the surface normal is called geodesic curvature $a_{TD} = \kappa_g$. Geodesic curvature is a measure of how close the curve is to a geometric geodesic (the shortest path given a constraint through space). The closer to zero it is, the closer the path is to a geodesic (with a geodesic having zero geodesic curvature). That is, we want any change in the tangent vector as we move along the surface to be due to surface changes only. These are sometimes called the intrinsic curvature κ_n and the extrinsic curvature κ_g because the intrinsic curvature is due to the surface's intrinsic properties. One could also call normal curvature the natural curvature (natural given the constraint), and it may serve as a useful mnemonic. The final coefficient is called the geodesic torsion or relative torsion, often denoted as τ_r or τ_g. As you might expect, there are relationships between the normal and geodesic curvature and the Frenet-Serret curvature, and a relationship between the geodesic torsion and the Frenet-Serret torsion. The easiest way to determine this relationship is to consider our Darboux coordinates $(\hat{\mathbf{T}}, \hat{\mathbf{n}}, \hat{\mathbf{D}})$ and the Frenet-Serret coordinates $(\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}})$. We have $\hat{\mathbf{T}}$ in common, and both are orthogonal right-handed coordinates so $\hat{\mathbf{n}}$ and $\hat{\mathbf{D}}$ can be rotated around $\hat{\mathbf{T}}$ at any particular position by an angle α to get $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$. That is,

$$\hat{\mathbf{n}} = \cos\alpha\hat{\mathbf{N}} + \sin\alpha\hat{\mathbf{B}} \tag{1.11.49}$$

$$\hat{\mathbf{D}} = -\sin\alpha\hat{\mathbf{N}} + \cos\alpha\hat{\mathbf{B}} \tag{1.11.50}$$

We can then take the arclength derivative via

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = -\sin\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s}\hat{\mathbf{N}} + \cos\alpha\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} + \cos\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s}\hat{\mathbf{B}} + \sin\alpha\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} \tag{1.11.51}$$

$$\frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s} = -\cos\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s}\hat{\mathbf{N}} - \sin\alpha\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} - \sin\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s}\hat{\mathbf{B}} + \cos\alpha\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} \tag{1.11.52}$$

and using the definitions for the Frenet-Serret formula yields

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = -\sin\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s}\hat{\mathbf{N}} + \cos\alpha\left[-\kappa\hat{\mathbf{T}} + \tau\hat{\mathbf{B}}\right] + \cos\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s}\hat{\mathbf{B}} + \sin\alpha\left[-\tau\hat{\mathbf{N}}\right] \tag{1.11.53}$$

$$\frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s} = -\cos\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s}\hat{\mathbf{N}} - \sin\alpha\left[-\kappa\hat{\mathbf{T}} + \tau\hat{\mathbf{B}}\right] - \sin\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}s}\hat{\mathbf{B}} + \cos\alpha\left[-\tau\hat{\mathbf{N}}\right] \tag{1.11.54}$$

which is more simply written as

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = -\kappa\cos\alpha\hat{\mathbf{T}} - \sin\alpha\left[\tau + \frac{\mathrm{d}\alpha}{\mathrm{d}s}\right]\hat{\mathbf{N}} + \cos\alpha\left[\tau + \frac{\mathrm{d}\alpha}{\mathrm{d}s}\right]\hat{\mathbf{B}} \tag{1.11.55}$$

$$\frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s} = \kappa\sin\alpha\hat{\mathbf{T}} - \cos\alpha\left[\tau + \frac{\mathrm{d}\alpha}{\mathrm{d}s}\right]\hat{\mathbf{N}} - \sin\alpha\left[\tau + \frac{\mathrm{d}\alpha}{\mathrm{d}s}\right]\hat{\mathbf{B}} \tag{1.11.56}$$

and using both (1.11.49) and (1.11.50) we then find

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = -\kappa\cos\alpha\hat{\mathbf{T}} - \left(\tau + \frac{\mathrm{d}\alpha}{\mathrm{d}s}\right)\hat{\mathbf{D}} \tag{1.11.57}$$

$$\frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s} = \kappa\sin\alpha\hat{\mathbf{T}} - \left(\tau + \frac{\mathrm{d}\alpha}{\mathrm{d}s}\right)\hat{\mathbf{n}} \tag{1.11.58}$$

This allows us to identify

$$\kappa_n \equiv a_{Tn} = \kappa\cos\alpha \tag{1.11.59}$$

$$\kappa_g \equiv a_{TD} = \kappa\sin\alpha \tag{1.11.60}$$

$$-\tau_g \equiv a_{nD} = -\tau - \frac{\mathrm{d}\alpha}{\mathrm{d}s} \tag{1.11.61}$$

which completes the Darboux description. We could have chosen a different sign convention, if we had desired. That is, the signs of κ_n and κ_g could be reversed and we would have an equivalent description with a different definition of what κ_n and κ_g are.[102]

To summarize, we now use the right-handed coordinate system $(\hat{\mathbf{T}}, \hat{\mathbf{n}}, \hat{\mathbf{D}})$ with $\hat{\mathbf{D}} = \hat{\mathbf{T}} \times \hat{\mathbf{n}}$ given by

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = \kappa_n\hat{\mathbf{n}} + \kappa_g\hat{\mathbf{D}} \tag{1.11.62}$$

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = -\kappa_n\hat{\mathbf{T}} - \tau_g\hat{\mathbf{D}} \tag{1.11.63}$$

$$\frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s} = -\kappa_g\hat{\mathbf{T}} + \tau_g\hat{\mathbf{n}} \tag{1.11.64}$$

with

$$\kappa_n = \kappa\cos\alpha = -\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s}\cdot\hat{\mathbf{T}} = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}\cdot\hat{\mathbf{n}} \tag{1.11.65}$$

$$\kappa_g = \kappa\sin\alpha = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}\cdot\hat{\mathbf{D}} = -\frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s}\cdot\hat{\mathbf{T}} \tag{1.11.66}$$

$$\kappa = \sqrt{\kappa_g^2 + \kappa_n^2} \tag{1.11.67}$$

$$\tau_g = \tau + \frac{\mathrm{d}\alpha}{\mathrm{d}s} = -\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s}\cdot\hat{\mathbf{D}} = \frac{\mathrm{d}\hat{\mathbf{D}}}{\mathrm{d}s}\cdot\hat{\mathbf{n}} \tag{1.11.68}$$

where α is the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{N}}$ (or $\hat{\mathbf{D}}$ and $\hat{\mathbf{B}}$).

1.11.3 Magnetic Field Line Trajectories

Then entire machinery completely translates when we make the identification $t \to \ell$, where ℓ is the length along a field line label defined by

$$B\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\ell} = \mathbf{B} \tag{1.11.69}$$

[102] Check the convention of whatever source you are using. I have used the usual mathematical convention here, but as we will see in the magnetic field case, other conventions for the signs of κ_g, κ_n, and τ_g are sometimes applied.

where $\mathbf{B}$ ($B = |\mathbf{B}|$) represents the magnetic field line and ***not*** the binormal vector. We note that we can rewrite this as

$$\mathbf{B} \cdot \nabla \mathbf{x} = B\hat{\mathbf{b}} \cdot \nabla \mathbf{x} = \mathbf{B} \tag{1.11.70}$$

$$\hat{\mathbf{b}} \cdot \nabla \mathbf{x} = \hat{\mathbf{b}} \tag{1.11.71}$$

by our definition of ℓ. It is then unfortunate that the binormal vector is usually also represented by a capital $\hat{\mathbf{B}}$ (Often the hat is left off, so that it is indistinguishable from our magnetic field $\mathbf{B}$. However, even if lower case letters are used we come into conflict with notation. Thus I will refer to the binormal direction with a new variable.). We will instead define our binormal as $\hat{\boldsymbol{\tau}}$.

Thus, we associate $(s, \hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}})$ with $(\ell, \hat{\mathbf{b}}, \hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{\beta}})$ to write

$$\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}\ell} = \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} = \kappa \hat{\boldsymbol{\kappa}} \tag{1.11.72}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{\kappa}}}{\mathrm{d}\ell} = \hat{\mathbf{b}} \cdot \nabla \hat{\boldsymbol{\kappa}} = -\kappa \hat{\mathbf{b}} + \tau \hat{\boldsymbol{\beta}} \tag{1.11.73}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{\beta}}}{\mathrm{d}\ell} = \hat{\mathbf{b}} \cdot \nabla \hat{\boldsymbol{\beta}} = -\tau \hat{\boldsymbol{\kappa}} \tag{1.11.74}$$

The choice of normal vector is informed by the magnetic curvature vector being defined as

$$\boldsymbol{\kappa} = \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \tag{1.11.75}$$

so that the normal vector $\hat{\boldsymbol{\kappa}}$ is given by

$$\boldsymbol{\kappa} = \kappa \hat{\boldsymbol{\kappa}} \tag{1.11.76}$$

with $\boldsymbol{\kappa}$ meaning the magnetic curvature and κ the magnitude of the magnetic curvature. We can define another vector based on the torsion. We use that the scalar torsion is given by $\tau = [\hat{\mathbf{b}} \cdot \nabla \hat{\boldsymbol{\beta}}] \cdot \hat{\boldsymbol{\kappa}}$. Therefore we define a torsion vector via

$$\boldsymbol{\tau} = -\tau \hat{\boldsymbol{\kappa}} \tag{1.11.77}$$

Note that it is not directly related to $\hat{\boldsymbol{\beta}}$, the binormal but to $\frac{\mathrm{d}\hat{\boldsymbol{\beta}}}{\mathrm{d}\ell} = \hat{\mathbf{b}} \cdot \nabla \hat{\boldsymbol{\beta}}$.

One should be on guard against all sorts of notations used for magnetic field line Frenet-Serret formula. The use of $\mathbf{B}$ for the binormal causes all sorts of headaches since $\mathbf{B}$ universally stands for magnetic fields. My notation is simple, but hardly universal. Whatever you can say about Frenet-Serret for particle trajectories can be said for my notation with the translation $(s, \hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}, \kappa, \tau) \rightarrow (\ell, \hat{\mathbf{b}}, \hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{\beta}}, \kappa, \tau)$. In one way, our equations are even simpler. There is no reason to parameterize with a time-like variable t for magnetic field lines.

To summarize, we have

$$\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}\ell} = \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} = \kappa\hat{\boldsymbol{\kappa}} \tag{1.11.78}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{\kappa}}}{\mathrm{d}\ell} = \hat{\mathbf{b}} \cdot \nabla\hat{\boldsymbol{\kappa}} = -\kappa\hat{\mathbf{b}} + \tau\hat{\boldsymbol{\beta}} \tag{1.11.79}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{\beta}}}{\mathrm{d}\ell} = \hat{\mathbf{b}} \cdot \nabla\hat{\boldsymbol{\tau}} = -\tau\hat{\boldsymbol{\kappa}} \tag{1.11.80}$$

$$\kappa = \left|\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}\ell}\right| = \left|\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}\right| \tag{1.11.81}$$

$$\tau = \left|\frac{\mathrm{d}\hat{\boldsymbol{\beta}}}{\mathrm{d}\ell}\right| = \left|\hat{\mathbf{b}} \cdot \nabla\hat{\boldsymbol{\beta}}\right| = \left|\hat{\mathbf{b}} \cdot \nabla\left(\hat{\mathbf{b}} \times \hat{\boldsymbol{\kappa}}\right)\right| = \left|\hat{\mathbf{b}} \times [\hat{\mathbf{b}} \cdot \nabla\hat{\boldsymbol{\kappa}}]\right| \tag{1.11.82}$$

$$\boldsymbol{\kappa} = \kappa\hat{\boldsymbol{\kappa}} \tag{1.11.83}$$

$$\boldsymbol{\tau} = -\tau\hat{\boldsymbol{\kappa}} = \hat{\mathbf{b}} \times [\hat{\mathbf{b}} \cdot \nabla\hat{\boldsymbol{\kappa}}] = \hat{\mathbf{b}} \times \tau\hat{\boldsymbol{\beta}} = \tau\hat{\mathbf{b}} \times \hat{\boldsymbol{\beta}} \tag{1.11.84}$$

with unit tangent vector $\hat{\mathbf{b}}$, unit normal vector $\hat{\boldsymbol{\kappa}}$, unit binormal vector $\hat{\boldsymbol{\beta}}$, magnetic curvature vector $\boldsymbol{\kappa} = \kappa\hat{\boldsymbol{\kappa}}$, torsion vector $\boldsymbol{\tau} = -\tau\hat{\boldsymbol{\kappa}}$, curvature κ, and torsion τ. Our magnetic field arclength (usually just called length along the field) ℓ has $\frac{\mathrm{d}}{\mathrm{d}\ell} \leftrightarrow \hat{\mathbf{b}} \cdot \nabla$.

1.11.4 Magnetic Darboux Frames

Magnetic Darboux frame notation suffers from the same problems as Frenet-Serret notation and is not completely standardized. In magnetic confinement, the surface that is of interest is a magnetic flux surface. The normal is usually given by $\nabla\psi$. We define the unit normal as $\hat{\mathbf{n}} \equiv \hat{\boldsymbol{\psi}} = \nabla\psi/|\nabla\psi|$, and form the right-handed coordinate system $(\hat{\mathbf{n}}, \hat{\mathbf{b}}, \hat{\boldsymbol{\eta}})$ where $\hat{\boldsymbol{\eta}} = \hat{\mathbf{n}} \times \hat{\mathbf{b}}$ and $\hat{\mathbf{b}}$ is the magnetic field tangent vector.

In an ideal world, the magnetic standard would follow the Darboux standard, but, alas, we do not live in such a world. In Section 1.11.2 our order $(\hat{\mathbf{T}}, \hat{\mathbf{n}}, \hat{\mathbf{D}})$ would correspond to $(\hat{\mathbf{b}}, \hat{\mathbf{n}}, \hat{\mathbf{b}} \times \hat{\mathbf{n}})$, which is different than the typical order used in magnetic coordinates $(\hat{\mathbf{n}}, \hat{\mathbf{b}}, \hat{\mathbf{n}} \times \hat{\mathbf{b}})$. In particular, the binormal vector direction is opposite that of our mathematical Darboux frame from before. Thus when doing any identifications we must use $\hat{\boldsymbol{\eta}} = -\hat{\mathbf{D}}$. The one other difference is that what is usually called the geodesic torsion τ_g is now called the normal torsion and has $\tau_n = -\tau_g$. This is simply because we find the normal torsion via $\hat{\mathbf{b}} \cdot \nabla\eta \cdot \hat{\mathbf{n}}$, or dotting $\hat{\mathbf{n}}$ into the derivative of the binormal vector $\hat{\boldsymbol{\eta}} = \hat{\mathbf{n}} \times \mathbf{b}$. We'd prefer it to be this rather than having another minus sign, and so it is simply defined this way. This is simply the way Darboux frames and Frenet-Serret trajectories work; there are arbitrary sign conventions and you should try to use what is most common.

This means our equations become

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}\ell} = \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{n}} = -\kappa_n\hat{\mathbf{b}} - \tau_n\hat{\boldsymbol{\eta}} \tag{1.11.85}$$

$$\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}\ell} = \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} = \kappa_n\hat{\mathbf{n}} - \kappa_g\hat{\boldsymbol{\eta}} \tag{1.11.86}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{\eta}}}{\mathrm{d}\ell} = \hat{\mathbf{b}} \cdot \nabla\hat{\boldsymbol{\eta}} = \tau_n\hat{\mathbf{n}} + \kappa_g\hat{\mathbf{b}} \tag{1.11.87}$$

with the same angle relations, though this time α is the angle between $\hat{\mathbf{n}}$ and $\hat{\boldsymbol{\kappa}}$ (or $-\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\beta}}$)

$$\kappa_n = -\kappa\cos\alpha = \boldsymbol{\kappa}\cdot\hat{\mathbf{n}} = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot\hat{\mathbf{n}} \tag{1.11.88}$$

$$\kappa_g = -\kappa\sin\alpha = -\boldsymbol{\kappa}\cdot\hat{\boldsymbol{\eta}} = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{n}}) \tag{1.11.89}$$

$$\kappa = \sqrt{\kappa_g^2+\kappa_n^2} \tag{1.11.90}$$

$$\tau_n = \tau + \frac{\mathrm{d}\alpha}{\mathrm{d}s} = -\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{n}}\cdot\hat{\boldsymbol{\eta}} = \hat{\mathbf{b}}\cdot\nabla\hat{\boldsymbol{\eta}}\cdot\hat{\mathbf{n}} \tag{1.11.91}$$

Remember that with this notation the geodesic curvature is the curvature along the magnetic surface while the normal curvature is the curvature perpendicular to magnetic surface. It can also have our geometric geodesic interpretation from the mathematical convention.

You might worry that τ_n is difficult to calculate via the angle definition as we need α all over the surface, but remember we can use that τ_n is given by

$$\begin{aligned}\tau_n = \hat{\mathbf{b}}\cdot\nabla\hat{\boldsymbol{\eta}}\cdot\hat{\mathbf{n}} &= \nabla(\frac{\hat{\mathbf{b}}\cdot\cancel{\hat{\boldsymbol{\eta}}}}{2})\cdot\hat{\mathbf{n}} + (\boldsymbol{\nabla}\times\hat{\mathbf{b}})\cdot(\hat{\boldsymbol{\eta}}\times\hat{\mathbf{n}}) + (\boldsymbol{\nabla}\times\hat{\boldsymbol{\eta}})\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{n}})\\ &= (\boldsymbol{\nabla}\times\hat{\mathbf{b}})\cdot(\hat{\boldsymbol{\eta}}\times\hat{\mathbf{n}}) + (\boldsymbol{\nabla}\times\hat{\boldsymbol{\eta}})\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{n}})\end{aligned} \tag{1.11.92}$$

If we used the right-handed coordinate system $(\nabla\psi, \mathbf{B}, \boldsymbol{\eta})$ (note it is orthogonal but not orthonormal) for $\mathbf{B}$ the magnetic field with a defined quantity $\boldsymbol{\eta} = \nabla\psi\times\mathbf{B}/|\nabla\psi|^2$ so that $\boldsymbol{\eta} = \hat{\boldsymbol{\eta}}\frac{B}{|\nabla\psi|}$ we can then write

$$\hat{\mathbf{b}}\cdot\nabla\hat{\boldsymbol{\eta}}\cdot\hat{\mathbf{n}} = \frac{\mathbf{B}\cdot\boldsymbol{\eta}\cdot\nabla\psi}{B|\nabla\psi||\boldsymbol{\eta}|} + \frac{\mathbf{B}\cdot\nabla(\frac{1}{|\boldsymbol{\eta}|})\cancel{\boldsymbol{\eta}\cdot\nabla\psi}}{B|\nabla\psi|} \tag{1.11.93}$$

$$= \frac{\mathbf{B}\cdot\nabla\boldsymbol{\eta}\cdot\nabla\psi}{B^2} \tag{1.11.94}$$

which then implies

$$\begin{aligned}B^2\tau_n &= (\boldsymbol{\nabla}\times\mathbf{B})\cdot(\boldsymbol{\eta}\times\nabla\psi) + (\boldsymbol{\nabla}\times\boldsymbol{\eta})\cdot(\mathbf{B}\times\nabla\psi)\\ &= \mu_0\mathbf{J}\cdot(|\nabla\psi|\frac{B}{|\nabla\psi|}\frac{\mathbf{B}}{B}) - (\boldsymbol{\nabla}\times\boldsymbol{\eta})\cdot|\nabla\psi|^2\boldsymbol{\eta}\end{aligned} \tag{1.11.95}$$

or

$$\tau_n = \frac{\mu_0\mathbf{B}\cdot\mathbf{J}}{B^2} - \frac{1}{|\boldsymbol{\eta}|^2}\boldsymbol{\eta}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\eta}) \tag{1.11.96}$$

The last term is defined to be the magnetic shear and is equivalent to $\hat{\boldsymbol{\eta}}\cdot(\boldsymbol{\nabla}\times\hat{\boldsymbol{\eta}})$ because $\hat{\boldsymbol{\eta}}\cdot\nabla|\boldsymbol{\eta}| = 0$. The first term is proportional to the parallel current (that is, the current parallel to the magnetic field).

1.11.5 Interpretation of Frenet-Serret Quantities

It is worth delving into the interpretations of quantities in the Frenet-Serret description. First, $\hat{\mathbf{T}} \leftrightarrow \hat{\mathbf{b}}$ is simply the tangent vector to the particle trajectory or magnetic field line, respectively. The normal vector $\hat{\mathbf{N}} \leftrightarrow \hat{\boldsymbol{\kappa}}$ is related to the curvature vector of the trajectory or field line. It points toward the center of the circle that could be fitted to that point along the trajectory. Then

κ, the curvature, itself, tells us the radius of that fitted circle via $\kappa = 1/R$. Finally, the binormal is simply the direction perpendicular to the tangent and normal directions. The torsion measures the rotation of the binormal vector, and so it is usually described as saying how "spring-like" the trajectory is. A curve with zero torsion is a planar curve (that is the curve is completely in a plane). Thus the smaller the torsion, the more slowly the trajectory departs from a plane. If the trajectory is helical-like then the trajectory can be pictured as being in a highly compressed spring.[103] For a large torsion with a near helical trajectory, the curve quickly departs from a plane and so the trajectory can be pictured as a more stretched out spring.

A helical trajectory shows this most easily, for it is a curve with constant curvature and constant torsion (both nonzero) and so is easy to picture. We can see some examples in Figures 1.10 and 1.11. Here $t = 0$ is at $(x, y, z) = (0, 1, 0)$.

The existence of a Darboux frame has the same limitations as the Frenet-Serret restrictions. The curve or trajectory cannot be degenerate. You will sometimes read that the Darboux frame is undefined at umbilical points,[104] but this is a different Darboux frame. This is a Darboux frame for a surface rather than for a curve. The curve allows us to uniquely pick out directions, even at umbilical points.

[103] The trajectory does not actually have to look like a spring, but for magnetic field lines we often care about situations like this. We care about nearly helical trajectories and so this is a viable interpretation.

[104] A delightful name that simply means in a small region near the point the surface is equivalent to a sphere, and so the curvature is the same along all directions.

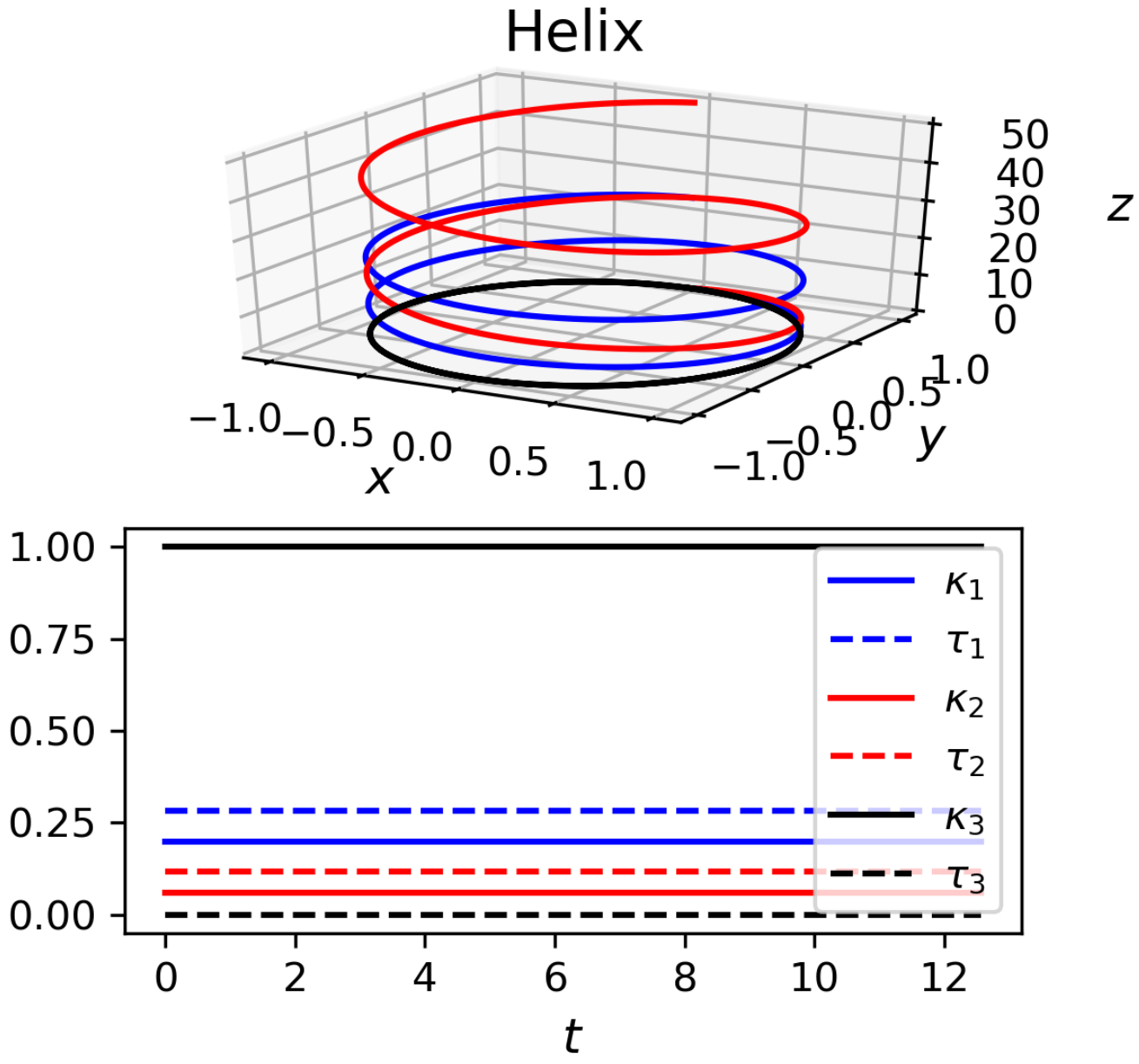

Figure 1.10: Helices with varying pitch.

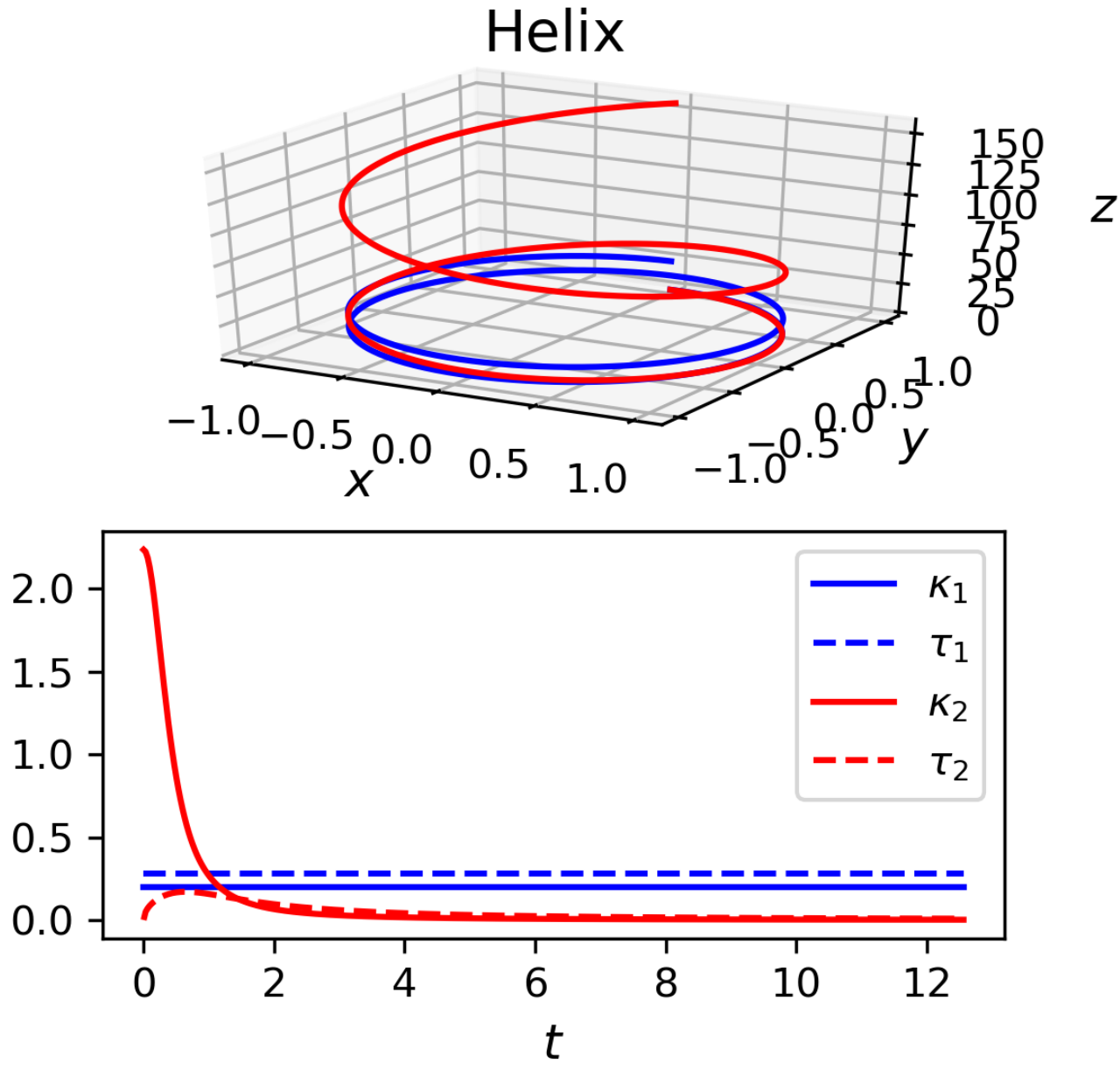

Figure 1.11: Helix with a given pitch against a curve that changes more rapidly in z.

1.11.6 Multiple Dimension Generalizations

You might wonder how one generalizes to higher dimensional analogues. For Euclidean spaces of n dimensions, we can still use the arclength parameterization. Thus, we have the unit tangent and unit normal defined the exact same way

$$\hat{\mathbf{e}}_1 \equiv \hat{\mathbf{T}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} \tag{1.11.97}$$

$$\hat{\mathbf{e}}_2 \equiv \hat{\mathbf{N}} = \frac{1}{\kappa}\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = \frac{1}{\kappa}\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}s^2} \tag{1.11.98}$$

with $\kappa = \left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}\right|$. We then form normals in a manner similar to Gram-Schmidt orthogonalization. We simply take another derivative with s and subtract off any portions along our previous directions. Thus the binormal, trinormal, quaternormal, quinquenormal, etc. can be formed.[105] Let $\hat{\mathbf{e}}_3$ be the binormal and $\hat{\mathbf{e}}_{n+1}$ the n-normal. The n can then also stand for the number of derivatives. Then the j-normal ($j > 2$) is formed by

$$\mathbf{e}_j = \frac{\mathrm{d}^j\mathbf{x}}{\mathrm{d}s^j} - \sum_{i=1}^{j-1}\frac{\mathrm{d}^j\mathbf{x}}{\mathrm{d}s^j}\cdot\hat{\mathbf{e}}_i\hat{\mathbf{e}}_i \tag{1.11.99}$$

$$\hat{\mathbf{e}}_j = \frac{\mathbf{e}_j}{|\mathbf{e}_j|} \tag{1.11.100}$$

Note how this may produce a different formula in three dimensions based on the definition of the cross product, but only by changing the sign on the torsion and the direction of the binormal. Now we know that $\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s}$ can have no component along $\hat{\mathbf{e}}_j$, but the other coefficients have to be calculated explicitly

$$\frac{\mathrm{d}\hat{\mathbf{e}}_1}{\mathrm{d}s} = \kappa\hat{\mathbf{e}}_2 \tag{1.11.101}$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_2}{\mathrm{d}s} = a_{21}\hat{\mathbf{e}}_1 + \sum_{i=3}^{n} a_{2i}\hat{\mathbf{e}}_i \tag{1.11.102}$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} = \sum_{i=1}^{j-1} a_{ji}\hat{\mathbf{e}}_i + \sum_{i=j+1}^{n} a_{ji}\hat{\mathbf{e}}_i \tag{1.11.103}$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_n}{\mathrm{d}s} = \sum_{i=2}^{n} a_{ni}\hat{\mathbf{e}}_i \tag{1.11.104}$$

We can also explicitly calculate $\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s}$ using

$$\frac{\mathrm{d}\mathbf{e}_j}{\mathrm{d}s} = \frac{\mathrm{d}|\mathbf{e}_j|}{\mathrm{d}s}\hat{\mathbf{e}}_j + |\mathbf{e}_j|\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} \tag{1.11.105}$$

$$\frac{\mathrm{d}\mathbf{e}_j}{\mathrm{d}s} = \frac{\mathbf{e}_j}{|\mathbf{e}_j|}\cdot\frac{\mathrm{d}\mathbf{e}_j}{\mathrm{d}s}\hat{\mathbf{e}}_j + |\mathbf{e}_j|\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} \tag{1.11.106}$$

$$\frac{\mathrm{d}\mathbf{e}_j}{\mathrm{d}s} = \hat{\mathbf{e}}_j\cdot\frac{\mathrm{d}\mathbf{e}_j}{\mathrm{d}s}\hat{\mathbf{e}}_j + |\mathbf{e}_j|\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} \tag{1.11.107}$$

$$\frac{\mathrm{d}\mathbf{e}_j}{\mathrm{d}s} = |\mathbf{e}_j|\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} \tag{1.11.108}$$

$$\tag{1.11.109}$$

[105] As you can see, these use Latin number terms before normal.

where the last line comes from using $\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} \cdot \hat{\mathbf{e}}_j = 0$ via

$$\hat{\mathbf{e}}_j \cdot \frac{\mathrm{d}\mathbf{e}_j}{\mathrm{d}s} = \overbrace{\frac{\mathrm{d}(\hat{\mathbf{e}}_j \cdot \mathbf{e}_j)}{\mathrm{d}s}}^{=0} - \mathbf{e}_j \cdot \frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} = -|\mathbf{e}_j| \overbrace{\hat{\mathbf{e}}_j \cdot \frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s}}^{=0} = 0 \tag{1.11.110}$$

We then have

$$\frac{\mathrm{d}\mathbf{e}_j}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}^j\mathbf{x}}{\mathrm{d}s^j} - \sum_{i=1}^{j-1} \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}^j\mathbf{x}}{\mathrm{d}s^j} \cdot \hat{\mathbf{e}}_i\hat{\mathbf{e}}_i\right] \tag{1.11.111}$$

$$= \frac{\mathrm{d}^{j+1}\mathbf{x}}{\mathrm{d}s^{j+1}} - \sum_{i=1}^{j-1} \left[\frac{\mathrm{d}^{j+1}\mathbf{x}}{\mathrm{d}s^{j+1}} \cdot \hat{\mathbf{e}}_i\hat{\mathbf{e}}_i + \frac{\mathrm{d}^j\mathbf{x}}{\mathrm{d}s^j} \cdot \frac{\mathrm{d}\hat{\mathbf{e}}_i}{\mathrm{d}s}\hat{\mathbf{e}}_i + \frac{\mathrm{d}^j\mathbf{x}}{\mathrm{d}s^j} \cdot \hat{\mathbf{e}}_i\frac{\mathrm{d}\hat{\mathbf{e}}_i}{\mathrm{d}s}\right] \tag{1.11.112}$$

$$= \mathbf{e}_{j+1} + \frac{\mathrm{d}^{j+1}\mathbf{x}}{\mathrm{d}s^{j+1}} \cdot \mathbf{e}_j\mathbf{e}_j - \sum_{i=1}^{j-1} \left[\frac{\mathrm{d}^j\mathbf{x}}{\mathrm{d}s^j} \cdot \frac{\mathrm{d}\hat{\mathbf{e}}_i}{\mathrm{d}s}\hat{\mathbf{e}}_i + \frac{\mathrm{d}^j\mathbf{x}}{\mathrm{d}s^j} \cdot \hat{\mathbf{e}}_i\frac{\mathrm{d}\hat{\mathbf{e}}_i}{\mathrm{d}s}\right] \tag{1.11.113}$$

The last line comes from the definition of $\mathbf{e}_{j+1}$ but seeing that we are missing one term $i = j$ in the summation. Now take $\cdot\hat{\mathbf{e}}_k$ to see what the kth component would be

$$\frac{\mathrm{d}\mathbf{e}_j}{\mathrm{d}s} \cdot \hat{\mathbf{e}}_k = |\mathbf{e}_{j+1}|\delta_{j+1,k} + \frac{\mathrm{d}^{j+1}\mathbf{x}}{\mathrm{d}s^{j+1}} \cdot \mathbf{e}_j\delta_{j,k} + \sum_{i=1}^{j-1} \left[\delta_{i,k}\frac{\mathrm{d}^j\mathbf{x}}{\mathrm{d}s^j} \cdot \frac{\mathrm{d}\hat{\mathbf{e}}_i}{\mathrm{d}s} + \frac{\mathrm{d}^j\mathbf{x}}{\mathrm{d}s^j} \cdot \hat{\mathbf{e}}_i\frac{\mathrm{d}\hat{\mathbf{e}}_i}{\mathrm{d}s} \cdot \hat{\mathbf{e}}_k\right] \tag{1.11.114}$$

which implies for $k > j$ the only contribution is from $k = j + 1$ and so

$$\frac{\mathrm{d}\mathbf{e}_j}{\mathrm{d}s} \cdot \hat{\mathbf{e}}_{j+1} = |\mathbf{e}_{j+1}| \tag{1.11.115}$$

which means that

$$\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} \cdot \hat{\mathbf{e}}_{j+1} = \frac{|\mathbf{e}_{j+1}|}{|\mathbf{e}_j|} \tag{1.11.116}$$

Remember that the contribution for $k = j$ must be zero, because we already know that $\hat{\mathbf{e}}_j \cdot \frac{\partial \mathbf{e}_j}{\partial s} = 0$ and so $\frac{\mathrm{d}^{j+1}\mathbf{x}}{\mathrm{d}s^{j+1}} \cdot \mathbf{e}_j = 0$ must be true.

For $k < j$ we can use the identity

$$\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} \cdot \hat{\mathbf{e}}_k = -\frac{\mathrm{d}\hat{\mathbf{e}}_k}{\mathrm{d}s} \cdot \hat{\mathbf{e}}_j \tag{1.11.117}$$

via $\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k = 0$ for $j \neq k$.

The only restrictions will then be that $1 \leq k \leq n$ for these. This implies that $a_{ij} = 0$ unless $i - j = \pm 1$ in which case it is given by $a_{ij} = (j - i)\chi_{\min(i,j)}$. We can make a simplification by defining

$$\chi_j = \frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} \cdot \hat{\mathbf{e}}_{j+1} \tag{1.11.118}$$

and find

$$\begin{aligned}
\frac{\mathrm{d}\hat{\mathbf{e}}_1}{\mathrm{d}s} &= \chi_1\hat{\mathbf{e}}_2 \\
\frac{\mathrm{d}\hat{\mathbf{e}}_2}{\mathrm{d}s} &= -\chi_1\hat{\mathbf{e}}_1 + \chi_2\hat{\mathbf{e}}_3 \\
\vdots &= \vdots \\
\frac{\mathrm{d}\hat{\mathbf{e}}_j}{\mathrm{d}s} &= -\chi_{j-1}\hat{\mathbf{e}}_{j-1} + \chi_j\hat{\mathbf{e}}_{j+1} \\
\vdots &= \vdots \\
\frac{\mathrm{d}\hat{\mathbf{e}}_n}{\mathrm{d}s} &= -\chi_{n-1}\hat{\mathbf{e}}_{n-1}
\end{aligned} \tag{1.11.119}$$

which in a mnemonic matrix form would read

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \vdots \\ \hat{\mathbf{e}}_j \\ \vdots \\ \hat{\mathbf{e}}_{n-1} \\ \hat{\mathbf{e}}_n \end{bmatrix} = \begin{bmatrix} 0 & \chi_1 & 0 & \dots & 0 & 0 & \dots \\ -\chi_1 & 0 & \chi_2 & \dots & 0 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 & \dots \\ 0 & 0 & -\chi_{j-1} & 0 & \chi_j & 0 & \dots \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \dots \\ 0 & \dots & \dots & \dots & -\chi_{n-1} & \dots & \chi_{n-1} \\ 0 & \dots & \dots & \dots & \dots & -\chi_{n-1} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \vdots \\ \hat{\mathbf{e}}_j \\ \vdots \\ \hat{\mathbf{e}}_{n-1} \\ \hat{\mathbf{e}}_n \end{bmatrix} \tag{1.11.120}$$

One final consideration is for relativistic calculations. The first problem we have is that the "arclength" in relativity can be negative. Thus, we must restrict ourselves to either purely timelike or purely spacelike curves, so that we can parameterize the surface easily by the "arclength". We will call the new arclength τ because the logical choice will turn out to be the proper time along the curve. We can use the spacetime difference $s = \mathbb{X}\cdot\mathbb{X}$ for $\mathbb{X}$ the position four-vector. Then we simply apply the exact same methods as above. For purely timelike curves[106] (the curves we'd expect of objects in our universe) we even now have the advantage of not worrying about degenerate curves because we are including the time coordinate in our description rather than using it directly as a parameterization. There are other subtleties when calculating, though. Derivatives must always be done with the proper total derivative (that is, covariant derivative) and never just take derivatives of the components, of course. Now, given a position vector, the $s < 0$ for a timelike curve with our metric and a spacelike curve has $s > 0$. The extra $-$ sign on the time direction ($\hat{\mathbf{e}}_0$) in dot products introduces an extra minus sign in the formulation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} \hat{\mathbf{e}}_0 \\ \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{bmatrix} = \begin{bmatrix} 0 & \chi_0 & 0 & 0 \\ \chi_0 & 0 & \chi_1 & 0 \\ 0 & -\chi_1 & 0 & \chi_2 \\ 0 & 0 & -\chi_2 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{e}}_0 \\ \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{bmatrix} \tag{1.11.121}$$

where the usual curvature $\chi_1 = \kappa$ and torsion $\chi_2 = \tau$ (this τ is torsion, not proper time; all other τ's from now on are proper times) in three dimensions reappear.

[106] Technically, we can handle null curves, as well. Thus as long as a curve is purely non-spacelike or purely non-timelike we can handle it.

The reason for the change in sign on χ_0 is that we have to be careful of doing dot products correctly now. Suppose we denote our standard tangent basis as $\mathbf{e}_\mu$. The tangent-reciprocal basis will then be $\mathbf{e}^\mu$ (remember this has $\mathbf{e}_0 = \hat{\mathbf{x}}_0$ and $\mathbf{e}^0 = -\hat{\mathbf{x}}_0$) with $\mathbf{e}_0 \cdot \mathbf{e}^0 = \hat{\mathbf{x}}_0 \cdot -\hat{\mathbf{x}}_0 = 1$ with our special dot product and so $\hat{\mathbf{x}}_0 \cdot \hat{\mathbf{x}}_0 = \mathbf{e}_0 \cdot \mathbf{e}_0 = -1$. That is, remember that the standard tangent basis and reciprocal-tangent basis have $\mathbf{e}_i = \mathbf{e}^i$ (the Latin i means only $1, 2, 3$) and only $\mathbf{e}_0 \neq \mathbf{e}^0$. To prevent confusion I will use e_μ below for the new basis set used by Frenet-Serret formulas and $\hat{\mathsf{e}}_\mu$ for the new unit vectors (I will make an exception for $\hat{\mathsf{e}}_0$ which isn't strictly a unit vector for the sake of uniform notation, the same as $\hat{\mathbf{x}}_0$). These are not the standard tangent basis vectors $\mathbf{e}_\mu$ so don't confuse $\mathbf{e}_\mu$ with $\hat{\mathsf{e}}_\mu$.

$$\mathsf{e}_0 = \frac{\mathrm{d}\mathbb{X}}{\mathrm{d}\tau} \tag{1.11.122}$$

$$\mathsf{e}_1 = \frac{\mathrm{d}^2\mathbb{X}}{\mathrm{d}\tau^2} - \cancel{\frac{\mathrm{d}^2\mathbb{X}}{\mathrm{d}\tau^2} \cdot \mathsf{e}_0}\mathsf{e}_0 \tag{1.11.123}$$

$$\mathsf{e}_2 = \frac{\mathrm{d}^3\mathbb{X}}{\mathrm{d}\tau^3} - \frac{\mathrm{d}^3\mathbb{X}}{\mathrm{d}\tau^3} \cdot [\mathsf{e}_0\mathsf{e}_0 + \mathsf{e}_1\mathsf{e}_1] \tag{1.11.124}$$

$$\mathsf{e}_3 = \frac{\mathrm{d}^4\mathbb{X}}{\mathrm{d}\tau^4} - \frac{\mathrm{d}^4\mathbb{X}}{\mathrm{d}\tau^4} \cdot [\mathsf{e}_0\mathsf{e}_0 + \mathsf{e}_1\mathsf{e}_1 + \mathsf{e}_2\mathsf{e}_2] \tag{1.11.125}$$

$$\tag{1.11.126}$$

Now we have to find out if e_0 is a unit vector (or something similar) to justify the cancellation in e_1. We will use the proper time τ as a parameterization, where σ is simply any time coordinate (one could choose the proper time for ease of calculation) and 0 to t parameterizes the curves endpoints

$$c\tau = \int_0^t \mathrm{d}\sigma \, \sqrt{-\frac{\mathrm{d}\mathbb{X}}{\mathrm{d}\sigma} \cdot \frac{\mathrm{d}\mathbb{X}}{\mathrm{d}\sigma}} \equiv \int_0^t \mathrm{d}\sigma \, \left|\frac{\mathrm{d}\mathbb{X}}{\mathrm{d}\sigma}\right| \tag{1.11.127}$$

(generically I will set $c = 1$) where we must remember that the dot product is now the relativistic one and so we need the extra minus sign when considering timelike curves.[107] Thus, we have for proper time τ that

$$\frac{\mathrm{d}\mathbb{X}}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}\mathbb{X}}{\mathrm{d}\tau} = \frac{\mathrm{d}\mathbb{X}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbb{X}}{\mathrm{d}t} \left|\frac{\mathrm{d}t}{\mathrm{d}\tau}\right|^2 = \frac{\frac{\mathrm{d}\mathbb{X}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbb{X}}{\mathrm{d}t}}{\left|\frac{\mathrm{d}\mathbb{X}}{\mathrm{d}t}\right|^2} \tag{1.11.128}$$

Now because this is a timelike curve then $\frac{\mathrm{d}\mathbb{X}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbb{X}}{\mathrm{d}t} < 0$, as we will show later. Then it is not a unit vector as the vector dotted into itself yields a -1. To see this explicitly, let's say we are measuring $\mathbb{X}$ in the proper time frame where the particle is simply stationary, then $\mathbb{X} = \tau\hat{\mathbf{x}}_0$ (setting $c = 1$ for convenience) and we have $\mathrm{d}\mathbb{X}/\mathrm{d}\tau \cdot \mathrm{d}\mathbb{X}/\mathrm{d}\tau = \hat{\mathbf{x}}_0 \cdot \hat{\mathbf{x}}_0 = -1$ hence the extra minus sign. It turns out that $\frac{\mathrm{d}\mathbb{X}}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}\mathbb{X}}{\mathrm{d}\tau}$ is invariant for τ the proper time, and negative with our sign convention so that timelike really does guarantee an overall minus sign (in fact it guarantees a minus sign for any generic time coordinate and for spacelike curves, as well). We must be conscious of this sign, as we will see, to account for the different signs for the Frenet-Serret formulas. Note that we just found $\hat{\mathsf{e}}_0 \cdot \hat{\mathsf{e}}_0 = -1$ in general where I put a hat on e_0 even though it is not a unit vector because it is close enough for us, and it simplifies our notation.

[107] If we were considering spacelike curves we would still need a positive sign. If we used the other metric signature $(+, -, -, -)$, we would have a positive sign.

Let's prove what I just asserted. We have $\mathbb{X} = ct\hat{\mathbf{x}}_0 + x_1\hat{\mathbf{x}}_1 + x_2\hat{\mathbf{x}}_2 + x_3\hat{\mathbf{x}}_3 \equiv ct\hat{\mathbf{x}}_0 + \mathbf{x}$ and defining $\mathbf{V}_\tau = \mathrm{d}\mathbf{x}/\mathrm{d}\tau$ so

$$\begin{aligned}\mathbb{V} \equiv \frac{\mathrm{d}\mathbb{X}}{\mathrm{d}\tau} &= c\frac{\mathrm{d}t}{\mathrm{d}\tau}\hat{\mathbf{x}}_0 + \frac{\mathrm{d}x_1}{\mathrm{d}\tau}\hat{\mathbf{x}}_1 + \frac{\mathrm{d}x_2}{\mathrm{d}\tau}\hat{\mathbf{x}}_2 + \frac{\mathrm{d}x_3}{\mathrm{d}\tau}\hat{\mathbf{x}}_3 \\ &= c\frac{\mathrm{d}t}{\mathrm{d}\tau}\hat{\mathbf{x}}_0 + \mathbf{V}_\tau\end{aligned} \tag{1.11.129}$$

$$\begin{aligned}\mathbb{V}\cdot\mathbb{V} &= \left(c\frac{\mathrm{d}t}{\mathrm{d}\tau}\hat{\mathbf{x}}_0 + \mathbf{V}_\tau\right)\cdot\left(c\frac{\mathrm{d}t}{\mathrm{d}\tau}\hat{\mathbf{x}}_0 + \mathbf{V}_\tau\right) \\ &= -c^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + V_\tau^2\end{aligned} \tag{1.11.130}$$

Now we need to remember that in General Relativity it is convention to assign the proper time the relation $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma$ with $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = V/c$ where $\mathbf{V}$ is the velocity associated with the time t and $V^2 = \mathbf{V}\cdot\mathbf{V}$.[108] We then use $\mathbf{V}_\tau = \frac{\mathrm{d}t}{\mathrm{d}\tau}\mathbf{V} = \gamma\mathbf{V}$ where $\mathbf{V} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$ so that $V_\tau^2 = \gamma^2 V^2$. We have then found

$$\mathbb{V}\cdot\mathbb{V} = -\gamma^2c^2 + \gamma^2V^2 = c^2\gamma^2\left(\beta^2-1\right) = c^2\frac{\beta^2-1}{1-\beta^2} = -c^2 \tag{1.11.131}$$

so that $\mathrm{d}\mathbb{X}/\mathrm{d}t\cdot\mathrm{d}\mathbb{X}/\mathrm{d}t = -c^2|\mathrm{d}\tau/\mathrm{d}t|^2 < 0$ always, as advertised. I will switch back to $c=1$ units so that we get -1 instead, for convenience. Note that this offers us a new way of writing our integral for the proper time (with σ_τ the proper time and σ_t some other frame's time)

$$c\tau = \int_0^t \mathrm{d}\sigma_\tau\ c = \int_0^t \frac{\mathrm{d}\sigma_t c}{\gamma(\sigma_t)} \tag{1.11.132}$$

Being a constant is also good enough for the cancellation above because we still have

$$\begin{aligned}\frac{\partial(\hat{\mathsf{e}}_0\cdot\hat{\mathsf{e}}_0)}{\partial\tau} &= \frac{\partial(-1)}{\partial\tau} = 0 \\ \hat{\mathsf{e}}_0\cdot\frac{\partial^2\mathbb{X}}{\partial\tau^2} &= \hat{\mathsf{e}}_0\cdot\frac{\partial\hat{\mathsf{e}}_0}{\partial\tau} = \frac{1}{2}\frac{\partial(\hat{\mathsf{e}}_0\cdot\hat{\mathsf{e}}_0)}{\partial\tau} = 0\end{aligned}$$

We can now multiply by $\cdot\mathsf{e}_1$ to find

$$\frac{\mathrm{d}\hat{\mathsf{e}}_0}{\mathrm{d}\tau}\cdot\mathsf{e}_1 = \frac{\mathrm{d}^2\mathbb{X}}{\mathrm{d}\tau^2}\cdot\mathsf{e}_1 = \mathsf{e}_1\cdot\mathsf{e}_1 \tag{1.11.133}$$

$$\frac{\mathrm{d}\hat{\mathsf{e}}_0}{\mathrm{d}\tau}\cdot\hat{\mathsf{e}}_1 = \frac{\mathsf{e}_1\cdot\mathsf{e}_1}{|\mathsf{e}_1|} = |\mathsf{e}_1| \equiv \chi_0 \tag{1.11.134}$$

By this we define $\hat{\mathsf{e}}_1 = \chi_0\mathsf{e}_1$ and so $\hat{\mathsf{e}}_1\cdot\hat{\mathsf{e}}_1 = 1$ explicitly by our definition. We don't have to worry about an extra minus sign because our norm $|\mathsf{e}_1|$ simply makes it a unit normal. Now we might ask ourselves what is the derivative of this previous quantity. So

$$\frac{\mathrm{d}\hat{\mathsf{e}}_0\cdot\hat{\mathsf{e}}_1}{\mathrm{d}\tau} = \frac{\mathrm{d}\hat{\mathsf{e}}_0}{\mathrm{d}\tau}\cdot\hat{\mathsf{e}}_1 + \frac{\mathrm{d}\hat{\mathsf{e}}_1}{\mathrm{d}\tau}\cdot\hat{\mathsf{e}}_0 \tag{1.11.135}$$

[108] This could be considered a definition of the parameterization τ. Note that if we are in a special relativistic situation then $\gamma(t) = \gamma$ with no time dependence so we get the time dilation given by $t = \tau\gamma$ as we should hope.

The above is by construction equal to zero since $\hat{\mathsf{e}}_0 \cdot \hat{\mathsf{e}}_1 = 0$ by construction. This means that we must have

$$\frac{\mathrm{d}\hat{\mathsf{e}}_1}{\mathrm{d}\tau} \cdot \hat{\mathsf{e}}_0 = -\frac{\mathrm{d}\hat{\mathsf{e}}_0}{\mathrm{d}\tau} \cdot \hat{\mathsf{e}}_1 = -\chi_0 \tag{1.11.136}$$

But remember that if we have

$$\frac{\mathrm{d}\hat{\mathsf{e}}_1}{\mathrm{d}\tau} = a_{10}\hat{\mathsf{e}}_0 + \sum_{i=1}^{3} a_{1i}\hat{\mathsf{e}}_i \tag{1.11.137}$$

then we have

$$\frac{\mathrm{d}\hat{\mathsf{e}}_1}{\mathrm{d}\tau} \cdot \hat{\mathsf{e}}_0 = -a_{10} = -\chi_0 \tag{1.11.138}$$

which means that $a_{10} = \chi_0$, accounting for the extra sign change due to $\hat{\mathsf{e}}_0 \cdot \hat{\mathsf{e}}_0 = -1$. For all other vectors we have no $\hat{\mathsf{e}}_0$ contribution and so we do not need to worry about sign changes due to our metric because we actually enforce unit normals via dividing by the norms (only with the first derivative of s for $\hat{\mathsf{e}}_0$ could we not enforce this unit normality because we needed the extra minus sign to ensure a positive real arclength when taking the derivative). So then everything else follows our multiple Euclidean dimension cases, and we find (1.11.121) is indeed the correct formula.

1.12 JWKB Approximation

> *How to organize the chaos that lies beyond the smallest numbers is therefore a problem that confronts the entire human race.*
>
> — W. W. Sawyer[29, p. 8]

This is a common approximation in physics, normally called the WKB approximation, neglecting the important J because physicists didn't look at what the mathematicians had already done. JWKB stands for Jeffreys-Wentzel-Kramers-Brillouin and is sometimes written WKBJ because WKB is the most known.[109] The method is rarely called a Liouville-Green method or LG method.

The method is a special case of multiple scale analysis, a general method of solving equations using a specially constructed perturbation series that allows one to easily remove unphysical solutions or terms.

The idea is simple. We have an equation for a function $f(\mathbf{x})$ of the form

$$\varepsilon\frac{\partial^n f}{\partial \mathbf{x}^n} + \mathbf{F}\left(\mathbf{x}, \frac{\partial f}{\partial \mathbf{x}}, \ldots, \frac{\partial^{n-1} f}{\partial \mathbf{x}^{n-1}}\right) = \mathbf{0} \tag{1.12.1}$$

where ε is a small parameter and $\mathbf{F}$ is some nth order tensor function of $\mathbf{x}$ and lower order derivatives of f (also $\mathbf{0}$ is the nth order zero tensor). We then assume a solution with a perturbation term δ such that the solution is of the form

$$f(\mathbf{x}) \approx \exp\left[\frac{1}{\delta}\sum_{i=0}^{\infty}\delta^i S_i(\mathbf{x})\right] \tag{1.12.2}$$

[109] I will call it JWKB to give credit where it is due to Jeffreys, who actually came up with a general method for linear second-order differential equations. WKB independently discovered the approximation for the Schrödinger equation and so also deserve credit.

as $\delta \to 0$. We determine what δ must be in terms of ε by solving the resulting equations in the asymptotic limit and then can refine our answer by solving higher order equations in δ or ε.

Thus, we put the solution into our equation and find

$$\varepsilon \frac{\partial^n}{\partial \mathbf{x}^n}\left(\exp\left[\frac{1}{\delta}\sum_{i=0}^{\infty}\delta^i S_i\right]\right) + \mathsf{F}\left(\mathbf{x}, \frac{\partial S_i}{\partial \mathbf{x}}, \ldots, \frac{\partial^n S_i}{\partial \mathbf{x}^n}\right) = \mathbf{0} \tag{1.12.3}$$

where we simply have to write out the $\frac{\partial^j f}{\partial \mathbf{x}^j}$ in terms of $\frac{\partial^k S_i}{\partial \mathbf{x}^k}$. The notation is ugly and so I simply rewrote the function as a function of the S_i,[110] so let's just look at a simple linear second order differential equation to get the idea. For example, we could write it as

$$\varepsilon \frac{\partial^2 f}{\partial \mathbf{x}} + \mathbf{a}_1(\mathbf{x})\frac{\partial f}{\partial \mathbf{x}} + \overset{\leftrightarrow}{\mathbf{a}}_0 = \overset{\leftrightarrow}{\mathbf{0}} \tag{1.12.4}$$

We then have

$$\frac{\partial}{\partial \mathbf{x}}\exp\left[\frac{1}{\delta}\sum_{i=0}^{\infty}\delta^i S_i\right] = \frac{\sum_{i=0}^{\infty}\delta^i \frac{\partial S_i}{\partial \mathbf{x}}}{\delta}\exp\left[\frac{1}{\delta}\sum_{i=0}^{\infty}\delta^i S_i\right] \tag{1.12.5}$$

$$\frac{\partial^2}{\partial \mathbf{x}^2}\exp\left[\frac{1}{\delta}\sum_{i=0}^{\infty}\delta^i S_i\right] = \frac{\delta\sum_{i=0}^{\infty}\delta^i \frac{\partial^2 S_i}{\partial \mathbf{x}^2} + \left(\sum_{i=0}^{\infty}\delta^i \frac{\partial S_i}{\partial \mathbf{x}}\right)\left(\sum_{i=0}^{\infty}\delta^i \frac{\partial S_i}{\partial \mathbf{x}}\right)}{\delta^2}\exp\left[\frac{1}{\delta}\sum_{i=0}^{\infty}\delta^i S_i\right] \tag{1.12.6}$$

and so if we divide out the $\exp\left[\frac{1}{\delta}\sum_{i=0}^{\infty}\delta^i S_i\right]$ we find

$$\varepsilon\frac{\delta\sum_{i=0}^{\infty}\delta^i \frac{\partial^2 S_i}{\partial \mathbf{x}^2} + \left(\sum_{i=0}^{\infty}\delta^i \frac{\partial S_i}{\partial \mathbf{x}}\right)\left(\sum_{i=0}^{\infty}\delta^i \frac{\partial S_i}{\partial \mathbf{x}}\right)}{\delta^2} + \mathbf{a}_1\frac{\sum_{i=0}^{\infty}\delta^i \frac{\partial S_i}{\partial \mathbf{x}}}{\delta} + \overset{\leftrightarrow}{\mathbf{a}}_0 = \overset{\leftrightarrow}{\mathbf{0}} \tag{1.12.7}$$

If we now consider only terms of $\mathcal{O}(1/\delta)$ and $\mathcal{O}(1/\delta^2)$ only, we find

$$\frac{\varepsilon}{\delta}\frac{\partial^2 S_0}{\partial \mathbf{x}^2} + \frac{\varepsilon}{\delta^2}\frac{\partial S_0}{\partial \mathbf{x}}\frac{\partial S_0}{\partial \mathbf{x}} + \frac{\varepsilon}{\delta}\frac{\partial S_0}{\partial \mathbf{x}}\frac{\partial S_1}{\partial \mathbf{x}} + \frac{\varepsilon}{\delta}\frac{\partial S_1}{\partial \mathbf{x}}\frac{\partial S_0}{\partial \mathbf{x}} + \frac{\mathbf{a}_1}{\delta}\frac{\partial S_0}{\partial \mathbf{x}} + \overset{\leftrightarrow}{\mathbf{a}}_0 = \overset{\leftrightarrow}{\mathbf{0}} \tag{1.12.8}$$

As $\delta \to 0$ a dominant balance argument implies that we must have either

$$\frac{\varepsilon}{\delta^2}\frac{\partial S_0}{\partial \mathbf{x}}\frac{\partial S_0}{\partial \mathbf{x}} + \overset{\leftrightarrow}{\mathbf{a}}_0 \sim \overset{\leftrightarrow}{\mathbf{0}} \tag{1.12.9}$$

with $\delta \sim \sqrt{\varepsilon}$, or

$$\frac{\mathbf{a}_1}{\delta}\frac{\partial S_0}{\partial \mathbf{x}} + \overset{\leftrightarrow}{\mathbf{a}}_0 \sim 0 \tag{1.12.10}$$

which would imply $\delta \sim 1$.

In the first case with $\delta \sim \sqrt{\varepsilon}$ we would have

$$\varepsilon^{1/2}\frac{\partial^2 S_0}{\partial \mathbf{x}^2} + \frac{\partial S_0}{\partial \mathbf{x}}\frac{\partial S_0}{\partial \mathbf{x}} + \varepsilon^{1/2}\frac{\partial S_0}{\partial \mathbf{x}}\frac{\partial S_1}{\partial \mathbf{x}} + \varepsilon^{1/2}\frac{\partial S_1}{\partial \mathbf{x}}\frac{\partial S_0}{\partial \mathbf{x}} + \varepsilon^{-1/2}\mathbf{a}_1\frac{\partial S_0}{\partial \mathbf{x}} + \overset{\leftrightarrow}{\mathbf{a}}_0 = \overset{\leftrightarrow}{\mathbf{0}} \tag{1.12.11}$$

[110] Faà di Bruno's formula would give us the correct answer, but writing it out is a pain. Usually we are not dealing with a large n so we can just do the calculation manually.

which would then mean the $\mathbf{a}_1$ term is dominant, a contradiction, and so it must not be correct.

In the case of $\delta \sim 1$, we simply have ε terms which are small, but note we do not have a useful power series for S_i, as we have to consider the infinite series. In this case the JWKB approximation is not very useful, and so we can see there are limitations to the approximation's usefulness. We could proceed by defining $S = \sum_i S_i$ and then we simply have the equation

$$\varepsilon\left(\frac{\partial^2 S}{\partial \mathbf{x}^2} + \frac{\partial S}{\partial \mathbf{x}}\frac{\partial S}{\partial \mathbf{x}}\right) + \mathbf{a}_1\frac{\partial S}{\partial \mathbf{x}} + \overleftrightarrow{\mathbf{a}}_0 = \overleftrightarrow{\mathbf{0}} \tag{1.12.12}$$

which actually looks more complicated than the original equation. There is a chance that this may be simpler to solve, however.

In any case, let's look more at our situation with $\delta \sim \sqrt{\varepsilon}$ where we try $\mathbf{a}_1 = \mathbf{0}$ and then we would have

$$\frac{\partial S_0}{\partial \mathbf{x}}\frac{\partial S_0}{\partial \mathbf{x}} = -\overleftrightarrow{\mathbf{a}}_0 \tag{1.12.13}$$

and the next order, $\mathcal{O}(\varepsilon^{1/2})$ then yields

$$\frac{\partial^2 S_0}{\partial \mathbf{x}^2} + \frac{\partial S_0}{\partial \mathbf{x}}\frac{\partial S_1}{\partial \mathbf{x}} + \frac{\partial S_1}{\partial \mathbf{x}}\frac{\partial S_0}{\partial \mathbf{x}} = \overleftrightarrow{\mathbf{0}} \tag{1.12.14}$$

We then simply solve the equations. If we were instead dealing with $\mathbb{1}:$ of the original equation (this is more common) we would find

$$\frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} + \overbrace{\mathbb{1}:\overleftrightarrow{\mathbf{a}}_0}^{\equiv a_0} = 0 \tag{1.12.15}$$

and so we get for our two equations that

$$\frac{\partial S_0}{\partial \mathbf{x}} \cdot \frac{\partial S_0}{\partial \mathbf{x}} = a_0 \tag{1.12.16}$$

$$\frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial S_0}{\partial \mathbf{x}} + 2\frac{\partial S_0}{\partial \mathbf{x}} \cdot \frac{\partial S_1}{\partial \mathbf{x}} = 0 \tag{1.12.17}$$

We can continue to more accurate approximations by going to $\mathcal{O}(\delta^2)$, the same as $\mathcal{O}(\varepsilon)$ and solving the next set of equations. The JWKB approximation is simply this process. Whether you can solve the resulting equations is a completely separate issue. You will notice that we changed a linear equation into sets of nonlinear equations which would usually be considered a step backwards. Therefore, you can already see the limitations from this simple demonstration. We see that the eikonal equation showed up in our case above with $\mathbf{a}_1 = \mathbf{0}$, and so it is worth seeing how we could solve such an equation.

1.12.1 Eikonal Equation

The eikonal equation comes (ultimately, through German *Eikonal*) from the Greek word for image. It is a non-linear partial differential equation that shows up in wave propagation. The traditional eikonal equation is written in the form

$$\left|\frac{\partial u}{\partial \mathbf{x}}\right| \equiv |\,\nabla u(\mathbf{x})| = \frac{1}{f(\mathbf{x})} \tag{1.12.18}$$

for $f(\mathbf{x})$ a given function with $u = 0$ on the boundary of the region we are looking at.

Sometimes you will see people call the slight generalization of (1.12.18) the eikonal equation as

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial \mathbf{x}} \cdot \frac{\partial S}{\partial \mathbf{x}} = g(\mathbf{x}) \tag{1.12.19}$$

where S is like u and g is like f in (1.12.18) (though we would require $S(\mathbf{x}, t) = S(\mathbf{x})$ and $g(\mathbf{x}) = [1/f(\mathbf{x})]^2$ to have them be the same) and we have S defined on a boundary. This is because the Hamilton-Jacobi equation and the eikonal equation can coincide with this interpretation.

Let's consider the case S is not a function of t first. One way of approaching this is to define $\mathbf{p} = \frac{\partial S}{\partial \mathbf{x}}$. We can then write

$$\mathbf{p} \cdot \mathbf{p} = g(\mathbf{x}) \tag{1.12.20}$$

as our equation. We can then introduce a new parameter s, such that $\mathbf{p} = \mathbf{p}(s)$ and $\mathbf{x} = \mathbf{x}(s)$. Now we can form a Hamiltonian for our S. This Hamiltonian $H(\mathbf{x}(s), \mathbf{p}(s)) = \mathbf{p} \cdot \mathbf{p} - g(\mathbf{x}) = 0$ must then satisfy

$$\frac{\partial H}{\partial \mathbf{x}} = -\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} \tag{1.12.21}$$

$$\frac{\partial H}{\partial \mathbf{p}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} \tag{1.12.22}$$

We define $s = 0$ so that $\mathbf{x}(0)$ and $\mathbf{p}(0)$ are our initial conditions, that is the data on our boundary. We then can solve these as an ODE for s. We clearly have

$$\frac{\mathrm{d}H}{\mathrm{d}s} = \frac{\partial H}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} + \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = -\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} \cdot \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = 0 \tag{1.12.23}$$

so that $H = 0$ is carried along these trajectories of s. Now if we want to find S we use

$$\frac{\mathrm{d}S}{\mathrm{d}s} = \frac{\partial S}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{p} \cdot \frac{\partial H}{\partial \mathbf{p}} \tag{1.12.24}$$

If we then integrate this along s' to s we see

$$\int_0^s \mathrm{d}s' \, \frac{\mathrm{d}S}{\mathrm{d}s'} = S(s) - S(0) = \int_0^s \mathrm{d}s' \, \mathbf{p} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s'} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}\mathbf{x}' \, \cdot \mathbf{p} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}\mathbf{x}' \, \cdot \frac{\partial S}{\partial \mathbf{x}'} \tag{1.12.25}$$

By the same reasoning on $\mathbf{p}$ we can show that our $\mathbf{p}$ is indeed $\frac{\partial S}{\partial \mathbf{x}}$ and so consistent by

$$\int_0^s \mathrm{d}s' \, \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s'} = \mathbf{p}(s) - \mathbf{p}(0) = -\int_0^s \mathrm{d}s' \, \frac{\partial H}{\partial \mathbf{x}} \tag{1.12.26}$$

Clearly $\mathbf{p}(0) = \left[\frac{\partial S}{\partial \mathbf{x}}\right]_{s=0}$ by construction. If we take the curl of this equation we see that the integral must be zero because

$$\frac{\partial^2 H}{\partial x_j \partial x_k} = \frac{\partial^2 H}{\partial x_k \partial x_j} \tag{1.12.27}$$

or equivalently that the curl of the gradient of H is clearly zero and so $\frac{\partial}{\partial \mathbf{x}} \times \mathbf{p}(s) = \mathbf{0}$ and so $\mathbf{p}(s)$ is indeed a gradient generally. We have then solved our problem of finding $S(\mathbf{x})$ given $g(\mathbf{x})$ via

$$S(\mathbf{x}(s)) - S(\mathbf{x}_0) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}\mathbf{x}' \, \cdot \mathbf{p} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}\mathbf{x}' \, \cdot \hat{\mathbf{k}} \sqrt{g(\mathbf{x})} \tag{1.12.28}$$

where $\mathbf{p} = \hat{\mathbf{k}}\sqrt{g(\mathbf{x})}$. We have to determine the direction of $\mathbf{p}$ as $\hat{\mathbf{k}}$ with either a good guess or using symmetries of the problem to determine the directions $\partial S/\partial \mathbf{x}$ could point. Otherwise, one must write $\hat{\mathbf{k}}$ as a generic unit vector and hope you can integrate it.

1.12.1.1 Time Dependent Eikonal Equation

Now if we have S a function of t so that $S = S(\mathbf{x}'(t'), t')$ where $\mathbf{x}'(0)$ is the initial $\mathbf{x}$ that is given. We could consider $t' = s$ from before and we would have

$$\frac{\mathrm{d}S}{\mathrm{d}t'} = \frac{\partial S}{\partial \mathbf{x}'} \cdot \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t'} + \frac{\partial S}{\partial t'} \tag{1.12.29}$$

If we use the identification $\frac{\partial S}{\partial \mathbf{x}'} = \mathbf{p}'$ again, and just plug in $\frac{\partial S}{\partial t'}$ we can define $H(\mathbf{x}'(t'), \mathbf{p}'(t'), t') = -\frac{\partial S}{\partial t'}$ and try the same as before with

$$\frac{\partial H}{\partial \mathbf{x}'} = -\frac{\mathrm{d}\mathbf{p}'}{\mathrm{d}t'} \tag{1.12.30}$$

$$\frac{\partial H}{\partial \mathbf{p}'} = \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t'} \tag{1.12.31}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t'} = \frac{\partial H}{\partial t'} \tag{1.12.32}$$

and so we have

$$\frac{\mathrm{d}S}{\mathrm{d}t'} = \frac{\partial S}{\partial \mathbf{x}'} \cdot \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t'} + \frac{\partial S}{\partial t'} = \mathbf{p}' \cdot \frac{\partial H}{\partial \mathbf{p}'} - H = \mathbf{p}' \cdot \frac{\partial H}{\partial \mathbf{p}'} - \overbrace{[\mathbf{p}' \cdot \mathbf{p}' - g(\mathbf{x}')]}^{=-\frac{\partial S}{\partial t'}} \tag{1.12.33}$$

So with the identify $H = \mathbf{p}' \cdot \mathbf{p}' - g(\mathbf{x}')$ (now the g and $\mathbf{p}'$ could be time dependent), we find

$$S(\mathbf{x}'(t), t) - S(\mathbf{x}'(0), 0) = \int_0^t \mathrm{d}t' \ \mathbf{p}' \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t'} - \int_0^t \mathrm{d}t' \ H(\mathbf{x}', \mathbf{p}', t') \tag{1.12.34}$$

$$S(\mathbf{x}'(t), t) - S(\mathbf{x}'(0), 0) = \int_{\mathbf{x}'(0)}^{\mathbf{x}'(t)} \mathrm{d}\mathbf{x} \ \cdot \frac{\partial S}{\partial \mathbf{x}} - \int_0^t \mathrm{d}t' \ \left[\frac{\partial S}{\partial \mathbf{x}'} \cdot \frac{\partial S}{\partial \mathbf{x}'} - g(\mathbf{x}')\right] \tag{1.12.35}$$

with

$$\mathbf{p}'(t) - \mathbf{p}'(0) = -\int_0^t \mathrm{d}t' \ \frac{\partial H}{\partial \mathbf{x}'} \tag{1.12.36}$$

again for the exact same reasons as in the time-independent case, which means our H actually does exist. The reason for using the primes, is that we want $S(\mathbf{x}, t)$, but our method assumes that $\mathbf{x}$ is related to t because of the form of the partial differential equation. That is, we are just moving parts of the original solution around as time progresses.

1.12.2 Schrödinger Equation

An example of solving an eikonal equation should be useful. If we assume we have

$$H = \frac{\hbar^2}{2m}\mathbf{p} \cdot \mathbf{p} + U(\mathbf{x}) \tag{1.12.37}$$

The Hamilton-Jacobi equation is simply given by

$$\frac{\partial S}{\partial \mathbf{x}} = \mathbf{p} \tag{1.12.38}$$

$$-\frac{\partial S}{\partial t} = H \tag{1.12.39}$$

where we must determine S. We can also translate H into S rather than ψ once we know the relationship between S and ψ. If we write $\psi = \psi_0 \exp\left(\frac{i}{\hbar}S(\mathbf{x},t)\right)$ for S and we see that

$$\frac{\partial\psi}{\partial t} = \frac{i}{\hbar}\frac{\partial S}{\partial t}\psi \tag{1.12.40}$$

and so we find

$$-\frac{\hbar}{i\psi}\frac{\partial\psi}{\partial t} = \frac{1}{2m}\mathbf{p}\cdot\mathbf{p} + U(\mathbf{x}) \tag{1.12.41}$$

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \frac{1}{2m}\mathbf{p}\cdot\mathbf{p}\psi + U(\mathbf{x})\psi \tag{1.12.42}$$

where we simply have to confront that $\mathbf{p}$ should be interpreted via

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{x}} = \frac{\hbar}{i\psi}\frac{\partial\psi}{\partial\mathbf{x}} \tag{1.12.43}$$

which means that we can recognize that

$$\mathbf{p}\psi = \frac{\hbar}{i}\frac{\partial\psi}{\partial\mathbf{x}} \tag{1.12.44}$$

and so if we regard $\mathbf{p}\psi$ as $\mathbf{p}$ operating on ψ we can consider $\mathbf{p} = \frac{\hbar}{i}\frac{\partial}{\partial\mathbf{x}}$ which allows

$$\mathbf{p}\cdot\mathbf{p}\psi = \frac{\hbar^2}{i^2}\frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial\psi}{\partial\mathbf{x}} = -\hbar^2\frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial\psi}{\partial\mathbf{x}} = -\hbar^2\,\nabla^2\psi \tag{1.12.45}$$

If you are uneasy about this identification, you should be. It is not rigorous. However, if we did the logical thing we would find

$$\mathbf{p}\cdot\mathbf{p} = \frac{\hbar^2}{i^2}\frac{1}{\psi^2}\frac{\partial\psi}{\partial\mathbf{x}} \tag{1.12.46}$$

which then yields

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \frac{1}{2m}\frac{\hbar^2}{i^2\psi^2}\frac{\partial\psi}{\partial\mathbf{x}}\cdot\frac{\partial\psi}{\partial\mathbf{x}}\psi + U(\mathbf{x})\psi \tag{1.12.47}$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{\partial\psi}{\partial\mathbf{x}}\cdot\frac{\partial\psi}{\partial\mathbf{x}} + U\psi \tag{1.12.48}$$

This appears to be something a little different from the Schrödinger equation.

One other thing that should be bothering you at this point is that ψ is complex, and so is S. Is what we are doing sensible? Remember that $\mathbf{x}$ remains real. This means, that we just can't assume the squares of things are real, but our derivatives with respect to $\mathbf{x}$ are fine, with no worries about the Cauchy-Riemann equations. However, you might further worry that with S complex that H, which now may be complex could have problems. Especially, $\frac{\partial H}{\partial\mathbf{p}}$, though we have $\partial H/\partial\mathbf{p}^* = \mathbf{0}$ and so we know that it is analytic. We can explicitly calculate this using $\mathbf{p} = \mathbf{p}^r + i\mathbf{p}^i = \sum_{j=1}^{3}\left(p_j^r + ip_j^i\right)\hat{\mathbf{x}}_j$

with the Einstein summation convention enforced (here superscript i means imaginary and is not an index; we use $\frac{\partial p_j^r}{\partial p_k^r} = \frac{\partial p_j^i}{\partial p_k^i} = \delta_{jk}$ with $\frac{\partial p_j^i}{\partial p_k^r} = \frac{\partial p_j^r}{\partial p_k^i} = 0$)

$$\begin{aligned}\frac{\partial H}{\partial \mathbf{p}^*} &= \frac{\partial H}{\partial p_j^*} = \frac{1}{2}\left(\frac{\partial}{\partial p_j^r} + i\frac{\partial}{\partial p_j^i}\right)\frac{1}{2m}\left([p_k^r]^2 - [p_k^i]^2 + 2ip_k^i p_k^r\right) \\ &= \frac{1}{4m}\left(2p_j^r + 2ip_j^i + i\left[-2p_j^i + 2ip_j^r\right]\right) \\ &= \frac{1}{4m}\left(2p_j^r + 2ip_j^i - 2ip_j^i - 2p_j^4\right) = 0\end{aligned} \tag{1.12.49}$$

and so we have $\frac{\partial H}{\partial \mathbf{p}^*} = \mathbf{0}$ and H is an analytic function.

Second, the above may not look like the Schrödinger equation, but we need to remember that $\mathbf{p}$ and $\mathbf{x}$ are considered independent functions in the Hamiltonian formulation.[111] Thus, we must have $\frac{\partial \mathbf{p}}{\partial \mathbf{x}} = \overleftrightarrow{\mathbf{0}}$. This means that $\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{p} = 0$, as well. Thus

$$\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{p} = \frac{\partial}{\partial \mathbf{x}} \cdot \left[\frac{\hbar}{i\psi}\frac{\partial \psi}{\partial \mathbf{x}}\right] = 0 \tag{1.12.50}$$

$$\frac{\partial}{\partial \mathbf{x}} \cdot \left[\frac{\hbar}{i\psi}\frac{\partial \psi}{\partial \mathbf{x}}\right] = \frac{\hbar}{i}\left[\frac{-1}{\psi^2}\frac{\partial \psi}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} + \frac{1}{\psi}\frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}}\right] \tag{1.12.51}$$

$$\begin{aligned}\frac{1}{\psi^2}\frac{\partial \psi}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} &= \frac{1}{\psi}\frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} \\ \frac{1}{\psi}\frac{\partial \psi}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} &= \frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} = \nabla^2\psi\end{aligned} \tag{1.12.52}$$

Thus, we can replace $\frac{1}{\psi}\frac{\partial \psi}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} = \nabla^2\psi$ and we indeed get the Schrödinger equation

$$\begin{aligned}i\hbar\frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{\partial \psi}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} + U\psi \\ i\hbar\frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi\end{aligned} \tag{1.12.53}$$

These preliminaries all just show that there is a strong connection between the classical Hamilton-Jacobi treatment and the Schrödinger equation with just a little bit of insight. It is strictly speaking tangential to the JWKB approximation, but it is good to see that there is a connection beyond the approximation itself.

So if we start with the Schrödinger equation[112] we can designate $\varepsilon = \hbar^2/(2m)$ and try out our solution of $\psi = \psi_0 \exp\left(\frac{1}{\delta}\sum_{j=0}^{\infty}\delta^j S_j\right)$ allowing S to be complex and having ψ_0 give the correct

[111] This is a very non-trivial fact. We have $S(\mathbf{x})$ and so you might expect $\frac{\partial^2 S}{\partial \mathbf{x}^2}$ to be nonzero. This is not true, because we consider $\mathbf{p} = \frac{\partial S}{\partial \mathbf{x}}$ to be an independent variable in the Hamilton-Jacobi equations, and in Hamiltonian mechanics in general. It may seem weird at first, but if we consider $\mathbf{p}(t)$ and $\mathbf{x}(t)$ to actually be functions of t, then $\mathbf{p}$ has no explicit $\mathbf{x}$ dependence anymore. This is simply a way of stating that in the space of all trajectories $\mathbf{x}$ and $\mathbf{p}$ are independent and only when we choose a specific trajectory is there a relationship between $\mathbf{p}$ and $\mathbf{x}$.

[112] I'll leave the "fun" of the time-dependent equation for you. We'll look at the time-independent case $\psi \propto \exp(-iEt/\hbar)$. So we replace $i\hbar\frac{\partial \psi}{\partial t}$ with $E\psi$. Technically, I told you how to handle the time-dependent case above, but in generic language.

units for ψ. We have (note that we are not assuming $\frac{\partial S}{\partial \mathbf{x}} = \mathbf{p}$ here so we cannot say $\frac{\partial^2 S}{\partial \mathbf{x}^2} = \overleftrightarrow{\mathbf{0}}$)

$$\frac{\partial \psi}{\partial \mathbf{x}} = \frac{\sum_{j=0} \delta^j \frac{\partial S_j}{\partial \mathbf{x}}}{\delta} \psi \tag{1.12.54}$$

$$\nabla^2 \psi = \frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial \psi}{\partial \mathbf{x}} = \frac{\sum_{j=0} \delta^{j+1} \nabla^2 S_j + \left(\sum_{j=0} \delta^j \frac{\partial S}{\partial \mathbf{x}}\right) \cdot \left(\sum_{j=0} \delta^j \frac{\partial S}{\partial \mathbf{x}}\right)}{\delta^2} \psi \tag{1.12.55}$$

as before and so if we take $\mathcal{O}(\delta^{-1})$ terms we find (let's just define $U - E = \widehat{U}$ for simplicity)

$$\frac{\varepsilon}{\delta} \nabla^2 S_0 + \frac{\varepsilon}{\delta^2} \frac{\partial S_0}{\partial \mathbf{x}} \cdot \frac{\partial S_0}{\partial \mathbf{x}} + \frac{2\varepsilon}{\delta} \frac{\partial S_0}{\partial \mathbf{x}} \cdot \frac{\partial S_1}{\partial \mathbf{x}} - \widehat{U}(\mathbf{x}) = 0 \tag{1.12.56}$$

This means we need to choose δ to balance $\widehat{U}(\mathbf{x})$ and $\frac{\varepsilon}{\delta^2} \frac{\partial S_0}{\partial \mathbf{x}} \cdot \frac{\partial S_0}{\partial \mathbf{x}}$ under dominant balance and be dimensionally consistent. We know S_0 has the same unit as δ, which means we need $\varepsilon/\delta = \hbar^2/(2m\delta)$ to have units of energy. If we give the unit of ε/δ the symbol $\widehat{d}$ then we can define $\widehat{U}(\mathbf{x})/\widehat{d} = u(\mathbf{x})$ as a dimensionless parameter corresponding to the potential energy minus the total energy $U - E$. We divide our equation through by $\widehat{d}$ and we have $\frac{\varepsilon}{\delta^2 \widehat{d}} \sim 1$ and we choose $\delta = \sqrt{\varepsilon/\widehat{d}} = \sqrt{\hbar^2/(2m\widehat{d})} = \hbar/(\sqrt{2m\widehat{d}}) = \hbar/(\sqrt{2m\widehat{d}})$. This is slightly different than the traditional ordering as I include $2m$ and $\widehat{d}$ in our ordering parameter.

$$\frac{\partial S_0}{\partial \mathbf{x}} \cdot \frac{\partial S_0}{\partial \mathbf{x}} = u(\mathbf{x}) \tag{1.12.57}$$

$$\nabla^2 S_0 + 2 \frac{\partial S_0}{\partial \mathbf{x}} \cdot \frac{\partial S_1}{\partial \mathbf{x}} = 0 \tag{1.12.58}$$

for S_0 and S_1. The first is simply an eikonal equation, so we can use that we previously found the solution as

$$S_0(\mathbf{x}) = S_0(\mathbf{x}_0) + \int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}\mathbf{x}' \cdot \frac{\partial S_0}{\partial \mathbf{x}'} \tag{1.12.59}$$

where $\mathbf{x}_0$ is an initial location (where the initial data is) and we then just calculate the line integral. We note that in fact we can use $\frac{\partial S_0}{\partial \mathbf{x}} = \sqrt{u}\hat{\mathbf{k}}$ where $\hat{\mathbf{k}}$ is the unit vector pointing in the $\frac{\partial S_0}{\partial \mathbf{x}}$ direction. If $u > 0$ ($U > E$) then $S_0 = \pm s(\mathbf{x})$ are both valid solutions for some s. If $u < 0$ ($E > U$) then because $\frac{\partial S}{\partial \mathbf{x}}$ is complex we know that $S_0 = \pm i s(\mathbf{x})$ for some s will satisfy the eikonal relation perfectly well, so there are two solutions. This can be viewed as simply changing the direction of $\hat{\mathbf{k}}$ and so our two solutions are

$$S_0(\mathbf{x}) = S_0(\mathbf{x}_0) \pm \int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}\mathbf{x}' \cdot \hat{\mathbf{k}}(\mathbf{x}') \sqrt{u(\mathbf{x}')} \tag{1.12.60}$$

Then we can solve our second equation via (I'll just assume $u > 0$ for simplicity, otherwise we can define $v = -u$ and use $v > 0$ with $\sqrt{u} = i\sqrt{v}$)

$$\frac{\partial S_0}{\partial \mathbf{x}} = \pm \hat{\mathbf{k}} \sqrt{u} \tag{1.12.61}$$

$$\nabla^2 S_0 = \pm \sqrt{u} \frac{\partial}{\partial \mathbf{x}} \cdot \hat{\mathbf{k}} \pm m \hat{\mathbf{k}} \cdot \frac{\frac{\partial u}{\partial \mathbf{x}}}{2\sqrt{u}} \tag{1.12.62}$$

and so

$$\frac{\partial S_1}{\partial \mathbf{x}} \cdot \pm\hat{\mathbf{k}}\sqrt{u} = \mp\sqrt{u(\mathbf{x})}\frac{\partial}{\partial \mathbf{x}} \cdot \hat{\mathbf{k}} \mp m\hat{\mathbf{k}} \cdot \frac{\frac{\partial u}{\partial \mathbf{x}}}{2\sqrt{u}} \tag{1.12.63}$$

If $\frac{\partial}{\partial \mathbf{x}} \cdot \hat{\mathbf{k}} = 0$ so that our unit vector has no divergence then the above equation simply states[113]

$$\frac{\partial S_1}{\partial \mathbf{x}} \cdot \pm\hat{\mathbf{k}} = \frac{\frac{\partial u}{\partial \mathbf{x}}}{4u} \cdot \pm\hat{\mathbf{k}} \tag{1.12.64}$$

$$\frac{\partial S_1}{\partial \mathbf{x}} = \frac{1}{4u}\frac{\partial u}{\partial \mathbf{x}} = \frac{1}{4}\frac{\partial \ln u}{\partial \mathbf{x}} \tag{1.12.65}$$

$$S_1 = \ln u^{1/4} + C_1 \tag{1.12.66}$$

for some constant C_1. We can make this a general result by using $S_1 = \ln |u|^{1/4}$. Note the crucial step is $\frac{\partial}{\partial \mathbf{x}} \cdot \hat{\mathbf{k}} = 0$ for this derivation which is not generically true and depends on the properties of $\frac{\partial S_0}{\partial \mathbf{x}}$. In any case, assuming this to be the case and taking $S_0(\mathbf{0}) = 0$ would then give our solution as

$$\begin{aligned}\psi \approx \psi_{0+} \exp\left[\left(\frac{\sqrt{2m\widehat{d}}}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}\mathbf{x}' \cdot \hat{\mathbf{k}}\sqrt{u}\right) - (\ln|u|^{1/4} + C_1)\right] \\ + \psi_{0-} \exp\left[\left(\frac{-\sqrt{2m\widehat{d}}}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}\mathbf{x}' \cdot \hat{\mathbf{k}}\sqrt{u}\right) + (\ln|u|^{1/4} + C_1)\right] \\ \approx \frac{C_+ \exp\left[\frac{1}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}\mathbf{x}' \cdot \hat{\mathbf{k}}\sqrt{2m(U-E)}\right] + C_- \exp\left[-\frac{1}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}\mathbf{x}' \cdot \hat{\mathbf{k}}\sqrt{2m(U-E)}\right]}{(2m|U-E|)^{1/4}}\end{aligned} \tag{1.12.67}$$

where C_+ and C_- have absorbed the necessary constants to give the $(2m|U-E|)^{1/4}$ in the denominator. That is $C_\pm = \exp(C_1)(2m\widehat{d})^{1/4}\psi_0$. If we look to dimensional analysis, then we see the denominator has units of the square root of momentum, $\mathrm{kg}^{1/2}\,\mathrm{m}^{1/2}\,\mathrm{s}^{-1/2}$, and so $C_\pm$ must have units of square root of momentum per volume for ψ for a three-dimensional wavefunction. In any case, so long as you give the correct units to $C_\pm$ or $\psi_{0\pm}$, this is the correct approximation.

We can then note that if $(U-E) < 0$ ($U > E$ so classically allowed) then we have oscillatory wave solutions and if we have $(U-E) > 0$ ($E < U$ so classically forbidden) we get exponential decay and growth as we would expect. This is easily adapted to the usual one-dimensional form where we do not need to worry about $\frac{\partial}{\partial \mathbf{x}} \cdot \hat{\mathbf{k}} = 0$ since we have $\hat{\mathbf{k}} = \hat{\mathbf{x}}_1 = \hat{\mathbf{x}}$ and so it is always divergence free.[114] We also note that $U - E = 0$ and its neighborhood lead to our solution blowing up, and so near classical turning points this solution method will fail.

The way to handle the $U - E \approx 0$ points is to perform asymptotic matching. While this is often considered part of the JWKB approximation, it is treated in Section 1.4.3.

1.13 Complex Contour Integration

The shortest path between two truths in the real domain passes through the complex domain.

[113] You can run through the case where $u < 0$ and see that you get the exact same formula for v as the i's cancel out. That is, you just get $S_1 = \ln v^{1/4} + C_1$.

[114] Many useful cases have $\hat{\mathbf{k}}$ divergence-free.

— Jacques Hadamard

I feel as if complex contour integration is taught quite well online and in textbooks. Thus, I will only go over the concepts quickly and without rigorously grounding the ideas mathematically. Instead, I will lay the groundwork and instead of formally showing the limits make sense, I will simply use words and intuition to explain why some things disappear and we can write out our contour integrals in a nice, simple way.

We will want to make use of the Cauchy integral formula which is valid for any analytic function $f(z)$ of the complex variable z. Most books introduce this via constructing simple curves and talking about the interior and exterior of the contour. However, as a joke points out, interior and exterior are only a point of view.[115] It is better to introduce the concept of a winding number, and then we simply don't have to worry about defining the interior as the "bounded" part and the exterior as the "unbounded" part, though it is very convenient. That is, the Jordan curve theorem does not need to be invoked (though we will use it). A winding number is very easy to explain in words. Draw a closed curve in the plane γ parameterized by s with $s_0 < s < s_e$ where the position given by s_e is the same as the position given by s_0. Then place a point at a anywhere in the plane. Then if you imagine standing on that point in the plane staring at $s = s_0$, you consider the angle of your line of sight (projected into the plane) to s_0 to be 0.[116] Then as you traverse the curve (stare at s as it goes around the curve turning your head with your line of sight), your line of sight with s will form an angle with the original line of sight. Consider counterclockwise (for example, your head turning left if you are "inside" the curve, but just be consistent) to be the positive direction. If you fully revolve around, you keep that angle (so you can have angles greater than 2π in radians). Then the winding number of a closed curve γ at a point a is $1/2\pi$ times the angle (in radians) your head rotated as you traversed all the way around the curve once, or the number of full revolutions made by your head with turning left positive and turning right negative. We denote the winding number for the closed curve γ and point a as $I(\gamma, a)$.

If you think about this in the complex plane, you will realize that we can easily find it thanks to properties of complex numbers. Imagine a is at the origin and we parameterize the curve by some complex number z. We can write $z = r\exp(i\theta)$ so that $\mathrm{d}z = \exp(i\theta)\,\mathrm{d}r + ir\exp(i\theta)\,\mathrm{d}\theta$. If we divide by this by z for a logarithmic differential we get

$$\frac{\mathrm{d}z}{z} = \frac{\exp(i\theta)\,\mathrm{d}r}{r\exp(i\theta)} + \frac{ir\exp(i\theta)\,\mathrm{d}\theta}{r\exp(i\theta)} \tag{1.13.1}$$

$$\frac{\mathrm{d}z}{z} = \mathrm{d}(\ln z) = \mathrm{d}(\ln r) + i\,\mathrm{d}\theta \tag{1.13.2}$$

As we have a closed curve, this means that $\mathrm{d}\ln r$ must be zero because we end up back at exactly where we came from. So any increase in r is balanced by a decrease in r to return to where we were. Notice that θ is not necessarily constrained by this, though, because a complex number can have $\theta + 2\pi$ equally represent it in exponential form. If a is not at the origin, we can remedy this by using $\zeta = z - a$ instead of z and find the same formula (since $\mathrm{d}a = 0$). So long as a is not

[115] There was an engineer, a physicist, and a mathematician. They were given a length of fence and told to enclose the greatest area. The engineer builds a square fence. The physicist reasons that a circle maximizes area for a given a perimeter and builds a circular fence. The mathematician makes a tiny circle of fence around himself and declares himself to be on the outside. (Hence, the rest of the earth is "enclosed" by his or her fence.)

[116] All I am saying is that when you stare at the point s_0 you consider yourself to have not revolved at all initially.

along the closed path we can define

$$I(\gamma, a) = \frac{1}{2\pi i} \oint_\gamma \frac{\mathrm{d}z}{z-a} \tag{1.13.3}$$

as the winding number of a for γ (sometimes called the index of a with respect to γ).[117]

Then we write for the points a_j (the a_j are singularities) which the function $g(z)$ are not defined for as

$$\int_\gamma \mathrm{d}z\ g(z) = 2\pi i \sum_j I(\gamma, a_j)\,\mathrm{Res}(g, a_j) \tag{1.13.4}$$

where the Res is the residue of f at a_j. Note this is for all the singularities, even those "outside" of γ, and so its proof does not require defining the inside and outside of a curve. The residue is defined by thinking about the singularity. It is the unique value R such that $g(z) - R/(z-a)$ has an analytic derivative near a. Alternatively, construct the Laurent series

$$g(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k \tag{1.13.5}$$

at the point a. Then the residue is simply the coefficient a_{-1}.

With this, we use that for a simple closed curve C and point a, that $I(C, a) = 1$ if a is in the bounded region of the curve and $I(C, a) = 0$ if it is in the unbounded region. This follows from the definition, but is easy to check visually. We can then use that we can change the contours into a sum of other contour integrals so long as the sum of the paths equals the original path. For $f(z)$ an analytic function for the bounded region the contour encloses (no singularities there) that

$$\oint_C \mathrm{d}z\ \frac{f(z)}{z-a} = \pm\pi i \begin{cases} 2f(a) & \text{if } a \text{ is inside } C \\ f(a) & \text{if } a \text{ is on } C \\ 0 & \text{if } a \text{ is outside } C \end{cases} \tag{1.13.6}$$

with C a simple closed curve with a a simple pole (of order one), and $+$ for C being oriented counterclockwise, and $-$ being for C oriented clockwise. This is then generalized to

$$\oint_C \mathrm{d}z\ \frac{f(z)}{(z-a)^{n+1}} = \pm\frac{\pi i}{n!} \begin{cases} 2f^{(n)}(a) & \text{if } a \text{ is inside } C \\ f^{(n)}(a) & \text{if } a \text{ is on } C \\ 0 & \text{if } a \text{ is outside } C \end{cases} \tag{1.13.7}$$

If $g(z) = f(z)/(z-a)^{n+1}$ we say that $g(z)$ has a pole of order n at a. I will explain how we arrived at the "on the contour" definition shortly.

The paths for the various pole locations are shown in Figure 1.12.

[117] The $2\pi i$ is to go from radians to revolutions, and to remove the i that comes from $ri\exp(i\theta)\,\mathrm{d}\theta$.

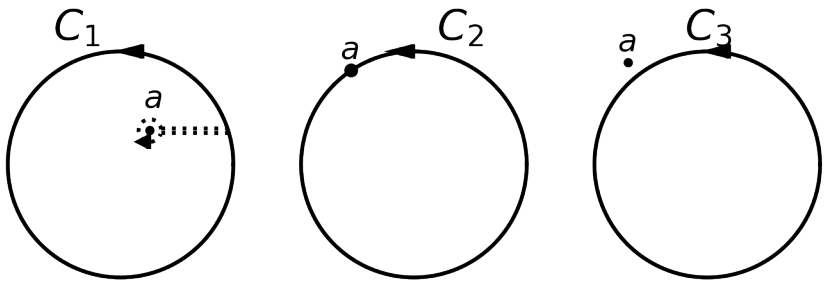

Figure 1.12: Referring to (1.13.6) (Cauchy Integral Formula for $n = 1$), the top case [a pole inside the contour] is the leftmost curve C_1, the + middle case [a pole on the contour] is for curve C_2, and the bottom case [a pole outside the contour] is C_3. Curves C_1 and C_2 show the equivalent integrations to be done with the dotted curves.

Justifying this is really a job for a mathematician, and they would almost certainly quibble about a pole being on the contour. In that case, the integral is not really properly defined, though the reasonable answer is what I gave above. We will give it this value that "makes most sense" which happens to have the name of the Sokhotski–Plemelj theorem (often just called the Plemelj formula).

The Plemelj formula along the real line are given via

$$\lim_{\epsilon\to 0^+}\int_a^b \mathrm{d}x\ \frac{f(x)}{x\pm i\epsilon} = \mp i\pi f(0) + \fint_a^b \mathrm{d}x\ \frac{f(x)}{x} \tag{1.13.8}$$

where $\fint$ means taking a Cauchy Principal Value defined by (with a singularity at m with $a < m < b$)

$$\fint_a^b \mathrm{d}x\ f(x) \equiv \lim_{\epsilon\to 0^+}\left[\int_a^{m-\epsilon}\mathrm{d}x\ f(x) + \int_{m+\epsilon}^b \mathrm{d}x\ f(x)\right] \tag{1.13.9}$$

with a hint of the complexities showing up as we see that how the poles approach the contour matters.

First, let's explain how the contours lead to Cauchy's integral formula above when a pole is not on the contour. We start with

$$I = \oint_{C_1}\mathrm{d}z\ \frac{f(z)}{z-a} \tag{1.13.10}$$

with C_1 a simple closed curve enclosing a (a is inside the contour) in the complex plane and $f(z)$ an analytic function. We know from complex analysis that an entire (that is, analytic or holomorphic) function has

$$\oint_C \mathrm{d}z\ f(z) = 0 \tag{1.13.11}$$

by Cauchy's integral theorem for any closed curve C. This is easily proved using that $f(z)$ must satisfy the Cauchy-Riemann equations, so you can do this yourself. For I above we create the

dotted path in C_1 from 1.12. If we make that path close to a perfect circle around a, then the two straight lines that depart from the original C_1 towards a will almost fully cancel and will be a negligible contribution. This new contour C_1' has no poles for $f(z)/(z-a)$ because it avoids the region in C_1 that has the pole entirely [Here C_1' uses the dotted path]. Then if we call the circle going counterclockwise around a the path C_a, when we add C_a to C_1' we will have simply created the original path C_1, and so we can evaluate C_a and C_1' separately to find the original C_1 integral. Thus

$$I = \oint_{C_1'} \mathrm{d}z\, \frac{f(z)}{z-a} + \oint_{C_a} \mathrm{d}z\, \frac{f(z)}{z-a} \tag{1.13.12}$$

Now the C_1' integral is zero by Cauchy's integral theorem. We can use $z = a + R\exp(i\theta)$ so $\mathrm{d}z = iR\exp(i\theta)\,\mathrm{d}\theta$ for the circle C_a with radius R. (Remember that this circle is constructed around a and is going counterclockwise so that it cancels the clockwise circle around a from C_1'). So

$$\oint_{C_a} \mathrm{d}z\, \frac{f(z)}{z-a} = \lim_{R\to 0} \int_0^{2\pi} \mathrm{d}\theta\, \frac{f\big(a+R\exp(i\theta)\big)iR\exp(i\theta)}{R\exp(i\theta)} = \lim_{R\to 0} \int_0^{2\pi} \mathrm{d}\theta\; if\big(a+R\exp(i\theta)\big) = 2\pi i f(a) \tag{1.13.13}$$

where I didn't rigorously prove that $f(z) \to f(a)$ but I hope that you can see that it is hardly a stretch when we take the limit. We are left with

$$I = \oint_{C_1} \mathrm{d}z\, \frac{f(z)}{z-a} = \cancel{\oint_{C_1'} \mathrm{d}z\, \frac{f(z)}{z-a}} + \oint_{C_a} \mathrm{d}z\, \frac{f(z)}{z-a} = 2\pi i f(a) \tag{1.13.14}$$

We now consider poles on the contour. One way to arrive at this answer is to view $f(a)$ like a function of a. This is the view of the Sokhotski–Plemelj formula (which is really a definition) that a contour integral actually creates two new functions (of a, with e for "exterior" and i for "interior") that satisfy

$$\varphi_e(a) = \frac{1}{2\pi i} \fint_{C_2} \mathrm{d}z\, \frac{f(z)}{z-a} - \frac{f(a)}{2} \tag{1.13.15}$$

$$\varphi_i(a) = \frac{1}{2\pi i} \fint_{C_2} \mathrm{d}z\, \frac{f(z)}{z-a} + \frac{f(a)}{2} \tag{1.13.16}$$

$$\frac{\varphi_e(a) + \varphi_i(a)}{2} = \fint_{C_2} \mathrm{d}z\, \frac{f(z)}{z-a} \tag{1.13.17}$$

$$\varphi_i(a) - \varphi_e(a) = f(a) \tag{1.13.18}$$

which give us the solutions for a on the contour. We simply have to complete the original $f(a)$ with one of the limiting values of φ_i or φ_e. The Cauchy principal value here means do the integral as if there were no pole on the contour. We define exterior to be the region along our curve that is to the right of the curve when we traverse the curve.[118] Here C_2^e means from the exterior side and so

$$\varphi_e(a) = \lim_{a\to C_2^e} \frac{1}{2\pi i} \int_{C_2} \mathrm{d}z\, \frac{f(z)}{z-a} \tag{1.13.19}$$

[118] Note that this immediately restricts this proof to functions where we can actually figure out the orientation of the curve in this simple manner. There are pathological curves that do not allow us to do this, but they should not show up in any physical situation.

and the interior function is defined similarly as to the left of the curve and C_2^i means from the interior side

$$\varphi_i(a) = \lim_{a \to C_2^i} \frac{1}{2\pi i} \int_{C_2} \mathrm{d}z \; \frac{f(z)}{z-a} \tag{1.13.20}$$

Note that this usually gives inconsistent values for $f(a)$ when we approach from different sides of the contour, which is why we get a $\pm$ in the Plemelj formula. This means the manner in which the pole approaches the contour matters, which may make you wonder how we got just a $+i\pi$ in our Cauchy's integral formula above when considering a counterclockwise curve. The answer is that we use $f_i(a)$ as our answer when we have a closed contour going counterclockwise. If the contour was closed going clockwise C_{2c}, then we would keep our interior and exterior labels the same, but we get the opposite sign.

If you don't like this, we can view this as coming from using an averaged limit of the two cases

$$\begin{aligned} \overbrace{\int_{C_2} \mathrm{d}z \; \frac{f(z)}{z-a}}^{a \text{ on } C} &= \frac{\overbrace{\int_{C_2} \mathrm{d}z \; \frac{f(z)}{z-a}}^{a \text{ outside } C} + \overbrace{\int_{C_2} \mathrm{d}z \; \frac{f(z)}{z-a}}^{a \text{ inside } C}}{2} \\ &= \frac{0 + \oint_{C_2} \mathrm{d}z \; \frac{f(z)}{z-a} + 2\pi i f(a) + \oint_{C_2} \mathrm{d}z \; \frac{f(z)}{z-a}}{2} = \int_{C_2} \mathrm{d}z \; \frac{f(z)}{z-a} + i\pi f(a) \end{aligned} \tag{1.13.21}$$

and so we are taking an "average" value of an integral that does not actually limit to a single value.

If the contour were oriented clockwise C_{2c} we'd use

$$\begin{aligned} \overbrace{\int_{C_{2c}} \mathrm{d}z \; \frac{f(z)}{z-a}}^{a \text{ on } C_{2c}} &= \frac{\overbrace{\int_{C_{2c}} \mathrm{d}z \; \frac{f(z)}{z-a}}^{a \text{ outside } C_{2c}} + \overbrace{\int_{C_{2c}} \mathrm{d}z \; \frac{f(z)}{z-a}}^{a \text{ inside } C_{2c}}}{2} \\ &= \frac{0 + \oint_{C_{2c}} \mathrm{d}z \; \frac{f(z)}{z-a} - 2\pi i f(a) + \oint_{C_{2c}} \mathrm{d}z \; \frac{f(z)}{z-a}}{2} = \int_{C_{2c}} \mathrm{d}z \; \frac{f(z)}{z-a} - i\pi f(a) \end{aligned} \tag{1.13.22}$$

where inside means bounded region. If we took it from the Plemelj formula definitions, this would be completing the $f(a)$ with the negative/exterior solution, because in those formulas the exterior is the region to the right of the oriented closed curve!

Figure 1.13: This shows the γ_+ contour for going "above" (positive/interior) the pole and the γ_- (negative/exterior) to a pole on the axis. In the limit the half-circle completely encloses the pole in one of the two curves.

If we use this for an integral along the real line axis (or any curve that is not closed), we simply have ambiguity in how we would prefer to complete the $f(a)$ function as we get to the curve

because we are not (usually) privileging either side of the contour. Thus, we are forced to use the Plemelj formula in their general form. This is often explained via doing an integral with a half-circle around the singularity, but if we are actually doing a limit, then this does not make much sense since then the singularity should be able to be fully avoided (indeed, one must wonder why one has to avoid the singularity in a way that only half encloses it; in the limit it should be able to fully enclose it). One can consider it again as the average limiting value, however. In this case, you have half-circles both ways[119] and take the average value of the two limits (See Figure 1.13 for the two different ways of deforming the curve around a singularity on the contour). This means that in the case that the pole comes from below the real axis we get (note that the contour with the straight line and the half-circle arc becomes a Cauchy principal value integral in the limit the half-circle arc becomes small)

$$\begin{aligned}
2\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x+i\delta} &\equiv \lim_{\delta\to 0^+}\left[\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}+\int_{\gamma_+}\mathrm{d}z\;\frac{f(z)}{z+i\delta}+\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}+\int_{\gamma_-}\mathrm{d}z\;\frac{f(z)}{z+i\delta}\right]\\
&= \lim_{\delta\to 0^+}\left[2\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}+\cancel{\int_{\gamma_+}\mathrm{d}z\;\frac{f(z)}{z+i\delta}}+\overbrace{\int_{\gamma_-}\mathrm{d}z\;\frac{f(z)}{z+i\delta}}^{=-2\pi i f(-i\delta)}\right]\\
&= 2\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}-2\pi i f(0)
\end{aligned}\tag{1.13.23}$$

$$\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x+i\delta}=\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}-\pi i f(0)\tag{1.13.24}$$

where the γ_- sign gives a $-2\pi i$ because it goes clockwise round the pole and we can apply Cauchy's integral theorem for both γ_- and γ_+ because the pole is not on the contour. On the other hand, if we have the $i\delta$ come from above then

$$\begin{aligned}
2\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x-i\delta} &\equiv \lim_{\delta\to 0^+}\left[\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}+\int_{\gamma_+}\mathrm{d}z\;\frac{f(z)}{z-i\delta}+\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}+\int_{\gamma_-}\mathrm{d}z\;\frac{f(z)}{z-i\delta}\right]\\
&= \lim_{\delta\to 0^+}\left[2\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}+\overbrace{\int_{\gamma_+}\mathrm{d}z\;\frac{f(z)}{z+i\delta}}^{2\pi i f(i\delta)}+\cancel{\int_{\gamma_-}\mathrm{d}z\;\frac{f(z)}{z+i\delta}}\right]\\
&= 2\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}+2\pi i f(0)
\end{aligned}\tag{1.13.25}$$

$$\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x-i\delta}=\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}+\pi i f(0)\tag{1.13.26}$$

which then gets us the Plemelj formula we saw earlier of

$$\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x\pm i\delta}=\mp i\pi f(0)+\fint_{-\infty}^{\infty}\mathrm{d}x\;\frac{f(x)}{x}\tag{1.13.27}$$

[119] Only one of which will enclose the pole.

For a singularity at x_0 on the real line, we can write for $a < m < b$ that

$$\lim_{\delta \to x_0^+} \int_a^b \mathrm{d}x \; \frac{f(x)}{x - x_0 \pm i\delta} = \mp i\pi f(x_0) + \fint_a^b \mathrm{d}x \; \frac{f(x)}{x - x_0} \tag{1.13.28}$$

the most general Plemelj formula on the real line. This of course works the exact same way if we are not on the real axis, but I will let you see how that works out. You do the exact same averaging of two ways, but you have to decide which region is the positive/interior and negative/exterior.

The crux of the matter is that you really need to think about how you can add different curves to get back your original integral. The nature of the pole is important for this.[120]

I will do one example where we extend the real line upwards into the complex plane and give you a flavor of how that arc is considered to go to zero.

We'll just use

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \; \frac{1}{x^4 + 1} \tag{1.13.29}$$

First we extend the contour to a semicircle. We see that the denominator can be written $z^4 + 1 = (z^2 + i)(z^2 - i) = (z + i\sqrt{i})(z - i\sqrt{i})(z - \sqrt{i})(z + \sqrt{i})$ [I am using the principal branch of the square root, so $\sqrt{i} = \exp(i\pi/2)^{1/2} = \exp(i\pi/4) = \sqrt{2} + \sqrt{2}i$]. The contour is made up of

$$\int_C \mathrm{d}z \; \frac{1}{z^4 + 1} = I + \lim_{R \to \infty} \int_0^{\pi} \mathrm{d}\theta \; \frac{Ri \exp(i\theta)}{R^4 \exp(i\theta) + 1} \tag{1.13.30}$$

We use Cauchy's integral theorem with $f(z) = (z + \sqrt{i})^{-1}(z + i\sqrt{i})^{-1}$ so that

$$\frac{1}{2\pi i} \int_C \mathrm{d}z \; \frac{1}{z^4 + 1} = \frac{1}{2\pi i} \int_C \mathrm{d}z \; \frac{f(z)}{(z - i\sqrt{i})(z - \sqrt{i})} = \left[\frac{f(\sqrt{i})}{\sqrt{i} - i\sqrt{i}} + \frac{f(i\sqrt{i})}{i\sqrt{i} - \sqrt{i}} \right] \tag{1.13.31}$$

$$= \frac{(\sqrt{i} + \sqrt{i})^{-1}(\sqrt{i} + i\sqrt{i})^{-1}}{\sqrt{i} - i\sqrt{i}} + \frac{(i\sqrt{i} + \sqrt{i})^{-1}(i\sqrt{i} + i\sqrt{i})^{-1}}{i\sqrt{i} - \sqrt{i}} \tag{1.13.32}$$

$$= \frac{1}{\sqrt{i}(1 - i)2i(1 + i)} + \frac{1}{\sqrt{i}(i - 1)2i\sqrt{i}\sqrt{i}(i + 1)} \tag{1.13.33}$$

$$= \frac{1}{\sqrt{i}4i} + \frac{1}{4\sqrt{i}} = \frac{-i + 1}{4\sqrt{2}(1 + i)} = \frac{1 - 2i + 1}{4\sqrt{2}} = \frac{1}{2\sqrt{2}i} \tag{1.13.34}$$

and so

$$\int_C \mathrm{d}z \; \frac{1}{z^4 + 1} = 2\pi i \left[\frac{1}{2\sqrt{2}i} \right] = \frac{\pi}{\sqrt{2}} \tag{1.13.35}$$

[120] In Landau damping, for example, remember that you start with functions with poles in frequency space satisfying $\Im(\omega) > 0$ and analytically continue the functions downward. This might make you think we have the poles coming from the negative/exterior in. However, the contour for the Laplace transform is oriented clockwise, so the poles are actually viewed as coming from the positive/interior in the Plemelj formula. For Cauchy's integral formula we get quite lucky and can ignore these subtleties in general if we think of ourselves as dealing with the "bounded" interior, as we saw above. We can just add $\pm i\pi$ (depending on counterclockwise or clockwise orientation) times the regular residue when a pole is on the contour curve.

And

$$\left|\lim_{R\to\infty}\int_0^\pi \mathrm{d}\theta\ \frac{Ri\exp(i\theta)}{R^4\exp(i\theta)+1}\right| \leq \lim_{R\to\infty}\left|\int_0^\pi \mathrm{d}\theta\ \frac{Ri\exp(i\theta)}{R^4\exp(i\theta)}\right| \tag{1.13.36}$$

$$\leq \lim_{R\to\infty}\int_0^\pi \mathrm{d}\theta\ \left|\frac{Ri\exp(i\theta)}{R^4\exp(i\theta)}\right| = \lim_{R\to\infty}\int_0^\pi \mathrm{d}\theta\ \left|R^{-3}i\right| = 0 \tag{1.13.37}$$

because $1/R^3 \to 0$ as $R \to \infty$. And so

$$I = \int_{-\infty}^{\infty}\mathrm{d}x\ \frac{1}{x^4+1} = \frac{\pi}{\sqrt{2}} \tag{1.13.38}$$

Essentially every way of contour integrating uses these tricks of finding arcs that go to zero or finding contours that cancel each other to rewrite integrals. You should always check that the extra arc is going to zero, however, for whatever function you are considering.

1.14 Analytic Continuation

Analytic continuation tends to be invoked as an incantation in physics. When you do contour integration and you want to integrate a function outside of its convergent region, you will hear someone say "we analytically continue the function, and perform the integration". I hope to demystify this concept. The basic idea is easy to understand. Suppose we have a function defined by

$$f(s) = \sum_{k=1}^{\infty} k^s \tag{1.14.1}$$

It is well-known that $s < -1$ is necessary for this series to converge. At $s = -1$, we have the infamous harmonic series. The above function is called the Riemann zeta function when it converges.[121] However, the Riemann zeta function is not $f(s)$, for the Riemann zeta function is defined for $s < -1$. In fact, it is defined for complex s. The Riemann zeta function is defined by

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}x\ \frac{x^{s-1}}{\exp(x)-1} \tag{1.14.2}$$

$$\Gamma(s) = \int_0^\infty \mathrm{d}x\ x^{s-1}\exp(x) \tag{1.14.3}$$

with $\Gamma(s)$ related to the factorial function via $x! = \Gamma(x+1)$ for integer x.

Analytic continuation says that if we are dealing with a function that has

$$\sum_{k=1}^{\infty} k^s \tag{1.14.4}$$

we can replace the divergent sum by the Riemann zeta definition which does converge. Thus, we were actually dealing with the Riemann zeta function when we were tackling our problem, but using a specific form for it.

[121] Sometimes it's called the Euler-Riemann zeta function.

There should be many questions about this tactic. The first should be whether it is unique. Are there multiple more complicated functions that could match on to the sum? Or is the Riemann zeta function the only one?

In fact, if we are dealing with complex numbers we can answer this problem. Suppose we are dealing with some open and connected subset[122] of the complex plane, call it S. Then we want to know given two functions $f(z)$ (which is only defined in S) and $g(z)$ (defined over a larger set than S but that matches $f(z)$ everywhere in S) whether $g(z)$ is the only function on this larger region that can match $f(z)$ on the smaller region. Because complex analytic functions have severe restrictions, it turns out that this is true. Consider a second $g_2(z)$ that also matches $f(z)$ in S. Then consider $g(z) - g_2(z)$. It is zero everywhere in S. But if two holomorphic (remember, these are complex functions that have derivatives to all orders) functions subtract to zero in some connected region S, then they must agree over their entire shared domains by the identity theorem of complex analysis. Thus $g_2 = g$, and there is a unique way of analytically continuing a complex function. Note that real-variable functions do not satisfy this. It only works in complex analysis.

As an example of how to do this in practice, suppose we do have a function defined by a power series. Say we start with $g(z) = 1/(z-1)$. We then define a new function $f(z)$ by the power series of $g(z)$ at $z = 0$ finding

$$f(z) = \sum_{k=0}^{\infty} c_k z^k = -\sum_{k=0}^{\infty} z^k \tag{1.14.5}$$

We see this does diverge at $z = 1$. Via the Cauchy-Hadamard theorem we know that $1/R_C = \limsup_{k\to\infty} |a_k|^{1/k}$ where R_C is the radius of convergence. Here $R_C = 1$. So a disk of radius one around the origin is our region of convergence. If we didn't know that $f(z)$ came from $g(z)$, we could expand the domain it is defined over by analytically continuing it. Let's then expand it at some other point within the region of convergence and find new coefficients for it. Our new power series will be centered at point z_0. Thus we have our new function

$$f_1(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k = \sum_{k=0}^{\infty} \frac{f_1^{(k)}(z_0)}{k!} (z - z_0)^k \tag{1.14.6}$$

We use the Cauchy-integral theorem to find the values of b_k

$$f_1^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \mathrm{d}z \; \frac{f_1(z)}{(z - z_0)^{n+1}} \tag{1.14.7}$$

And so

$$b_n = \frac{1}{2\pi i} \oint_C \mathrm{d}z \; \frac{f_1(z)}{(z - z_0)^{n+1}} \tag{1.14.8}$$

Our closed curve C will be a circle about $z = z_0$ with radius r. Here z_0 is any point in our original radius of convergence. We will then later check if this gives a larger region of convergence, or convergence over a different region of the complex plane. We will try $f_1(z) = f(z)$ over the new

[122] There is no avoiding mathematical terminology here. Open means that the boundary is not included. For real numbers, think of an open interval like $(0, 1)$ which does not contain 0 or 1.

region to see if our analytic continuation works. We calculate

$$\begin{aligned} b_n &= \frac{1}{2\pi i}\oint_C \mathrm{d}z\ \frac{f(z)}{(z-z_0)^{n+1}} \\ &= \frac{-1}{2\pi i}\oint_C \mathrm{d}z\ \frac{\sum_{k=0}^{\infty} z^k}{(z-z_0)^{n+1}} \\ &= \frac{-1}{2\pi i}\sum_{k=0}^{\infty}\oint_C \mathrm{d}z\ \frac{z^k}{(z-z_0)^{n+1}} \end{aligned} \tag{1.14.9}$$

we write $z - z_0 = r\exp(i\theta)$ so $\mathrm{d}z = ir\exp(i\theta)\,\mathrm{d}\theta$ for our curve with constant r. Then we find

$$\begin{aligned} b_n &= \frac{-1}{2\pi}\sum_{k=0}^{\infty}\int_0^{2\pi}\mathrm{d}\theta\ r\exp(i\theta)\frac{(r\exp(i\theta)+z_0)^k}{(r\exp(i\theta))^{n+1}} \\ &= \frac{-1}{2\pi}\sum_{k=0}^{\infty}\int_0^{2\pi}\mathrm{d}\theta\ \frac{(r\exp(i\theta)+z_0)^k}{(r\exp(i\theta))^{n}} \\ &= \frac{-1}{2\pi}\sum_{k=0}^{\infty}\int_0^{2\pi}\mathrm{d}\theta\ \frac{\sum_{j=0}^{k} C_j^k z_0^{k-j}(r\exp(i\theta))^j}{(r\exp(i\theta))^{n}} \end{aligned} \tag{1.14.10}$$

with C_j^n meaning n choose j from combinatorics. The integrand will clearly be zero for all but the $j = n$ case, when there is no θ dependence because it cancels out. Then

$$b_n = \frac{-1}{2\pi}\sum_{k=0}^{\infty} C_n^k z_0^{k-n}\int_0^{2\pi}\mathrm{d}\theta\ = -\sum_{k=0}^{\infty} C_n^k z_0^{k-n} = -z_0^{-n}\sum_{k=0}^{\infty} C_n^k z_0^k \tag{1.14.11}$$

It can be shown that

$$\sum_{k=0}^{\infty} C_n^k z_0^k = \frac{z_0^n}{(1-z_0)^{n+1}} \tag{1.14.12}$$

and so

$$b_n = -z_0^{-n}\frac{z_0^n}{(1-z_0)^{n+1}} = -(1-z_0)^{-n-1} \tag{1.14.13}$$

So

$$f_1(z) = \sum_{k=0}^{\infty} -\frac{1}{1-z_0}\left(\frac{z-z_0}{1-z_0}\right)^k = \frac{-1}{1-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{1-z_0}\right)^k \tag{1.14.14}$$

The radius of convergence of the final sum can then be found to be $|1 - z_0|$. Remember we can choose any $|z_0| < 1$. So we can clearly chose z_0 such that

$$|1 - z_0| > 1 \tag{1.14.15}$$

by choosing $\Re(z_0) \leq 0$. Thus, we have a larger radius of convergence and we have indeed continued the analytic function to a larger region. This is fairly obvious when we have chosen originally $g(z) = 1/(z-1)$. By choosing a different point we simply get the series at a different location which is farther from the singularity $z = 1$ and so has a larger radius of convergence.

You might think this is unimpressive, but if we only knew $f(z)$ and were asked what $f(-1)$ is, we would say

$$f(-1) = -\sum_{k=0}^{\infty}(-1)^k = ? \tag{1.14.16}$$

has no ultimate limit and so does not converge. However, if we chose $z_0 = -0.5$, we'd now be able to say

$$\begin{aligned} f_1(-1) &= \frac{-1}{1-(-0.5)}\sum_{k=0}^{\infty}\left(\frac{-1-(-0.5)}{1-(-0.5)}\right)^k = \frac{-1}{1.5}\sum_{k=0}^{\infty}\left(\frac{-0.5}{1.5}\right)^k \\ &= \frac{-2}{3}\sum_{k=0}^{\infty}(-3)^{-k} = \frac{-2}{3}\frac{1}{1+\frac{1}{3}} = \frac{-2}{\cancel{3}}\frac{\cancel{3}}{4} = \frac{-1}{2} \end{aligned} \tag{1.14.17}$$

Via this analytic continuation (because we're actually dealing with $g(-1) = 1/[(-1)-1] = -1/2$) it looks like we have given a divergent series a value! This is in fact the negative of Grandi's series, and we have explained why it can said to "sum" to $-1/2$. Note that via further analytic continuation we could find $f(2) = g(2) = 1/(2-1) = 1$ [the first equals sign means analytically continued] which would be the analytically continued value of $f(z) = -\sum_{k=0}^{\infty} z^k$. In which case, we see that a series that contains only negative terms somehow "sums" to a positive value. Of course, it does not sum, the new value is simply associated with our original series via the analytic continuation. For the general case, we can simply repeat the process of choosing new points and calculating the new coefficients until we have a domain as large as we desire, so long as we can actually calculate the new coefficients.

We can see an example with a red dot representing a singularity that is a simple pole of order one in Figure 1.14.

But nothing is ever quite so simple. Suppose we had analytically continued $\ln z$. This function[123] is multi-valued for complex values. So our analytic continuation of it would also be multi-valued depending on which branch of $\ln z$ we choose. Then we must be careful to analytically continue the function so that it is on the branch we desire. If as we analytically continue we choose a path that goes all the way around the singularity (that is the path goes 2π radians around the singularity) we get a different branch and will have added a factor of $2\pi i$. So we are no longer on the principal branch. Thus, analytic continuation works, but we must remember to think about whether the analytic continuation of a function is multi-valued and how to stay on the branch we desire.

[123] Technically it is not a function, but a multi-valued function. The important thing to realize is that by restricting ourselves to branches of a multi-valued function we can get a true function.

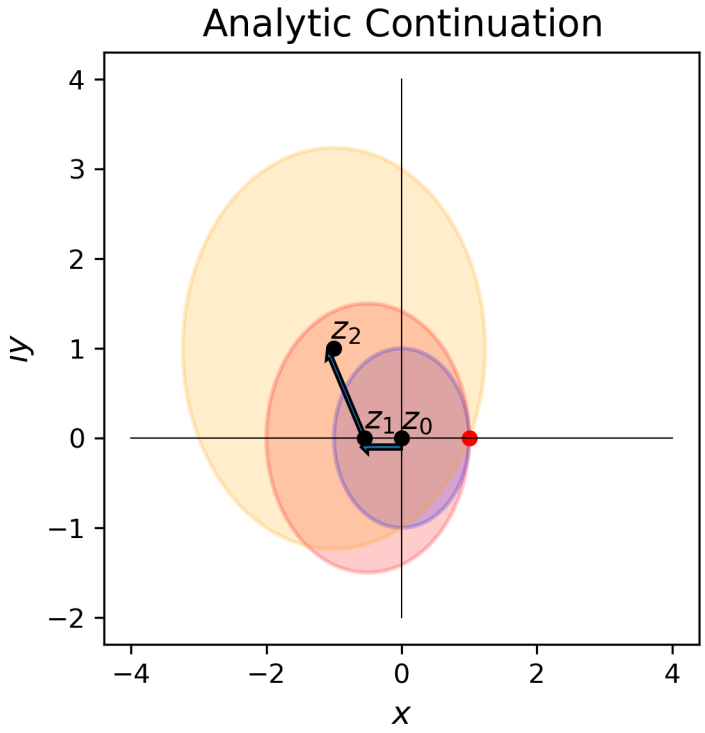

Figure 1.14: A simple example of analytic continuation. We start at z_0 and have a radius of convergence determined by the red dot singularity. We then continue to z_1 within our convergence radius, and recalculate. We can repeat this process with z_2 within our new radius of convergence from z_1, etc.

1.15 Special Functions

> *I do not wish to imply that the hypergeometric function is the only function about which mathematicians know anything. That is far from being true. There are other fertile valleys with which the wooden ploughs of the twentieth century can cope; but the valley inhabited by schoolboys, by engineers, by physicists, and by students of elementary mathematics, is the valley of the Hypergeometric Function, and its boundaries are (but for one or two small clefts explored by pioneers) virgin rock.*
>
> — W. W. Sawyer[29, p. 64]

This will simply explain some interesting features of some special functions that show up in plasma physics or in physics in general. They are worth knowing a little bit about so that you can understand how they behave when they show up. I will go into a depth that I think is reasonable for a book of this kind. There are often books entirely dedicated to the properties of a single special function, so know that I am giving only a superficial treatment compared to the depth of knowledge available.

1.15.1 Hypergeometric Series

Truly, there are no series and functions as little taught as the hypergeometric series in physics, yet so ubiquitous in physics examples. The name seems to form a barrier to understanding, but the name really comes from the fact that the geometric series is a special case. A hypergeometric series is a natural generalization. So given a series in terms of a complex/real variable z, a hypergeometric series is written as

$$\sum_{n=0}^{\infty} \beta_n z^n \tag{1.15.1}$$

where

$$\frac{\beta_{n+1}}{\beta_n} = \frac{A(n)}{B(n)} \tag{1.15.2}$$

with $A(n)$ and $B(n)$ polynomials in n. This is often said in words as β_{n+1}/β_n is a rational function of n. When defining a hypergeometric series, it is conventional to assign $\beta_0 = 1$, and to factor the polynomials into linear terms (thus introducing complex coefficients) so that we write

$$\frac{A(n)}{B(n)} = \frac{\prod_{i=1}^{N}(a_i + n)}{\prod_{j=1}^{M}(b_j + n)} \tag{1.15.3}$$

where N is the highest power of n in $A(n)$ and M is the highest power of n in $B(n)$. For some reason, it is also traditional to assume that one of the factors of $B(n)$ is $(1 + n)$ so the above is actually written

$$\frac{A(n)}{B(n)} = \frac{c\prod_{i=1}^{N}(a_i + n)}{d(1 + n)\prod_{j=1}^{M-1}(b_j + n)} \tag{1.15.4}$$

where c and d are chosen so that we actually get back $A(n)/B(n)$. They can be viewed as the correct leading order coefficients. Then we can write the series as

$$1+\frac{\prod_{i=1}^{N}a_i}{(1+0)\prod_{j=1}^{M-1}b_j}\frac{cz}{d}+\frac{\prod_{i=1}^{N}a_i}{(1+0)\prod_{j=1}^{M-1}b_j}\frac{\prod_{i=1}^{N}(a_i+1)}{(1+1)\prod_{j=1}^{M-1}(b_j+1)}\left(\frac{cz}{d}\right)^2+\cdots \tag{1.15.5}$$

We can introduce $\widehat{z}=cz/d$ and then ignore the hat to rewrite the series in the form

$$1+\frac{\prod_{i=1}^{N}a_i}{\prod_{j=1}^{M-1}b_j}\frac{z}{1}+\frac{\prod_{i=1}^{N}a_i}{\prod_{j=1}^{M-1}b_j}\frac{\prod_{i=1}^{N}(a_i+1)}{\prod_{j=1}^{M-1}(b_j+1)}\frac{z^2}{2!}+\cdots \tag{1.15.6}$$

We then write this series as

$${}_NF_{M-1}(a_1,\ldots,a_N;b_1,\ldots,b_{M-1};z)=\sum_{n=0}^{\infty}\beta_n z^n \tag{1.15.7}$$

with the appropriately defined β_n. Traditionally, we set $N=p$ and $M-1=q$ and use the definition of Pochhammer's symbol given by

$$(a)_n\equiv\begin{cases}1 & n=0\\ a(a+1)(a+2)\cdots(a+n-1) & n>0\end{cases}=\frac{\Gamma(a+n)}{\Gamma(a)} \tag{1.15.8}$$

and then we write

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)=\sum_{n=0}^{\infty}\beta_n z^n=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} \tag{1.15.9}$$

Sometimes the symbolism is modified so that there is less chance of error by writing

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)={}_pF_q\left(\begin{array}{c}a_1,\ldots,a_p\\ b_1\ldots,b_q\end{array}\middle|z\right) \tag{1.15.10}$$

or something similar. Note that if this is well-defined, the series defines a hypergeometric function. To be well-defined, then the radius of convergence in z must be nonzero, and the series must converge. The derivative is given via

$$\frac{\mathrm{d}}{\mathrm{d}z}\,{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)=\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}\,{}_pF_q(a_1+1,\ldots,a_p+1;b_1+1,\ldots,b_q+1;z) \tag{1.15.11}$$

To simply notation I will write

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)\equiv w_{pq}(\mathtt{a};\mathtt{b},z) \tag{1.15.12}$$

Then the above relation follows from the definition as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}w_{pq}(\mathtt{a};\mathtt{b},z)&=\frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}=\sum_{n=1}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^{n-1}}{(n-1)!}\\&=\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}(a_i+n-1)}{\prod_{j=1}^{q}(b_i+n-1)}\frac{(a_1)_{n-1}\cdots(a_p)_{n-1}}{(b_1)_{n-1}\cdots(b_q)_{n-1}}\frac{z^{n-1}}{(n-1)!}\end{aligned} \tag{1.15.13}$$

we can then use $m = n-1$ or $n = m+1$ and this becomes

$$\frac{\mathrm{d}}{\mathrm{d}z} w_{pq}(\mathtt{a};\mathtt{b};z) = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i+m)}{\prod_{j=1}^{q}(b_i+m)} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{z^m}{(m)!} \tag{1.15.14}$$

We can then use from the definition[124] that

$$(a_j)_m(a_j+m) = (a_j)_{m+1} = a_j(a_j+1)_m \tag{1.15.15}$$

so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z} w_{pq}(\mathtt{a};\mathtt{b},z) &= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{q} b_i} \frac{(a_1+1)_m \cdots (a_p+1)_m}{(b_1+1)_m \cdots (b_q+1)_m} \frac{z^m}{(m)!} = \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{q} b_i} \sum_{n=0}^{\infty} \frac{(a_1+1)_n \cdots (a_p+1)_n}{(b_1+1)_n \cdots (b_q+1)_n} \frac{z^n}{n!} \\ &= \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{q} b_j} w_{pq}(\mathtt{a}+1;\mathtt{b}+1,z)\end{aligned} \tag{1.15.16}$$

where $\mathtt{a}+1$ means add 1 to each element of $\mathtt{a}$ [i.e., $\mathtt{a}+1 = (a_1+1, a_2+1, \ldots, a_p+1)$].

One can also find

$$z\frac{\mathrm{d}}{\mathrm{d}z} w_{pq}(\mathtt{a};\mathtt{b};z) = z\frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{(n-1)!} \tag{1.15.17}$$

We can then add $a_i w_{pq}(\mathtt{a};\mathtt{b};z)$ to the previous result to find

$$\begin{aligned}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_i\right) w_{pq}(\mathtt{a};\mathtt{b};z) &= \sum_{n=0}^{\infty} \left(\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{(n-1)!} + a_i \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \frac{n+a_i}{n} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{(n-1)!} \\ &= \sum_{n=0}^{\infty} a_i \frac{(a_1)_n \cdots (a_{i-1})_n (a_i+1)_n (a_{i+1})_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \\ &= a_i \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{i-1})_n (a_i+1)_n (a_{i+1})_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \\ &= a_i w_{pq}(\mathtt{a}'_i;\mathtt{b};z)\end{aligned} \tag{1.15.18}$$

where we used the relationship from (1.15.15) with $\mathtt{a}'_i$ the same as $\mathtt{a}$ except in the ith entry where $a'_i = a_i + 1$.

Exactly analogously we find

$$\begin{aligned}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + b_k - 1\right) w_{pq}(\mathtt{a};\mathtt{b};z) &= \sum_{n=0}^{\infty} \left(\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{(n-1)!} + (b_k-1) \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \frac{n+(b_k-1)}{n} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{(n-1)!}\end{aligned} \tag{1.15.19}$$

[124] This Pochhammer symbol notation is sometimes called the rising factorial.

We now use

$$\frac{(b_j)_n}{(b_j+n+1)} = (b_j)_{n-1} = \frac{(b_j-1)_n}{b_j-1} \tag{1.15.20}$$

and the above becomes

$$\begin{aligned}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+b_k-1\right)w_{pq}(\mathtt{a};\mathtt{b};z) &= \sum_{n=0}^{\infty}(b_j-1)\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_{j-1})_n(b_j-1)_n(b_{j+1})^n\cdots(b_q)_n}\frac{z^n}{n!}\\ &= (b_j-1)w_{pq}(\mathtt{a};\mathtt{b}'_k;z)\end{aligned} \tag{1.15.21}$$

where $\mathtt{b}'_k$ is the same as $\mathtt{b}$ except the kth entry has $b'_k = b_k - 1$.

This means

$$\begin{aligned}\prod_{n=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+a_n\right)w_{pq}(\mathtt{a};\mathtt{b};z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\prod_{n=1}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+b_n-1\right)w_{pq}(\mathtt{a};\mathtt{b};z)\right)\\ \prod_{n=1}^{p}a_n w_{pq}(\mathtt{a}+1;\mathtt{b};z) &= \frac{\mathrm{d}}{\mathrm{d}z}\prod_{n=1}^{q}(b_n-1)w_{pq}(\mathtt{a};\mathtt{b}-1;z)\\ \prod_{n=1}^{p}a_n w_{pq}(\mathtt{a}+1;\mathtt{b};z) &= \frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}(b_j-1)}\prod_{n=1}^{q}(b_n-1)w_{pq}(\mathtt{a}+1;\mathtt{b};z)\\ \prod_{n=1}^{p}a_n w_{pq}(\mathtt{a}+1;\mathtt{b};z) &= \prod_{n=1}^{p}a_n w_{pq}(\mathtt{a}+1;\mathtt{b};z)\end{aligned} \tag{1.15.22}$$

We can also look at specific cases

$${}_0F_0(;;z) = \exp(z) \tag{1.15.23}$$

$${}_1F_0(a;;z) = (1-z)^{-a} \tag{1.15.24}$$

Then ${}_0F_1(;b;z)$ is related to the Bessel functions by

$$J_\alpha(x) = \frac{\left(\frac{x}{2}\right)^\alpha}{\Gamma(\alpha+1)}\,{}_0F_1(:\alpha+1;-\frac{1}{4}x^2) \tag{1.15.25}$$

$$I_\alpha(x) = \frac{\left(\frac{x}{2}\right)^\alpha}{\Gamma(\alpha+1)}\,{}_0F_1(:\alpha+1;\frac{1}{4}x^2) \tag{1.15.26}$$

If we use our formula above, we find

$$w_{01}(;b;z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+b-1\right)w_{01}(;b;z) \tag{1.15.27}$$

$$w_{01}(;b;z) = \left(\frac{\mathrm{d}}{\mathrm{d}z}+z\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(b-1)\frac{\mathrm{d}}{\mathrm{d}z}\right)w_{01}(;b;z) \tag{1.15.28}$$

or abbreviating $w = w_{01}(;b;z)$ we find the governing differential equation to be

$$z\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+b\frac{\mathrm{d}w}{\mathrm{d}z}-w = 0 \tag{1.15.29}$$

Then we have the confluent hypergeometric functions of the first kind, often written $w_{11}(a;b;z) = M(a;b;z)$ They satisfy

$$z\frac{\mathrm{d}w}{\mathrm{d}z} + aw = \left(z\frac{\mathrm{d}^2}{\mathrm{d}z^2} + b\frac{\mathrm{d}}{\mathrm{d}z}\right) w \tag{1.15.30}$$

$$z\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + (b-z)\frac{\mathrm{d}w}{\mathrm{d}z} - aw = 0 \tag{1.15.31}$$

For $w_{11}(-n : b; z)$ with $n > 0$ then $w_{11}(-n; b; z)$ are related to the Laguerre polynomials, and so to Hermite polynomials.

We can also calculate the differential equations whose solutions are $w_{20}(a_1, a_2; ; z)$.

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_1\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_2\right) w = \frac{\mathrm{d}}{\mathrm{d}z}w \tag{1.15.32}$$

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_1\right)\left(z\frac{\mathrm{d}w}{\mathrm{d}z} + a_2w\right) = \frac{\mathrm{d}w}{\mathrm{d}z} \tag{1.15.33}$$

$$z\frac{\mathrm{d}w}{\mathrm{d}z} + z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + za_2\frac{\mathrm{d}w}{\mathrm{d}z} + za_1\frac{\mathrm{d}w}{\mathrm{d}z} + a_1a_2w = \frac{\mathrm{d}w}{\mathrm{d}z} \tag{1.15.34}$$

$$z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + z[1 + a_1 + a_2]\frac{\mathrm{d}w}{\mathrm{d}z} + a_1a_2w = \frac{\mathrm{d}w}{\mathrm{d}z} \tag{1.15.35}$$

$$z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + (z[1 + a_1 + a_2] - 1)\frac{\mathrm{d}w}{\mathrm{d}z} + a_1a_2w = 0 \tag{1.15.36}$$

The final hypergeometric series we will consider are for $w_{21}(a_1, a_2; b; z)$. The differential equation will be

$$z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + z[1 + a_1 + a_2]\frac{\mathrm{d}w}{\mathrm{d}z} + a_1a_2w = z\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + b\frac{\mathrm{d}w}{\mathrm{d}z} \tag{1.15.37}$$

$$z(z-1)\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + (z[1 + a_1 + a_2] - b)\frac{\mathrm{d}w}{\mathrm{d}z} + a_1a_2w = 0 \tag{1.15.38}$$

These series w_{21} have the confusing name of hypergeometric functions. It is often called Gauss's hypergeometric functions or classical hypergeometric functions. The confusion is that these hypergeometric series are often called generalized hypergeometric series so that we do not confuse them with basic hypergeometric series (sometimes called q-hypergeometric series).

This means that there are at least five uses of hypergeometric series or functions. First, the generalized hypergeometric series (i.e., "just" a hypergeometric series), the basic or q-hypergeometric series, the confluent hypergeometric function, and the hypergeometric function. You will have to confront this needless mess of terms by remembering that each one is precisely defined and you must precisely say what you mean.

Finally, I should summarize when the hypergeometric series converge.

When $p \leq q$ then the summation converges and we have an analytic function defined for all z.

When $p = q + 1$ the case is more complicated.[125] If there is an $a_i < 0$, then clearly at some point $(a_i)_n = 0$ and the series terminates leaving us a polynomial. If all the $a_i > 0$ then the radius

[125] It seems mathematical functions, like life, does not enjoy making simple statements.

of convergence is 1 and we only have convergence in $|z| < 1$. One can then proceed to define an analytic continuation outside of $|z| < 1$ and define that to be the generalized hypergeometric function corresponding to the series. There is a principal branch associated with this, and so one must consult a reference[126] to see which integral defines this analytic continuation. For $p > q+1$, unless $a_i < 0$ the series is undefined everywhere except $z = 0$. When $a_i < 0$ for some i, then we are left with a polynomial again.

Finally, I should mention some alternate notation. The generalized hypergeometric series are sometimes represented as

$$ {}_pF_q(\mathtt{a};\mathtt{b};z) = {}_pF_q\left(\begin{array}{c}\mathtt{a}\\ \mathtt{b}\end{array};z\right) = {}_pF_q\left(\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array};z\right) = {}_pF_q\left(\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array}\middle| z\right) \tag{1.15.39} $$

and that there is a generalization of these generalized hypergeometric series when they are functions called the Meijer G-function which is denoted with the notation

$$ G^{m,n}_{p,q}\left(\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array}\middle| z\right) \equiv \frac{1}{2\pi i}\int_L \mathrm{d}s\, \frac{\left[\prod_{j=1}^{m}\Gamma(b_j-s)\right]\left[\prod_{j=1}^{n}\Gamma(1-a_j+s)\right]}{\left[\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\right]\left[\prod_{j=n+1}^{p}\Gamma(a_j-s)\right]} z^s \tag{1.15.40} $$

$$ G^{m,n}_{p,q}\left(\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array}\middle| z\right) = G^{m,n}_{p,q}\left(\begin{array}{c}\mathtt{a}\\ \mathtt{b}\end{array}\middle| z\right) = G^{m,n}_{p,q}\left(\begin{array}{c}\mathtt{a}\\ \mathtt{b}\end{array};z\right) \tag{1.15.41} $$

where L is the path of the integral which is rather complicated to describe and so I will simply point you to the relevant section of the DLMF[10]. The advantage of the Meijer G-function is that because of the path choice of L you actually create an analytic function with a couple of restrictions on the $\mathtt{a}$, $\mathtt{b}$, and using integer m, n, p, and q. See the definition in the DLMF for what the restrictions are. I won't talk anymore of this, as there are even further generalizations beyond this, but I have never encountered the Meijer G-functions in practice (except in the sense that generalized hypergeometric series are Meijer G-functions), and the generalized hypergeometric series are enough work to understand.

1.15.2 Confluent Hypergeometric Series

The confluent in confluent hypergeometric series refers to a confluence of solutions. This follows from an identity for the $p \leq q$ case. We have

$$ \lim_{|\alpha|\to\infty} {}_{p+1}F_q(\mathtt{a},\alpha;\mathtt{b};z/\alpha) = {}_pF_q(\mathtt{a};\mathtt{b};z) \tag{1.15.42} $$

This is easy to show because

$$ \lim_{|\alpha|\to\infty} {}_{p+1}F_q(\mathtt{a},\alpha;\mathtt{b};z/\alpha) = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_{p+1})_n(\alpha)_n}{(b_1)_n\cdots(b_q)_n(n!)}\left(\frac{z}{\alpha}\right)^n \tag{1.15.43} $$

and so looking only at the α parts we see

$$ \lim_{|\alpha|\to\infty}\frac{(\alpha)_n}{\alpha^n} = \lim_{|\alpha|\to\infty}\frac{\alpha^n+\mathcal{O}(\alpha^{n-1})}{\alpha^n} = \lim_{|\alpha|\to\infty}\left(1+\mathcal{O}(\alpha^{-1})\right) = 1 \tag{1.15.44} $$

[126] The DLMF[10] is your friend.

which means

$$\lim_{|\alpha|\to\infty} {}_{p+1}F_q(\mathtt{a},\alpha;\mathtt{b};z/\alpha) = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_{p+1})_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} = {}_pF_q(\mathtt{a};\mathtt{b};z) \tag{1.15.45}$$

Note that if we have $p = q$, then on the left hand side of (1.15.42) we have "regular" singularities at α and ∞ whereas on the right, these two singularities have to have combined into an "irregular" singularity at ∞. Thus, a confluence of solutions has occurred. Given a differential equation of the form

$$\frac{\mathrm{d}^n f}{\mathrm{d}z^n} + \sum_{i=1}^{n-1} g_i(z)\frac{\mathrm{d}^i f}{\mathrm{d}z^i} + g_0(z)f = 0 \tag{1.15.46}$$

then a regular singularity at z_0 satisfies $(z-z_0)^{n-i}g_i(z_0)$ is analytic for all i. If not (that is the previous expression is not analytic for all i), then z_0 is an irregular singularity. For $z_0 = \infty$ you can test the regularity by taking $z \to 1/\widehat{z}$ and trying $\widehat{z} = 0$.

What this means in non-mathematics speak is that a regular singularity can be treated as we do in complex analysis with poles. That is, we can work with regular singularities to find the solutions of the entire equation rather easily by containing the singularity, so to speak. When we have an irregular singularity, there is no containing it in a simple manner, and so we have to construct a solution in a different way.

Thus, when we look at the above equation on the left hand side of (1.15.42), we can find the regular singularities by looking at the governing differential equation. By considering (1.15.22) to find the generic differential equation for a hypergeometric series solution with $p = q$, we write out the differential equation for ${}_{q+1}F_q(\mathtt{a},\alpha;\mathtt{b};z/\alpha)$. We can then use $\widehat{z} = z/\alpha$ and see that the coefficient of the highest order derivative $\frac{\mathrm{d}^{q+1}}{\mathrm{d}\widehat{z}^{q+1}}$ is given by

$$\widehat{z}^{q+1} - \widehat{z}^q = \frac{z^q}{\alpha^q}\left(\frac{z}{\alpha} - 1\right) \tag{1.15.47}$$

if we then switch to $\frac{\mathrm{d}}{\mathrm{d}z}$ we use $\frac{\mathrm{d}}{\mathrm{d}\widehat{z}} = \frac{\mathrm{d}z}{\mathrm{d}\widehat{z}}\frac{\mathrm{d}}{\mathrm{d}z} = \alpha\frac{\mathrm{d}}{\mathrm{d}z}$. Then the coefficient of the highest order derivative $\frac{\mathrm{d}^{q+1}}{\mathrm{d}z^{q+1}}$ is given by

$$z^q(z-\alpha) \tag{1.15.48}$$

If we divide through by this factor to get our differential equation in the form of (1.15.46), we see that there will be singularities at $z = \infty$ and $z = \alpha$ for the $g_i(z)$, if the $g_i(z)$ do not equal zero at ∞ and α.

On the other hand, the right hand side of (1.15.42) with $p = q$ will only have z^{q+1} as a coefficient for the highest order derivative, and so only has a singularity at $z = \infty$. The so-called confluence has occurred and since these two differential equations are equivalent, an irregular singularity must occur at ∞. Remember that irregular singularities basically say that near these singularities that some terms in your differential equation completely dominate and so you are essentially solving a different differential equation.

The confluent hypergeometric function is given by ${}_1F_1(a;b;z)$. There is a confluence for other p and q, but those functions generally have a different name. The confluent hypergeometric function

is often called Kummer's function (of the first kind) and denoted $M(a,b,z)$. It is a limiting case of the hypergeometric function, so that

$$M(a,b,z) = \lim_{b\to\infty} {}_2F_1(a,b;c;z/b) \tag{1.15.49}$$

Our governing differential equation must have two solutions, so the second solution is called the Tricomi confluent hypergeometric function $U(a,b,z)$, or sometimes simply the confluent hypergeometric function of the second kind. It is determined by what a and b are similarly to $M(a,b,z)$, but has the property that

$$U(a,b,z) \overset{z\to\infty}{\longrightarrow} z^{-a} \tag{1.15.50}$$

Finally, there are other forms of the confluent generalized hypergeometric functions that are used. The above are called the Kummer functions, while one can reparameterize the equations to get the Whittaker functions. Here we take a solution to Kummer's differential equation w and switch to $W = \exp(-z/2)z^{\frac{1}{2}+\mu}w$, $\kappa = b/2 - a$, and $\mu = (b-1)/2$ and get

$$M_{\kappa,\mu}(z) = \exp\left(-z/2\right) z^{\frac{1}{2}+\mu} M\left(\frac{1}{2}+\mu-\kappa, 1+2\mu, z\right) \tag{1.15.51}$$

$$W_{\kappa,\mu}(z) = \exp\left(-z/2\right) z^{\frac{1}{2}+\mu} U\left(\frac{1}{2}+\mu-\kappa, 1+2\mu, z\right) \tag{1.15.52}$$

The confluent hypergeometric functions are related to a surprising number of useful functions. We have

$$M\left(a, a+1, -z\right) = \exp(-z) M(1, a+1, z) = a z^{-a}\gamma(a,z) \tag{1.15.53}$$

$$U\left(a, a, z\right) = z^{1-a} U(1, 2-a, z) = z^{1-a}\exp(z) E_a(z) = \exp(z)\Gamma(1-a,z) \tag{1.15.54}$$

$$M\left(\frac{1}{2}, \frac{3}{2}, -z^2\right) = \frac{\sqrt{\pi}}{2z}\operatorname{erf}(z) \tag{1.15.55}$$

$$U\left(\frac{1}{2}, \frac{3}{2}, z^2\right) = \sqrt{\pi}\exp(z^2)\operatorname{erfc}(z) \tag{1.15.56}$$

$$M\left(\nu+\frac{1}{2}, 2\nu+1, 2z\right) = \Gamma(1+\nu)\exp(z)\left(\frac{z}{2}\right)^{-\nu} I_\nu(z) \tag{1.15.57}$$

$$U\left(\nu+\frac{1}{2}, 2\nu+1, 2z\right) = \frac{\exp(z)}{\sqrt{\pi}}\left(2z\right)^{-\nu} K_\nu(z) \tag{1.15.58}$$

$$U\left(\frac{5}{6}, \frac{5}{3}, \frac{4}{3}z^{3/2}\right) = \sqrt{\pi}\frac{3^{5/6}}{2^{2/3}}\frac{\exp\left(\frac{2}{3}z^{3/2}\right)}{z}\operatorname{Ai}(z) \tag{1.15.59}$$

$$M\left(\frac{a}{2}+\frac{1}{4}, \frac{1}{2}, \frac{1}{2}z^2\right) = \frac{2^{\frac{a}{2}-\frac{3}{4}}\Gamma\left(\frac{a}{2}+\frac{1}{4}\right)\exp\left(\frac{z^2}{4}\right)}{\sqrt{\pi}}\left[U(a,z)+U(a,-z)\right] \tag{1.15.60}$$

$$U\left(\frac{a}{2}+\frac{1}{4}, \frac{1}{2}, \frac{1}{2}z^2\right) = \frac{2^{\frac{a}{2}-\frac{5}{4}}\Gamma\left(\frac{a}{2}+\frac{1}{4}\right)\exp\left(\frac{z^2}{4}\right)}{z\sqrt{\pi}}\left[U(a,-z)-U(a,z)\right] \tag{1.15.61}$$

$$M\left(-n, \frac{1}{2}, z^2\right) = (-1)^n\frac{n!}{(2n)!}H_{2n}(z) \tag{1.15.62}$$

$$M\left(-n,\frac{3}{2},z^2\right)=(-1)^n\frac{n!}{(2n+1)!(2z)}H_{2n+1}(z) \tag{1.15.63}$$

$$U\left(\frac{1-n}{2},\frac{3}{2},z^2\right)=\frac{H_n(z)}{z(2^n)} \tag{1.15.64}$$

$$U(-n,\alpha+1,z)=(-1)^\alpha(\alpha+1)_nM(-n,\alpha+1,z)=(-1)^\alpha(n!)L_n^{(\alpha)}(z) \tag{1.15.65}$$

$$U(-n,z-n+1,a)=(-z)_nM(-n,z-n+1,a)=a^nC_n(z;a) \tag{1.15.66}$$

$$U(a,b,z)=z^{-a}\,{}_2F_0(a,a-b+1,;;-z^{-1}) \tag{1.15.67}$$

$$M(\ell+1\mp i\eta,2\ell+2,\pm2i\rho)=\frac{(2\ell+1)!}{2^\ell\exp\left(\frac{-\pi\eta}{2}\right)}\rho^{-\ell-1}\exp(\pm i\rho)F_\ell(\eta,\rho) \tag{1.15.68}$$

$$U(\ell+1\mp i\eta,2\ell+2,\pm2i\rho)=\frac{\exp\left(\mp i\left[\rho-\eta\ln(2\rho)-\frac{\ell\pi}{2}+\arg(\Gamma(\ell+1+i\eta))\right]\right)}{(\mp2i\rho)^{\ell+1\pm i\eta}}H_l^{\pm}(\eta,\rho) \tag{1.15.69}$$

Here $\gamma(a,z)$ and $\Gamma(a,z)$ are the incomplete gamma functions, $E_a(z)$ is a generalized exponential integral, erf is the error function, erfc is the complementary error function, $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions, $\mathrm{Ai}(z)$ is the Airy function, $U(a,z)$ is the parabolic cylinder function, $\Gamma(z)$ is the gamma function, $H_n(z)$ is a Hermite polynomial of order n, $L_n^{(\alpha)}(z)$ are the generalized Laguerre polynomials (sometimes associated Laguerre polynomials), $C_n(z;a)$ are the Charlier polynomials, and $F_\ell(\eta,\rho)$ and $H_l^{\pm}(\eta,\rho)$ are the Coulomb functions.

The Coulomb functions are those that solve

$$\frac{\mathrm{d}^2w}{\mathrm{d}\rho^2}+\left(1-\frac{2\eta}{\eta}-\frac{\ell(\ell+1)}{\rho^2}\right)w=0 \tag{1.15.70}$$

and the choice of $\pm$ in $F_\ell(\eta,\rho)$ makes no difference so long as you always choose the top or the bottom consistently.

Unfortunately, you may see the Coulomb functions in a different guise as a solution to the equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}r^2}+\left(\epsilon+\frac{2}{r}-\frac{\ell(\ell+1)}{r^2}\right)w=0 \tag{1.15.71}$$

where $r=-\eta\rho$ and $\epsilon=1/\eta^2$. The solutions are then written as

$$f(\epsilon;\ell;r)=\frac{\kappa^{\ell+1}M_{\epsilon,\ell+\frac{1}{2}}(2r/\kappa)}{(2\ell+1)!} \tag{1.15.72}$$

$$f(\epsilon;\ell;r)=(2r)^{\ell+1}\exp\left(-r/\kappa\right)\frac{M(\ell+1-\kappa,2\ell+2,2r/\kappa)}{(2\ell+1)!} \tag{1.15.73}$$

$$\kappa=\begin{cases}(-\epsilon)^{-1/2} & \epsilon<0,r>0\\ -(-\epsilon)^{-1/2} & \epsilon<0,r<0\\ \pm i\epsilon^{-1/2} & \epsilon>0\end{cases} \tag{1.15.74}$$

for the regular solution (the choice of $\pm$ makes no difference to the solution, as it leads to the same

solution). The intensely ugly

$$h(\varepsilon;\ell;r) = \frac{\Gamma(\ell+1-\kappa)}{\pi\kappa^\ell}\left[W_{\kappa,\ell+\frac{1}{2}}(2r/\kappa) + (-1)^\ell S(\varepsilon,r)\frac{\Gamma(\ell+1+\kappa)}{2(2\ell+1)!}M_{\kappa,\ell+\frac{1}{2}}(2r/\kappa)\right] \tag{1.15.75}$$

$$S(\varepsilon,r) = \begin{cases} 2\cos\left(\pi|\varepsilon|^{-1/2}\right) & \varepsilon<0, r>0 \\ 0 & \varepsilon<0, r<0 \\ \exp\left(\pi\varepsilon^{-1/2}\right) & \varepsilon>0, r>0 \\ \exp\left(-\pi\varepsilon^{-1/2}\right) & \varepsilon>0, r<0 \end{cases} \tag{1.15.76}$$

is the corresponding irregular solution.

As you can see, you can go a long way with the right confluent hypergeometric series.

1.15.3 Error Functions

The error function is fondly called erf,[127] and is related to integrals of Gaussians. Therefore, it shows up in a variety of circumstances, because Gaussians show up all over the place thanks to the Central Limit Theorem in probability. It is defined by

$$\operatorname{erf}(z) = \frac{1}{\sqrt{\pi}}\int_{-z}^{z} \mathrm{d}t\ \exp(-t^2) = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{d}t\ \exp(-t^2) \tag{1.15.77}$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{n!(2n+1)} \tag{1.15.78}$$

$$\overline{\operatorname{erf}(z)} = \operatorname{erf}(\overline{z}) \tag{1.15.79}$$

$$\frac{\mathrm{d}\operatorname{erf}(z)}{\mathrm{d}z} = \frac{2\exp(-z^2)}{\sqrt{\pi}} \tag{1.15.80}$$

$$\frac{\mathrm{d}^n\operatorname{erf}(z)}{\mathrm{d}z^n} = \frac{2(-1)^{n-1}}{\sqrt{\pi}}H_{n-1}(z)\exp(-z^2) = \frac{2}{\sqrt{\pi}}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left[\exp(-z^2)\right] \tag{1.15.81}$$

$$\int \mathrm{d}z\ \operatorname{erf}(z) = z\operatorname{erf}(z) + \frac{\exp(-z^2)}{\sqrt{\pi}} \tag{1.15.82}$$

The integral relationship comes from integration by parts. We can write a differential equation for erf by

$$\frac{\mathrm{d}^2\operatorname{erf}(x)}{\mathrm{d}x^2} = -2x\frac{\mathrm{d}\operatorname{erf}(x)}{\mathrm{d}x} \tag{1.15.83}$$

The error function's cousin, the complementary error function erfc is defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{d}t\ \exp(-t^2) = 1-\operatorname{erf}(z) \tag{1.15.84}$$

with similar properties. You may also see erfi(z), the imaginary error function defined via $\operatorname{erfi}(z) = -i\operatorname{erf}(iz)$. This means that

$$\frac{\mathrm{d}^2\operatorname{erfi}(x)}{\mathrm{d}x^2} = 2x\frac{\mathrm{d}\operatorname{erfi}(x)}{\mathrm{d}x} \tag{1.15.85}$$

[127] Pronounced like Nerf without the "N".

A useful asymptotic expansion of $\operatorname{erfc}(z)$ as $z \to \infty$ yields a good approximation for $\operatorname{erf}(z) = 1 - \operatorname{erfc}(z)$. This expansion is given as

$$\operatorname{erfc}(z) \sim \frac{\exp(-z^2)}{z\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{(2m-1)!!}{(2z^2)^m} = \frac{\exp(-z^2)}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\right)_m}{z^{2m+1}} \tag{1.15.86}$$

$$\operatorname{erfc}(-z) \sim 2 - \frac{\exp(-z^2)}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\right)_m}{z^{2m+1}} \tag{1.15.87}$$

where double factorial means every other so

$$n!! = \begin{cases} n(n-2)(n-4)\cdots(4)(2) & n \text{ even} \\ n(n-2)(n-4)\cdots(3)(1) & n \text{ odd} \\ 1 & n = 0 \end{cases} \tag{1.15.88}$$

1.15.4 Plasma Dispersion Function

The plasma dispersion function is essentially an error function, but in a form more often seen in plasma physics. It is defined as

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\exp(-x^2)}{x - \zeta} \tag{1.15.89}$$

for $\Im(\zeta) > 0$. For $\Im(\zeta) < 0$, use the analytic continuation of the function. This leads to the generally true expression

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{C} \mathrm{d}x \, \frac{\exp(-x^2)}{x - \zeta} \tag{1.15.90}$$

where you have to figure out what the correct integration path C is given your ζ. You might be saying to yourself, "Wait, that doesn't quite look like an error function. There's that denominator in there. How do you get there?" This is not an obvious road. First we note

$$Z'(\zeta) = \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial \zeta} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\exp(-x)^2}{x - \zeta} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{-\exp(-x^2)}{(x-\zeta)^2} \tag{1.15.91}$$

We can integrate such a thing by parts with $u = \exp(-x^2)$ and $\mathrm{d}v = \mathrm{d}x/(x-\zeta)^2$ so $\mathrm{d}u = -2x\exp(-x^2)\,\mathrm{d}x$ and $v = -1/(x-\zeta)$ to find

$$\sqrt{\pi} Z'(\zeta) = \cancel{-\frac{\exp(-x^2)}{x-\zeta}\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{-2x\exp(-x^2)}{-(x-\zeta)} \tag{1.15.92}$$

$$= -\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{2x\exp(-x^2)}{x-\zeta} \tag{1.15.93}$$

We can then recognize that if we add and subtract ζ in the numerator we have

$$Z'(\zeta) = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{(x+\zeta-\zeta)\exp(-x^2)}{x-\zeta} = -\frac{2}{\sqrt{\pi}} \mathrm{d}x \, \exp(-x^2) + \frac{2\zeta}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\exp(-x^2)}{x-\zeta} \tag{1.15.94}$$

$$Z'(\zeta) = 2[-1 - \zeta Z(\zeta)] = -2[1 + \zeta Z(\zeta)] \tag{1.15.95}$$

which is the governing differential equation.

If we took another derivative we find

$$Z''(\zeta) = -2Z(\zeta) - 2\zeta Z'(\zeta) \tag{1.15.96}$$

This looks suspiciously like the error function's differential equation. The difference is we have an extra $-2Z(\zeta)$ term. By trying a new function $G(\zeta)\exp(-\zeta^2) = Z(\zeta)$ we see that the above becomes

$$Z'(\zeta) = G'(\zeta)\exp(-\zeta^2) - 2\zeta\exp(-\zeta^2)G(\zeta) \tag{1.15.97}$$

$$Z''(\zeta) = G''(\zeta)\exp(-\zeta^2) - 4\zeta\exp(-\zeta^2)G'(\zeta) - 2G(\zeta)\left[\exp(-\zeta^2) - 2\zeta^2\exp(-\zeta^2)\right] \tag{1.15.98}$$

$$\tag{1.15.99}$$

which means we have

$$G''(\zeta) - 4\zeta G'(\zeta) - 2G(\zeta)\left[1 - 2\zeta^2\right] = \cancel{-2G(\zeta)} - 2\zeta\left[G'(\zeta) - \cancel{2\zeta G(\zeta)}\right] \tag{1.15.100}$$

$$G''(\zeta) = [4\zeta - 2\zeta]G'(\zeta) = 2\zeta G'(\zeta) \tag{1.15.101}$$

which is simply the differential equation for $\operatorname{erfi}(x)$ as stated in (1.15.85). Thus $G(\zeta)$ is $\operatorname{erfi}(\zeta) = -i\operatorname{erf}(i\zeta)$ up to a constant. Thus, we see that

$$Z(\zeta) = C_0\exp(-\zeta^2)(-i\operatorname{erf}(i\zeta) + C_1) \tag{1.15.102}$$

for constants C_0 and C_1. We can then use the original differential equation $Z' = -2[1 + \zeta Z]$ to write

$$Z' = -iC_0\left(\exp(-\zeta^2)\frac{\mathrm{d}\operatorname{erf}(i\zeta)}{\mathrm{d}\zeta} + -2\zeta\exp(-\zeta^2)\operatorname{erf}(i\zeta)\right) \tag{1.15.103}$$

$$= -iC_0\left(i\exp(-\zeta^2)\frac{2\exp(\zeta^2)}{\sqrt{\pi}} - 2\zeta\exp(-\zeta^2)\operatorname{erf}(i\zeta)\right) \tag{1.15.104}$$

$$= -iC_0\left(i\frac{2}{\sqrt{\pi}} - 2\zeta\exp(-\zeta^2)\operatorname{erf}(i\zeta)\right) = \frac{2C_0}{\sqrt{\pi}} - 2\zeta Z \tag{1.15.105}$$

which implies $C_0 = -\sqrt{\pi}$. Then we use that we want $Z(0) = i\sqrt{\pi}$ (as we will show below) to determine C_1. This means

$$Z(0) = \sqrt{\pi}\exp(-0^2)\left(i\operatorname{erf}(i0) - C_1\right) = -\sqrt{\pi}C_1 = i\sqrt{\pi} \tag{1.15.106}$$

so $C_1 = -i$ and we get

$$Z(\zeta) = i\sqrt{\pi}\exp(-\zeta^2)(\operatorname{erf}(i\zeta) + 1) \tag{1.15.107}$$

Alternatively, we can go about this the hard way, which does not require an inspired guess. We use that (1.15.95) is a simple ODE and so using an integrating factor method, we see the solution is

$$\nu(\zeta) = \int_0^\zeta \mathrm{d}\zeta'\ 2\zeta' = \zeta^2 \tag{1.15.108}$$

$$Z(\zeta)\exp\left(\nu(\zeta)\right) = C + \int_0^\zeta \mathrm{d}\zeta'\ \exp(\nu(\zeta'))(-2) = C - 2\int_0^\zeta \mathrm{d}\zeta'\ \exp((\zeta')^2) \tag{1.15.109}$$

for some constant C. Define $t = i\zeta'$ and we see that

$$\int_0^\zeta \mathrm{d}\zeta' \ \exp\left((\zeta')^2\right) = -i\int_0^{i\zeta} \mathrm{d}t \ \exp(-t^2) = -i\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(i\zeta) \tag{1.15.110}$$

yielding

$$Z(\zeta) = \exp(-\zeta^2)\left(C + i\sqrt{\pi}\,\mathrm{erf}(i\zeta)\right) \tag{1.15.111}$$

We have

$$Z(0) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}t \ \exp(-t^2)/t \tag{1.15.112}$$

using the Plemelj formula with the pole at $t = 0$ will yield[128]

$$Z(0) = \frac{\pi i\exp(-0^2)}{\sqrt{\pi}} = \sqrt{\pi}i \tag{1.15.113}$$

and so $C = \sqrt{\pi}i$

$$Z(\zeta) = \exp(-\zeta^2)\sqrt{\pi}\,(i + i\,\mathrm{erf}(i\zeta)) \tag{1.15.114}$$

There is yet one more twist. We use

$$\int_{-\infty}^{0} \mathrm{d}t \ \exp(-t^2) = \sqrt{\pi}/2 \tag{1.15.115}$$

so that

$$Z(\zeta) = \exp(-\zeta^2)\left(2i\int_{-\infty}^{0} \mathrm{d}t \ \exp(-t^2) + 2i\int_0^{i\zeta} \mathrm{d}t \ \exp(-t^2)\right) \tag{1.15.116}$$

$$Z(\zeta) = 2i\exp(-\zeta^2)\int_{-\infty}^{i\zeta} \mathrm{d}t \ \exp(-t^2) = i\sqrt{\pi}\exp\left(-\zeta^2\right)[1 + \mathrm{erf}(i\zeta)] \tag{1.15.117}$$

We then have the following properties

$$Z(\bar{\zeta}) = -\overline{Z(-\zeta)} \tag{1.15.118}$$

$$Z(\bar{\zeta}) = \overline{Z(\zeta)} + 2i\sqrt{\pi}\exp(-\bar{\zeta}^2) \quad \Im(\zeta) > 0 \tag{1.15.119}$$

with power series around $\zeta = 0$ of

$$Z(\zeta) = i\sqrt{\pi}\exp(-\zeta^2) - \sqrt{\pi}\zeta\sum_{n=0}^{\infty}\frac{(-\zeta^2)^n}{\Gamma(n+\frac{3}{2})} \tag{1.15.120}$$

[128]Since the pole is on our contour, and the denominator is of the form $t - \zeta$, we consider our answer as $\zeta \to 0$ from above the real axis, so the denominator is of the form $t - i\delta$ with $\delta > 0$. Then we get a + sign from the Plemelj formula.

and for $|\zeta| \to \infty$ we have the asymptotic series

$$Z(\zeta) \sim i\sigma\sqrt{\pi}\exp(-\zeta^2) - \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\zeta^{2n+1}} \tag{1.15.121}$$

$$\sigma \equiv \begin{cases} 0 & \Im(\zeta) > \frac{1}{|\Re(\zeta)|} \\ 1 & |\Im(\zeta)| < \frac{1}{|\Re(\zeta)|} \\ 2 & \Im(\zeta) < -\frac{1}{|\Re(\zeta)|} \end{cases} \tag{1.15.122}$$

You will sometimes see people using the Faddeeva function $w(\zeta) = Z(\zeta)/(i\sqrt{\pi})$ which is just normalizing the plasma dispersion function. Thus

$$w(\zeta) = \exp(-\zeta^2)\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^{\zeta} \mathrm{d}t\ \exp(t^2)\right) = \exp(-\zeta^2)\left(1 + \operatorname{erf}(i\zeta)\right) \tag{1.15.123}$$

1.15.5 Gaussians

Integrals with Gaussians are ubiquitous. Sometimes called the normal distribution, a Gaussian distribution is defined by

$$\exp(-\alpha x^2) \tag{1.15.124}$$

for some α. Statisticians prefer $\alpha = 1/2$ while physicists generally prefer $\alpha = 1$ for x in a normalized variable. I will stick to the physicists prescription because this is a physics text, and I do not mind if our physicists' variance σ_p^2 is one half of the statistician's variance σ^2 (That is $\sigma_p^2 = \frac{\sigma^2}{2}$).

Usually we want to integrate the above over some interval. When that interval is all real values, we have for $\alpha > 0$

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \exp(-\alpha x^2) \tag{1.15.125}$$

which has a tidy trick to get the answer. We first assume that the answer is finite.[129] First we square the entire expression and call it A^2

$$A^2 = \left(\int_{-\infty}^{\infty} \mathrm{d}x\ \exp(-\alpha x^2)\right)^2 = \left(\int_{-\infty}^{\infty} \mathrm{d}x\ \exp(-\alpha x^2)\right)\left(\int_{-\infty}^{\infty} \mathrm{d}x\ \exp(-\alpha x^2)\right) \tag{1.15.126}$$

then we take the second expression and change $x \to y$

$$A^2 = \left(\int_{-\infty}^{\infty} \mathrm{d}x\ \exp(-\alpha x^2)\right)\left(\int_{-\infty}^{\infty} \mathrm{d}y\ \exp(-\alpha y^2)\right) \tag{1.15.127}$$

Because these are finite, we can then take the y integral inside of the x integral

$$A^2 = \left(\int_{-\infty}^{\infty} \mathrm{d}x\ \exp(-\alpha x^2)\left(\int_{-\infty}^{\infty} \mathrm{d}y\ \exp(-\alpha y^2)\right)\right) \tag{1.15.128}$$

[129] This is an important first step or else everything we do will be nonsense like squaring infinity and adding and subtracting infinity. It is also fairly easy to show that this must be a finite value. For $x > 1$ clearly $x\exp(-\alpha x^2)$ will be larger than $\exp(-\alpha x^2)$ and for the $1 > x > 0$ we can just replace $\exp(-\alpha x^2)$ with 1. If we do these we see that the answer must be less than $2(1 + \exp(-a)/a)$.

We can then use that $\exp(-\alpha x^2)$ is constant with respect to the y integral and so can place it inside of the y integral

$$A^2 = \int_{-\infty}^{\infty} \mathrm{d}x \ \int_{-\infty}^{\infty} \mathrm{d}y \ \exp(-\alpha x^2)\exp(-\alpha y^2) \tag{1.15.129}$$

and use $\exp(-\alpha x^2)\exp(-\alpha y^2) = \exp(-\alpha[x^2+y^2])$. Now we can change to polar coordinates with $\mathrm{d}x\,\mathrm{d}y = \mathcal{J}\,\mathrm{d}r\,\mathrm{d}\theta$ with $r^2 = x^2+y^2$ and $\mathcal{J} = r$ so that we find

$$A^2 = \int_0^{\infty} \mathrm{d}r \ \int_0^{2\pi} \mathrm{d}\theta \ r\exp(-\alpha r^2) = 2\pi\int_0^{\infty} \mathrm{d}r \ r\exp(-\alpha r^2) \tag{1.15.130}$$

Try $t = \alpha r^2$ so $\mathrm{d}t = 2\alpha r\,\mathrm{d}r$ and we have

$$A^2 = \frac{\pi}{\alpha}\int_0^{\infty} \mathrm{d}t \ \exp(-t) = \frac{\pi}{\alpha}\frac{\exp(-\infty)-\exp(0)}{-1} = \frac{\pi}{\alpha} \tag{1.15.131}$$

and so

$$\int_{-\infty}^{\infty} \mathrm{d}x \ \exp(-\alpha x^2) = \sqrt{\frac{\pi}{\alpha}} \tag{1.15.132}$$

Because the integrand is even we then also know that

$$\int_0^{\infty} \mathrm{d}x \ \exp(-\alpha x^2) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \tag{1.15.133}$$

We can find the answer to

$$\int_0^{\infty} \mathrm{d}x \ x^{2n}\exp(-\alpha x^2) \tag{1.15.134}$$

by noticing

$$\frac{\partial^n}{\partial\alpha^n}\int_0^{\infty} \mathrm{d}x \ \exp(-\alpha x^2) = \int_0^{\infty} \mathrm{d}x \ (-1)^n x^{2n}\exp(-\alpha x^2) \tag{1.15.135}$$

so that

$$\int_0^{\infty} \mathrm{d}x \ x^{2n}\exp(-\alpha x^2) = \frac{\sqrt{\pi}(-1)^n}{2}\frac{\partial^n}{\partial\alpha^n}\alpha^{-1/2} = \frac{\sqrt{\pi}n!!}{2^n\alpha^{(2n+1)/2}} \tag{1.15.136}$$

That, or we can recognize that the above is a gamma function integral and find

$$\int_0^{\infty} \mathrm{d}x \ x^{\beta}\exp(-\alpha t^2) = \frac{\Gamma\left(\frac{\beta+1}{2}\right)}{2\alpha^{\frac{\beta+1}{2}}} \tag{1.15.137}$$

1.15.6 Gamma Functions

The gamma functions come in two kinds: complete and incomplete. Traditionally they are denoted with the Greek symbol gamma. The complete being $\Gamma(z)$ and the incomplete being one of either $\Gamma(s,z)$ for the upper incomplete and $\gamma(s,z)$ for the lower incomplete, yielding a logical naming system.[130] We then have the definitions

$$\Gamma(a,z) = \int_z^\infty \mathrm{d}t\ t^{a-1}\exp(-t) \tag{1.15.138}$$

$$\gamma(a,z) = \int_0^z \mathrm{d}t\ t^{a-1}\exp(-t) \tag{1.15.139}$$

with the relationship to the complete gamma function given by

$$\Gamma(a) = \gamma(a,z) + \Gamma(a,Z) \tag{1.15.140}$$

We have that

$$\Gamma(n+1) = n! \tag{1.15.141}$$

for n an integer. For $z \to \infty$ we find

$$\ln(\Gamma(z)) \sim \left(z - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)z^{2k-1}} \tag{1.15.142}$$

with B_k the Bernoulli numbers[131] defined by $x = 0$ for the Bernoulli polynomials given by $B_k(x)$. These polynomials (and hence the numbers) are given by

$$\frac{t\exp(xt)}{\exp(t)-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \tag{1.15.143}$$

$$\sum_{k=0}^{m-1}\binom{m}{k} B_k(x) = nx^{n-1} \tag{1.15.144}$$

with $B_0(x) = 1$.

You will sometimes see $\psi(z) = \Gamma'(z)/\Gamma(z)$ used, as well. Its properties clearly follow from $\Gamma(z)$, but it is named the polygamma function.

The Beta function $B(a,b)$ is also related to the gamma function via

$$B(a,b) = \int_0^1 \mathrm{d}t\ t^{a-1}(1-t)^{b-1} = \int_0^\infty \mathrm{d}t\ \frac{t^{a-1}}{(1+t)^{a+b}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{1.15.145}$$

The other important properties of the gamma function are

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \tag{1.15.146}$$

[130] It feels as if this is one of the few examples of fairly logical naming system having widespread use, but we do not notice the logical systems as much, either.

[131] There are fascinating connections between Bernoulli numbers and Pascal's triangle. You should definitely look them up.

when z is not an integer, called Euler's reflection formula. Thus

$$\Gamma(\alpha - n) = (-1)^{n-1}\frac{\Gamma(-\alpha)\Gamma(1+\alpha)}{\Gamma(n+1-\alpha)} \tag{1.15.147}$$

One other useful property is the Legendre duplication formula

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z) \tag{1.15.148}$$

These last two identities are often of great use in simplifying expressions when gamma functions do show up. Finally, one can generally use

$$\prod_{k=0}^{m-1}\Gamma(z+\frac{k}{m}) = (2m)^{(m-1)/2}m^{\frac{1}{2}-mz}\Gamma(mz) \tag{1.15.149}$$

$$\overline{\Gamma(z)} = \Gamma(\overline{z}) \tag{1.15.150}$$

Finally, I may as well mention $\Gamma(1/2) = \sqrt{\pi}$ which allows one to find many of the other half-integer values.

1.15.7 Exponential Integrals

The exponential integrals are a class of functions defined by integrating the exponential. We see the logic of the naming convention, but it is ruined by having more than one of them. The definition for general exponential integrals is given by

$$E_n(z) = z^{n-1}\Gamma(1-n,z) = z^{n-1}\int_z^\infty \mathrm{d}t\ \frac{\exp(t)}{t^n} \tag{1.15.151}$$

In fact, there are other representations that are common, such as

$$E_n(z) = \int_1^\infty \mathrm{d}t\ \frac{\exp(-zt)}{t^n} \tag{1.15.152}$$

which requires $|\arg(z) < \pi/2|$ or

$$E_n(z) = \frac{z^{n-1}\exp(-z)}{\Gamma(n)}\int_0^\infty \mathrm{d}t\ \frac{t^{n-1}\exp(-zt)}{1+t} \tag{1.15.153}$$

which requires $\Re(n) > 0$ and $|\arg(z)| < \pi/2$.

However, what you would think would be *the* exponential integral would be $n = 1$ given by

$$E_1(z) = \int_z^\infty \mathrm{d}t\ \frac{\exp(-t)}{t} = \int_1^\infty \mathrm{d}t\ \frac{\exp(-tz)}{t} \tag{1.15.154}$$

but for historical reasons, you actually have the exponential integral defined as

$$\mathrm{Ei}(x) = -\int_{-x}^\infty \mathrm{d}t\ \frac{\exp(-t)}{t} = -E_1(-x) \tag{1.15.155}$$

for positive x values. So you have to actually make sure you are talking about the "right" exponential integral whenever you deal with it.

We have the series expansion

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}\left(\psi(n) - \ln z\right) - \sum_{\substack{k=1 \\ k\neq n-1}}^{\infty} \frac{(-z)^k}{k!(1-n+k)} \tag{1.15.156}$$

and asymptotic series

$$E_1(z) \sim \frac{\exp(-z)}{z}\sum_{n=0}^{N-1}\frac{n!}{(-z)^n} + \mathcal{O}(N!z^{-N}) \tag{1.15.157}$$

It is often more useful to use the analytic function $\mathrm{Ein}(z)$ rather than $E_1(z)$ or $\mathrm{Ei}(x)$ with definition

$$\mathrm{Ein}(z) = \int_0^z \mathrm{d}t\ \frac{1-\exp(-t)}{t} \tag{1.15.158}$$

leading to

$$E_1(x) = -\gamma - \ln z + \mathrm{Ein}(z) \quad |\arg(z)| < \pi \tag{1.15.159}$$

$$\mathrm{Ei}(x) = \gamma + \ln x - \mathrm{Ein}(-x) \quad x > 0 \tag{1.15.160}$$

with $\gamma = \psi(1)$ the Euler-Mascheroni constant.

We also have

$$E_n'(z) = -E_{n-1}(z) \tag{1.15.161}$$

which is a very useful property.

Finally, there is a relationship between the exponential integral and sine [$\mathrm{Si}(z)$] and cosine [$\mathrm{Ci}(z)$] integrals

$$E_1(ix) = i\left[\frac{-\pi}{2} + \mathrm{Si}(x)\right] - \mathrm{Ci}(x) \quad x > 0 \tag{1.15.162}$$

$$\mathrm{Si}(z) = \int_0^z \mathrm{d}t\ \frac{\sin(t)}{t} \tag{1.15.163}$$

$$\mathrm{Ci}(z) = -\int_z^\infty \mathrm{d}t\ \frac{\cos(t)}{t} \tag{1.15.164}$$

$$\mathrm{Cin}(z) = \int_0^z \mathrm{d}t\ \frac{1-\cos(t)}{t} \tag{1.15.165}$$

$$\mathrm{Ci}(z) = -\mathrm{Cin}(z) + \ln z - \gamma \tag{1.15.166}$$

where $\mathrm{Cin}(z)$ is an even analytic function of z.

It is sometimes useful to know

$$\lim_{z\to\infty}\mathrm{Si}(z) = \frac{\pi}{2} \tag{1.15.167}$$

$$\lim_{z\to\infty}\mathrm{Ci}(z) = 0 \tag{1.15.168}$$

1.15.8 Bessel Functions

The various types of Bessel functions all come from solutions to differential equations that are of a similar form. They also have somewhat analogous properties and forms. They show up in spherical and cylindrically symmetric systems, and so are of special interest to physicists.

1.15.8.1 Bessel Functions

We can start with the regular Bessel functions which are the solution to the differential equation for $f(z)$ given by

$$z^2\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + z\frac{\mathrm{d}f}{\mathrm{d}z} + (z^2 - \nu^2)f = 0 \tag{1.15.169}$$

There are two main kinds,[132] corresponding to the two solutions. The Bessel function of the first kind are denoted $J_\nu(z)$ given by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\nu + k + 1)} \tag{1.15.170}$$

The Bessel function of the second kind $Y_\nu(z)$ (sometimes called Weber's function) is given by

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \tag{1.15.171}$$

If ν is an integer we say

$$Y_n(z) = \frac{1}{\pi}\frac{\partial J_\nu(z)}{\partial \nu}\bigg|_{\nu=n} + \frac{(-1)^n}{\pi}\frac{\partial J_\nu(z)}{\partial \nu}\bigg|_{\nu=n} \tag{1.15.172}$$

Then there are Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, sometimes called Bessel functions of the third kind. They are given by

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) \tag{1.15.173}$$
$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z) \tag{1.15.174}$$

with the property that as $z \to \infty$

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\exp\left(i\left[z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right]\right) \tag{1.15.175}$$
$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\exp\left(-i\left[z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right]\right) \tag{1.15.176}$$

for $-\pi < \arg(z) < 2\pi$. One then chooses among these functions as solutions depending on the region we are dealing with so that we don't have to deal with headaches. For example, $J_\nu(z)$ and $Y_\nu(z)$ are used when $z = x$ is real and $0 < x < \infty$. The DLMF[10] lists the suitable pairs.

[132] In fact, there are three kinds because Hankel functions are disguised Bessel functions.

Some important properties are

$$J_0(z) \xrightarrow{z\to 0} 1 \tag{1.15.177}$$

$$Y_0(z) \xrightarrow{z\to 0} \frac{2}{\pi}\ln z \tag{1.15.178}$$

$$H_0^{(1)}(z) \xrightarrow{z\to 0} -H_0^{(2)}(z) \xrightarrow{z\to 0} \frac{2i}{\pi}\ln z \tag{1.15.179}$$

$$J_\nu(z) \xrightarrow{z\to 0} \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \tag{1.15.180}$$

$$Y_\nu(z) \xrightarrow{z\to 0} \frac{-\left(\frac{z}{2}\right)^{-\nu}}{\pi}\Gamma(\nu) \tag{1.15.181}$$

$$Y_{-\nu}(z) \xrightarrow{z\to 0} \frac{-\left(\frac{z}{2}\right)^{-\nu}}{\pi}\cos(\nu\pi)\Gamma(\nu) \tag{1.15.182}$$

$$H_\nu^{(1)}(z) \xrightarrow{z\to 0} -H_\nu^{(2)}(z) \xrightarrow{z\to 0} -\frac{i}{\pi}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} \tag{1.15.183}$$

with $J_\nu(z)$ not okay for ν a negative integer, the formula for $Y_\nu(z)$ only valid for $\Re(\nu) > 0$ or ν a negative half-integer, and the formula $Y_{-\nu}(z)$ only valid for $\Re(\nu) > 0$ and ν not a positive half-integer.

We also have for $z \to \infty$ that

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{1.15.184}$$

$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{1.15.185}$$

One important way of representing Bessel functions of the first kind is through an integral form given by (n is an integer)

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\ \cos(z\sin\theta - n\theta) = \frac{i^{-n}}{\pi}\int_0^\pi \mathrm{d}\theta\ \exp(iz\cos\theta)\cos(n\theta) \tag{1.15.186}$$

$$J_0(z) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\ \cos(z\sin\theta) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\ \cos(z\cos\theta) \tag{1.15.187}$$

In addition, the generating function for $J_\nu(z)$ is given by

$$\exp\left(\frac{z}{2}\left[t - \frac{1}{t}\right]\right) = \sum_{n=-\infty}^{\infty} t^n J_n(z) \tag{1.15.188}$$

which immediately implies for $t = 1$ that

$$\exp\left(\frac{z}{2}\left[1 - \frac{1}{1}\right]\right) = \exp(0) = 1 = \sum_{n=-\infty}^{\infty} J_n(z) \tag{1.15.189}$$

One can also use the generating function to prove

$$1 = \sum_{n=-\infty}^{\infty} [J_n(z)]^2 \tag{1.15.190}$$

Sometimes the generating function is written using $t = \exp(i\theta)$

$$\exp\left(iz\sin\theta\right) = \sum_{n=-\infty}^{\infty} \exp(in\theta) J_n(z) \tag{1.15.191}$$

1.15.8.2 Modified Bessel Functions

Next we consider modified Bessel functions. These are essentially the same as Bessel functions, but using $\widehat{z} = \pm iz$ instead of z. This gives a differential equation

$$z^2 \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + z\frac{\mathrm{d}f}{\mathrm{d}z} - (z^2 + \nu^2) f = 0 \tag{1.15.192}$$

There are now only two kinds. The first kind, sometimes called the standard solution is denoted $I_\nu(z)$ and given by

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\nu + k + 1)} \tag{1.15.193}$$

The second kind $K_\nu(z)$ is defined by

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} \tag{1.15.194}$$

for non-integer ν and

$$K_\nu(z) = \frac{(-1)^{n-1}}{2}\left(\left(\frac{\partial I_\nu}{\partial \nu}\right)_{\nu=n} + \left(\frac{\partial I_\nu}{\partial \nu}\right)_{\nu=-n}\right) \tag{1.15.195}$$

For $z \to \infty$ we then have [valid for $|\arg(z)| < \pi/2$ for $I_\nu(z)$ and for $|\arg(z)| < \frac{3\pi}{2}$ for $K_\nu(z)$]

$$I_\nu(z) \sim \frac{\exp(z)}{\sqrt{2\pi z}} \tag{1.15.196}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\exp(-z) \tag{1.15.197}$$

For z with $-\pi < \arg(z) < \pi/2$ we have

$$J_\nu(iz) = \exp\left(\frac{i\nu\pi}{2}\right) I_\nu(z) \tag{1.15.198}$$

$$Y_\nu(iz) = \exp\left(\frac{i(\nu+1)\pi}{2}\right) I_\nu(z) - \frac{2\exp\left(\frac{-i\nu\pi}{2}\right)}{\pi} K_\nu(z) \tag{1.15.199}$$

We then have

$$I_\nu(z) \xrightarrow{z\to 0} \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \quad \nu \neq -1, -2, \ldots \tag{1.15.200}$$

$$K_\nu(z) \xrightarrow{z\to 0} \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} \tag{1.15.201}$$

$$K_0(z) \xrightarrow{z\to 0} -\ln z \tag{1.15.202}$$

$$I_\nu(z) \xrightarrow{z\to\infty} \frac{\exp(z)}{\sqrt{2\pi z}} \tag{1.15.203}$$

$$K_\nu(z) \xrightarrow{z\to\infty} \frac{\exp(-z)\sqrt{\pi}}{\sqrt{2z}} \tag{1.15.204}$$

The modified Bessel function of the first kind has generating function

$$\exp\left(\frac{z}{2}\left[t+\frac{1}{t}\right]\right)=\sum_{n=-\infty}^{\infty}t^n I_n(t) \tag{1.15.205}$$

which may be written with $t=\exp(i\theta)$ as

$$\exp\left(z\cos\theta\right)=\sum_{n=-\infty}^{\infty}\exp(in\theta)I_n(t) \tag{1.15.206}$$

1.15.8.3 Spherical Bessel Functions

Finally, we consider spherical Bessel functions. They are classified as regular Bessel functions with analogous first, second, and third kinds. The first and second kind solve the differential equation

$$z^2\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}+2z\frac{\mathrm{d}f}{\mathrm{d}z}+(z^2-n(n+1))f=0 \tag{1.15.207}$$

while the third kind solve

$$z^2\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}+2z\frac{\mathrm{d}f}{\mathrm{d}z}-(z^2+n(n+1))f=0 \tag{1.15.208}$$

Here we make n a nonnegative integer. This is for convenience because if n is allowed to be negative then $n(n+1)$ will be positive unless $n=-1$, but that case is the same as $n=0$. Thus $-2(-2+1)=-2(-1)=2=1(1+1)$ or for $m=-|n|$ we have $m(m+1)=-|n|(-|n|+1)=|n|(|n|-1)$ which simply is the same as the positive case but with $|n|$ a positive integer. The first and second kind are denoted $y_n(z)$ and $y_n(z)$, respectively. They satisfy

$$j_n(z)=\sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z)=(-1)^n\sqrt{\frac{\pi}{2z}}Y_{-n-\frac{1}{2}}(z) \tag{1.15.209}$$

$$y_n(z)=\sqrt{\frac{\pi}{2z}}Y_{n+\frac{1}{2}}(z)=(-1)^{n+1}\sqrt{\frac{\pi}{2z}}J_{-n-\frac{1}{2}}(z) \tag{1.15.210}$$

while the third kind are given by $h_n^{(1)}(z)$ and $h_n^{(2)}(z)$ with

$$h_n^{(1)}(z)=j_n(z)+iy_n(z) \tag{1.15.211}$$

$$h_n^{(2)}(z)=j_n(z)-iy_n(z) \tag{1.15.212}$$

$$h_n^{(1)}(z)=\sqrt{\frac{\pi}{2z}}H^{(1)}_{n+\frac{1}{2}}(z)=(-1)^{n+1}i\sqrt{\frac{\pi}{2z}}H^{(1)}_{-n-\frac{1}{2}}(z) \tag{1.15.213}$$

$$h_n^{(2)}(z)=\sqrt{\frac{\pi}{2z}}H^{(2)}_{n+\frac{1}{2}}(z)=(-1)^{n}i\sqrt{\frac{\pi}{2z}}H^{(2)}_{-n-\frac{1}{2}}(z) \tag{1.15.214}$$

One can define analogous modified spherical Bessel functions via

$$i_n^{(1)}=\sqrt{\frac{\pi}{2z}}I_{n+\frac{1}{2}}(z) \tag{1.15.215}$$

$$i_n^{(2)}=\sqrt{\frac{\pi}{2z}}I_{-n-\frac{1}{2}}(z) \tag{1.15.216}$$

$$k_n^{(2)}=\sqrt{\frac{\pi}{2z}}I_{n+\frac{1}{2}}(z)=\sqrt{\frac{\pi}{2z}}K_{-n-\frac{1}{2}}(z) \tag{1.15.217}$$

with the reflection formulas

$$j_n(-z) = (-1)^n j_n(z) \tag{1.15.218}$$
$$y_n(-z) = (-1)^{n+1} y_n(z) \tag{1.15.219}$$
$$h_n^{(1)}(-z) = (-1)^n h_n^{(2)}(z) \tag{1.15.220}$$
$$h_n^{(2)}(-z) = (-1)^n h_n^{(1)}(z) \tag{1.15.221}$$
$$i_n^{(1)}(-z) = (-1)^n i_n^{(1)}(z) \tag{1.15.222}$$
$$i_n^{(2)}(-z) = (-1)^{n+1} i_n^{(2)}(z) \tag{1.15.223}$$
$$k_n(-z) = -\frac{\pi}{2}\left(i_n^{(1)}(z) + i_n^{(2)}(z)\right) \tag{1.15.224}$$

And we have

$$j_n(z), i_n^{(1)}(z) \stackrel{z\to 0}{\longrightarrow} \frac{z^n}{(2n+1)!!} \tag{1.15.225}$$
$$-y_n(z), ih_n^{(1)}(z), -ih_n^{(2)}(z), (-1)^n i_n^{(2)}(z), \frac{2}{\pi}k_n(z) \stackrel{z\to 0}{\longrightarrow} \frac{(2n-1)!!}{z^{n+1}} \tag{1.15.226}$$
$$j_n(z) \stackrel{z\to\infty}{\longrightarrow} \frac{\sin\left(z - \frac{n\pi}{2}\right)}{z} \tag{1.15.227}$$
$$y_n(z) \stackrel{z\to\infty}{\longrightarrow} -\frac{\cos\left(z - \frac{n\pi}{2}\right)}{z} \tag{1.15.228}$$
$$h_n^{(1)}(z) \stackrel{z\to\infty}{\longrightarrow} i^{-n-1}\frac{\exp(iz)}{z} \tag{1.15.229}$$
$$h_n^{(2)}(z) \stackrel{z\to\infty}{\longrightarrow} i^{n+1}\frac{\exp(-iz)}{z} \tag{1.15.230}$$
$$i_n^{(1)}(z) \stackrel{z\to\infty}{\longrightarrow} \frac{\exp(z)}{2z} \quad |\arg(z)| < \pi/2 \tag{1.15.231}$$
$$i_n^{(2)}(z) \stackrel{z\to\infty}{\longrightarrow} \frac{\exp(z)}{2z} \quad |\arg(z)| < \pi/2 \tag{1.15.232}$$
$$k_n(z) \stackrel{z\to\infty}{\longrightarrow} \frac{\pi\exp(-z)}{2z} \tag{1.15.233}$$

The spherical Bessel function generating functions are given by

$$\frac{1}{z}\cos\left(\sqrt{z^2 - 2zt}\right) = \frac{\cos z}{z} + \sum_{n=1}^{\infty}\frac{t^n}{n!}j_{n-1}(z) \tag{1.15.234}$$
$$\frac{1}{z}\sin\left(\sqrt{z^2 - 2zt}\right) = \frac{\sin z}{z} + \sum_{n=1}^{\infty}\frac{t^n}{n!}y_{n-1}(z) \tag{1.15.235}$$
$$\frac{1}{z}\cosh\left(\sqrt{z^2 - 2zt}\right) = \frac{\cosh z}{z} + \sum_{n=1}^{\infty}\frac{(it)^n}{n!}i_{n-1}^{(1)}(z) \tag{1.15.236}$$
$$\frac{1}{z}\sinh\left(\sqrt{z^2 - 2zt}\right) = \frac{\sinh z}{z} + \sum_{n=1}^{\infty}\frac{(it)^n}{n!}i_{n-1}^{(2)}(z) \tag{1.15.237}$$
$$\frac{1}{z}\exp\left(-\sqrt{z^2 - 2zt}\right) = \frac{\exp(-z)}{z} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-it)^n}{n!}k_{n-1}(z) \tag{1.15.238}$$

1.16 Probability and Statistics Warnings

Henry Petroski wrote that in engineering, "success is foreseeing failure," a useful thought in many contexts.

— Paul Rosenbaum[27, p. 171]

Probability is not an easy subject. Combinatorics, statistics, and probability are easier to do mathematically than to apply to any real world situations. Part of the problem is that talking about probability is confusing. If I tell you there is a 50% chance a fair coin will land heads, that is probability. If I say I think there is a 50% chance person A committed a crime, is that probability? If I say there's a 50% chance that it will rain tomorrow, is that probability? It seems like they are testing different things, doesn't it? The fair coin is splitting up a configuration space in a clear way. The crime situation seems to be more related to the evidence I have available than on any true property about the world. The weather probability is especially galling. Does it mean that in 50% of days like this, it rained in the past? or 50% of weather simulations rained? or does it mean something entirely different, like 50% of an area will have rain?[133] These different ideas are all packed up into a single mathematical framework, though. There are no easy answers, and you really need to think about what someone means when they give a probability. Indeed, some philosophers and mathematicians would not consider some of those examples above to be probabilities because they don't split up a configuration space unambiguously.

The other problem is that assuming that statistics is just Gaussians is wrong. The real world has uncertainties that are not at all like the rapidly decaying tails of Gaussians. You need to always be on guard that one of these problematic distributions will show up in your work. Otherwise, when a problematic distribution (a fat-tailed distribution) shows up, it will destroy your ability to predict things accurately which can be disastrous if you are counting on only a certain very small fraction of errors being possible.

I, personally, think that one of the confusing things about probabilities is that they often seem divorced from physical reality and simply state that something has probability x, without stating the conditions that are required. For this reason, I am against unconditional probabilities, because I think they are half of the problem. If you are forced to give what your conditions are for your probability, it often makes things much clearer. Therefore, I will try to stick to conditional probabilities, and explain why I think unconditional probabilities simply are not necessary.

The other half of the problem is that humans are better at thinking probabilities in a specific context. One great tip for understanding statistics and probability problems is, essentially, a psychological one. Instead of talking about probabilities, think of making the problem into ones of natural frequencies. That is if a problem says that a person has a 1% chance, think 1 in 100. It is amazing how presenting the problem with nice whole numbers will make problems easier to understand.

For example, consider testing for disease X. The test will always give the right answer when a person has disease X, but is only accurate 98% of the time when the person does not have disease X [that is, 2% of the time the test will say a healthy person has disease X]. We also know that for

[133] The answer given by the US NWS is that probability of precipitation means that more than 0.25 mm of precipitation will fall in a single spot, averaged over forecast area. This means the confidence is based on weather simulations. So if you had 80% confidence it would rain in 20% of the forecast area, then the probability of precipitation would be 16%.

the population we are talking about, the prevalence of the disease is 0.01%. Someone from this population takes the test and gets a positive result. What is the probability they have disease X? Rather hard, don't you think? You could use Bayes' (conditional) Theorem with X_h meaning "has disease X", T_+ meaning "tests positive", and P meaning "is a person in the population we have statistics for", then we have

$$\begin{aligned} P(X_h|T_+ \text{ and } P) &= \frac{P(X_h|P)}{P(T_+|P)} P(T_+|X_h \text{ and } P) \\ &= \frac{P(X_h|P)P(T_+|X_h \text{ and } P)}{P(T_+|X_h \text{ and } P) + P(T_+|[\text{not } X_h] \text{ and } P)} \\ &= \frac{0.0001}{1(0.0001) + .02(0.9999)} 1 \approx 0.00498 \end{aligned} \tag{1.16.1}$$

which is not very intuitive (at least to me).

If we instead write this with nice whole numbers initially,[134] it is much easier for me to imagine. Let's say we have 10 000 of the given population and have all of them take the test. We know that (on average) 1 person in this population has disease X. Of the 9999 that do not, 2% or about 199.98 people on average, will falsely test positive. Then the chance that we will pick the person actually with disease X among these approximately 201 people that test positive is $1/200.98 \approx 0.004\,98$ or about 0.5%, just as before. This way makes the steps much, much clearer. At least for me.

1.16.1 Central Limit Problems

If you have taken any courses on statistics, you will have learned the central limit theorem. In fact, there are numerous central limit theorems, but the one that most people learn says something like this.

Given a set of random variables[135] $X_1, X_2, \ldots, X_N$, that were independently and identically distributed[136] with expected mean μ and finite variance σ^2 [so each random variable has the same mean μ and variance σ^2], then as n approaches infinity, the sample mean S_n given by

$$S_n \equiv \frac{X_1 + \cdots + X_n}{n} \tag{1.16.2}$$

can be used to form a random variable $\sqrt{n}(S_n - \mu)$ which converges to a normal distribution with mean 0 and variance σ^2.

This is usually translated to say that for any distribution, if we sample it long enough we get a Gaussian or normal distribution when we sum the random variables associated with that distribution. And so, practically everything converges to a Gaussian. This interpretation is false.

What is usually glossed over is the assumption of finite variance σ^2 and finite mean μ. You may think this is a mild assumption, but it is actually more restrictive than it seems at first glance. In fact, the more general central limit theorems acknowledge this. They do not require a finite variance and mean, but then find that the S_n will converge to one of several stable distributions.

[134] This is called writing it with natural frequencies.

[135] A random variable is a variable whose outcome depends on a "random" phenomenon. That is, it can be viewed as the value one gets from sampling a particular distribution associated with that variable.

[136] That is, they all have the same distribution function associated with them, and each variable was chosen independently [they do not depend on each other].

Indeed, let's examine a case, the Lorentzian, sometimes called the Cauchy distribution.[137] It is given by

$$f(x;x_0,\gamma) = \left(\pi\gamma\left[1+\frac{(x-x_0)^2}{\gamma^2}\right]\right)^{-1} = \frac{1}{\pi\gamma}\frac{\gamma^2}{(x-x_0)^2+\gamma^2} \tag{1.16.3}$$

Here x_0 gives the peak and 2γ characterizes the size of the full width at half maximum. The γ also is equal to half of the interquartile range and is called the probable error. Suppose $\gamma = 1$ and $x_0 = 1$ and we get a harmless looking function

$$f(x;1,1) = \left(\frac{1}{\pi}\right)\frac{1}{1+x^2} \tag{1.16.4}$$

You can easily check that

$$\int_{-\infty}^{\infty} \mathrm{d}x\ f(x;1,1) = \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}x\ \frac{1}{1+x^2} = 1 \tag{1.16.5}$$

This requires finding

$$\int_{-\infty}^{\infty}\mathrm{d}x\ \frac{1}{1+x^2} = \arctan(x)\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \frac{-\pi}{2} = \pi \tag{1.16.6}$$

However any moment x^α with integer $\alpha > 0$ is either divergent or undefined. You might expect that odd moments are zero (which is the Cauchy principal value, of course), but in fact there is no answer. This is because the first moment, or mean, is proportional to the integral

$$\int_{-\infty}^{\infty}\mathrm{d}x\ \frac{x}{1+x^2} \tag{1.16.7}$$

This has a defined value only if

$$\lim_{a,b\to\infty}\left[\int_{-b}^{a}\mathrm{d}x\ \frac{x}{1+x^2}\right] \tag{1.16.8}$$

is well-defined. If we use $u = 1+x^2$ then

$$\lim_{a,b\to\infty}\left[\int_{1+b^2}^{1+a^2}\mathrm{d}u\ \frac{1}{2u}\right] = \lim_{a,b\to\infty}\left[\frac{\ln(1+a^2)-\ln(1+b^2)}{2}\right] = \lim_{a,b\to\infty}\left[\frac{1}{2}\ln\left(\frac{1+a^2}{1+b^2}\right)\right] \tag{1.16.9}$$

Clearly this depends on how a and b go to infinity, and so the limit is not well-defined.[138] For higher odd moments, this process repeats itself. This integral therefore is not well-defined. For higher even moments, it is, in a sense, better. They "only" diverge. For the variance we'd see

$$\int_{-\infty}^{\infty}\mathrm{d}x\ \frac{x^2}{1+x^2} = \int_{-\infty}^{\infty}\mathrm{d}x\ \left[1-\frac{1}{1+x^2}\right] \tag{1.16.10}$$

[137] In fact, it is called the Lorentz distribution, the Cauchy-Lorentz distribution, the Lorentz function, the Lorentzian function, or the Breit-Wigner function in various contexts. I was inspired by Filip Piękniewski and his post on his website.

[138] If you do not believe this, see what you get for an answer when $b = a$ and when $a = 2b$ with $\lim_{b\to\infty}$.

and so

$$\begin{aligned}\lim_{a,b\to\infty}\left[\int_{-b}^{a}\mathrm{d}x\ 1-\int_{-b}^{a}\mathrm{d}x\ \frac{1}{1+x^2}\right] &= \lim_{a,b\to\infty}\left[(a+b)+\arctan(a)-\arctan(-b)\right]\\ &= \lim_{a,b\to\infty}\left[(a+b)+\arctan(a)+\arctan(b)\right]\end{aligned} \tag{1.16.11}$$

which clearly diverges for any way that a and b approach ∞.

It is worth delving slightly into "fat" and "heavy" tails so that you will not be confused when reading these in the literature.

You will often hear fat-tailed and heavy-tailed, but they have conflicting uses in the literature. These terms can then apply to the left, right, or both tails. A fat-tailed distribution is always a heavy-tailed distribution, but not the other way around. When such distinctions arise, it is common to only use heavy-tailed, long-tailed,[139] and subexponential distributions as the monikers for different types of distributions.

A heavy-tailed distribution has its tail not exponentially bounded, that is, the tail can have a probability greater than that of the exponential distribution in the tail. A long-tailed distribution is one that given that you have chosen a value from your distribution greater than x, then the probability it is greater than $x+t$ for $t>0$ approaches one as $x\to\infty$. That is, if you know x is a large value, it is actually likely that it is more than likely an extremely large value. A long-tailed distribution is a heavy distribution, but not necessarily vice versa. Finally, a subexponential distribution is a distribution that given n independent random variables X_i from such a distribution, we have the probability of $\sum_i X_i > x$ approaches the probability of $\max(X_1,\ldots,X_n) > x$ as $x\to\infty$.[140] All subexponential distributions are long-tailed distributions (but a long-tailed distribution is not necessarily a subexponential distribution). Finally, a fat-tailed distribution is one that has the probability density go to zero like x^{-a} as $x\to\infty$ for $a>0$.

Clearly we see that all fat-tailed distributions are heavy since a power of x will be greater than an exponential of x.

Our Lorentzian is a fat-tailed distribution, and so we cannot depend on the original central limit theorem to help us. In fact, the Lorentzian $x_0=0$ and $\gamma=1$ is a type of a student's t distribution. It destroys most of the common ideas we have that come from the central limit theorem. For example, the idea that if you take enough samples, you will converge on the mean. The Lorentzian distribution has no mean, so you will never get a convergence. A Lorentzian distribution also means that "rare" events are not nearly as rare as they seem when thinking through the lens of a Gaussian or normal distribution.

The main lesson here is that you need to make sure that you are working with a sensible model and that to do so you cannot assume that you are working only with Gaussian distributions! It will often be the case, but make sure that you have no reason to be working with a Lorentzian or any other stable[141] distribution! In other words, if you have a power law distribution (a fat-tailed distribution), then don't use intuitions honed on Gaussians.

[139] As if heavy-tailed and fat-tailed aren't enough!

[140] This is sometimes called the big jump or catastrophe principle since it says that the probability of a rare event occurring after n random samples is basically the probability of the worst case happening in any one sample.

[141] Stable in this sense means that the sum of two random independent variables with the same distribution also has that same distribution up to location and scale parameters.

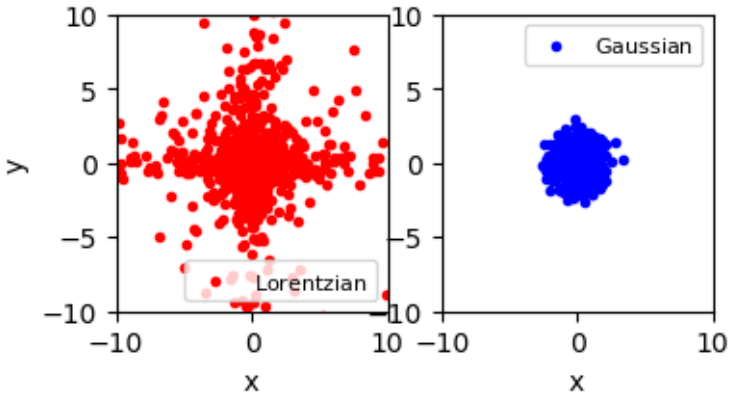

Figure 1.15: This shows the Lorentzian $\left(f(x;0,1), f(y\ :\ 0,1)\right)$ and Gaussian $\left(f(x;0,\frac{2}{2.355} = 0.85), f(y;0,\frac{2}{2.355} = 0.85)\right)$ so that the full-width half max is 2 for both cases. Note how we are then going nearly 12 standard deviations out with our scale. The Gaussian is neatly contained, but the Lorentzian seems to be doing something else.

The following Figures 1.15 and 1.16 show that rare events are perhaps not so rare with a Lorentzian, at least when compared against a Gaussian.

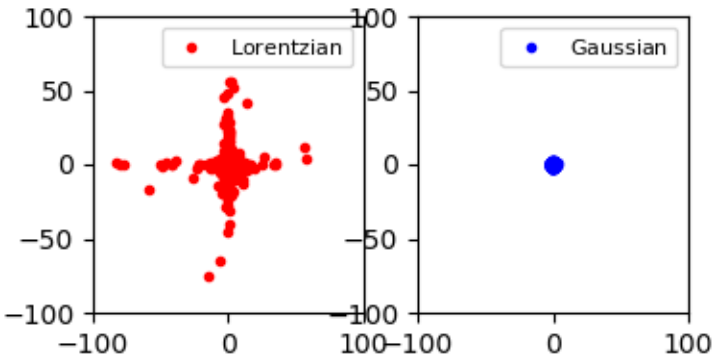

Figure 1.16: This shows the Lorentzian $(f(x;0,1), f(y\ :\ 0,1))$ and Gaussian $(f(x;0,\frac{2}{2.355} = 0.85), f(y;0,\frac{2}{2.355} = 0.85))$ so that the full-width half max is 2 for both cases. Note how we are then going nearly 120 standard deviations out with our scale! The Gaussian is neatly contained, but the Lorentzian now looks like a cross. This is because it is rare to have extreme events both in x and in y. You can see how what seem like rare events in a Gaussian become not so rare with a Lorentzian.

1.16.2 Probability Thoughts

If you have studied probability, you have probably been exposed to probability interpretations. People speak of "probabilities" as if it is clear what they mean, but I tend to think other than by fiat, it is not at all clear what a probability means. Remember the example when I tell you the weather will be rainy tomorrow with probability 1/2. How should this be interpreted? If you are using a series of events from the real world, the question then becomes, how do you decide what is "like" the situation? If I tell you 1/2 of days like this are rainy, a lot of work is being done by "days like this". This is the reference class problem, and it afflicts anyone who talks of an unconditional probability (that is without stating the conditions. In probability circumstances, these are often stated following the word "given"). I am indebted to the thinking and work by Hájek[19][18][20] for making me aware of the problem and introducing me to thinking of probabilities as really being conditional probabilities.

First, though, let me outline what is usually meant by probability. Many probability tutorials start with sets and introduce an operator Pr that translates sets into probabilities. Then the sets act as events that can be translated into the real world. We impose for admissible sets A that $\Pr(A) \geq 0$ to ensure non-negativity. If C represents all possible sets (that is everything that is possible/admissible) then $\Pr(C) = 1$ so that we have a normalization. Finally, we have for all admissible A and B with $A \cap B \equiv \emptyset$ (the empty set) that

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) \tag{1.16.12}$$

Here $\cup$ is a union operator, so the largest possible set made up of elements of A and B. The way to think about this is easier to understand with an example. Consider a coin that we will flip once with only two sides. We could represent all possible options with the set C as $0, 1$ where 0 represents heads and 1 represents tails. Then we must require

$$1 = \Pr(0,1) = \Pr(0 \cup 1) = \Pr(0) + \Pr(1) \tag{1.16.13}$$

If we have reasons to believe that there is no bias[142] then we can assign $\Pr(0) = \Pr(1) = 1/2$. In more complicated but finite examples the belief in no bias may or may not be satisfied, but it leads to the usual analyses of an n sided die, for example. Note that these rules do not tell you what you should assign as probabilities, however, just how to deal with them once you have.

If you work through all the possibilities and consider "and" to mean $\cap$ (this is the intersection, and so $A \cap B$ means all the elements in common for A and B) and "or" to mean $\cup$, and "not" to mean complement ^{C} (given the entire set C with B a subset of C, then $B^C = A$ is the set of all elements of C that are not in B) then

$$\Pr(A^C) \equiv \Pr(\text{not } A) = 1 - \Pr(A) \tag{1.16.14}$$
$$\Pr(A \cup B) \equiv \Pr(A \text{ or } B) = \Pr(A) - \Pr(B) - \Pr(A \cap B) \tag{1.16.15}$$
$$\Pr(A \cap B) \equiv \Pr(A \text{ and } B) = \Pr(A|B)\Pr(B) \quad A \text{ and } B \text{ not independent} \tag{1.16.16}$$
$$\Pr(A \cap B) \equiv \Pr(A \text{ and } B) = \Pr(A)\Pr(B) \quad A \text{ and } B \text{ independent} \tag{1.16.17}$$
$$\Pr(A|B) \equiv \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)} \quad \Pr(B) > 0 \tag{1.16.18}$$

where the last line is Bayes' theorem proved via $A \cap B = B \cap A$ so

$$\Pr(A|B)\Pr(B) = \Pr(A \cap B) = \Pr(B \cap A) = \Pr(B|A)\Pr(A) \tag{1.16.19}$$
$$\Pr(A|B)\Pr(B) = \Pr(B|A)\Pr(B) \tag{1.16.20}$$
$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(B)}{\Pr(B)} \tag{1.16.21}$$
$$\tag{1.16.22}$$

We can "conditionalize" it using $[\Pr(C) \neq 0]$

$$\Pr(A \cap B \cap C) = \Pr(A \cap [B \cap C]) = \Pr(A|B \cap C)\Pr(B \cap C) = \Pr(A|B \cap C)\Pr(B|C)\Pr(C) \tag{1.16.23}$$
$$= \Pr(B \cap A \cap C) = \Pr(B \cap [A \cap C]) = \Pr(B|A \cap C)\Pr(A \cap C) = \Pr(B|A \cap C)\Pr(A|C)\Pr(C) \tag{1.16.24}$$
$$\Pr(A|B \cap C)\Pr(B|C)\Pr(C) = \Pr(B|A \cap C)\Pr(A|C)\Pr(C) \tag{1.16.25}$$
$$\Pr(A|B \cap C) = \frac{\Pr(B|A \cap C)\Pr(A|C)\Pr(C)}{\Pr(B|C)\Pr(C)} = \frac{\Pr(B|A \cap C)\Pr(A|C)}{\Pr(B|C)} \tag{1.16.26}$$
$$\Pr(A|B \text{ and } C) = \frac{\Pr(B|A \text{ and } C)\Pr(A|C)}{\Pr(B|C)} \tag{1.16.27}$$

Later I will try to help this be more memorable by using different variables, but, for now, this is fine. For finite sets with $\Pr(B) = 0$ then $\Pr(A|B)$ is undefined, since B cannot happen. When we consider infinite sets, this becomes problematic. Any competent probability textbook will teach you that $\Pr(B) = 0$ does not mean that something does not happen when you deal with continuum values. For example, if you throw a dart at a dartboard and it hits the board at a random location, the probability of hitting any location (x, y) is zero. But it will hit a location! Say it hits $(0, 1)$.

[142] In fact, building a biased coin is very difficult if the coin is flipped well and the coin is caught before it hits the floor and starts to spin.[15]

We know that $\Pr\big((0,1)\big) = 0$. This also tells us that with infinite sets that $\Pr(B) = 1$ does not mean the event will certainly happen. For in the previous case we would say $\Pr\big(\text{not}(0,1)\big) = 1$ yet it would still be possible to hit $(0,1)$. The term "almost sure" is then defined to mean that a probability 1 event is almost sure if the complement of the set (the "not") in the probability space is not the empty set. So $(0,1)$ above is not the empty set and so $\Pr(\text{not}(0,1)) = 1$ is an almost sure event. Note that an impossible event will still have $\Pr = 0$, and a certain event will have $\Pr = 1$, but that they are no the only events with these probabilities.

However, there are events we cannot say things about with the above formalism that seem like they have straightforward answers. Suppose we know that by magic that the dart will hit either $(0,1)$ or $(1,0)$ with them being equally likely. Doesn't it seem like $\Pr\big((0,1)|(0,1)\,\text{or}(1,0)\big)$ should be $1/2$? But our rule says no, it is undefined because the probability is 0. Of course, if you go through the weeds of mathematical sophistication, one can construct a new meaning for $\Pr(A|B)$ that gives the "right" answer, but most texts simply ignore these problems. As we will see later, there is an alternate formulation that avoids this problem altogether.

Indeed, suppose you have a coin you believe is fair. If you flip it, what is the probability that it lands on heads if I toss it fairly? It seems like it should be $1/2$, but our rule tells us

$$\Pr(\text{heads}|\text{fair toss}) = \frac{\Pr(\text{heads} \cap \text{fair toss})}{\Pr(\text{fair toss})} \tag{1.16.28}$$

But what is the probability that I fairly toss the coin? Should it even matter for how I posed the question?

All of this comes down to saying, whenever you deal with an unconditional probability, you should ask yourself what is implicitly being used as a conditional. This will help you ensure you are using the right data and makes it all the easier to understand what the probability even means. At the very least, when you are given a probability, the set C (of all possibilities) you are talking about should be given.

If you want to only work with conditional probabilities, consider using Popper functions. These are traditionally given using statements (in propositional languages) A, B, or C rather than sets[143] and use the following axioms

$$\Pr(A|A) = 1 \tag{1.16.29}$$
$$\Pr(A|C \text{ and } B) = \Pr(A|B \text{ and } C) \tag{1.16.30}$$
$$\Pr(B \text{ and } A|C) = \Pr(A \text{ and } B|C) \tag{1.16.31}$$
$$\tag{1.16.32}$$

For a given B we have either for all A that

$$\Pr(A|B) + \Pr(\text{not } A|B) = 1 \tag{1.16.33}$$

or for all C

$$\Pr(C|B) = 1 \tag{1.16.34}$$

[143] If you are worried about the math, these can be related to non-Archimedean probability functions.[6] You can also change this back into sets, if you feel uncomfortable with statements.

This is saying that if $\Pr(\text{not } B|B) = 1$ then $\Pr(C|B) = 1$ for all C. That is if we have a contradiction as true, everything is true. Finally, we have

$$\Pr(A \text{ and } B|C) = \Pr(A|B \text{ and } C)\Pr(B|C) \tag{1.16.35}$$

We can now prove a conditional Bayes' theorem using (1.16.31) and (1.16.35). We have

$$\Pr(A \text{ and } B|C) = \Pr(A|B \text{ and } C)\Pr(B|C) \tag{1.16.36}$$

$$\Pr(B \text{ and } A|C) = \Pr(B|A \text{ and } C)\Pr(A|C) \tag{1.16.37}$$

$$\Pr(A|B \text{ and } C)\Pr(B|C) = \Pr(B|A \text{ and } C)\Pr(A|C)\Pr(A|B \text{ and } C) \quad = \frac{\Pr(B|A \text{ and } C)\Pr(A|C)}{\Pr(B|C)} \tag{1.16.38}$$

This can be more easily remembered by saying using H for hypothesis instead of A, E for evidence instead of B, and B for background conditions instead of C. Then we have

$$\Pr(H|E \text{ and } B) = \frac{\Pr(E|H \text{ and } B)\Pr(H|B)}{\Pr(E|B)} \tag{1.16.39}$$

which says that the probability of the hypothesis given the evidence and background conditions is related to the probability of the evidence given the hypothesis and background conditions, the probability of the hypothesis given the background conditions alone, and the probability of the evidence given the background conditions alone.

This brings up one final "problem". What (background) conditions should you actually use? This is the reference class problem, and the answer is generically use what is useful for the situation. If your situation gives you certain conditions, use those. You will not find a probability unconditioned out there.

One may apply to what I will call the unconditional probability Euthyphro dilemma[144] to see how there are no justifiable unconditional probabilities. Suppose someone has an unconditional probability that they believe is justified. If we ask them why they think it is justified they have essentially three responses available, which eventually must collapse to two. They could give a reason by stating a principle that is at play in this situation. In this case, whatever that principle is, it is a (background) condition, and so they actually have a conditional probability.[145] Second they could simply say they just assume or believe it. At this point, one should question whether that is any justification at all. Third, they could refer to an expert, but then we ask the same question to the expert. Eventually we must get an answer of conditions or sheer belief. If it truly were an infinite loop of experts, then this would not be a justification, but an impressive consensus. But it would not really be a justification if expert A cites expert B who eventually cites expert A as the "reason" for their belief in an unconditional probability. At the very least, I wouldn't be confident in a statement if a loop of experts gave as justification that each of them believe each other has a valid reason, but no one can produce it, but can only say another expert believes it.

[144]This is a famous morality dilemma (though there are those who dispute it is a dilemma). It's modern form asks, "Is something good because God commands it? Or does God command it because it is good?".

[145]Note that if they say look at the results, then they are referring to a principle. That the it is consistent with the *given* data. Certainly the data must have some sort of principles guiding how it was measured. At the very least, one must outline the potential possibilities for the probability to mean something, and what is possible is a condition.

To end, let's bring this back to physics. A lot plasma physics uses kinetics, and statistical mechanic-like ideas. You should keep in mind that probabilities are notoriously difficult with thermodynamic ideas, and it helps to spell out the assumptions used when given a probability. Conditional probabilities do this explicitly. If someone says that a particular state is unlikely, they mean given the macroscopic physical conditions we view, very few microstates are compatible with this. Of course, one could ask why those particular physical conditions are important (the reference class problem). Usually the answer is that these are what are useful to measure and calculate with, which is as good of an answer as can be given.

1.17 Further Reading

There is a wealth of texts on curvilinear coordinates. W. D. D'haeseleer[9] is of course a good read for curvilinear coordinates in a magnetic confinement plasma physics context. General relativity textbooks also provide some similar content, though generally lean heavily into differential forms, differentiable manifolds, and more abstract mathematical approaches. If you would like to see tensors generically investigated in a more mathematical context, Lovelock[23] is also a good reference. There are a wealth of textbooks on tensor analysis, and so one should not find too much trouble finding other references. Asymptology has quite a few texts that can be found under that name and asymptotics. A text I enjoyed was deBruijn[7]. For a general treatment of mathematics for physical problems, Bender and Orzsag is a classic[2]. Fourier series, Fourier transforms, Laplace transforms, and Taylor series should be found in common undergraduate mathematics textbooks, but a delightful coverage of them in spectral methods is given by Boyd[5]. Ballooning transforms are well-described in Connor, Hastie, Taylor[8], and there are some review papers available. Variational Calculus is covered in classical physics texts, but functional analysis, especially for numerical work, is excellently covered in Sawyer[28]. For the Hamiltonian nature of the magnetic fields, Morrison[25] has a good review article, though the subject goes back to Kerst[22] in 1962, and, in fact, the idea was already there in the late 1950's. However, most of the credit should really go to Boozer[4] in 1981 and many papers thereafter. Unsurprisingly, the Frenet-Serret formulas come from Frenet and Serret who independently discovered them. I have found online resources to be more than sufficient for understanding them. The JWKB approximation also has a wealth of resources to choose from online and in texts. Complex analysis texts are abundant, and contour integration is usually treated excellently. Analytic continuation is treated in any good complex variables textbook, as well, and also has fairly good treatments online. Special functions are subjects on their own. I quite like the DLMF[10] (Digital Library of Mathematical Functions) provided by NIST. This is similar to Abramowitz and Stegun[1] which is available online. The probability and statistics section can be supplemented with online resources, textbooks, and if you want more on conditional probabilities being the most basic, Hájek[19][18][20].

1.18 Problem Set

1.1. For Section 1.1.

1.1.1. Consider a function $f(E, y, z) = E+z-y$ with $E = x^2+y^2$. What is $\left(\frac{\partial f}{\partial E}\right)_{y,z}$? How about $\left(\frac{\partial f}{\partial y}\right)_{E,z}$ and $\left(\frac{\partial f}{\partial z}\right)_{E,y}$? What if we substitute $f(E, y, z) = f(x, y, z) = x^2 + y^2 + z - y$? What are $\left(\frac{\partial f}{\partial x}\right)_{y,z}$, $\left(\frac{\partial f}{\partial x}\right)_{y,z}$, $\left(\frac{\partial f}{\partial x}\right)_{y,z}$? If we compare our results, is there a simple way to

translate something like $\left(\frac{\partial f}{\partial x}\right)_{y,z}$ in terms of $\left(\frac{\partial f}{\partial E}\right)_{y,z}$ and $\left(\frac{\partial E}{\partial x}\right)_{y}$? How about for $\left(\frac{\partial f}{\partial y}\right)_{x,z}$ in terms of $\left(\frac{\partial f}{\partial y}\right)_{E,z}$, $\left(\frac{\partial E}{\partial y}\right)_{x}$, and $\left(\frac{\partial f}{\partial E}\right)_{y,z}$?

1.1.2. Show that for a function $f(x,y,z) = g(a,b,c)$ with $a = a(x,y,z)$, $b = b(x,y,z)$, and $c = c(x,y,z)$ that

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \left(\frac{\partial g}{\partial a}\right)_{b,c}\left(\frac{\partial a}{\partial x}\right)_{y,z} + \left(\frac{\partial g}{\partial b}\right)_{a,c}\left(\frac{\partial b}{\partial x}\right)_{y,z} + \left(\frac{\partial g}{\partial c}\right)_{a,b}\left(\frac{\partial c}{\partial x}\right)_{y,z} \tag{1.18.1}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,z} = \left(\frac{\partial g}{\partial a}\right)_{b,c}\left(\frac{\partial a}{\partial y}\right)_{x,z} + \left(\frac{\partial g}{\partial b}\right)_{a,c}\left(\frac{\partial b}{\partial y}\right)_{x,z} + \left(\frac{\partial g}{\partial c}\right)_{a,b}\left(\frac{\partial c}{\partial y}\right)_{x,z} \tag{1.18.2}$$

$$\left(\frac{\partial f}{\partial z}\right)_{x,y} = \left(\frac{\partial g}{\partial a}\right)_{b,c}\left(\frac{\partial a}{\partial z}\right)_{x,y} + \left(\frac{\partial g}{\partial b}\right)_{a,c}\left(\frac{\partial b}{\partial z}\right)_{x,y} + \left(\frac{\partial g}{\partial c}\right)_{a,b}\left(\frac{\partial c}{\partial z}\right)_{x,y} \tag{1.18.3}$$

The slightly more confusing notation, but perfectly valid way, is to write $f(x,y,z) = f(a,b,c)$ instead can be summarized as (here x_i is a one of $x = x_1$, $y = x_2$, or $z = x_3$ and x_{jj} stands for the other two, not equal to i, and similarly for a_i and a_{jj} with $a = a_1$, $b = a_2$, and $c = a_3$)

$$\left(\frac{\partial f}{\partial x_i}\right)_{x_{jj}} = \sum_{i=1}^{3}\left(\frac{\partial f}{\partial a_i}\right)_{a_{jj}}\left(\frac{\partial a_i}{\partial x_i}\right)_{x_{jj}} \tag{1.18.4}$$

The other way is also perfectly valid, though it requires us to find $x = x(a,b,c)$, $y = y(a,b,c)$, and $z = z(a,b,c)$ to find

$$\left(\frac{\partial f}{\partial a_i}\right)_{a_{jj}} = \sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i}\right)_{x_{jj}}\left(\frac{\partial x_i}{\partial a_i}\right)_{a_{jj}} \tag{1.18.5}$$

Test this on our previous function $f(E(x,y),y,z) = E + z - y$ with $E = x^2 + y^2$ and $f(x,y,z) = x^2 + y^2 + z - y$. One can write $x = x(E,y)$ and $y = y(E,x)$ to use the "inverse" formula.

1.1.3. Let $\overleftrightarrow{\mathbf{T}} = xy^2\hat{\mathbf{x}}\hat{\mathbf{y}}$. Find $\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}$ a volume integral around the unit cube centered at the origin. That is, find

$$\int_{-1}^{1}\mathrm{d}z\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}\mathrm{d}x\;\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}$$

Then find the same value via Gauss's theorem (so six different surface integrals)

$$\int_{-1}^{1}\mathrm{d}z\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}\mathrm{d}x\;\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}} = \oiint \mathrm{d}S\;\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{T}}$$

Now what do you get if you were to use the other convention for integral order? That is what do you get for

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \oiint\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

And what would you get if with the other integral order convention used the alternate divergence $\boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{T}}^{\intercal}$?

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{T}}^{\intercal} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \oiint \overleftrightarrow{\mathbf{T}}^{\intercal} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

1.1.4. Prove the generalized chain rule $\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{f}}{\partial \mathbf{y}}$. Make sure you understand why $\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \neq \frac{\partial \mathbf{f}}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{x}}$. As a hint, write out everything with Einstein summation notation remembering that $\frac{\partial}{\partial \mathbf{x}} = \mathbf{e}^i \frac{\partial}{\partial \xi^i}$.

1.2. For Section 1.2

1.2.1. Reproduce Figure 1.2 using $\xi^1 = x - y$ and $\xi^2 = y$. Calculate $\frac{\partial \mathbf{x}}{\partial \xi^i}$ (using $\mathbf{x} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$) and $\frac{\partial \xi^i}{\partial \mathbf{x}}$. Make sure you keep things constant that should be in the partial derivatives. It is easiest to find the tangent basis vectors by solving $x = x(\xi^1, \xi^2)$ and $y = y(\xi^1, \xi^2)$.

1.2.2. Try finding the tangent basis vectors and reciprocal basis vectors for

$$\xi^1 = x - y \tag{1.18.6}$$
$$\xi^2 = y - z \tag{1.18.7}$$
$$\xi^3 = x + y + z \tag{1.18.8}$$

With your favorite 3D plotting software plot the $\xi^i = c_{ij}$ for constants c_{ij}. See the normal directions to these surfaces. Then let $\xi^{j'} = c_{j'}$ and $\xi^{k'} = c_{k'}$ and see how $\xi^{i'}$ varies to find the tangent directions.

1.2.3. Show how the reciprocal basis defined by (1.2.33) are satisfied by (1.2.30)-(1.2.32). Try it with the Cartesian basis $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. Do you notice anything about the reciprocal basis and the tangent basis?

1.2.4. Calculate the tangent basis and reciprocal basis for cylindrical coordinates (use either C.2 or C.3), spherical coordinates (use C.4), and primitive toroidal coordinates (see C.5). Feel free to use Appendix C for the definitions. The answers are also there, but try not to use them unless you have no choice.

1.2.5. For cylindrical, spherical, or primitive toroidal coordinates, is there a special relationship between the tangent basis vectors, themselves? That is, what is $\frac{\partial \mathbf{x}}{\partial \xi^{i'}} \cdot \frac{\partial \mathbf{x}}{\partial \xi^{j'}}$. What about the reciprocal basis vectors $\nabla\xi^{i'} \cdot \nabla\xi^{j'}$? Are they orthogonal? What does that tell us immediately about the metric coefficients g_{ij} and g^{ij}? Consider whether $\nabla\xi^{i'} \cdot \nabla\xi^{j'} = f(\xi^k)\delta^{i'j'}$ for some function $f(\xi^k)$ implies that $\frac{\partial \mathbf{x}}{\partial \xi^{i'}} \cdot \frac{\partial \mathbf{x}}{\partial \xi^{j'}} = g(\xi^k)\delta^{i'j'}$ and vice versa. Similarly what if only $\nabla\xi^1 \cdot \nabla\xi^2 = \nabla\xi^1 \cdot \nabla\xi^3 = 0$ but $\nabla\xi^2 \cdot \nabla\xi^3 \neq 0$? And if only $\nabla\xi^1 \cdot \nabla\xi^3 = 0$ for the basis vectors? This tells us what partial orthogonality gives us. You should find something interesting for two pairs of basis vectors orthogonal, but little of interest for only one pair orthogonal.

1.2.6. Consider cylindrical coordinates from C.3. Consider a vector representing velocity in "standard" form $\mathbf{V} = S_{VR}\hat{\mathbf{R}} + S_{V\theta}\hat{\boldsymbol{\theta}} + S_{VZ}\hat{\mathbf{Z}}$. What would the contravariant components V^R, V^θ and V^Z be in terms of the standard components? What are the units of each of the contravariant components? What are the units of the tangent basis vectors? What

about the covariant components V_R, V_θ, and V_Z? What are the units of the reciprocal basis vectors? What happens when we multiply a contravariant component by a tangent basis vector? Or a covariant component by a reciprocal basis vector?

1.2.7. Consider primitive toroidal coordinates from C.5. Consider a vector representing velocity in "standard" form $\mathbf{V} = S_{Vr}\hat{\mathbf{r}} + S_{V\theta}\hat{\boldsymbol{\theta}} + S_{V\zeta}\hat{\boldsymbol{\zeta}}$. What would the contravariant components V^r, V^θ and V^ζ be in terms of the standard components? What are the units of each of the contravariant components? What are the units of the tangent basis vectors? What about the covariant components V_r, V_θ, and V_ζ? What are the units of the reciprocal basis vectors? What happens when we multiply a contravariant component by a tangent basis vector? Or a covariant component by a reciprocal basis vector?

1.2.8. Suppose we used as a Jacobian matrix $\boldsymbol{\mathcal{J}}^\intercal$. Would this change the Jacobian determinant? Would using this definition be consistent with acting on the closest vector to our right for a matrix?

1.2.9. Show that δ^i_j is an order two type (1,1) tensor component. This will show that the identity tensor $\mathbb{1}$ is indeed a tensor and is given in Einstein notation as the delta function. Show that δ_{ij} and δ^{ij} are not components of an order two tensor.

1.2.10. Show that ϵ_{ijk} and ϵ^{ijk} are not components of an order 3 tensor. You will find that the reason they are not has to do with a factor of $\mathcal{J}$. Would $\mathcal{J}\epsilon^{ijk}$ and $\epsilon_{ijk}/\mathcal{J}$ be components of an order 3 tensor? They should be. Call this new order three tensor E the permutation tensor. Sometimes the Levi-Civita symbol is called a tensor density since the Jacobian determinant simply needs to multiply it in order for it to become a tensor.

1.2.11. The Christoffel symbols in the form Γ^i_{jk} and $\Gamma_{i,jk}$ may suggest they are order three tensors. Prove that they are not.

1.2.12. If you want practice with Einstein summation or index notation, the best practice is to get a bunch of vector identities and do the translation into Cartesian components, manipulate them to a new form, translate back into abstract vectors and see if you get the same identity. These usually use $\epsilon_{ijk}\epsilon_{ijl}$ identities, and such. Consult Appendix B. Some good ones to derive yourself are (B.7)-(B.10), (B.16)-(B.19), (B.22), (B.25), -, (B.132)-(B.136), or (B.145)-(B.159). I would definitely recommend proving

$$\begin{aligned}
\nabla(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A} + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) \\
\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) \\
\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{B}\cdot\nabla\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{B} \\
\boldsymbol{\nabla}\cdot(\mathbf{AB}) &= (\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{B} + \mathbf{A}\cdot\nabla\mathbf{B} \\
\boldsymbol{\nabla}\times(\mathbf{AB}) &= (\boldsymbol{\nabla}\times\mathbf{A})\mathbf{B} + \mathbf{A}\times\nabla\mathbf{B}
\end{aligned}$$

1.2.13. We discussed dyadics a little bit. We showed that not every tensor can be constructed from just a single dyad. Let's think about how many dyads are required to generally represent a second order tensor. Clearly we could arrange to sum nine dyads to form any second order tensor. That is $\overleftrightarrow{\mathbf{T}} = \sum_{i,j} T^{ij}\mathbf{e}_i\mathbf{e}_j$, and so we could define 9 vectors and

append a basis vector. That is we could have (no summation) $\mathbf{A}_{i,j} = T^{ij}\mathbf{e}_i$ We then append $\mathbf{e}_j$ as a vector and we have the necessary dyads, $\overleftrightarrow{\mathbf{T}} = \sum_{i,j} \mathbf{A}_{i,j}\mathbf{e}_j$. So

$$\begin{aligned}\mathbf{A}_{1,1} &= T_{11}\mathbf{e}_1\\ \mathbf{A}_{1,2} &= T_{12}\mathbf{e}_1\\ \mathbf{A}_{2,1} &= T_{21}\mathbf{e}_2\\ \vdots &= \vdots\end{aligned}$$

Can we do better? Is there a way to use fewer than 9 dyads to form an order two tensor?

1.2.14. What should we do with triadics, quadradics, and polyadics generally? Can we use fewer than 3^n nth order polyadics to represent an nth order tensor?

1.2.15. Test some of the laws of nature to see if they respect parity inversion ($\mathbf{x} \to -\mathbf{x}$). Try Newton's Law and Maxwell's equations, for example. The Lorentz force law is also a good test. If they don't respect parity inversion, what if we do parity inversion and time reversal ($t \to -t$) at the same time?

1.2.16. Construct the Maxwell Stress Tensor in abstract tensor form. Begin with the Lorentz Force Law for a charged particle

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B} + \mathbf{E})$$

We can work with a unit volume instead for convenience if we want to work with a bunch of particles in a continuum limit.

$$\begin{aligned}\frac{\mathbf{F}}{V} &= \frac{q}{V}(\mathbf{v} \times \mathbf{B} + \mathbf{E})\\ \mathbf{f} &= \rho_q(\mathbf{v} \times \mathbf{B} + \mathbf{E})\\ \mathbf{f} &= \mathbf{J} \times \mathbf{B} + \rho_q\mathbf{E}\end{aligned}$$

Now use Maxwell's Laws (with the Maxwell-Ampère Law) to replace ρ_q and $\mathbf{J}$. Change the time derivative so that it acts on $\mathbf{E}\times\mathbf{B}$. Use Faraday's Law to remove the resulting $\frac{\partial \mathbf{B}}{\partial t}$. Define $\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}$. You then collect $\mathbf{E}$ and $\mathbf{B}$ into

$$\mathbf{f} = \epsilon_0\left[(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{E} - \mathbf{E}\times\boldsymbol{\nabla}\times\mathbf{E}\right] + \frac{1}{\mu_0}\left[-\mathbf{B}\times\boldsymbol{\nabla}\times\mathbf{B}\right] - \mu_0\epsilon_0\frac{\partial \mathbf{S}}{\partial t}$$

Now use vector identities and $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ to rewrite this as

$$\mathbf{f} = \epsilon_0\left[(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{E} - \mathbf{E}\cdot\,\nabla\mathbf{E}\right] + \frac{1}{\mu_0}\left[(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{B} - \mathbf{B}\cdot\,\nabla\mathbf{B}\right] - \frac{1}{2}\nabla\left(\epsilon_0\mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right) - \mu_0\epsilon_0\frac{\partial \mathbf{S}}{\partial t}$$

The Maxwell Stress Tensor is then defined by

$$\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\sigma}} = \epsilon_0\left[(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{E} - \mathbf{E}\cdot\,\nabla\mathbf{E}\right] + \frac{1}{\mu_0}\left[(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{B} - \mathbf{B}\cdot\,\nabla\mathbf{B}\right] - \frac{1}{2}\nabla\left(\epsilon_0\mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right)$$

so that we write the force law in a conservative form

$$\mathbf{f} = \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\sigma}} - \mu_0 \epsilon_0 \frac{\partial \mathbf{S}}{\partial t}$$

Using tensor identities show that we then have

$$\overleftrightarrow{\boldsymbol{\sigma}} = \epsilon_0 \left(\mathbf{E}\mathbf{E} - \frac{\mathbb{1}}{2}\mathbf{E} \cdot \mathbf{E} \right) + \frac{1}{\mu_0} \left(\mathbf{B}\mathbf{B} - \frac{\mathbb{1}}{2}\mathbf{B} \cdot \mathbf{B} \right)$$

1.2.17. Show that the Maxwell stress tensor is a polar tensor and that it is even under time reversal symmetry.

1.2.18. In special relativity and general relativity, one can rewrite Maxwell's equations as a single tensor law. Define the four-vector electromagnetic potential in 4D with our sign convention (see Section 1.2.6) as $\mathbb{A}$ as (using the standard basis which I took to be $\mathbf{e}^0 = -\mathbf{x}_0$)

$$\mathbb{A} = \frac{\phi}{c}\mathbf{x}_0 + A^1\mathbf{x}_1 + A^2\mathbf{x}_2 + A^3\mathbf{x}_3 = \frac{\phi}{c}\mathbf{x}_0 + \mathbf{A}$$

where the components for A^1, A^2, and A^3 are the same as the (3D) magnetic vector potential. Remember that four-vectors have their own dot product rule, and the reciprocal basis $\mathbf{e}^i$ have $\mathbf{e}^0 = -\mathbf{x}_0$. It is conventional to use Greek letters rather than Roman letters as indices for running from 0 to 3. Roman letters as indices run from 1 to 3. Then we can write

$$\mathbb{A} = A^\mu \mathbf{e}_\mu = A_\mu \mathbf{e}^\mu \tag{1.18.9}$$

Then we can write the electromagnetic tensor (sometimes called the electromagnetic field tensor, field strength tensor, Faraday tensor, or Maxwell bivector) as

$$\overleftrightarrow{\mathbb{F}} = \Box\mathbb{A} - (\Box\mathbb{A})^\intercal$$

with $\Box$ indicating $\mathbf{e}^\mu \frac{\partial}{\partial \xi^\mu}$ over all 4 indices ($\xi^0 = t$ in the standard basis and $\mathbf{e}^0 = -\mathbf{x}_0$). Or using indices to write the components out, we find

$$\overleftrightarrow{\mathbb{F}} = F^{\mu\nu}\mathbf{e}_\mu\mathbf{e}_\nu = F_{\mu\nu}\mathbf{e}^\mu\mathbf{e}^\nu = F^\mu_{\cdot\nu}\mathbf{e}_\mu\mathbf{e}^\nu = F_\mu^{\cdot\nu}\mathbf{e}^\mu\mathbf{e}_\nu$$

When using a contravariant representation with the standard basis ($\hat{\mathbf{x}}_i$)

$$\overleftrightarrow{\mathbb{F}} = F^{\mu\nu}\mathbf{e}_\mu\mathbf{e}_\nu = \mathbf{e}^\mu\partial_\mu(A^\nu\mathbf{e}_\nu) - \mathbf{e}^\nu\partial_\nu(A^\mu\mathbf{e}_\mu)$$

find the components $F^{\mu\nu}$ and $F_{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$. Note by definition that $\overleftrightarrow{\mathbb{F}}^\intercal = -\overleftrightarrow{\mathbb{F}}$, and that using the standard basis means that the basis set is unaffected by derivatives.

1.2.19. Using the electromagnetic tensor, show that $\epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E} = \rho_q$ and $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$ can be represented as

$$\Box \cdot \overleftrightarrow{\mathbb{F}} = \mu_0 \mathbb{J}$$

with $\mathbb{J} = c\rho_q \hat{\mathbf{x}}_0 + \mathbf{J}$ [$\mathbf{J}$ is in Cartesian form]. Use the contravariant representation of $\overleftrightarrow{\mathbb{F}}$ to do so. Then show that $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and $\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E}$ is also represented via

$$\Box\overleftrightarrow{\mathbb{F}} + (\Box\overleftrightarrow{\mathbb{F}}^{\intercal})^{\intercal} - (\Box\overleftrightarrow{\mathbb{F}})^{\intercal} \tag{1.18.10}$$

where the transpose only operates on the outermost indices so $G^{\intercal}_{\alpha\beta\gamma} = G_{\gamma\beta\alpha}$. This uses $\overleftrightarrow{\mathbb{F}}^{\intercal} = -\overleftrightarrow{\mathbb{F}}$ and so in textbooks it is often written in a covariant representation as

$$\partial_\gamma F_{\alpha\beta}\mathbf{e}^\gamma\mathbf{e}^\alpha\mathbf{e}^\beta + \partial_\alpha F_{\beta\gamma}\mathbf{e}^\alpha\mathbf{e}^\beta\mathbf{e}^\gamma + \partial_\beta F_{\gamma\alpha}\mathbf{e}^\beta\mathbf{e}^\gamma\mathbf{e}^\alpha$$

That is, this is the antisymmetric tensor and so could be written $6\partial_{[a}F_{\beta,\gamma]}$. We could define a new symbol $^\sharp$ such that it adds each even cyclic permutation of the basis vectors divided by the factorial of the order of the tensor. Then

$$\begin{aligned} 3!(\Box\overleftrightarrow{\mathbb{F}})^\sharp &= \sum_{\alpha,\beta,\gamma}\left(\mathbf{e}^\gamma\frac{\partial}{\partial x^\gamma}\left[F^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta\right]\right)^\sharp \\ &= \sum_{\alpha,\beta,\gamma}\left(\mathbf{e}^\gamma\frac{\partial}{\partial x^\gamma}\left[F^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta\right] + \mathbf{e}^\alpha\frac{\partial}{\partial x^\alpha}\left[F^{\beta\gamma}\mathbf{e}_\beta\mathbf{e}_\gamma\right] + \mathbf{e}^\beta\frac{\partial}{\partial x^\beta}\left[F^{\gamma\alpha}\mathbf{e}_\gamma\mathbf{e}_\alpha\right]\right) \end{aligned}$$

And then note for order two that it simply yields

$$2(\overleftrightarrow{\mathbf{T}})^\sharp = T^{ij}\mathbf{e}_i\mathbf{e}_j + T^{ji}\mathbf{e}_j\mathbf{e}_i = 2\overleftrightarrow{\mathbf{T}}_{\mathrm{S}} \tag{1.18.11}$$

1.3. For Section 1.3

1.3.1. Convince yourself that

$$\int_{-\pi}^{\pi}\mathrm{d}x\ \exp(ijkx)\exp(ij'kx) = 2\pi\delta_{j,-j'}$$

and see how it follows that

$$\int_{-L/2}^{L/2}\mathrm{d}x\ \exp(2ij\pi x/L)\exp(2ij'\pi x/L) = 2\pi\delta_{j,-j'} \tag{1.18.12}$$

What does this imply about $\sin(2j\pi x/L)$ and $\cos(2j\pi x/L)$ or $\sin(jkx)$ and $\cos(jkx)$?

1.3.2. Directly calculate

$$\int_{-\pi}^{\pi}\mathrm{d}x\ \sin(jkx)\sin(\pm j'kx) = \pm 2\pi\delta_{j,j'} \tag{1.18.13}$$

$$\int_{-\pi}^{\pi}\mathrm{d}x\ \sin(jkx)\cos(j'kx) = 0 \tag{1.18.14}$$

$$\int_{-\pi}^{\pi}\mathrm{d}x\ \cos(jkx)\cos(\pm j'kx) = 2\pi\delta_{j,j'} \tag{1.18.15}$$

It is fastest to divide into two cases. One with $j = -j'$ and one without. Note we are integrating even or odd functions over a symmetric interval, as well.

1.3.3. Find the Fourier coefficients for the function $f(x) = x$ for $L = 1/2$. Calculate N (the number of coefficients) for $N = 1, 2, 3, 4$. Note that x is not periodic over this domain. What do you notice when you plot the approximation $\sum_{j=-N}^{N} f_j \exp(2\pi ijx/L)$ for these values?

1.3.4. Find the Fourier coefficients for the function $f(x) = |x|$ for $L = 1$. Calculate N (the number of coefficients) for $N = 1, 2, 3, 4$. What does this look like over several periods. Sometimes this is called a sawtooth wave. What do you notice when you plot the approximation $\sum_{j=-N}^{N} f_j \exp(2\pi ijx/L)$ for these values? Is the accuracy better than you expected?

1.3.5. I defined the Fourier series for $f(x)$ periodic in domain $[-L/2, L/2]$. What if you you have $f(x)$ periodic in domain $[L_1, L_2]$? Let $L_2 > L_1$ and $L_2 - L_1 = \Delta L$. What would $y = x - L_1 - (\Delta L/2)$ yield as the new domain of periodicity in y? That is calculate y for $x = L_2$ and $x = L_1$ and note what we can say about the limits for the new y integral. Would this yield the properties necessary for my definition with $\Delta L \to L/2$?

1.3.6. Suppose we have a complex function $f(z)$. It is traditional to use the exponential form since one has to translate sin and cos into this form when dealing with complex arguments. Do we still have $f_{-j} = f_j^*$ or is this freedom now lost? Use $f(z) = g(x, y) + ih(x, y)$ for $z = x + iy$. However, our interval will always be over the real line so that $f(z) \to f(x) = g(x, 0) + ih(x, 0)$. So show that the following use for complex f with $s = 2\pi x/L$ is fine:

$$
\begin{aligned}
f(x) &= \sum_{j=-N}^{N} f_j \exp(2\pi ijx/L) \\
\Re(f_j) &= \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x\ \Re(f(x)) \exp(-2i\pi jx/L) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}s\ g(s) \exp(-ijs) \\
\Im(f_j) &= \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x\ \Im(f(x)) \exp(-2i\pi jx/L) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}s\ h(s) \exp(-ijs) \\
f_j &= \Re f_j + i\Im f_j = \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x\ f(s) \exp(-2i\pi jx/L)
\end{aligned}
$$

1.3.7. Create plots for a toroidal cross section (i.e., θ constant, or for $\theta = 0$ we also have $Z = 0$ if we are using primitive toroidal coordinates) equivalent to those in Figure 1.4. For a torus, this plot will look like an annulus. To do so, you will need to plot a function like $f(x, y) = R\cos(-n\zeta)$ with $\tan(-\zeta) = y/x$ and R the radius at which we want the perturbation.

1.3.8. Use plotting software to reproduce Figures 1.6 and 1.7. Use (1.3.24) and (1.3.25) to do so.

$$
\begin{aligned}
x &= \sin(\theta)\left[R' + r'\sin\left(m\theta - \frac{nz}{R'}\right)\right] \\
y &= \cos(\theta)\left[R' + r'\sin\left(m\theta - \frac{nz}{R'}\right)\right]
\end{aligned}
$$

1.3.9. Consider the function $f(\theta) = \exp(i\theta)$ and compute the ballooning transform $\widehat{f}$. This should be very easy.

1.3.10. Consider the differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\theta^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}\theta} = \lambda f(\theta)$$

Find f by the usual linear ODE means over the interval θ from 0 to 2π. Then solve this with the ballooning tranformation angle η from $-\infty$ to ∞. What do you notice about using the solution in this form? Do you think it has something to do with our ODE solution form?

1.4. For Section 1.4.

1.4.1. Find an asymptotic expression ($x \to \infty$) for the integral

$$\int_0^\pi \mathrm{d}t \ \frac{\sin(t)}{t} \exp(-xt)$$

Why does this work when Watson's lemma seems to imply $t^{-1/2}$ is a necessary prerequisite? [Hint: What's nice about $\sin(t)$ near $t = 0$?]

1.4.2. Find an asymptotic expression ($x \to \infty$) for the integral

$$\int_0^\infty \mathrm{d}t \ \frac{\exp(-xt)}{1+t}$$

1.4.3. Find an asymptotic expression ($x \to \infty$) for the integral

$$\int_{-\infty}^\infty \mathrm{d}t \ \frac{\exp(-xt^2)}{1+t}$$

1.4.4. Find an asymptotic expression ($x \to \infty$) for the integral

$$\int_0^{\pi/2} \mathrm{d}t \ \exp(-x\sin(t))$$

1.4.5. Find an asymptotic expression ($x \to 0$) for

$$\sqrt{x^4 + 2x^3 + 4x^2}$$

1.4.6. Find an asymptotic expression ($x \to \infty$) for the inverse function of

$$x^2 \exp(x)$$

1.4.7. Find an asymptotic expression ($x \to \infty$) for the inverse function of

$$x + \exp(x)$$

1.4.8. Find the inverse function of

$$x + \exp(x)$$

in terms of the Lambert W function. Does this align with what you found in the previous problem?

1.4.9. Asymptotic matching across a resistive layer is a classic problem tackled by FKR (Furth, Killeen, Rosenbluth)[14]. Work through the asymptotic matching procedure in their paper.

1.5. For Section 1.5.

1.5.1. Consider $f(z) = \exp(z)$ for $z = x + iy$. What is $\frac{\partial f}{\partial \bar{z}}$? Does this satisfy the Cauchy-Riemann equations?

1.5.2. Consider the function $f(z_1, z_2) = \exp(z_1)\sin(z_2)$. Now what are the Wirtinger derivatives $\frac{\partial f}{\partial \bar{z}_j}$ for both? How would you test that z_1 and z_2 satisfy the Cauchy-Riemann equations otherwise [in this case it should be simple since z_1 and z_2 are easily separable]?

1.5.3. Consider the function $f(z_1, z_2) = \exp(z_1 z_2)$. Now what are $\frac{\partial f}{\partial \bar{z}_j}$ for both? How would you test that z_1 and z_2 satisfy the Cauchy-Riemann equations otherwise?

1.5.4. Take $f(\mathbf{x}) = \mathbf{x}\cdot\mathbf{c} + \exp(\mathbf{x}\cdot\mathbf{k})$ and develop the Taylor series for $\mathbf{x}_0$.

1.6. For Section 1.6.

1.6.1. Linearize the Navier-Stokes equation

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = -\nabla p + \boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}} + \mathbf{f}$$

$$\rho = \sum_{j=0}^{\infty}\rho_j\delta^j \qquad \mathbf{V} = \sum_{j=0}^{\infty}\mathbf{V}_j\delta^j \qquad p = \sum_{j=0}^{\infty}p_j\delta^j$$

$$\overleftrightarrow{\mathbf{T}} = \sum_{j=0}^{\infty}\overleftrightarrow{\mathbf{T}}_j\delta^j \qquad \mathbf{f} = \sum_{j=0}^{\infty}\mathbf{f}_j\delta^j$$

Solve for $\mathcal{O}(1)$, $\mathcal{O}(\delta)$ and $\mathcal{O}(\delta^2)$.

1.7. For Section 1.7.

1.7.1. Consider the Van der Pol oscillator

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \mu(1-x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

Perform a perturbation series analysis and then a multiple scale time analysis.

1.7.2. Consider writing the kinetic equation in the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = C$$

Then introduce a collision time τ_c. We then expect C to be an important term on this time scale and so write $G = C/\tau_c$. Suppose there is a periodic motion on a long time scale and give this periodic motion a time period of τ_b, the bounce time. Then let $\epsilon = \tau_b/\tau_c \gg 1$. We non-dimensionalize the above equation via introducing $s = t/\tau_b$ and write

$$\frac{1}{\tau_b}\frac{\mathrm{d}f}{\mathrm{d}s} = G/\tau_c \Rightarrow \frac{\mathrm{d}f}{\mathrm{d}s} = \epsilon G$$

We can then consider f and C to be in a non-dimensional form (for example, divide by f_0 on both sides, if you truly desire this). Perform multiple scale analysis with $\tau_0 = s$ and $\tau_1 = \epsilon s$on

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \epsilon G$$

to fully solve at zeroth order in ϵ. You will use the idea that there can be no secular terms so the solvability criterion will be

$$\int_0^{\tau_b} \mathrm{d}\tau_0 \, \frac{\mathrm{d}f}{\mathrm{d}\tau_1} = \int_0^{\tau_b} \mathrm{d}\tau_0 \; G$$

This enforces that there is no growing in τ_0 (time) solution. Rewrite the solvability criterion in the original time t.

1.8. For Section 1.8.

1.8.1. Derive (1.8.59) below for a functional S, an integral of $L(\mathtt{f}, \ldots, \mathtt{f}^{(j)})$

$$\frac{\partial L}{\partial \mathtt{f}} + \sum_i \lambda_i \frac{\partial G_i}{\partial \mathtt{f}} + \sum_{j>0} \left[(-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \frac{\partial L}{\partial \mathtt{f}^{(j)}} \right] = 0 \tag{1.8.59}$$

1.8.2. Find the variational derivative $\frac{\delta F}{\delta f}$ for $F[f]$ below. If there is no $\frac{\delta F}{\delta f}$, say so and compute δF.

$$\begin{aligned}
F[f] &= [f(x)]^2 \\
F[f] &= [\sin(f(x))]^2 \\
F[f] &= \int_0^x \mathrm{d}t \; t[f(t)]^2 \\
F[f] &= f(x) f'(x) \\
F[f] &= \int_0^1 \mathrm{d}t \; [f'(t)]^2
\end{aligned}$$

1.8.3. Use the Rayleigh-Ritz method discussed near (1.8.85) for

$$S[\psi] = \frac{\int \mathrm{d}r \; \psi^* H \psi}{\int \mathrm{d}r \; \psi^* \psi} \tag{1.18.16}$$

with

$$H = \frac{-\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} + V(r) \tag{1.18.17}$$

$$V = -\frac{e^2}{4\pi\epsilon_0 r} \tag{1.18.18}$$

Try it first with a polynomial r^α. Then try $\exp(r)$. For the ground state energy the actual answer is

$$E_1 = -\frac{\mu e^4}{32\pi^2 \epsilon_0^2 \hbar^2} \tag{1.18.19}$$

1.9. For Section 1.9.

1.9.1. Show that Newton's second law is given by the Lagrangian

$$\mathcal{L} = \frac{\dot{x}^2}{2m} - V(x)$$

Calculate Hamilton's equations for this, and show that they also lead to Newton's second law. You have shown that all three formulations (Newton's, Lagrange's, and Hamilton's) are equivalent for this problem statement.

1.9.2. For a harmonic oscillator $\mathcal{H} = p^2 + x^2$ (In appropriately normalized coordinates), calculate Hamilton's principal function S. Also show how one can use the generating function types on this particular equation to form all sorts of canonical variables.

1.10. For Section 1.10.

1.10.1. Consider $\mathbf{B} = B_Z\hat{\mathbf{Z}} + B_\zeta\hat{\boldsymbol{\zeta}}$ with B_Z and B_ζ constant. Use the magnetic field line equations to see the field line trajectories. Then find the appropriate magnetic toroidal and poloidal fluxes. Here the toroidal flux is across a constant Z plane and the poloidal flux is across a constant ζ plane. Show that Hamilton's equatons are satisfied.

1.11. For Section 1.11.

1.11.1. Choose a $\kappa(t)$ and $\tau(t)$ for $t > 0$ and plot the trajectory thus created.

1.11.2. Choose a helical parameterization

$$\begin{aligned} x &= \cos(t) \\ y &= \sin(t) \\ z &= \alpha t \end{aligned}$$

and find κ and τ. Do $\kappa + \tau$ sum to something?

1.11.3. Choose some other parameterization, say

$$\begin{aligned} x &= \cos(t) \\ y &= \sin(t) \\ z &= \alpha t^3 \end{aligned}$$

and find $\kappa(t)$ and $\tau(t)$. Do $\kappa + \tau$ now sum to anything? Or are they independent?

1.12. For Section 1.13.

1.12.1. Use contour integration to evaluate

$$\int_0^\infty \mathrm{d}x \; \frac{1}{x^2 + 1}$$

1.12.2. Use contour integration to evaluate

$$\int_0^\infty \mathrm{d}x \; \frac{\exp(itx)}{x^2 + 1}$$

1.12.3. Use contour integration to evaluate

$$\int_{-\pi}^{\pi} \mathrm{d}x \; \frac{1}{a + b(\sin(t))^2}$$

Use the substitution $z = \exp(it)$ so that $\sin(t) = \frac{z-z^*}{2i}$ with $z^* = \exp(-it) = 1/z$.

1.12.4. Consider the function $\log z$ for $z = r\exp(i\theta)$. You will remember this traditionally has a branch cut on the negative real axis, where on the positive imaginary side it has the value $\log|r| + i\theta$ and on the negative imaginary side it has the value $\log|r| - i\theta$. If we integrate over the branch cut, we will have problems [so never integrate over a region with a branch cut. Get arbitrarily close, but not on it]. What is happening is we are switching branches as we go around in θ when we do this, and so we have a discontinuity. If we were to stay on the same branch then we would have a multi-valued but continuous function. What other common functions have branch cuts?

1.12.5. Consider the function $\sqrt{z}\sqrt{z-1}$. What is it's value for $z = 1-i\epsilon$? How about $z = 1+i\epsilon$? Suppose you want there to be only one value on the real axis. One could "glue" the solutions together there. What region makes sense as the branch point then? The problem above is simply that $z^{1/2} = \pm|\sqrt{z}|$ and similarly for $(z-1)^{1/2} = \pm|\sqrt{z-1}|$. If we choose inconsistent values on the $\pm$ we have multiple answers. If we instead force $+$ on both, then we have a branch cut there.

1.12.6. Use contour integration to evaluate

$$\int_0^{\infty} \mathrm{d}x \; \frac{x^\lambda}{1+x}$$

for $-1 < \lambda < 0$. This requires choosing a branch cut for x^λ. The branch cut is where we will have the function undefined, and so we need to create a contour integral path that can approach but not be on the branch cut. The "keyhole contour" along the positive real axis (a small near circle of radius ϵ around the origin, then a straight line off the real axis [above it in th ecomplex plane] with an arc going from $+\infty$ nearly 2π around in the complex plane and coming back a small distance off of the real line from $+\infty$ back to the nearly complete circle near the origin) is convenient. Remember you can choose where you want your branch cut to be. For this case choosing the real axis is nice.

1.13. For Section 1.14.

1.13.1. Try analytically continuing

$$f(z) = \sum_{j=0}^{\infty} j z^j \tag{1.18.20}$$

which comes from $1/(z-1)^2$ expanded around $z = 0$. Analytically continue it from $z_1 = -0.5$. You should get

$$f(z_1) = \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^{2+j} (1+n) \left(\frac{1}{2} + z_1\right)^j$$

1.13.2. I said it could be shown that

$$\sum_{k=0}^{\infty} C_n^k z_0^k = \frac{z_0^n}{(1-z_0)^{n+1}}$$

Show it's true for $n = 0$, and then use induction to prove it generally.

References

[1] M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Applied mathematics series. U.S. Government Printing Office, 1970. URL: https://books.google.com/books?id=3X55fTnM848C.

[2] C.M. Bender and S.A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*. Springer New York, 2013. ISBN: 9781475730692. URL: https://books.google.com/books?id=xz0mBQAAQBAJ.

[3] K. Bird and M.J. Sherwin. *American Prometheus: The Triumph and Tragedy of J. Robert Oppenheimer*. Vintage. Knopf Doubleday Publishing Group, 2007. ISBN: 9780307424730. URL: https://books.google.com/books?id=jfSn2RJZI9EC.

[4] Allen H Boozer. "Plasma equilibrium with rational magnetic surfaces". In: *The Physics of Fluids* 24.11 (1981), pp. 1999–2003.

[5] J.P. Boyd. *Chebyshev and Fourier Spectral Methods: Second Revised Edition*. Dover Books on Mathematics. Dover Publications, 2013. ISBN: 9780486141923. URL: https://books.google.com/books?id=b4TCAgAAQBAJ.

[6] Hazel Brickhill and Leon Horsten. "Popper functions, lexicographical probability, and non-Archimedean probability". In: *arXiv preprint arXiv:1608.02850* (2016).

[7] N.G. de Bruijn. *Asymptotic Methods in Analysis*. Bibliotheca mathematica. Dover Publications, 1981. ISBN: 9780486642215. URL: https://books.google.com/books?id=0qj9AgAAQBAJ.

[8] JW Connor, RJ Hastie, and JB Taylor. "Shear, periodicity, and plasma ballooning modes". In: *Physical Review Letters* 40.6 (1978), p. 396.

[9] W.D. D'haeseleer. *Flux coordinates and magnetic field structure: a guide to a fundamental tool of plasma structure*. Springer series in computational physics. Springer-Verlag, 1991. ISBN: 9783540524199. URL: https://books.google.com/books?id=X3x5AAAAIAAJ.

[10] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.26 of 2020-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. URL: http://dlmf.nist.gov/.

[11] S. Drobot et al. *Mathematical Models in Physical Sciences: Proceedings*. Prentice-Hall, 1963. URL: https://books.google.com/books?id=OLmxAAAAIAAJ.

[12] MR Flannery. "The enigma of nonholonomic constraints". In: *American Journal of Physics* 73.3 (2005), pp. 265–272.

[13] Mattias Flygare. *Holonomic versus nonholonomic constraints*. 2012. URL: http://www.diva-portal.org/smash/record.jsf?pid=diva2%3A528335.

[14] Harold P Furth, John Killeen, and Marshall N Rosenbluth. "Finite-resistivity instabilities of a sheet pinch". In: *The physics of Fluids* 6.4 (1963), pp. 459–484.

[15] Andrew Gelman and Deborah Nolan. "You can load a die, but you can't bias a coin". In: *The American Statistician* 56.4 (2002), pp. 308–311.

[16] A. Gottlieb. *The Dream of Reason: A History of Western Philosophy from the Greeks to the Renaissance*. W.W. Norton, 2016. ISBN: 9780393352986. URL: https://books.google.com/books?id=PVskjgEACAAJ.

[17] D.J. Griffiths. *Introduction to Electrodynamics*. Pearson international edition. Prentice Hall, 1999. ISBN: 9780139199608. URL: https://books.google.com/books?id=x0akQgAACAAJ.

[18] Alan Hájek et al. "Conditional probability is the very guide of life". In: *Probability is the very guide of life: The philosophical uses of chance* (2003), pp. 183–203.

[19] Alan Hájek. "Interpretations of Probability". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2019. Metaphysics Research Lab, Stanford University, 2019.

[20] Alan Hájek. "Symmetry Is the Very Guide of Life". In: *Accessed online at http://www. socsc. smu. edu. sg/events/Paper/hajek2. pdf* (2007), pp. 7–23.

[21] P. Helander and D.J. Sigmar. *Collisional Transport in Magnetized Plasmas*. Cambridge Monographs on Plasma Physics. Cambridge University Press, 2005. ISBN: 9780521020985. URL: https://books.google.com/books?id=nm-V01E0H2MC.

[22] DW Kerst. "The influence of errors on plasma-confining magnetic fields". In: *Journal of Nuclear Energy. Part C, Plasma Physics, Accelerators, Thermonuclear Research* 4.4 (1962), p. 253.

[23] D. Lovelock and H. Rund. *Tensors, Differential Forms, and Variational Principles*. Dover Books on Mathematics. Dover Publications, 2012. ISBN: 9780486131986. URL: https://books.google.com/books?id=uUi8AQAAQBAJ.

[24] J.S. Mill. *On Liberty*. Ticknor and Fields, 1863. URL: https://books.google.com/books?id=7DcoAAAAYAAJ.

[25] PJ Morrison. "Magnetic field lines, Hamiltonian dynamics, and nontwist systems". In: *Physics of Plasmas* 7.6 (2000), pp. 2279–2289.

[26] G. Polya and J.H. Conway. *How to Solve It: A New Aspect of Mathematical Method*. Penguin mathematics. Princeton University Press, 2004. ISBN: 9780691119663. URL: https://books.google.com/books?id=z%5C_hsbu9kyQQC.

[27] P.R. Rosenbaum. *Observation and Experiment*. Harvard University Press, 2017. ISBN: 9780674975576. URL: https://books.google.com/books?id=NnOuDwAAQBAJ.

[28] W.W. Sawyer. *A First Look at Numerical Functional Analysis*. Dover Books on Mathematics. Dover Publications, 2010. ISBN: 9780486478821. URL: https://books.google.com/books?id=gZeRDQAAQBAJ.

[29] W.W. Sawyer. *Vision in Elementary Mathematics*. Dover Books on Mathematics. Dover Publications, 2012. ISBN: 9780486143620. URL: https://books.google.com/books?id=LaK8AQAAQBAJ.

[30] A. Tocqueville, G. Bevan, and I. Kramnick. *Democracy in America: And Two Essays on America*. Penguin classics. Penguin Books Limited, 2003. ISBN: 9780140447606. URL: https://books.google.com/books?id=af3tmwYlgBEC.

[31] K. Vogtmann, A. Weinstein, and V.I. Arnol'd. *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics. Springer New York, 2013. ISBN: 9781475720631. URL: https://books.google.com/books?id=5OQlBQAAQBAJ.

Chapter 2

Plasma Physics

> *People say: "what good does it do to point out the obvious?" A great deal of good; for we sometimes know facts without paying attention to them.*
>
> — Lucius Annaeus Seneca

This chapter serves as fairly basic introduction to plasma physics theory. It only touches on a couple of topics, but should give you a good grounding for magnetic confinement problems. It begins with explaining what plasmas are so that we know what it is we are studying. It then details the magnetic field structure and its implications for confining a plasma. This includes a fairly robust discussion of straight field line coordinates (flux coordinates). I briefly list some of the major research areas for magnetic confinement devices, and then I explain how we can create fluid theories of plasmas from the underlying kinetic behavior of particles in electromagnetic fields. After deriving one such theory, magnetohydrodynamics (MHD), I talk about the various incarnations of MHD that are typically used. I go into a bit more detail on Ideal MHD, where we consider the plasma a perfect conductor. Plasma waves and instabilities is an enormous topic, and I only touch on it through an MHD prism. The technique is the same, though, whatever your underlying equations. This naturally leads to questions of discontinuous interfaces. Finally, I take a somewhat unorthodox approach to particle drifts. It seems to me that particle drifts are often taught without explaining their significance, and so I try to give a broader discussion that explains what particle drifts are actually good for in theory.

2.1 What Plasmas Are

> *As will be emphasized on more than one occasion, children often fail to solve problems because they cannot understand what the problems are. Naturally, if you do not know what a question means, you cannot be expected to give the correct answer.*
>
> — W. W. Sawyer[17, p. 6]

With some of the mathematical subtleties behind us, we can finally get into what the plasma physics field is about. To do so, we must first answer a question that dogs any field: What is it that we are studying? Since we are doing research, to give a too rigorous answer may exclude research that we would really rather consider a part of plasma physics, while giving too broad of criteria makes it mysterious what is special about plasma physics as opposed to just physics. I will

give you an answer as to what a plasma is, but you should not take it as some given-down truth that settles things. It is simply one way to come up with a quantitative answer to help you get a sense of what really is a plasma and what really is not. Determining if something is a plasma is more a question of how the phenomena can be explained rather than whether it fulfills some exacting set of properties that corresponds to a plasma.

In words, plasmas are usually defined as ionized gases. We have given the components of a gas enough energy that the nuclei and the electrons are no longer stuck to each other. This leads to all sorts of fascinating effects because now the components are affected by electromagnetic fields much more strongly, which leads to collective behavior of particles very different from that of gases. From now on I will only refer to "neutral plasmas", because non-neutral plasmas[1] would require a bunch of caveats to all of my statements. So if we look at the particles very closely, we don't care about the constituents themselves. That is, if we are looking too closely at the particles, we won't see collective behavior, we will just see particles moving around due to Maxwell's and Newton's laws (ignoring quantum effects). Calculating the environment for the single particle may be difficult on a practical level, but it is not difficult on a conceptual level. Thus we need a length scale that is larger than single particles in order to say that we are studying a "plasma" rather than just a bunch of charged particles. The sort of length scale we are looking for will describe at about what scale single particle dynamics are substantially altered by the presence of many other particles nearby.

2.1.1 Screening Lengths

To see this, consider the electrostatic potential ϕ for a volume of particles and a single test particle of charge q_T at the origin. It will satisfy Poisson's equation

$$-\nabla^2\phi = \frac{\rho_q}{\epsilon_0} + \frac{q_T\delta(\mathbf{x})}{\epsilon_0} \tag{2.1.1}$$

where ρ_q is the net charge of all particles in a unit volume. For a plasma, we can separate the charged particles by species charge q_s and number density n_s via

$$\rho_q = \sum_s q_s n_s \tag{2.1.2}$$

Now, for normal plasmas, we expect quasineutrality to hold. This means that $\sum_s q_s n_s \simeq 0$. This statement *should not* be read as the charge being exactly zero, but is really a statement about the importance of the electric field $\mathbf{E}$ on the particles. We are saying that $|q\mathbf{E}| \ll |\mathbf{F}_{\text{other}}|$; the force on a particle is not dominated by the electric field, and in fact the force due to the electric field is sub-dominant to some other forces. We now need to think about the distribution of the number density. From statistical mechanics, we would expect a Boltzmann distribution near equilibrium, where the number density is stratified via the electrostatic energy divided by the thermal energy. We will consider an equilibrium so that the temperature for each species T_s is constant. Then $n_s = n_{0s}\exp(-q_s\phi/(k_B T_s))$.[2] The important thing is that $q_s\phi/(k_B T_s) \ll 1$, so that the thermal energy is dominant over the electrostatic potential energy (note how this is one way of saying that

[1] Non-neutral plasmas are plasmas of a single species, such as electrons, and so do not satisfy the quasineutrality conditions of usual plasmas. Non-neutral plasmas are fascinating, but beyond the scope of this book focusing on fusion relevant plasmas.

[2] This is an equation using all SI units with k_B the Boltzmann constant.

electric forces are not dominant). Then we can use the handy Taylor series approximation around this solution

$$-\epsilon_0 \nabla^2\phi = q_T\delta(\mathbf{x}) + \sum_s q_s n_s = \sum_s q_s n_{0s} \exp\left(\frac{-q_s\phi}{k_B T_s}\right) \approx \sum_s q_s n_{0s}\left[1 - \frac{q_s\phi}{k_B T_s} + \mathcal{O}\left(\left\{\frac{q_s\phi}{k_B T_s}\right\}^2\right)\right] \tag{2.1.3}$$

$$-\nabla^2\phi \approx \frac{q_T}{\epsilon_0}\delta(\mathbf{x}) + \cancel{\sum_s \frac{n_{0s}q_s}{\epsilon_0}} - \sum_s q_s n_{s0}\frac{q_s\phi}{\epsilon_0 k_B T_s} = q_T\delta(\mathbf{x}) - \phi\underbrace{\left(\sum_s \frac{n_{0s}q_s^2}{\epsilon_0 k_B T_s}\right)}_{\equiv\lambda_D^{-2}} \tag{2.1.4}$$

Notice that the term in parentheses in (2.1.4) is constant with respect to the potential variation. Notice that it also has units of inverse meters squared and so we define the Debye length λ_D as one over the square root of the term in parentheses. If we now consider only radial variation, consistent with a Boltzmann distribution, we can make some progress. We can consider a spherical distribution so that in physicist's spherical coordinates (see Appendix C.4) (r, θ, φ) we have

$$\nabla^2\phi = \boldsymbol{\nabla}\cdot\nabla\phi = \boldsymbol{\nabla}\cdot\left(\frac{\partial\phi}{\partial r}\nabla r + \cancel{\frac{\partial\phi}{\partial\theta}\nabla\theta} + \cancel{\frac{\partial\phi}{\partial\varphi}\nabla\varphi}\right) = \nabla\left(\frac{\partial\phi}{\partial r}\right)\cdot\nabla r + \frac{\partial\phi}{\partial r}\boldsymbol{\nabla}\cdot\nabla r \tag{2.1.5}$$

The last term uses that $\mathbf{r}$ is the same as the position vector $\mathbf{x}$ and so

$$\begin{aligned}\boldsymbol{\nabla}\cdot\nabla r = \boldsymbol{\nabla}\cdot\frac{\mathbf{r}}{r} &= \nabla\left(\frac{1}{r}\right)\cdot\mathbf{r} + \frac{1}{r}\boldsymbol{\nabla}\cdot\mathbf{r} \\ &= \frac{-\mathbf{r}}{r^3}\cdot\mathbf{r} + \frac{3}{r} = \frac{-1}{r} + \frac{3}{r} = \frac{2}{r}\end{aligned} \tag{2.1.6}$$

whereas the first term satisfies

$$\nabla\left(\frac{\partial\phi}{\partial r}\right) = \frac{\partial\,\nabla\phi}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial r}\nabla r\right) = \frac{\partial^2\phi}{\partial r^2}\nabla r + \frac{\partial\phi}{\partial r}\frac{\partial\,\nabla r}{\partial r} = \frac{\partial^2\phi}{\partial r^2}\nabla r + \frac{\partial\phi}{\partial r}\nabla\cancel{\left(\frac{\partial r}{\partial r}\right)} \tag{2.1.7}$$

Thus,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2}\nabla r\cdot\nabla r + \frac{2}{r}\frac{\partial\phi}{\partial r} = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} \tag{2.1.8}$$

and we have as our differential equation

$$-\frac{\partial^2\phi}{\partial r^2} - \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{\phi}{\lambda_D^2} = \frac{q_T\delta(r)}{\epsilon_0} \tag{2.1.9}$$

$$-\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2\phi\right) + \frac{\phi}{\lambda_D^2} = \frac{q_T\delta(r)}{\epsilon_0} \tag{2.1.10}$$

One can find the solution is (try using $\Phi = r\phi$ and solve the resulting equation for Φ)

$$\phi = \frac{q_T \exp\left(\frac{-r}{\lambda_D}\right)}{4\pi\epsilon_0 r} \tag{2.1.11}$$

Now we have shown that there is a screening distance of

$$\lambda_D^{-1} = \sqrt{\sum_s \frac{n_{0s} q_s^2}{\epsilon_0 k_B T_s}} \tag{2.1.12}$$

called the Debye length over which the plasma will actually screen out the charge. This is exactly what we are looking for. If we look too closely we will just see the bare charge, but if we look at a scale $L > \lambda_D$, then we will be looking at the effects that are due to the quasineutral plasma. Thus, the first criterion is that for L a characteristic length of the system we are looking at, we want $\lambda_D \ll L$. Note that in many cases people talk of the Debye length, but mean the electron Debye length

$$\lambda_{D_e} = \sqrt{\frac{\epsilon_0 k_B T_e}{n_{0e} e^2}}$$

with e the elementary charge. When there are only two species (ions i and electrons e), and the temperatures $T_i = T_e \equiv T$ are the same, then (using quasineutrality the ion charge must be $+e$ and so $n_{0i} = n_{0e} \equiv n_0/2$)

$$\lambda_D = \sqrt{\frac{1}{\frac{n_{0e}e^2}{\epsilon_0 k_B T} + \frac{n_{0i}e^2}{\epsilon_0 k_B T}}} = \sqrt{\frac{\epsilon_0 k_B T}{(n_{0i} + n_{0e})e^2}} = \sqrt{\frac{2n_{0e}e^2}{\epsilon_0 k_B T}} = \sqrt{2}\lambda_{D_e} \tag{2.1.13}$$

and the $\sqrt{2}$ factor is a fairly negligible correction for most calculations.

2.1.2 Many Particles in a Debye Sphere

The next criterion seems like a similar way of saying the same thing, but we again want there to be something more than just particle dynamics. Thus we want a lot of particles in whatever region we are considering. For the screening process to even make sense, we need there to be a sea of ions and electrons around our test charge so that we can reasonably call it a plasma. So if we take a sphere with the Debye length as its radius, and we call the number of particles in this Debye sphere N, we want

$$\frac{N}{\frac{4}{3}\pi\lambda_D^3} \gg 1 \,\text{particle/volume} \tag{2.1.14}$$

or in other words, we want the number density to be much larger than an inverse Debye sphere. Thus, this is often written for number density n as

$$\frac{4}{3}\pi n \lambda_D^3 \gg 1 \tag{2.1.15}$$

Because these are rough approximations, the $4\pi/3$ is often dropped. So then $\Lambda \equiv 4\pi n \lambda_D^3$ is often defined as the plasma parameter. Note that the factor of 4π is essentially for convention, and not too much weight should be put in the specific pre-factor. Unfortunately, the inverse of this is sometimes also called the plasma parameter but denoted g

$$g \equiv \frac{1}{n\lambda_D^3} \ll 1 \tag{2.1.16}$$

In some literature the Debye number $N_D \equiv \frac{4}{3}\pi n\lambda_D^3$ is used instead of the plasma parameter. It would not surprise me if this was partially to offset the confusion from the term plasma parameter. One final comment on this terminology is that the plasma coupling factor Γ is often defined. This is a ratio of the electrostatic energy (for a radius of approximately $n^{-1/3}$) to thermal energy

$$\Gamma \equiv \frac{E_C}{E_T} \approx \frac{\frac{q^2}{4\pi\epsilon_0 n^{-1/3}}}{k_B T} = \frac{q^2 n^{1/3}}{4\pi\epsilon_0 k_B T} = \frac{1}{4\pi n^{2/3}\lambda_D^2} \propto \left(\frac{1}{n\lambda_D^3}\right)^{2/3} \propto \Lambda^{-2/3} \tag{2.1.17}$$

and so the plasma coupling factor and the plasma parameter are describing similar things.

In addition, we can see that this approximation is necessary via our our Taylor series expansion (2.1.4) from earlier. We needed $\frac{\phi}{\epsilon_0 k_B T} \ll 1$ and so using an electron as our test particle we'd find

$$\frac{-e\phi}{\epsilon_0 k_B T} \approx \frac{e^2 \exp\left(\frac{-r}{\lambda_D}\right)}{4\pi\epsilon_0 k_B T r} = \frac{n_{0e}e^2}{\epsilon_0 k_B T}\frac{\exp(-r/\lambda_D)}{4\pi n_{0e} r} = \frac{1}{\lambda_{D_e}^2}\frac{\exp(-r/\lambda_D)}{4\pi n_{0e} r} \tag{2.1.18}$$

we can use that $n_{0e}^{1/3}$ functions as a scale factor on r and we could then write

$$\frac{-e\phi}{\epsilon_0 k_B T} \approx \frac{\exp\left(\frac{-r n_{0e}^{1/3}}{(n_{0e}\lambda_{D_e}^3)^{1/3}}\right)}{4\pi(n_{0e}\lambda_{D_e})^{2/3}(n_{0e}^{1/3} r)} \tag{2.1.19}$$

If we expect $n_{0e}^{1/3} r \sim 1$ then the above expression simplifies to

$$\frac{-e\phi}{\epsilon_0 k_B T} \sim \frac{\exp\left(\frac{-1}{(n_{0e}\lambda_{D_e}^3)^{1/3}}\right)}{4\pi(n_{0e}\lambda_{D_e})^{2/3}} \tag{2.1.20}$$

From this it is again easy to see that $n\lambda_{D_e}^3 \gg 1$ will ensure that $e\phi/(\epsilon_0 k_B T)$ is small.

2.1.3 Two Other Criteria

There are two further considerations that are used when talking of plasmas. First, we will not consider scales where quantum effects on particle dynamics are important. That is we cannot allow our scale to be smaller than an atom, so we need $n_0^{-1/3} \gg a_0$ where a_0 is the Bohr radius, given in SI by

$$a_0 \equiv \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = 5.3 \times 10^{-11}\,\text{m} \tag{2.1.21}$$

Thus, we desire

$$L \gg \lambda_D \gg n_0^{-1/3} \gg a_0 \tag{2.1.22}$$

The final consideration is for that of neutrals. We do not want neutrals dominating the behavior by thermally bouncing into our ionized gas and disrupting the plasma dynamics with atomic processes. For a simple plasma, we can see what a typical frequency for plasma motion is and so determine a frequency criterion. Consider a plasma with uniform density $n = n_{0e}/2 = n_{0i}/2$. We want to look at the force on a small portion of the plasma. For simplicity, consider a very thin slab, and say the

positive particles are shifted out of this thin slab a distance Δx. So then on the right side of the slab we will have a surface charge density $\sigma = ne\Delta x$. On the other side of the slab, we will have a surface charge density of $-\sigma$ because we initially had no net charge. Inside the slab we would have an electric field of σ/ϵ_0 directed towards the left due to a uniformly charged plane. Using Newton's 2nd law with the Coulomb force $\mathbf{F}_q = q\mathbf{E}$ we'd then find for a single particle of mass m and charge e in the slab that

$$m\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\Delta x\right) = -\frac{ne^2\Delta x}{\epsilon_0} \tag{2.1.23}$$

which for n constant, is an equation for Δx

$$\frac{\mathrm{d}^2\Delta x}{\mathrm{d}x^2} = -\frac{\overbrace{ne^2}^{\omega_p^2}}{m\epsilon_0}\Delta x \tag{2.1.24}$$

$$\Delta x = A\exp(-i\omega_p x) + B\exp(i\omega_p x) \tag{2.1.25}$$

or simple harmonic motion with the plasma frequency of species s defined by

$$\omega_{ps} = \sqrt{\frac{n_s q_s^2}{m_s\epsilon_0}} \tag{2.1.26}$$

So these are the oscillations that we expect from a plasma. If something else dominates this process, we expect that the system will no longer be a plasma. So if we use this as a characteristic frequency for plasma motion, we want the plasma species-neutral collision frequency ν_{sn} to be much less frequent.

Then the final criterion is

$$\frac{\omega_{ps}}{\nu_{sn}} \equiv \omega_{ps}\tau_{sn} \gg 1 \tag{2.1.27}$$

2.1.4 Other Considerations

The four criteria are listed in Table 2.1. As stated before, these should really only be thought of as guiding principles. People will argue about what a plasma is by applying the criteria, but the only real answer to the question is if the equations commonly used in plasma physics describe the

Criterion Number	Approximation
1	$L \gg \lambda_D$
2	$n\lambda_D^3 \gg 1$
3	$n^{-1/3} \gg a_0$
4	$\omega_{ps}\tau_{sn} \gg 1$

Table 2.1: The four plasma criteria. Here L is characteristic length for the system we are looking at, $\lambda_D = \left(\sum_s [n_{0s}q_s^2]/[\epsilon_0 k_B T_s]\right)^{-1/2}$ is the Debye length, n is the number density of the plasma species, a_0 is the Bohr radius, $\omega_{ps} = \sqrt{[n_{0s}q_s^2][m\epsilon_0]}$ is the plasma frequency of species s and $\tau_{sn} = n_n\sigma_s^{s|n}\sqrt{k_B T_s/m_s}$ is the species-neutral collision time. $\sigma_s^{s|n}$ is the cross section for a collision and n_n is the neutral density.

physical behavior. If the physical behavior is described, then you may as well call it a plasma for the purposes of that application. If you find that equations used to model plasmas do not apply then it is not a plasma.

Finally, I want to emphasize the limitations of the criteria again. These do not apply to non-neutral plasmas; they do not apply to strongly coupled plasmas.[3] I am considering only a certain type of plasma that is common in the universe[4] and explaining when we can expect our models to accurately describe the physical behavior. This is ultimately what we care about, and so no single definition for a plasma should have too much emphasis. Otherwise we get trapped in word games rather than whether we can predict or explain what is going to happen.

This may appear to be tautological, in that only things that are described by plasma models are considered plasmas, but there is not too much to worry about. If you wish to be more rigorous, then you start with what the particles[5] are doing and you can come up with approximations that divide the world into plasmas and non-plasmas. There is a degree of arbitrariness in the exact criteria, but it is usually clear if something is a borderline case or not. If we consider a boundary in some parameter space where physics is well-described by plasma models, we will consider scenarios inside that boundary as having plasmas in them. These are situations when our small parameters are actually small for the problem at hand. If it is not a borderline case, then the model will be broken [outside the boundary], or it will work excellently [inside the boundary]. Near the that boundary edge, the plasma model will only partially work, or work for some behavior but not others. If you want to call phenomena of interest in the boundary plasma behavior depends on what you consider to be important to model correctly. This is simply accepting that our coarse-grained categories are useful approximations, not something that nature necessarily respects. Figure 2.1 shows a diagram for a plasma model space. There are regions where it simply is not clear if the model accurately describes the physics of the situation. If our plasma model does not accurately describe the physics, then whatever we are studying is not a plasma for those circumstances. Perhaps a different plasma physics model will let the situation be called a plasma.[6]

[3] Quark-gluon plasmas are one such plasma. There the $\Lambda \gg 1$, $\Gamma \ll 1$ criterion is relaxed to allow $\Lambda \lesssim 1$, $\Gamma \gtrsim 1$.

[4] And that is applicable to magnetic confinement fusion.

[5] If you are a true stickler, then you could go down to the fields used in the field theory of the Standard Model.

[6] For example, a plasma may not act like an electrically conducting fluid and so is not an MHD plasma. But it may be easy to represent with plasma kinetic theory.

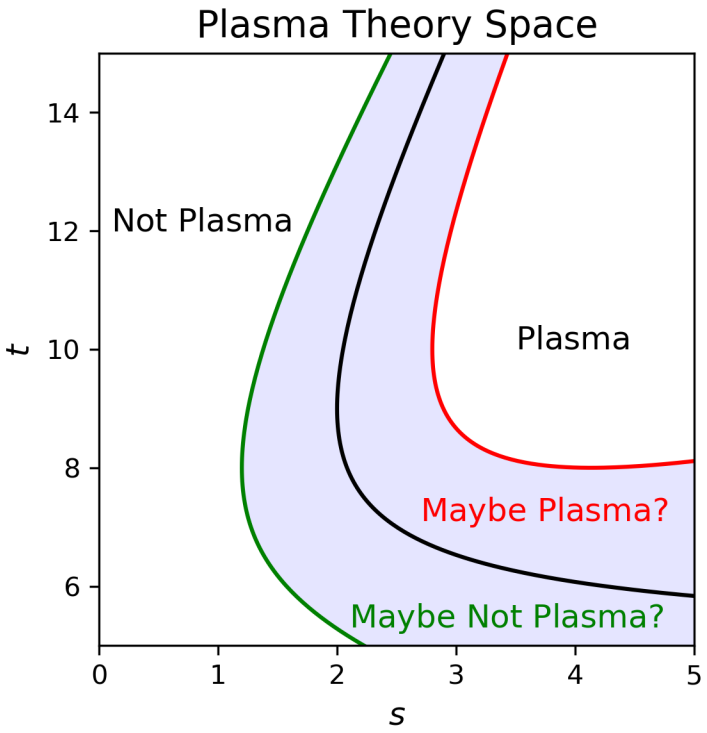

Figure 2.1: Consider a plasma theory that uses parameters s and t. This theory has only a certain regime of validity, so that if s and t are particular values, it no longer works, and so we would question whether the system at those parameters are a plasma. This is a fuzzy question, as is indicated by the large shaded region where it's not clear if the object is a plasma because the model isn't fully reliable there. Even here, the crisp clean lines demarcating the shaded region are more clear than it is in reality.

2.2 Magnetic Field Topology

> *If you have trouble thinking of a new project, take your old project and just add a magnetic field.*
>
> — Unknown origin, attributed to some Russian physics group

When considering how to analyze plasmas, one of the most important considerations is whether magnetic fields are important for the analysis of the plasma configuration. This is usually measured by a term called the magnetization parameter δ_m, and sometimes by another parameter δ_c. These are given by

$$\delta_m \equiv \frac{\rho}{L} \tag{2.2.1}$$

$$\delta_c \equiv \frac{\nu}{\Omega} \tag{2.2.2}$$

Here $\Omega = eB/m$ is the gyrofrequency[7] so ρ is a characteristic Larmor radius,[8] v is a characteristic velocity, L is a characteristic length, and ν is the collision frequency.[9] The ρ/L parameter is more commonly used for approximations, though both need to be small, generally speaking, for accurate results. For a plasma to be magnetized, we desire the particle's orbit to be significantly altered by the presence of the magnetic field, which means that ρ, the radius of a gyrorbit should be smaller than a characteristic length. That is, we require $\rho/L \ll 1$ for a magnetized plasma. If $\rho \gtrsim L$ then the particle has barely gone through the gyroorbit over the characteristic length scale L and so its effects on particle trajectories (and hence the plasma at the scale we care about) will be minimal because the particle is hardly gyrating. The other magnetization parameter δ_c needs $\delta_c \ll 1$, as well for it to be a magnetized plasma. This is a way of stating that collisions are not so overly dominant that the particles go through a couple of gyroorbits before they undergo a collision and so the magnetic field is affecting the trajectories more than simple collisions. Note that δ_c and δ_L are independent of each other. In most cases of interest, however, when one parameter is small, the other is as well.

Once these conditions are met, then the structure or topology of the magnetic field will be an important consideration in the dynamics of the plasma.

2.2.1 Flux Coordinates

If we have a magnetized plasma and it is possible to form nested flux surfaces, then there are a couple of coordinate systems that obey properties that are incredibly useful. But first, what is a flux surface? When talking about the Hamiltonian nature of magnetic fields (see Section 1.10) we mentioned toroidal fluxes and poloidal fluxes and this will lead into what a flux surface is. First, let's look at what a flux is. A flux for a given vector quantity $\mathbf{g}$, which I will denote $\Phi_{\mathbf{g}}$, over a given surface S_f with unit normal $\hat{\mathbf{n}}$ is given by

$$\Phi_{\mathbf{g}} = \iint_{S_f} \mathrm{d}S\ \hat{\mathbf{n}} \cdot \mathbf{g} \tag{2.2.3}$$

[7] The gyrofrequency is often called the Larmor frequency or the cyclotron frequency.

[8] Similarly, the Larmor radius is often called the gyroradius, cyclotron radius, or radius of gyration.

[9] Do not confuse this collision frequency with the neutral collision frequency. This is the frequency of collisions between plasma species.

Then a flux surface is a surface with normal $\hat{\mathbf{n}}$ such that $\mathbf{g}\cdot\hat{\mathbf{n}} = 0$. Thus, over a flux surface the corresponding flux is automatically zero. Because magnetic flux is the most common type I will omit the $\mathbf{B}$ and simply write Φ for the flux. Often ψ is used for a flux or for a normalized flux $\psi = \Phi/(2\pi)$, and the surface is often chosen to have a disk-like structure. The poloidal or toroidal designation for a flux tells us the surface to consider. A poloidal flux is the flux over a surface at a constant poloidal angle and a toroidal flux is a flux over a constant toroidal angle. They are often written as ψ_p, Φ_p and ψ_t, Φ_t for poloidal flux and toroidal flux, respectively. Whenever you see a flux, you should realize that a factor of 2π could be lurking in the definition somewhere.

2.2.2 Flux Integrals and Surfaces

A flux surface is then a surface where the flux Φ is constant along that surface (since $\mathbf{B}\cdot\hat{\mathbf{n}} = 0$ on this surface by definition). From before, this is equivalent to saying it is a surface such that the surface normal is everywhere parallel to $\nabla\Phi$ or $\hat{\mathbf{n}} = \nabla\Phi/|\nabla\Phi|$. It is a contour of constant Φ in space. If the magnetic field creates nested flux surfaces, this means that each constant Φ surface is closed and completely contained within a different flux surface. This can be broken if there are magnetic nulls, for example, because then the magnetic flux surface may not be closed and so there is no sense in which one flux surface is "inside" of another. For a torus, we simply can imagine a doughnut. If we consider the surface of the doughnut a flux surface, it is clear we could form nested flux surfaces by shaving off layers of the doughnut and defining that as a flux surface. Because each shaved off layer can fit into a previous layer, the flux surfaces would be nested. The importance of the nested flux surfaces is that we can then label a radial coordinate via a flux surface. As we go outward radially, Φ either increases or decreases and so Φ can label where we are radially.[10] Now let's use cylindrical coordinates (R, Z, ζ) and the primitive toroidal coordinates (r, θ, ζ) to give a simple example that can be easily visualized. Here θ is a poloidal angle and ζ is a toroidal angle. Let's assume for plotting purposes that our flux surfaces are simple (circular) torii centered around $r = 0$. Then to find a toroidal flux we simply need to integrate the circular cross section in the form of a disk. This is seen in Figure 2.2. So using the dummy (minor radius) radial variable s we write

$$\begin{aligned}\Phi_t(r) &= \iint \mathrm{d}S\ \hat{\boldsymbol{\zeta}}\cdot\mathbf{B} = \int_0^r \mathrm{d}s\ s\int_0^{2\pi}\mathrm{d}\theta\ \hat{\boldsymbol{\zeta}}\cdot\mathbf{B}\\ &= \iint \mathrm{d}S\ \mathcal{J}\,\nabla\zeta\cdot\mathbf{B}\end{aligned} \tag{2.2.4}$$

When it comes to a poloidal flux there are two options. A ring flux or a disk flux. The disk flux is analogous to the toroidal flux above, it integrates from $R = 0$ out to a disk at some other radius R as seen in Figure 2.3. This is called the disk flux Φ_p^d and given by (using dummy radial variable S_r for the major radius)

$$\begin{aligned}\Phi_p^d(R) &= \iint_{S_d} \mathrm{d}S\ \hat{\boldsymbol{\theta}}\cdot\mathbf{B} = \int_0^R \mathrm{d}S_r\ S_r\int_0^{2\pi}\mathrm{d}\zeta\ \hat{\boldsymbol{\theta}}\cdot\mathbf{B}\\ &= \iint_{S_d} \mathrm{d}S\ \mathcal{J}\,\nabla\theta\cdot\mathbf{B}\end{aligned} \tag{2.2.5}$$

[10] You may worry about whether this is a one-to-one function, but for plasma experiments it often is. If it isn't then one must be a little more careful when exchanging r for Φ.

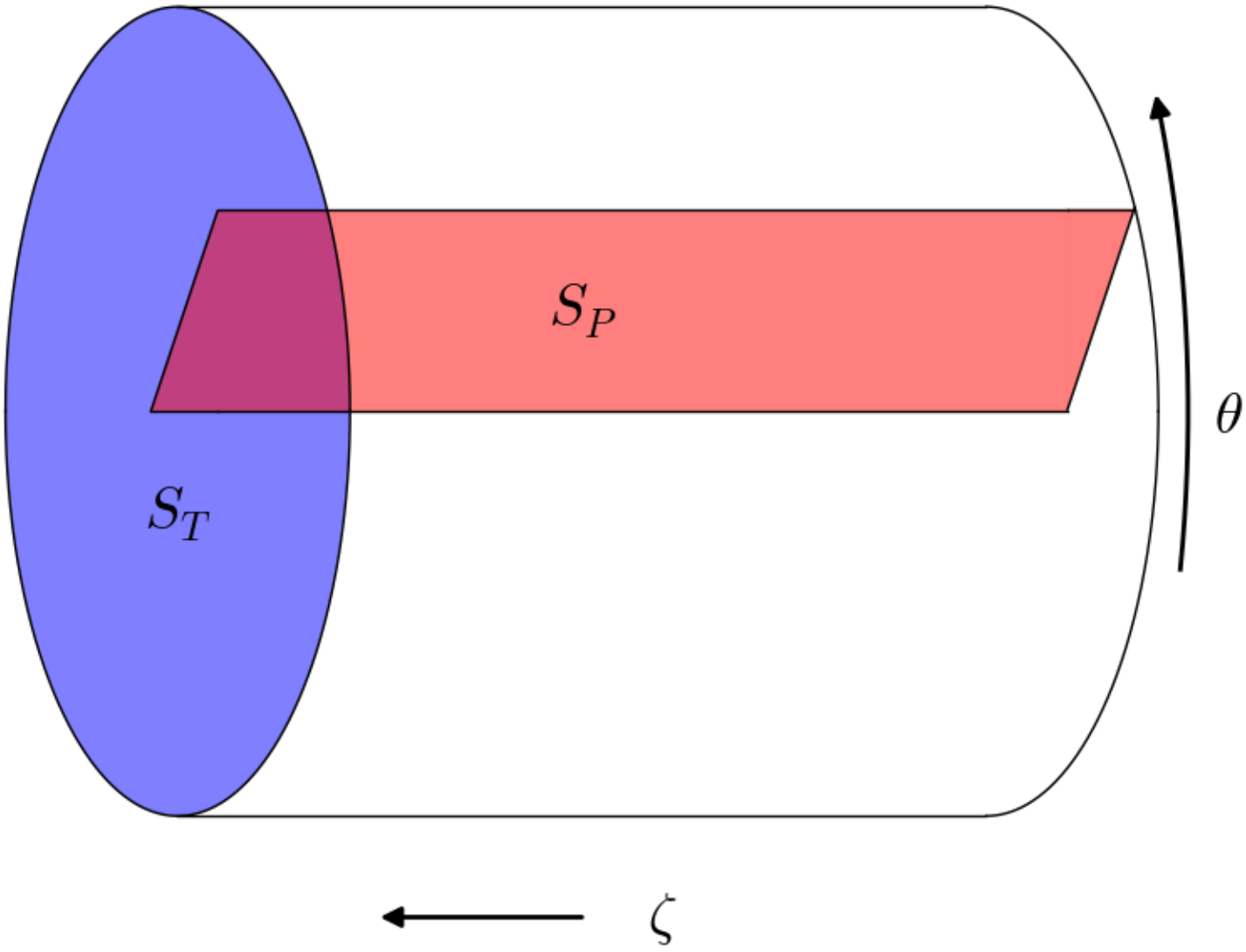

Figure 2.2: The toroidal flux S_T and the ring poloidal flux S_P through a portion of a toroidal device is shown.

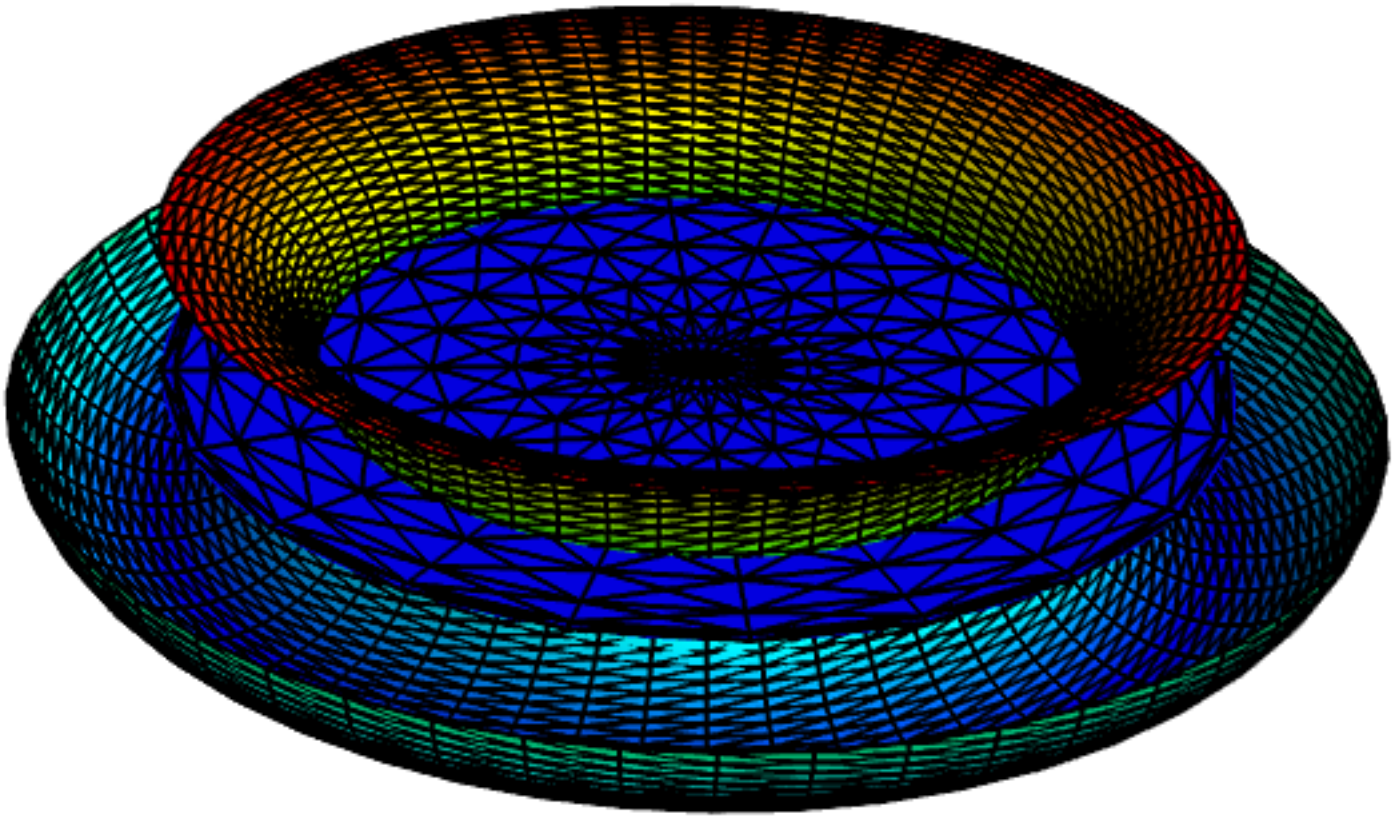

Figure 2.3: The surface to use for integration with a poloidal disk flux is the blue disk shown. The torus itself is cut open to see the shape of the poloidal disk flux.

Note that this form requires thinking about the magnetic field in two different coordinate systems, as we are using R and θ together. If this is undesirable, one can use that $R = R_0 + r\sin\theta$ and the above integral is equivalent to $\theta = \pi$ and

$$\begin{aligned}\Phi_p^d(r) &= \iint_{S_d} \mathrm{d}S\ \hat{\boldsymbol{\theta}}\cdot\mathbf{B} = \int_{R_0}^{R-R_0}\mathrm{d}s\ s\int_0^{2\pi}\mathrm{d}\zeta\ \hat{\boldsymbol{\theta}}\cdot\mathbf{B} \\ &= \iint_{S_d}\mathrm{d}S\ \mathcal{J}\,\nabla\theta\cdot\mathbf{B}\end{aligned} \tag{2.2.6}$$

You may find the sign flipped above so that the integral is from 0 to R_0 in some cases depending on whether the person desires $\Phi_p^d(r)$ to increase in the same direction as $\Phi_p^d(R)$.[11] Again, one must look carefully at the particular application to see what convention is chosen.

When using a ring flux, it is conventional to choose $\theta = \pi$, although in principle any angle could be chosen when the nested flux surfaces exist. $\theta = \pi$ is most convenient, however, because then there is a simple relationship between the ring flux and the disk flux versions. The way to create a ring flux is to go from the magnetic axis (we'll say it is $r = 0$ in our simple system) out to some other minor radius, as seen in Figure 2.2. Then it is easy to show

$$\Phi_p^d(R = R_0) = \phi_p^d(r = (R - R_0)/\sin\theta) + \phi_p^r(r) \tag{2.2.7}$$

That is a disk flux for a certain radius (or flux surface) and a ring flux to that same radius (or flux surface) will always sum to the disk flux for the magnetic axis ($R = R_0$ or $r = 0$).

One last comment is that one can rewrite the above equation for a generic flux from a surface integral into a volume integral. It is sometimes claimed this is easier or more useful, but if one were to actually calculate the flux it is not clear there are any savings. To change to a volume integral, one must integrate over the volume enclosed by a flux surface. This means you need to know the form of the flux surface (at which point, why not just do the surface integral?), but it is sometimes useful from a theoretical point of view.

The way is clever and not at all obvious. First consider a ribbon flux[12] at constant θ. Then we can form the volume by following flux surfaces around from the constant $\theta = 0$ surface around to the $\theta = 2\pi$ surface. We can consider the constant $\theta = 0$ surface S_0 and consider the surface at 2π [which is physically the same location and so forms a full volume] $S_{2\pi}$ with S_f for the surface that follows the flux surface defined between S_0 and $S_{2\pi}$ surfaces. The volume enclosed is then given by $\mathcal{V}_\theta$. Even though there is in a sense, no room between S_0 and $S_{2\pi}$, we can use them as two separate surfaces that happen to be essentially the same surface. This is like splitting doughnut but not unfurling it. We then find

$$\iiint_{\mathcal{V}_\theta}\mathrm{d}^3x\ \mathbf{B}\cdot\nabla\theta = \iiint_{\mathcal{V}_\theta}\mathrm{d}^3x\ [\boldsymbol{\nabla}\cdot(\theta\mathbf{B}) - \cancel{\theta\boldsymbol{\nabla}\cdot\mathbf{B}}] = \iiint_{\mathcal{V}_\theta}\mathrm{d}^3x\ \boldsymbol{\nabla}\cdot(\theta\mathbf{B}) = \oiint\mathrm{d}S\ \hat{\mathbf{n}}\cdot\theta\mathbf{B} \tag{2.2.8}$$

By construction the surface integral is given by

$$\oiint\mathrm{d}S\ \hat{\mathbf{n}}\cdot\theta\mathbf{B} = \iint\mathrm{d}S_0\ \hat{\mathbf{n}}\cdot\theta\mathbf{B} + \iint\mathrm{d}S_{2\pi}\ \hat{\mathbf{n}}\cdot\theta\mathbf{B} + \iint\mathrm{d}S_f\ \hat{\mathbf{n}}\cdot\theta\mathbf{B} \tag{2.2.9}$$

[11] This is saying we can write either $f(r)$ or $g(r)$ if $f(r) = g(-r)$ and so usually one chooses $f(r)$.
[12] The disk flux is more complicated to handle but an analogous form can be derived.

both S_0 and $S_{2\pi}$ have $\hat{\mathbf{n}} = \hat{\boldsymbol{\theta}}$, while S_f has $\hat{\mathbf{n}}$ pointing perpendicular to $\mathbf{B}$ because it is a flux surface by construction. We can also note that for S_0 we have $\theta = 0$ throughout the integral (because $\mathrm{d}S_0$ and $\mathrm{d}S_{2\pi}$ only vary with r and ζ) and for $S_{2\pi}$ we have $\theta = 2\pi$ throughout the integral and so we can replace the θ. Thus

$$\oiint \mathrm{d}S\ \hat{\mathbf{n}} \cdot \theta \mathbf{B} = \cancel{\iint \mathrm{d}S_0\ \hat{\mathbf{n}} \cdot (0)\mathbf{B}} + \iint \mathrm{d}S_{2\pi}\ \hat{\mathbf{n}} \cdot 2\pi\mathbf{B} + \cancel{\iint \mathrm{d}S_f\ \hat{\mathbf{n}} \cdot \theta\mathbf{B}} \tag{2.2.10}$$

$$\oiint \mathrm{d}S\ \hat{\mathbf{n}} \cdot \theta \mathbf{B} = 2\pi \iint \mathrm{d}S_0\ \hat{\mathbf{n}} \cdot \mathbf{B} = 2\pi\Phi_p^r(r) \tag{2.2.11}$$

Similarly, for the toroidal flux we can cut at $\zeta = 0$ for S_0 and $\zeta = 2\pi$ for $S_{2\pi}$ and create a toroidal volume by following flux surface to connect these surfaces S_f. This time S_0 and $S_{2\pi}$ are two circular disks and, if we unfurled the torus would form the top and bottom of a cylinder. Again, we use that the S_0 and $S_{2\pi}$ are essentially the same location and it is as if we made a slice but didn't pull apart the doughnut. If the total contained volume is $\mathcal{V}_\zeta$ then we use

$$\iiint_{\mathcal{V}_\zeta} \mathrm{d}^3x\ \mathbf{B} \cdot \nabla\zeta = \iiint_{\mathcal{V}_\zeta} \mathrm{d}^3x\ [\boldsymbol{\nabla} \cdot (\zeta\mathbf{B}) - \cancel{\zeta \boldsymbol{\nabla} \cdot \mathbf{B}}] = \iiint_{\mathcal{V}_\zeta} \mathrm{d}^3x\ \boldsymbol{\nabla} \cdot (\zeta\mathbf{B}) = \oiint \mathrm{d}S\ \hat{\mathbf{n}} \cdot \zeta\mathbf{B} \tag{2.2.12}$$

By construction the surface integral is given by

$$\oiint \mathrm{d}S\ \hat{\mathbf{n}} \cdot \zeta \mathbf{B} = \iint \mathrm{d}S_0\ \hat{\mathbf{n}} \cdot \zeta\mathbf{B} + \iint \mathrm{d}S_{2\pi}\ \hat{\mathbf{n}} \cdot \zeta\mathbf{B} + \iint \mathrm{d}S_f\ \hat{\mathbf{n}} \cdot \zeta\mathbf{B} \tag{2.2.13}$$

both S_0 and $S_{2\pi}$ have $\hat{\mathbf{n}} = \hat{\boldsymbol{\zeta}}$, while S_f has $\hat{\mathbf{n}}$ pointing perpendicular to $\mathbf{B}$ because it is a flux surface. We can also note that for S_0 we have $\zeta = 0$ throughout the integral and for $S_{2\pi}$ we have $\zeta = 2\pi$ throughout the integral and so we can replace the ζ. Thus

$$\oiint \mathrm{d}S\ \hat{\mathbf{n}} \cdot \zeta \mathbf{B} = \cancel{\iint \mathrm{d}S_0\ \hat{\mathbf{n}} \cdot (0)\mathbf{B}} + \iint \mathrm{d}S_{2\pi}\ \hat{\mathbf{n}} \cdot 2\pi\mathbf{B} + \cancel{\iint \mathrm{d}S_f\ \hat{\mathbf{n}} \cdot \zeta\mathbf{B}} \tag{2.2.14}$$

$$\oiint \mathrm{d}S\ \hat{\mathbf{n}} \cdot \zeta \mathbf{B} = 2\pi \iint \mathrm{d}S_0\ \hat{\mathbf{n}} \cdot \mathbf{B} = 2\pi\Phi_t(r) \tag{2.2.15}$$

We then have altogether that

$$2\pi\Phi_t(r) = \iiint_{\mathcal{V}_\zeta} \mathrm{d}^3x\ \mathbf{B} \cdot \nabla\zeta \tag{2.2.16}$$

$$2\pi\Phi_p^r(r) = \iiint_{\mathcal{V}_\theta} \mathrm{d}^3x\ \mathbf{B} \cdot \nabla\theta \tag{2.2.17}$$

as alternate forms for finding the toroidal flux and the ribbon poloidal flux.

Note that the only real property we used was that the surface was periodic with θ or ζ over a period of 2π. The verbal descriptions assumed a circular torus shape, but the mathematics did not. Thus this is general for any periodic θ and ζ with nested flux surfaces.

2.2.3 The Properties and Construction of Flux Coordinates

Now the flux surfaces can be much more complicated then the simple torii, and in those cases we can construct coordinates that take into account the shapes of the flux surfaces. The pay

off is simpler expressions for the magnetic field, and often simpler dynamical equations to solve. Because these coordinates are often difficult to visualize, however, one often loses the ability to easily translate a result into Cartesian coordinates.

In general, it is possible to construct flux surfaces[9, p. 100], although these flux surfaces are not necessarily nested. If we just assume that they are, then we can use that given a flux, we can identify the radial position. When the flux is nested like this, we can use that toroidal flux ψ_t and poloidal flux ψ_p are related such that $\psi_t = \psi_t(\psi_p)$ or $\psi_p = \psi_p(\psi_t)$ without any needed angle information. For now we'll just call the flux Φ and not worry whether it is toroidal or poloidal. All we require is that it is a flux that corresponds to a radial position. We will later use nested flux surfaces so that we can simplify our magnetic field representation. Then a flux function is a function that satisfies $f = f(\Phi)$, that is, f is constant on flux surfaces. Remember that by definition we have for a flux surface that $\mathbf{B}\cdot\nabla\Phi = 0$. Then flux surfaces are created such that $\mathbf{B}$ is always perpendicular to them. This means that

$$\mathbf{B}\cdot\nabla f = \mathbf{B}\cdot\nabla\Phi\frac{\partial f}{\partial\Phi} = 0 \tag{2.2.18}$$

is another way of determining that f is a flux function or flux label. Then one way of proceeding is to construct a way of describing the magnetic field in relation to our fluxes.

We can construct a surface potential[13][13] as $G(\Phi,\theta,\zeta)$. We let $\theta = \theta(\mathbf{x})$ and $\zeta = \zeta(\mathbf{x})$ be general multi-valued functions that determine a flux surface which can be labeled by a radial variable (Φ here). Then we require (1) $\nabla G\times\nabla\Phi$ is single-valued on every path and (2) G is single-valued along paths that do not encircle a magnetic axis or geometric axis (along flux surfaces). This is a surface potential because what matters is ∇G, not G itself.

We can then show that given a flux function/label f and a vector function $\mathbf{W}(\mathbf{x})$ satisfying $\nabla f\cdot\boldsymbol{\nabla}\times\mathbf{W} = 0$, then there exists a surface potential G such that $\nabla f\times\mathbf{W} = \nabla f\times\nabla G$. We proceed by defining a point $\mathbf{z}$ on the flux surface defined by f that contains the point $\mathbf{x}$ that we are constructing our function for. In other words, find the flux surface that a chosen $\mathbf{x}$ is on and find another point on that flux surface $\mathbf{z}$. Thus given this $\mathbf{x}$, we find a $\mathbf{z}$ on a surface of f so that

$$G(\mathbf{x}) = \int_{\mathbf{z}(f)}^{\mathbf{x}} \mathrm{d}\mathbf{x}'\ \cdot\mathbf{W}(\mathbf{x}') \tag{2.2.19}$$

with the path from $\mathbf{z}(f)$ to $\mathbf{x}$ fully on the surface of f containing $\mathbf{z}$ and $\mathbf{x}$. We can explicitly parameterize this with s so that

$$G(\mathbf{x}) = \int_{s_1}^{s_2}\mathrm{d}s\ \frac{\partial\mathbf{x}'}{\partial s}\cdot\mathbf{W}(\mathbf{x}'(s)) \equiv \int_{s_1(s)}^{s_2(s)}\mathrm{d}s'\ k(s') \tag{2.2.20}$$

We use the Leibniz integral rule for differentiating an integral:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)}\mathrm{d}t\ f(x,t) = f(x,b(x))\frac{\mathrm{d}b}{\mathrm{d}x} - f(x,a(x))\frac{\mathrm{d}a}{\mathrm{d}x} + \int_{a(x)}^{b(x)}\mathrm{d}t\ \frac{\partial f(x,t)}{\partial x} \tag{2.2.21}$$

[13]This is sometimes called a stream function or generating function. Whatever we call it, the important thing is that it has properties (1) and (2) as described above.

and so

$$\begin{aligned}\frac{\partial G}{\partial \mathbf{x}} &= \frac{\partial}{\partial \mathbf{x}}\int_{s_1(\mathbf{z})}^{s_2(\mathbf{x})} \mathrm{d}s'\; k(s') = k(s_2)\frac{\partial s_2}{\partial \mathbf{x}} - k(s_1)\frac{\partial s_1}{\partial \mathbf{x}} = \mathbf{W}(\mathbf{x})\cdot\overbrace{\frac{\partial \mathbf{x}}{\partial s_2}\frac{\partial s_2}{\partial \mathbf{x}}}^{=\mathbb{1}} - \mathbf{W}(\mathbf{z})\cdot\frac{\partial \mathbf{z}}{\partial s_1}\frac{\partial s_1}{\partial \mathbf{x}} \\ &= \mathbf{W}(\mathbf{x}) - \frac{\partial s_1}{\partial \mathbf{x}}\frac{\partial \mathbf{z}}{\partial s_1}\cdot\mathbf{W}(\mathbf{z}) = \mathbf{W}(\mathbf{x}) - \frac{\partial s_1}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial s_1}\cdot\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\cdot\mathbf{W}(\mathbf{z}) \\ &= \mathbf{W}(\mathbf{x}) - \overbrace{\frac{\partial s_1}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial s_1}}^{\mathbb{1}}\cdot\overbrace{\frac{\partial \mathbf{z}}{\partial \mathbf{x}}}^{\frac{\partial f}{\partial \mathbf{x}}\frac{\partial \mathbf{z}}{\partial f}}\cdot\mathbf{W}(\mathbf{z}) = \mathbf{W}(\mathbf{x}) - \frac{\partial f}{\partial \mathbf{x}}\left[\frac{\partial \mathbf{z}}{\partial f}\cdot\mathbf{W}(\mathbf{z})\right] \\ &= \mathbf{W}(\mathbf{x}) - C(\mathbf{z})\frac{\partial f}{\partial \mathbf{x}}\end{aligned} \tag{2.2.22}$$

with $C(\mathbf{z})$ some function dependent only on $\mathbf{z}$. We had to be very careful above, as we required the $\mathbf{z}$ part of $\partial\mathbf{z}/\partial\mathbf{x}$ to be dotted into $\mathbf{W}(\mathbf{z})$, which is why $\partial\mathbf{z}/\partial\mathbf{x}$ is moved to the left of $\mathbf{W}$. The last term is in the direction $\nabla f = \frac{\partial f}{\partial \mathbf{x}}$ and so

$$\nabla f \times (\mathbf{W} - \nabla G) = \nabla f \times (\cancel{\mathbf{W}} - \cancel{\mathbf{W}} + C\,\nabla f) = 0 \tag{2.2.23}$$

Or equivalently

$$\nabla f \times \mathbf{W} = \nabla f \times \nabla G \tag{2.2.24}$$

Because $\mathbf{W}$ is single-valued then $\nabla f \times \nabla G$ must be single-valued along flux surface paths as required for property (1). Then we would like property (2) to hold as well for this G. We consider a closed path (along a flux surface) with no loops encircling a magnetic or geometric axis. Then we have

$$\oint_C \mathrm{d}\boldsymbol{\ell}\cdot\mathbf{W} = \iint \mathrm{d}S\;\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\mathbf{W} = \iint \mathrm{d}S\;\frac{\nabla f}{|\nabla f|}\cdot\boldsymbol{\nabla}\times\mathbf{W} = \iint \mathrm{d}S\;0 = 0 \tag{2.2.25}$$

where we have used the property (that must be checked in general) that $\nabla f\cdot\boldsymbol{\nabla}\times\mathbf{W} = 0$. This path is essentially a deformed circle inscribed on a flux surface. If we encircled the geometric or magnetic axis, however, then the surface enclosed by the bounding curve C would not just be a flux surface and so $\hat{\mathbf{n}}$ would not point in the ∇f direction.

We can now write the magnetic field as

$$\mathbf{B} = \nabla f \times \mathbf{W} \tag{2.2.26}$$

Because $\nabla f \propto \nabla r \propto \nabla\Phi$ and $\mathbf{B}\cdot\nabla\Phi = 0$ then for a generic $\mathbf{W}$ it must be possible to write the magnetic field as above. We use

$$0 = \boldsymbol{\nabla}\cdot\mathbf{B} = \boldsymbol{\nabla}\cdot(\nabla f\times\mathbf{W}) = \cancel{(\boldsymbol{\nabla}\times\nabla f)}\cdot\mathbf{W} - \nabla f\cdot\boldsymbol{\nabla}\times\mathbf{W} \tag{2.2.27}$$

We then have $\nabla f\cdot\boldsymbol{\nabla}\times\mathbf{W} = 0$ and our above theorem applies so we can replace $\mathbf{W}$ with ∇G. This means that

$$\mathbf{B} = \nabla f \times \nabla G \tag{2.2.28}$$

Do not confuse this with the more general Clebsch representation, usually shown as

$$\mathbf{B} = \nabla\alpha \times \nabla\beta \tag{2.2.29}$$

In our case $f = f(\Phi)$ while $\alpha = \alpha(\Phi, \theta, \zeta)$ and $\beta = \beta(\Phi, \theta, \zeta)$. That is neither α nor β are flux functions/labels. Our form (2.2.28) is more specific to our needs and therefore is more limiting on (and so tells us a great deal more about) the form of $\mathbf{B}$.

What is the form of our surface potential? Clearly $G = G(\Phi, \theta, \zeta)$ is the most general, but we desire it to describe things on flux surfaces. Therefore, we want it to depend on θ and ζ with coefficients dependent on the particular flux surface labeled by Φ. We can then write

$$G(\Phi, \theta, \zeta) = a(\Phi)\theta + b(\Phi)\zeta + G_0(\Phi, \theta, \zeta) \tag{2.2.30}$$

where $G_0(\Phi, \theta, \zeta) = G_0(\Phi, \theta + 2\pi, \zeta + 2\pi)$ is an arbitrary periodic function of θ and ζ. Note how property (1) enforces that only linear in θ and ζ terms are possible to keep G single-valued on flux-surface paths. For example assume it depended on $c(\Phi)\zeta^n$. Then

$$\nabla G \times \nabla\Phi = nc(\Phi)\zeta^{n-1}\nabla\zeta \times \nabla\Phi + \zeta^n \cancel{\nabla c \times \nabla\Phi} \tag{2.2.31}$$

Then ζ^{n-1} being a function of ζ (for $n \neq 1$) means that $\nabla G \times \nabla\Phi$ depends on the multi-valued function ζ, and so it cannot be single valued. Therefore $n = 1$ is the only allowable function. The same argument can be used for θ. You might worry that this is not enough to rule out more exotic functional relationships. Suppose we did try a new function $h(\theta, \zeta)$ which is not periodic in θ and ζ. We can still form an arbitrarily small path and perform a Taylor series expansion of h along the path. This will reduce h to a polynomial to whatever accuracy is desired. We then use the same argument as above and show that only a linear contribution is allowable, and so the function h is not allowable. The enforcement of G_0 being periodic in θ and ζ ensures that G is single-valued for all other functions.

So we replace G with expression (2.2.30) and can write

$$\begin{aligned}\mathbf{B} = \nabla f \times \nabla G &= \nabla f \times \nabla[a(\Phi)\theta + b(\Phi)\zeta + G_0(\Phi, \theta, \zeta)] \\ &= a\nabla f \times \nabla\theta + b\nabla f \times \nabla\zeta + \nabla f \times \nabla G_0\end{aligned} \tag{2.2.32}$$

We can of course choose a new θ coordinate to eliminate G_0 dependence. Simply use $\theta' = \theta - G_0/a$ and we find

$$\nabla\theta' = \nabla\theta - \frac{1}{a}\nabla G_0 + \frac{G_0}{a^2}\nabla a \tag{2.2.33}$$

which with ∇a pointing in the same direction as ∇f (the $\nabla\Phi$ direction) implies

$$a\nabla f \times \nabla\theta' = a\nabla f \times \nabla\theta' - \nabla f \times \nabla G_0 + \cancel{\frac{G_0}{a^2}\nabla f \times \nabla a} \tag{2.2.34}$$

and so we could just as easily write

$$\mathbf{B} = a\nabla f \times \nabla\theta' + b\nabla f \times \nabla\zeta \tag{2.2.35}$$

Note that this is of the same form as (1.10.8) for the canonical form of $\mathbf{B}$ if we identify

$$a(\Phi)\nabla f \to \nabla\psi_t \tag{2.2.36}$$

$$b(\Phi)\nabla f \to \nabla\psi_p \tag{2.2.37}$$

Can we do this identification? First of all, remember that ψ_t is actually $1/(2\pi)$ times the toroidal flux Φ_t and similarly $\psi_p = \Phi_p/(2\pi) = \Phi/(2\pi)$. When there are magnetic flux surfaces, we know that ψ_t (or Φ_t) and ψ_p (or Φ_p) are directly related to each other. Here we have $\Phi = 2\pi\psi_p = \Phi_p$. If we choose $b = -1$ then clearly we can use $f = \psi_p^r = \Phi_p^r/(2\pi)$, for the ribbon flux definition of poloidal flux as our flux label f. We then require

$$a(\Phi)\,\nabla f = \frac{a(\Phi)}{2\pi}\,\nabla\Phi = \nabla\psi_t \tag{2.2.38}$$

We use nested flux surfaces to write $\Phi_t = \Phi_t(\Phi_p^r)$ and so $\nabla\Phi_t = \frac{\mathrm{d}\Phi_t}{\mathrm{d}\Phi_p^r}\nabla\Phi_p^r$. Thus if we choose $a(\Phi_p^r) = \frac{\mathrm{d}\Phi_t}{\mathrm{d}\Phi_p^r}$ we find

$$a(\Phi)\,\nabla f = \frac{1}{2\pi}\frac{\mathrm{d}\Phi_t}{\mathrm{d}\Phi_p^r}\,\nabla\Phi_p^r = \frac{1}{2\pi}\,\nabla\Phi_t = \nabla\psi_t \tag{2.2.39}$$

Thus, we can make the identification by construction. This then implies

$$\begin{aligned}\mathbf{B} &= \nabla\psi_t \times \nabla\theta' - \nabla\psi_p^r \times \nabla\zeta = \nabla\psi_t \times \nabla\theta' + \nabla\zeta \times \nabla\psi_p^r \\ \mathbf{B} &= \frac{\nabla\Phi_t \times \nabla\theta' + \nabla\zeta \times \nabla\Phi_p^r}{2\pi}\end{aligned} \tag{2.2.40}$$

which shows we can express the magnetic field in terms of its fluxes. We can also bring up the idea of the safety factor. The safety factor in these straight-field line flux coordinates has a simple interpretation. We could have kept $\nabla\psi_t$ in terms of $\nabla\Phi_p^r = \nabla\Phi$ and then defined

$$q(\Phi) = \frac{\mathrm{d}\Phi_t}{\mathrm{d}\Phi_p^r} \tag{2.2.41}$$

as the safety factor and written

$$2\pi\mathbf{B} = q\,\nabla\Phi_p^r \times \nabla\theta' + \nabla\zeta \times \nabla\Phi_p^r = \nabla\Phi_p^r \times \nabla(q\theta') + \nabla\zeta \times \nabla\Phi_p^r \tag{2.2.42}$$

$$2\pi\mathbf{B} = \nabla\Phi_p^r \times \nabla[q\theta' - \zeta] \tag{2.2.43}$$

We can then identify $q\theta' - \zeta$ as constant along field $\mathbf{B}$ and call this a field line label. Sometimes it is called *the* field line label. This is somewhat poor terminology since it is not constant along field lines because the field lines can go past 2π in either angle. But $q\theta' - \zeta$ is only constant on a cut from 0 to 2π in θ' or ζ. If a field line goes more than 2π in θ' or ζ without meeting itself, then $q\theta' - \zeta$ can change value. For rational field lines it does not matter much because they connect back on themselves and so going past will simply take you to a self-similar rational field line.

We can also identify poloidal ($\mathbf{B}_p$) and toroidal ($\mathbf{B}_t$) components of $\mathbf{B}$ from (2.2.40) via defining

$$\mathbf{B}_p = \frac{\nabla\zeta \times \nabla\Phi_p^r}{2\pi} = \nabla\zeta \times \nabla\psi_p^r \tag{2.2.44}$$

$$\mathbf{B}_t = \frac{\nabla\Phi_t \times \nabla\theta}{2\pi} = \nabla\psi_t \times \nabla\zeta \tag{2.2.45}$$

2.2.3.1 The Safety Factor

Finally, it is worth delving a bit into the safety factor q. In our flux coordinates we can note that the field line pitch is given by (see Section 1.10 from (1.10.14) to (1.10.30))

$$\frac{\mathrm{d}\theta'}{\mathrm{d}\zeta} = \frac{\mathbf{B}\cdot\nabla\theta'}{\mathbf{B}\cdot\nabla\zeta} \tag{2.2.46}$$

We can use (2.2.43) to write

$$\frac{\mathrm{d}\theta'}{\mathrm{d}\zeta} = \frac{-\nabla\theta\cdot\nabla\Phi_p^r\times\nabla\zeta}{\nabla\zeta\cdot\nabla\Phi_p^r\times\nabla(q\theta)} = \frac{\cancel{\nabla\zeta\cdot\nabla\Phi_p^r\times\nabla\theta}}{q(\Phi_p^r)\cancel{\nabla\zeta\cdot\nabla\Phi_p^r\times\nabla\theta}} = \frac{1}{q(\Phi_p^r)} \tag{2.2.47}$$

which shows the field line pitch is a constant on flux surfaces and $1/q(\Phi_p^r)$. Note that only when using straight-field line (flux) coordinates is it true that the safety factor is a flux surface quantity equal to the field-line pitch. Flux coordinates are straight-(magnetic)-field line coordinates and vice versa as these terms are used interchangeably.

In general, q is a flux-surface quantity, but it is *not* the field-line pitch. To see this we will assume we do have angles θ and ζ, but that they are not necessarily flux coordinates. We will still have that Φ_p and Φ_t are flux labels, and that therefore they can be thought of as functions of each other. The derivative of $\Phi_p = \Phi_p(\Phi_t)$ is called the rotational transform[14] and is given by

$$\iota = 2\pi\frac{\mathrm{d}\Phi_p}{\mathrm{d}\Phi_t} \tag{2.2.48}$$

The factor of 2π is there so that as ζ varies from 0 to 2π periodically, ι gives the change in θ when ζ finishes one period. This quantity is for a single field line, and so it can change as one goes from the first toroidal transit to the second, and so on. However, as you increase the number of transits, one finds that average change in θ per toroidal transit approaches a consistent value given by $\iota/2\pi$ or $\imath$.[15] Note that $\iota = 0$ implies a purely toroidal line (i.e., no changes in θ for each toroidal transit in ζ) while $\iota = \infty$ implies a purely poloidal line (there never is a change in ζ and so the denominator of ι blows up).

To get an averaged rotational transform, we consider the field line over a large number of toroidal transits. We wait until the field line returns to the original location (or within some arbitrary area of it related to the error we are willing to tolerate) after n toroidal circuits. Each toroidal circuit will be associated with some change in θ and will therefore create m poloidal circuits. Then we can say on average the field line pitch (just $\overline{\mathrm{d}\Phi_p/\mathrm{d}\Phi_t}$ without 2π and the line over the quantity indicating an average over toroidal transits) is

$$\overline{\frac{\mathrm{d}\Phi_p}{\mathrm{d}\Phi_t}} = \frac{n}{m} \tag{2.2.49}$$

This form seems to imply that n/m is always rational, but this is not actually true unless we let the arbitrary area around our initial location become the exact initial location (so that the error

[14] Unfortunately terminology is muddled once again. The factor of 2π is often omitted when calling something the "rotational transform". But you rarely hear normalized rotational transform. You will hear iota (rotational transform) and iota-bar (rotational transform, but iota divided by 2π).

[15] The use of iota ι is also rather unfortunate since in actual writing ι looks like an i without the dot and so could be so confused. $\imath$ is hardly better, as if you are not careful it will look like a t. The notation is standard now, and so it is not worth trying to change.

tolerated goes to zero). For then the field line has actually returned to its initial location and it is a rational flux surface.[16] In the generic case we have only found a rational number close to the desired irrational average value by not requiring the line to close on itself exactly.

Generically we define ι and ι as this average, but omit the averaging line over them since the non-averaged quantity is usually not useful. Also remember that ι and ι are for a specific flux surface so the quantities should be evaluated at a specific radial location. Thus

$$\iota = \lim_{Z\to\infty} \frac{\int_0^Z \mathrm{d}\zeta \, \frac{\mathbf{B}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\zeta}}{Z} \tag{2.2.50}$$

$$\iota = 2\pi \lim_{Z\to\infty} \frac{\int_0^Z \mathrm{d}\zeta \, \frac{\mathbf{B}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\zeta}}{Z} \tag{2.2.51}$$

One can then follow the exact same idea for the safety factor q as an average value given by $1/\iota$. In practice, one usually instead considers the number of poloidal transits m to toroidal transits n and so defines

$$q = \lim_{\Theta\to\infty} \frac{\int_0^\Theta \mathrm{d}\theta \, \frac{\mathbf{B}\cdot\nabla\zeta}{\mathbf{B}\cdot\nabla\theta}}{\Theta} \tag{2.2.52}$$

which gives the average pitch of $\mathrm{d}\Phi_t/\mathrm{d}\Phi_p$. These are equivalent descriptions of the problem. Here $q = 0$ implies that the average field line at that surface is purely poloidal, while $q = \infty$ implies that the average field line at that surface is purely toroidal.

The use of q or ι is a matter of taste in some sense, but you will find that in the literature q is more often used for tokamaks ans ι is more often used for stellarators. The radial logarithmic derivative of q is often given the name magnetic shear and given the variable s to represent it. Thus

$$s = \frac{r}{q}\frac{\mathrm{d}q}{\mathrm{d}r} = \frac{\mathrm{d}\ln q}{\mathrm{d}\ln r} = \frac{\mathrm{d}(-\ln\iota)}{\mathrm{d}\ln r} = -\frac{r}{\iota}\frac{\mathrm{d}\iota}{\mathrm{d}r} = -\frac{r}{\iota}\frac{\mathrm{d}\iota}{\mathrm{d}r} \tag{2.2.53}$$

The safety factor is related to plasma stability and so that is where its name comes from. Generally speaking, higher q indicates more stability, however, when q is rational, instabilities often occur near that location. When using reverse field pinches (RFPs), one reason to prefer q to ι is that the field becomes purely poloidal and so it is easier to deal with $q = 0$ than $\iota = \infty$ numerically or analytically.

I have often heard people say that "it makes sense" that tokamak literature uses q and stellarator literature uses ι. None of the answers have seemed especially convincing to me, other than for RFPs. The association of high with good may make the safety factor easier to interpret, but I do not think that is an especially strong answer. The shear is usually opposite for tokamaks and stellarators, so if one wanted, what one usually calls negative shear in our definition would be more common in stellarators and so it might make sense to write $s = \mathrm{d}\ln\iota/\mathrm{d}\ln r$ for stellarators and consider rotational transform. The only other reason I can think of is that with stellarators you are often thinking of how the plasma shape changes as you go along the toroidal direction, and so ι seems more natural. There is nothing wrong with the literature diverging in preferred notation, but I have never felt that anyone gave a great reason for it.

[16] In reality, it's more like a rational field line, but if we have many field lines it forms a surface of closed field lines.

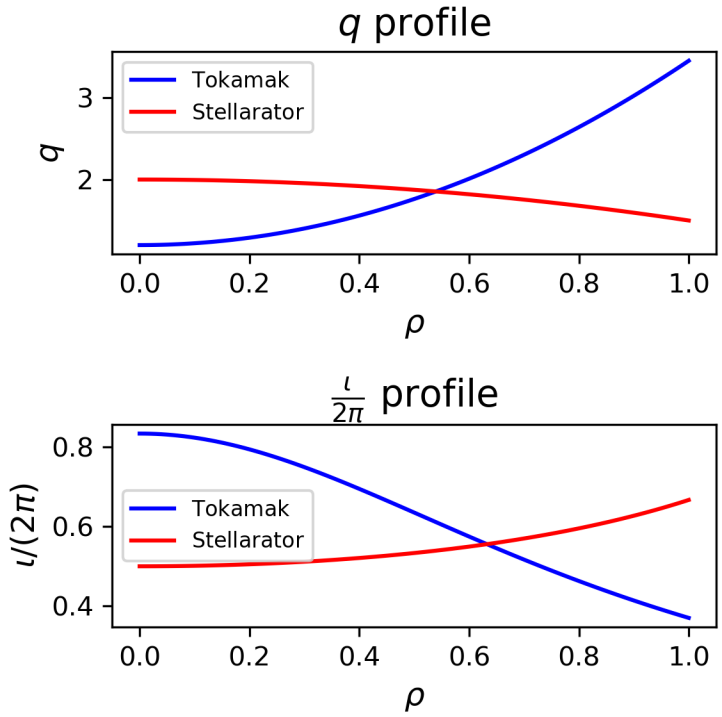

Figure 2.4: A typical profile shape is shown for a tokamak and stellarator for the safety factor q on top, and for rotational transform $\iota = \iota/(2\pi)$ on bottom. Obviously the profile shapes can change quite a bit depending on the tokamak or stellarator and the desires of the operators of the device, but these are rather typical shapes and numbers. Tokamaks like to have a safety factor of greater than 1 at the center and rapidly rising (generically speaking). Stellarators usually have the opposite sign of shear and the absolute value of the shear is less than that of a tokamak generically.

In any case, a typical shape and somewhat typical numbers are shown for a stellarator and tokamak case in Figure 2.4.

One other thing to note is that one usually cares about the edge q. In a limited tokamak, this is simply whatever q is at the limiter location, but in a diverted tokamak, the edge has $q \to \infty$. Thus, people usually quote q_{95} or something similar where q_{95} indicates the value of q at 95% of the radial edge value. That is, if a is the edge of the plasma, then $q_{95} = q(0.95a)$, which does not suffer from being infinite.

2.2.3.2 Flux Surface Averaging

We can now advance to flux surface averaging. The conventional definition is to define for some quantity Q the flux surface averaged value $\langle Q \rangle$ given by

$$\langle Q \rangle \equiv \frac{\iiint \mathrm{d}^3x' \; Q(\mathbf{x}')\delta(r'-r)}{\iiint \mathrm{d}^3x' \; \delta(r'-r)} = \frac{\oiint_{S_r} \mathrm{d}S \; Q(\mathbf{x}')}{\oiint_{S_r} \mathrm{d}S} \tag{2.2.54}$$

where r is some radial variable, the primes denote dummy integration variables, and S_r is the surface picked out by radial value r. Note that for any flux label $f = f(r)$, this ensures

$$\langle f \rangle = \frac{\iiint \mathrm{d}^3x'\ f(r')\delta(r'-r)}{\iiint \mathrm{d}^3x'\ \delta(r'-r)} = f(r)\frac{\cancel{\iiint \mathrm{d}^3x'\ \delta(r'-r)}}{\cancel{\iiint \mathrm{d}^3x'\ \delta(r'-r)}} = f \tag{2.2.55}$$

so that flux labels/functions are unaffected by flux surface averaging, as we'd expect.

We can form a volume of a flux surface via

$$V(r) = \iiint_V \mathrm{d}^3x' \ = \int \mathrm{d}r \ \int \mathrm{d}\theta \ \int \mathrm{d}\zeta \ \mathcal{J} \tag{2.2.56}$$

where the limits of r, θ, and ζ are given through them covering the volume $V(r)$. If we define the surface given by θ and ζ as S_r then we can realize

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\int \mathrm{d}r \ \int \mathrm{d}\theta \ \int \mathrm{d}\theta \ \mathcal{J} = \oiint_{S_r} \mathrm{d}\theta \ \mathrm{d}\zeta \ \mathcal{J} = \oiint_{S_r} \mathrm{d}S \tag{2.2.57}$$

where we have used that $\frac{\mathrm{d}}{\mathrm{d}r}$ essentially removes the $\int \mathrm{d}r$.

This means we can write

$$\langle f \rangle = \frac{1}{\frac{\mathrm{d}V}{\mathrm{d}r}} \oiint_{S_r} \mathrm{d}\theta \ \mathrm{d}\zeta \ \mathcal{J}f = \frac{1}{\frac{\mathrm{d}V}{\mathrm{d}r}} \oiint_{S_r} \mathrm{d}S \ f \tag{2.2.58}$$

Note that if our radial coordinate r is in fact the flux surface volume V, then we can write

$$\langle f \rangle = \oiint_{S_r} \mathrm{d}\theta \ \mathrm{d}\zeta \ \mathcal{J}f = \frac{\mathrm{d}}{\mathrm{d}V}\int \mathrm{d}V \ \oiint_{S_r} \mathrm{d}\theta \ \mathrm{d}\zeta \ \mathcal{J}f = \frac{\mathrm{d}}{\mathrm{d}V}\iiint_V \mathrm{d}^3x' \ f \tag{2.2.59}$$

Consider vector $\mathbf{A}$. Then using Gauss's Law we find

$$\langle \boldsymbol{\nabla}\cdot\mathbf{A} \rangle = \frac{\mathrm{d}}{\mathrm{d}V}\iiint \mathrm{d}^3x \ \boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\mathrm{d}}{\mathrm{d}V}\oiint \mathrm{d}S_r \ \hat{\mathbf{n}}\cdot\mathbf{A} \tag{2.2.60}$$

The unit vector is given by $\hat{\mathbf{n}} = \nabla V/|\nabla V|$, and we can use that $\mathrm{d}S_r\hat{\mathbf{n}} = \mathrm{d}\theta\,\mathrm{d}\zeta\mathcal{J}\nabla V$. We can then apply (2.2.58) and so

$$\langle \boldsymbol{\nabla}\cdot\mathbf{A} \rangle = \frac{\mathrm{d}}{\mathrm{d}V}\int \mathrm{d}\theta \ \int \mathrm{d}\zeta \ \nabla V\cdot\mathbf{A} = \frac{\mathrm{d}}{\mathrm{d}V}\langle \nabla V\cdot\mathbf{A}\rangle \tag{2.2.61}$$

Note that if we choose $g\mathbf{B}$ then

$$\frac{\mathrm{d}}{\mathrm{d}V}\langle \nabla V\cdot g\mathbf{B}\rangle = 0 \tag{2.2.62}$$

This follows from $\nabla V\cdot\mathbf{B} = 0$ because ∇V points perpendicular to flux surfaces and $\mathbf{B}$ lies only on flux surfaces and so by definition of a flux surface we must have $\nabla V\cdot\mathbf{B} = 0$. From this we use for arbitrary function g

$$\langle \boldsymbol{\nabla}\cdot(g\mathbf{B})\rangle = \langle g\cancel{\boldsymbol{\nabla}\cdot\mathbf{B}} + \mathbf{B}\cdot\nabla g\rangle = \langle \mathbf{B}\cdot\nabla g\rangle = \frac{\mathrm{d}}{\mathrm{d}V}\langle \nabla V\cdot g\mathbf{B}\rangle = 0 \tag{2.2.63}$$

$$\langle \mathbf{B}\cdot\nabla g\rangle = 0 \tag{2.2.64}$$

Finally, we can use for arbitrary vector $\mathbf{C}$ that

$$\boldsymbol{\nabla}\cdot(\nabla r\times\mathbf{C}) = \cancel{\boldsymbol{\nabla}\times\nabla r}\cdot\mathbf{A} - \boldsymbol{\nabla}\times\mathbf{C}\cdot\nabla r \tag{2.2.65}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{C}\times\nabla r) = (\boldsymbol{\nabla}\times\mathbf{C})\cdot\nabla r \tag{2.2.66}$$

Thus we find replacing $\mathbf{A}$ with $\mathbf{C}\times\nabla r$ and using $\nabla V = \frac{\partial V}{\partial r}\nabla r$ that

$$\langle\nabla r\cdot\boldsymbol{\nabla}\times\mathbf{C}\rangle = \langle\boldsymbol{\nabla}\cdot(\mathbf{C}\times\nabla r)\rangle = \frac{\mathrm{d}}{\mathrm{d}V}\langle\nabla V\cdot\cancel{[\mathbf{C}\times\nabla r]}\rangle = 0 \tag{2.2.67}$$

The importance of these results is that $\langle\mathbf{B}\cdot\nabla g\rangle = 0$ means that $\langle\cdot\rangle$ is an annihilator for the operation $\mathbf{B}\cdot\nabla g$ because $\langle\mathbf{B}\cdot\nabla g\rangle$ is automatically zero. That is, if you have any equation you want to flux surface average, you know that $\mathbf{B}\cdot\nabla$ terms can be annihilated by the flux surface averaging operation. This is often useful in problems where the operator $\mathbf{B}\cdot\nabla$ shows up, because it leads to great simplifications in the flux averaged equations. It is also worth stating that a similar trick can work for a generic dyad $\mathbf{BA}$ with $\mathbf{B}$ the magnetic field. Then

$$\langle\boldsymbol{\nabla}\cdot(\mathbf{BA})\rangle = \frac{\mathrm{d}}{\mathrm{d}V}\int\mathrm{d}\theta\int\mathrm{d}\zeta\ \nabla V\cdot\mathbf{BA} = \frac{\mathrm{d}}{\mathrm{d}V}\langle\nabla V\cdot\mathbf{BA}\rangle = \mathbf{0} \tag{2.2.68}$$

because again $\nabla V\cdot\mathbf{B} = 0$. We then have

$$\langle\boldsymbol{\nabla}\cdot(\mathbf{BA})\rangle = \langle\cancel{(\boldsymbol{\nabla}\cdot\mathbf{B})}\mathbf{A} + \mathbf{B}\cdot\nabla\mathbf{A}\rangle = \langle\mathbf{B}\cdot\nabla\mathbf{A}\rangle = \frac{\mathrm{d}}{\mathrm{d}V}\langle\nabla V\cdot\mathbf{BA}\rangle = \mathbf{0} \tag{2.2.69}$$

$$\langle\mathbf{B}\cdot\nabla\mathbf{A}\rangle = \mathbf{0} \tag{2.2.70}$$

so that the generalization also works. However, remember that $\mathbf{BA}\neq\mathbf{AB}$ so that

$$\langle\boldsymbol{\nabla}\cdot(\mathbf{AB})\rangle = \frac{\mathrm{d}}{\mathrm{d}V}\int\mathrm{d}\theta\int\mathrm{d}\zeta\ \nabla V\cdot\mathbf{AB} = \frac{\mathrm{d}}{\mathrm{d}V}\langle\nabla V\cdot\mathbf{AB}\rangle \tag{2.2.71}$$

and there is no guarantee of this vanishing because we have not put any restrictions on $\mathbf{A}$.

2.2.3.3 Flux Coordinate Systems

Finally, let's consider a couple of different flux coordinate systems (i.e., flux coordinates). We can begin by reminding ourselves of some notation. I will use r is a radial coordinate (any flux label will do, though I will enforce the Jacobian determinant to be positive), θ can be a generic poloidal coordinate, ζ a generic toroidal coordinate, and the subscript f implies that the coordinates are now flux coordinate angles. We remind ourselves that we can construct the surface potential

$$G(r,\theta,\zeta) = a(r)\theta + b(r)\zeta + G_0(r,\theta,\zeta) \tag{2.2.72}$$

with two properties satisfied. (1) $\nabla G\times\nabla r$ is single-valued on every path and (2) G is single-valued along paths that do not encircle a magnetic axis or geometric axis (along flux surfaces). We can also write

$$\mathbf{B} = \nabla r\times\nabla G \tag{2.2.73}$$

Note how this automatically implies

$$B^r = \nabla r\cdot\mathbf{B} = \nabla r\cdot\nabla r\times\nabla G = 0 \tag{2.2.74}$$

as required for a flux function/label r. We can then write

$$\mathbf{B} = B^\theta \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta = \mathcal{J} B^\theta \, \nabla\zeta \times \nabla r + \mathcal{J} B^\zeta \, \nabla r \times \nabla\theta \tag{2.2.75}$$

Because $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ this implies

$$\frac{\partial}{\partial\theta}\left(\mathcal{J} B^\theta\right) + \frac{\partial}{\partial\zeta}\left(\mathcal{J} B^\zeta\right) = 0 \tag{2.2.76}$$

and so we can recognize that we can use our surface potential to get the contravariant representation of the components of $\mathbf{B}$. Thus, we use

$$B^\theta = -\frac{1}{\mathcal{J}}\frac{\partial G}{\partial\theta} \tag{2.2.77}$$

$$B^\zeta = \frac{1}{\mathcal{J}}\frac{\partial G}{\partial\zeta} \tag{2.2.78}$$

which satisfies the $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ constraint. Using this with $\nabla r \times \frac{\partial G}{\partial r} \nabla r = 0$ we can write

$$\mathbf{B} = \nabla r \times \nabla G \tag{2.2.79}$$

as we began. We can then again remind ourselves that the form of G is given by

$$\frac{\partial G}{\partial\theta} = a(r) + \frac{\partial G_0}{\partial\theta} \tag{2.2.80}$$

$$\frac{\partial G}{\partial\zeta} = b(r) + \frac{\partial G_0}{\partial\zeta} \tag{2.2.81}$$

We can then use that

$$\begin{aligned}
\frac{\mathrm{d}\Phi_t}{\mathrm{d}r} &= \frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{2\pi}\iiint \mathrm{d}^3x \; \mathbf{B}\cdot\nabla\zeta = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\zeta \; \mathcal{J}\mathbf{B}\cdot\nabla\zeta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\zeta \; \mathcal{J}B^\zeta = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\zeta \; \frac{\partial G}{\partial\zeta} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\zeta \; \left[b(r) + \cancel{\frac{\partial G_0}{\partial\zeta}}\right] = 2\pi b(r)
\end{aligned} \tag{2.2.82}$$

where I have used the periodicity of G_0 for the integral of $\frac{\partial G_0}{\partial\zeta}$. Analogously,

$$\begin{aligned}
\frac{\mathrm{d}\Phi_p}{\mathrm{d}r} &= \frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{2\pi}\iiint \mathrm{d}^3x \; \mathbf{B}\cdot\nabla\theta = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\theta \; \mathcal{J}\mathbf{B}\cdot\nabla\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\theta \; \mathcal{J}B^\theta = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\theta \; [-\frac{\partial G}{\partial\theta}] \\
&= -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\theta \; \left[a(r) + \cancel{\frac{\partial G_0}{\partial\theta}}\right] = -2\pi a(r)
\end{aligned} \tag{2.2.83}$$

Thus

$$G(r,\theta,\zeta) = \frac{1}{2\pi}\left[\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}\theta - \frac{\mathrm{d}\Phi_p}{\mathrm{d}r}\zeta\right] + G_0(r,\theta,\zeta) \tag{2.2.84}$$

Now, if $G_0 = 0$ or is a flux function only, then θ and ζ are considered flux coordinates. If not, we can construct flux coordinates by absorbing G_0 into θ or ζ as we did before. This comes from seeing that G on a flux surface is a constant. Because $\mathbf{B} = \nabla r \times \nabla G$, then if we hold r at r_0 (so stay on a single flux surface), then $\mathbf{B}$ must be perpendicular to ∇G and $\nabla r|_{r=r_0}$. Because $\mathbf{B}$ lies on flux surfaces, then G must be constant on flux surfaces because $\mathbf{B} = \nabla r \times \nabla G$ says that $\mathbf{B}$ lies on surfaces of constant r and constant G. So G must be constant on a flux surface. So when G_0 is a flux surface quantity, then $G - G_0$ is a constant and we have

$$\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}\theta - \frac{\mathrm{d}\Phi_p}{\mathrm{d}r}\zeta = C \tag{2.2.85}$$

This is of course just the equation of a straight line in (θ, ζ) coordinates. Note that we could write

$$\zeta = \frac{\mathrm{d}\Phi_t}{\mathrm{d}\Phi_p}\theta - C = q\theta - C \tag{2.2.86}$$

or

$$\theta = \frac{\mathrm{d}\Phi_p}{\mathrm{d}\Phi_t}\zeta + C = \iota\theta + C \tag{2.2.87}$$

which on a flux surface is the simple equation of a line and hence the field lines are straight. Thus, in this case we have $\theta = \theta_f$ and $\zeta = \zeta_f$. Generically, this will not happen, of course. We can use

$$\begin{aligned} \theta_f &= \theta + 2\pi \frac{G_0}{\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}} \\ \zeta_f &= \zeta \end{aligned} \tag{2.2.88}$$

or

$$\begin{aligned} \theta_f &= \theta \\ \zeta_f &= \zeta + 2\pi \frac{G_0}{\frac{\mathrm{d}\Phi_p}{\mathrm{d}r}} \end{aligned} \tag{2.2.89}$$

to produce flux coordinates. In either case we find

$$2\pi G(r, \theta_f, \zeta_f) = \frac{\mathrm{d}\Phi_t}{\mathrm{d}r}\theta_f - \frac{\mathrm{d}\Phi_p}{\mathrm{d}r}\zeta_f \tag{2.2.90}$$

which for constant G on a flux surface also gives straight lines with field line pitch of the safety factor or rotational transform. The new magnetic field coordinates are then

$$B^r = 0 \tag{2.2.91}$$

$$B^{\theta_f} = \frac{1}{2\pi\mathcal{J}_f}\frac{\mathrm{d}\Phi_p}{\mathrm{d}r} \tag{2.2.92}$$

$$B^{\zeta_f} = \frac{1}{2\pi\mathcal{J}_f}\frac{\mathrm{d}\Phi_t}{\mathrm{d}r} \tag{2.2.93}$$

We can create different flux coordinates (with subscript F) via the transformation

$$\theta_F = \theta_f + \frac{\mathrm{d}\Phi_p}{\mathrm{d}r}G_1(r, \theta_f, \zeta_f) \tag{2.2.94}$$

$$\zeta_F = \zeta_f + \frac{\mathrm{d}\Phi_t}{\mathrm{d}r}G_1(r, \theta_f, \zeta_f) \tag{2.2.95}$$

which implies

$$\begin{aligned} 2\pi G(r,\theta_F,\zeta_F) &= \frac{\mathrm{d}\Phi_t}{\mathrm{d}r}[\theta_F - \frac{\mathrm{d}\Phi_p}{\mathrm{d}r}G_1(r,\theta_f,\zeta_f)] - \frac{\mathrm{d}\Phi_p}{\mathrm{d}r}[\zeta_F + \frac{\mathrm{d}\Phi_t}{\mathrm{d}r}G_1(r,\theta_f,\zeta_f)] \\ &= \frac{\mathrm{d}\Phi_t}{\mathrm{d}r}\theta_F - \cancel{\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}\frac{\mathrm{d}\Phi_p}{\mathrm{d}r}G_1} - \frac{\mathrm{d}\Phi_p}{\mathrm{d}r}\zeta_F + \cancel{\frac{\mathrm{d}\Phi_p}{\mathrm{d}r}\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}G_1} \\ &= \frac{\mathrm{d}\Phi_t}{\mathrm{d}r}\theta_F - \frac{\mathrm{d}\Phi_p}{\mathrm{d}r}\zeta_F \end{aligned} \tag{2.2.96}$$

which allows us freedom through G_1 to change our flux coordinates while keeping magnetic field lines straight.

2.2.3.4 Symmetry Coordinates

The first set of flux coordinates to consider are called symmetry coordinates. As we will see, they should be called axisymmetry coordinates, but that is neither here nor there at this point. They are symmetry coordinates because we define ζ via an extra symmetry in our problem; that it is independent of ζ, namely

$$\frac{\partial Q}{\partial \zeta} = 0 \tag{2.2.97}$$

for any scalar quantity Q. Notice that vectors cannot obey this identity because the basis vectors may change as ζ changes. For example in cylindrical coordinates ∇R and $\nabla\zeta$ change as ζ changes. A tokamak ideally satisfies axisymmetry in the conventional geometric toroidal direction, however, and so looking at symmetric coordinates is useful. It is traditional to use subscript zero or a lower case "o" on these flux coordinates, and so I will use ζ_o as the symmetry coordinate.

We want to know how the basis vectors (tangent basis vectors and reciprocal basis vectors) associated with ζ_o are related to each other. Ideally, they would point in the same direction so that the cross product between them is zero. We can then look at

$$\nabla\zeta_o \times \frac{\partial \mathbf{x}}{\partial \zeta_o} = \mathcal{J}\,\nabla\zeta \times (\,\nabla r_o \times\, \nabla\theta_o) = \mathcal{J}(\,\nabla\zeta_o \cdot\, \nabla\theta_o)\,\nabla r_o - \mathcal{J}(\,\nabla r_o \cdot\, \nabla\zeta_o)\,\nabla\theta_o \tag{2.2.98}$$

For simplicity, say

$$\nabla\zeta_o \times \frac{\partial \mathbf{x}}{\partial \zeta_o} = \mathbf{A} \tag{2.2.99}$$

$$A_{r_o} = \mathcal{J}(\,\nabla\zeta_o \cdot\, \nabla\theta_o) = \mathcal{J} g^{\zeta_0\theta_o} \tag{2.2.100}$$

$$A_{\theta_o} = \mathcal{J}(\,\nabla r_o \cdot\, \nabla\zeta_o) = \mathcal{J} g^{r_o\zeta_o} \tag{2.2.101}$$

$$A_{\zeta_o} = 0 \tag{2.2.102}$$

If $g^{r_o\theta_o} \neq 0$, this implies that $\nabla\zeta_o$ has a component along ∇r_o. We can then write $\nabla\zeta_o$ in a contravariant representation

$$\nabla\zeta_o = \nabla\zeta_o \cdot\, \nabla r_o \frac{\partial \mathbf{x}}{\partial r_o} + \nabla\zeta_o \cdot\, \nabla\theta_o \frac{\partial \mathbf{x}}{\partial \theta_o} + \nabla\zeta_o \cdot\, \nabla\zeta_o \frac{\partial \mathbf{x}}{\partial \zeta_o} = g^{r_o\zeta_o}\frac{\partial \mathbf{x}}{\partial r_o} + g^{\theta_o\zeta_o}\frac{\partial \mathbf{x}}{\partial \theta_o} + g^{\zeta_o\zeta_o}\frac{\partial \mathbf{x}}{\partial \zeta_o} \tag{2.2.103}$$

Unfortunately, the angular symmetry implies nothing to help us with this problem. We must instead specify a ζ_o and determine if it has the desired property of having $\nabla\zeta_o \times \partial\mathbf{x}/\partial\zeta_o = \mathbf{0}$.

In the case of using a cylindrical azimuthal angle ζ_o via $\tan(-\zeta_o) = y/x$. One can then find $\nabla\zeta_o = (y\hat{\mathbf{x}} - x\hat{\mathbf{y}})/\sqrt{x^2+y^2}$. If we assume

$$x = a(r_o, \theta_o)\cos\zeta_o \tag{2.2.104}$$
$$y = -a(r_o, \theta_o)\sin\zeta_o \tag{2.2.105}$$
$$z = b(r_o, \theta_o) \tag{2.2.106}$$

then

$$\frac{\partial \mathbf{x}}{\partial \zeta_o} = -a\sin\zeta_o\hat{\mathbf{x}} - a\cos\zeta_o\hat{\mathbf{y}} = y\hat{\mathbf{x}} - x\hat{\mathbf{y}} \tag{2.2.107}$$

and so

$$\nabla\zeta_o \times \frac{\partial \mathbf{x}}{\partial \zeta} = \hat{\mathbf{z}}\left(\frac{-yx + yx}{\sqrt{x^2+y^2}}\right) = \mathbf{0} \tag{2.2.108}$$

which indeed is zero. Axisymmetry implies symmetry around an axis, and so symmetry coordinates really should be called axisymmetric coordinates.

It is important to realize symmetry coordinates require two assumptions. One is that we are using a geometric angle such that $\frac{\partial \mathbf{x}}{\partial \zeta_o} \times \nabla\zeta_o = \mathbf{0}$ so that we have true axisymmetry and not just symmetry. Also, that if we rotate our coordinates around the axis of symmetry by an arbitrary angle, then the components of the new vector or tensor remain the same.[17] For any scalar this says $Q(r_o, \theta_o, \zeta_0 = a) = Q(r_o, \theta_o, \zeta_o = b)$ for any a and b and so $\partial Q/\partial\zeta_o = 0$. Note because the basis vectors can still depend on ζ_o, we cannot write things like $\partial\mathbf{B}/\partial\zeta_o = \mathbf{0}$. For a vector we can write $\mathbf{B} = B^i(r_o, \theta_0, \zeta_o)\mathbf{e}_i(r_o, \theta_o, \zeta_o) = B^{i\prime}(r_o, \theta_o, \zeta_o + c_\zeta)\mathbf{e}'_i(r_o, \theta_o, \zeta_o + c_\zeta)$ where $\mathbf{e}_i$ is some basis vector set and $\mathbf{e}'_i$ is the basis vector set rotated by an angle c_ζ. Then axisymmetry says that $B^i = B^{i\prime}$ which means that B^i are independent of ζ_o and so $\frac{\partial B^i}{\partial \zeta_o} = 0$. When books say things like $\left(\frac{\partial Q}{\partial \zeta_o}\right)_{r_o,\theta_o} = 0$ for any physical quantity Q, we need to remember that this means components of vectors and tensors, rather than the vectors or tensors themselves.

If you take away only one thing, remember that symmetry coordinates should be axisymmetry coordinates. A simple angular symmetry is not enough, as I will show.

A simple counterexample that angular symmetry is not enough is that instead of the ζ_o constant surface being a plane purely in the z direction, it could tilt, so that at $\zeta_o = 0$ it is a plane extending in y and z with an angle α from the $z = 0$ plane. We could then rotate this plane around the origin for each ζ_o constant plane. Clearly $\frac{\partial \mathbf{x}}{\partial \zeta_o}$ will be the same, but now $\nabla\zeta_o$ will not point in $\partial\mathbf{x}/\partial\zeta_o$ by construction. See Figure 2.5 for an example of these tilted constant ζ planes.

However, once we have $g^{\zeta_o r_o} = g^{\zeta_o\theta_o} = 0$ we can find $g_{\zeta_o r_o} = g_{\zeta_o\theta_o} = 0$ by using $\nabla\zeta_o = H(r_o, \theta_o, \zeta_o)\partial\mathbf{x}/\partial\zeta_o$ and the reciprocal relations.

We now consider the symmetry coordinates for cylindrical (R, Z, ζ) system for a tokamak with axisymmetric $\zeta = \zeta_o$. We previously showed that this system's angle satisfies the necessary constraints. We can find $g_{\zeta_o\zeta_o}$ in this system through the usual manipulations for coordinate systems,

[17] Some textbooks say all physical quantities q must have $\frac{\partial q}{\partial \zeta_o} = 0$ or worse that $\frac{\partial \mathbf{B}}{\partial \zeta_o} = \mathbf{0}$. The first suffers from vagueness (are vectors physical quantities or are there components?) while the latter is too restrictive. Think about it. If $\frac{\partial \mathbf{B}}{\partial \zeta_o} = \mathbf{0}$ that means that $\mathbf{B}$ cannot change directions as we change ζ_o. What we actually want is that the components of $\mathbf{B}$, when rotated around the symmetry axis, are invariant.

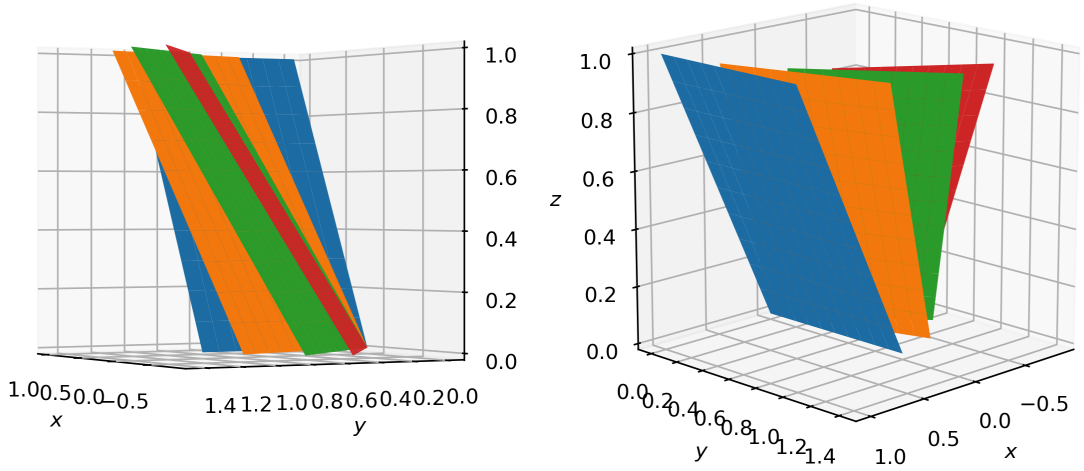

Figure 2.5: This shows planes of constant ζ, when they are tilted off of pure planes in z. We can clearly still have ζ going around the axis, but now the normals of the planes of constant zeta do not point tangent to ζ.

for example in Appendix C.8.2. Then $|\nabla\zeta_o|^2 = 1/R^2$ and so $g^{\zeta_o\zeta_o} = R^2$ and $g_{\zeta_o\zeta_o} = 1/R^2$ using partial orthogonality.

We can then write Ampère's Law for the r_o component

$$\mu_o J^{r_o} = (\boldsymbol{\nabla}\times\mathbf{B})^{r_o} = \frac{1}{\mathcal{J}}\left(\frac{\partial}{\partial\theta_o}B_{\zeta_o} - \frac{\partial}{\partial\zeta_o}B_{\theta_o}\right) \tag{2.2.109}$$

The $\partial B_{\theta_o}/\partial\zeta_o$ term is zero by axisymmetry. In ideal MHD we have $\mathbf{J}\times\mathbf{B} = \nabla p$ with p a flux label. $\mathbf{J}\cdot\nabla p = 0$, and because p is a flux label this implies $\mathbf{J}\cdot\nabla r_o\frac{\partial p}{\partial r_o} = 0$ so that $J^{r_o} = 0$. Thus we find $\partial B_{\zeta_o}/\partial\theta_o = 0$ so that B_ζ is not a function of θ_o or ζ_o and is thus a flux function. Then

$$B^{\zeta_o} = \cancel{g^{\zeta_o r_o}}B_{r_o} + \cancel{g^{\zeta_o\theta_o}}B_{\theta_o} + g^{\zeta_o\zeta_o}B_{\zeta_o} = \frac{B_{\zeta_o}}{R^2} \tag{2.2.110}$$

If we define $I(r_o) = B_{\zeta_o}$ then this says $B^{\zeta_o} = I/R^2$. We then would like to find $\mathcal{J}$. We can do this via our generic representation (2.2.93) which says

$$B^{\zeta_o} = \frac{1}{2\pi\mathcal{J}}\frac{\mathrm{d}\Phi_t}{\mathrm{d}r_o} = \frac{I}{R^2} \tag{2.2.111}$$

Thus

$$\mathcal{J} = \frac{R^2 I}{2\pi}\frac{\mathrm{d}\Phi_t}{\mathrm{d}r_o} = q\frac{R^2 I}{2\pi}\frac{\mathrm{d}\Phi_p}{\mathrm{d}r_o} \tag{2.2.112}$$

for this coordinate system. It is worth noting that R is not a flux label and so $\mathcal{J}$ is not a flux label. Finally, we can rewrite the magnetic field using the poloidal flux $\Phi_p = r_o$. Then

$$2\pi\mathbf{B} = q\,\nabla\Phi_p\times\nabla\theta_o + \nabla\zeta_o\times\nabla\Phi_p \tag{2.2.113}$$

with $q = \mathrm{d}\Phi_t/\mathrm{d}\Phi_p$. We know that $\mathcal{J}\,\nabla\Phi_p \times \nabla\theta_o = \frac{\partial \mathbf{x}}{\partial \zeta_o} = \nabla\zeta_o/R^2$ and so the form is

$$2\pi\mathbf{B} = q\frac{\nabla\zeta_o}{\mathcal{J}R^2} + \nabla\zeta_o \times \nabla\Phi_p = 2\pi I\,\nabla\zeta_o + \nabla\zeta_o \times \nabla\Phi_p \tag{2.2.114}$$

2.2.3.5 Hamada Coordinates

The next coordinate system straightens the **J** current density field lines as well as the magnetic field lines. These are called Hamada coordinates. In order to do so we can follow the same procedure for straightening current lines as we did for the magnetic field since we will require for the straight **J** coordinates that $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$ and $\mathbf{J}\cdot\nabla r = 0$. It is not obvious that r can be chosen so that both **J** and **B** are straight, that is, that the magnetic flux surfaces coincide with the current density flux surfaces. We can follow the same procedure outlined by (2.2.76) with $G_{\mathbf{J}}$ instead of the G used for the magnetic fields. Thus we have

$$\mathbf{J} = \nabla r \times \nabla G_{\mathbf{J}} \tag{2.2.115}$$

and we define I_t and I_p to be the flux of the current density through a constant toroidal or poloidal surface, respectively. Note that the flux of the current density is simply the current through that surface. We'll use I_p^r for the ribbon flux, and so we define

$$I_p^r(r) = \frac{1}{2\pi}\iiint_{V(r)} \mathrm{d}^3x\ \mathbf{J}\cdot\nabla\theta = \iint_{S_p} \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{J} = \iint_{S_p} \mathrm{d}r\mathrm{d}\theta\ \mathcal{J}\,\nabla\theta\cdot\mathbf{J} \tag{2.2.116}$$

$$I_t(r) = \frac{1}{2\pi}\iiint_{V(r)} \mathrm{d}^3x\ \mathbf{J}\cdot\nabla\zeta = \iint_{S_t} \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{J} = \iint_{S_t} \mathrm{d}r\mathrm{d}\zeta\ \mathcal{J}\,\nabla\zeta\cdot\mathbf{J} \tag{2.2.117}$$

We can use that

$$G_{\mathbf{J}}(r,\theta,\zeta) = \frac{1}{2\pi}\left[\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}\theta - \frac{\mathrm{d}\Phi_p}{\mathrm{d}r}\zeta\right] + G_{0,\mathbf{J}}(r,\theta,\zeta) \tag{2.2.118}$$

Then we use the same transformation laws

$$\begin{aligned}\theta_f &= \theta + 2\pi\frac{G_{0,\mathbf{J}}}{\frac{\mathrm{d}I_t}{\mathrm{d}r}}\\ \zeta_f &= \zeta\end{aligned} \tag{2.2.119}$$

or

$$\begin{aligned}\theta_f &= \theta\\ \zeta_f &= \zeta + 2\pi\frac{G_{0,\mathbf{J}}}{\frac{\mathrm{d}I_p}{\mathrm{d}r}}\end{aligned} \tag{2.2.120}$$

to create straight (current density) field lines. Then the equation of a current density line is given by

$$\frac{\mathrm{d}I_t}{\mathrm{d}r}\theta_f - \frac{\mathrm{d}I_p^r}{\mathrm{d}r}\zeta_f = C_o \tag{2.2.121}$$

for a constant $C_o = G_{\mathbf{J}} - G_{0,\mathbf{J}}$.

Now what if we want $G_{0,\mathbf{J}}$ to be a flux function in the sense that $\mathbf{B}\cdot\nabla G_{0,\mathbf{J}}=0$? This is what we would need for straight current density and straight magnetic field lines to coincide with the same coordinate system. It can be shown that for p a flux function and the ideal $\mathbf{J}\times\mathbf{B}=\nabla p$ that this can be rewritten as

$$\frac{\frac{\mathrm{d}I_p}{\mathrm{d}r}\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}-\frac{\mathrm{d}I_t}{\mathrm{d}r}\frac{\mathrm{d}\Phi_p}{\mathrm{d}r}}{4\pi^2\mathcal{J}}-\mathbf{B}\cdot\nabla G_{0,\mathbf{J}}=\frac{\mathrm{d}p}{\mathrm{d}r} \tag{2.2.122}$$

$$\mathbf{B}\cdot\nabla G_{0,\mathbf{J}}=\frac{\frac{\mathrm{d}I_p}{\mathrm{d}r}\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}-\frac{\mathrm{d}I_t}{\mathrm{d}r}\frac{\mathrm{d}\Phi_p}{\mathrm{d}r}}{4\pi^2\mathcal{J}}-\frac{\mathrm{d}p}{\mathrm{d}r}\equiv Q \tag{2.2.123}$$

so that we write $\mathbf{B}\cdot\nabla G_{0,\mathbf{J}}=Q$. If you're curious how you get this, you use the contravariant representations of the magnetic and current density vector fields

$$\mathbf{B}=\frac{1}{2\pi\mathcal{J}}\left[\frac{\mathrm{d}\Phi_p}{\mathrm{d}r}\frac{\partial\mathbf{x}}{\partial\theta_f}+\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}\frac{\partial\mathbf{x}}{\partial\zeta_f}\right] \tag{2.2.124}$$

$$\mathbf{J}=\frac{1}{2\pi\mathcal{J}}\left[\left(\frac{\mathrm{d}I_p}{\mathrm{d}r}-2\pi\frac{\partial G_{0,\mathbf{J}}}{\partial\theta_f}\right)\frac{\partial\mathbf{x}}{\partial\theta_f}+\left(\frac{\mathrm{d}I_t}{\mathrm{d}r}-2\pi\frac{\partial G_{0,\mathbf{J}}}{\partial\zeta_f}\right)\frac{\partial\mathbf{x}}{\partial\zeta_f}\right] \tag{2.2.125}$$

For this to occur we must have

$$\langle\mathbf{B}\cdot\nabla G_{0,\mathbf{J}}\rangle=\langle Q\rangle=0 \tag{2.2.126}$$

$$\oint\frac{\mathrm{d}l}{B}Q=0 \tag{2.2.127}$$

where l is a parameter that gives the length along a magnetic field line and $B=|\mathbf{B}|$. One can then do the flux surface average taking into account flux functions and find a solvability criterion of

$$\frac{\mathrm{d}I_p}{\mathrm{d}r}\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}-\frac{\mathrm{d}I_t}{\mathrm{d}r}\frac{\mathrm{d}\Phi_p}{\mathrm{d}r}=\frac{\mathrm{d}p}{\mathrm{d}r}\frac{\mathrm{d}V}{\mathrm{d}r} \tag{2.2.128}$$

in addition to the closed line integral. We put this result in our integral and find

$$\oint\frac{\mathrm{d}l}{B}\left[\frac{1}{4\pi^2\mathcal{J}}\frac{\mathrm{d}p}{\mathrm{d}r}\frac{\mathrm{d}V}{\mathrm{d}r}-\frac{\mathrm{d}p}{\mathrm{d}r}\right]=0 \tag{2.2.129}$$

We can use then use that field line equations imply $\mathrm{d}\ell/B=\mathrm{d}\zeta_f B^{\zeta_f}$ and use $B^{\zeta_f}=\Phi_t/(2\pi\mathcal{J})$ to find

$$\oint\frac{\mathrm{d}\zeta_f}{\frac{\Phi_t}{2\pi\mathcal{J}}}\frac{1}{4\pi^2\mathcal{J}}\frac{\mathrm{d}p}{\mathrm{d}r}\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{\mathrm{d}p}{\mathrm{d}r}\oint\frac{\mathrm{d}\ell}{B} \tag{2.2.130}$$

$$\frac{1}{2\pi\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}}\frac{\mathrm{d}p}{\mathrm{d}r}\frac{\mathrm{d}V}{\mathrm{d}r}\oint\mathrm{d}\zeta_f=\frac{\mathrm{d}p}{\mathrm{d}r}\oint\frac{\mathrm{d}\ell}{B} \tag{2.2.131}$$

If the field line closes after N toroidal transits then $\oint\mathrm{d}\zeta_f=\int_0^{2\pi N}\mathrm{d}\zeta_f=2N\pi$ and so

$$N\frac{\frac{\mathrm{d}p}{\mathrm{d}r}\frac{\mathrm{d}V}{\mathrm{d}r}}{\frac{\mathrm{d}\Phi_t}{\mathrm{d}r}}=\frac{\mathrm{d}p}{\mathrm{d}r}\oint\frac{\mathrm{d}\ell}{B} \tag{2.2.132}$$

For $\mathrm{d}p/\mathrm{d}r \neq 0$ and using $\frac{\mathrm{d}V}{\mathrm{d}r}/\frac{\mathrm{d}\Phi_t}{\mathrm{d}r} = \mathrm{d}V/\mathrm{d}\Phi_t$ we can write this condition as

$$\frac{1}{N}\oint \frac{\mathrm{d}\ell}{B} = \frac{\mathrm{d}V}{\mathrm{d}\Phi_t} \tag{2.2.133}$$

The left hand side integral is sometimes called the proper length of a field line. Thus on a flux surface, we require all closed field lines to have the same proper length. This condition is true under axisymmetry, but is not in general true. Thus Hamada coordinates can have troubles in stellarators, but work well in tokamak settings. We can then find $\mathcal{J}$ via

$$\left\langle \frac{1}{4\pi^2\mathcal{J}}\frac{\mathrm{d}p}{\mathrm{d}r}\frac{\mathrm{d}V}{\mathrm{d}r} - \frac{\mathrm{d}p}{\mathrm{d}r}\right\rangle = 0 \tag{2.2.134}$$

and so (using $\frac{\mathrm{d}p}{\mathrm{d}r} \neq 0$)

$$\left\langle \frac{1}{\mathcal{J}}\right\rangle = \frac{4\pi^2}{\frac{\mathrm{d}V}{\mathrm{d}r}} \tag{2.2.135}$$

Note that if we choose $r = V$ as our radial coordinate then

$$\left\langle \frac{1}{\mathcal{J}}\right\rangle = 4\pi^2 \tag{2.2.136}$$

We then simply make the choice $\mathcal{J} = 1/(4\pi^2)$ and we have Hamada coordinates. This way $\mathcal{J}$ is a flux label and even better, a constant.

Thus if we put a subscript H on Hamada coordinates we have

$$r_H = V \tag{2.2.137}$$

$$\mathcal{J}_H = \frac{1}{4\pi^2} = \frac{1}{(2\pi)^2} \tag{2.2.138}$$

$$\langle Q\rangle = \oiint \frac{\mathrm{d}\theta_H \mathrm{d}\zeta_h}{(2\pi)^2} Q(r_H, \theta_H, \zeta_H) \tag{2.2.139}$$

which is very simple looking.

2.2.3.6 Boozer Coordinates

These are the straight field line flux coordinates introduced by Boozer. These coordinates divide the magnetic field into $\mathbf{B} = \mathbf{B}_o + \mathbf{B}_\mathbf{J}$ where

$$\boldsymbol{\nabla}\times\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{B}_0 + \boldsymbol{\nabla}\times\mathbf{B}_\mathbf{J} = \mathbf{0} + \mu_0\mathbf{J} \tag{2.2.140}$$

so that $\mathbf{B}_0$ is the vacuum field and because it is curl-free we write it as

$$\mathbf{B}_0 = \nabla\mu \tag{2.2.141}$$

for scalar potential μ, called the magnetic scalar potential. Then one can eventually write

$$\mathbf{B} = \nabla\chi\times\nabla r \tag{2.2.142}$$

$$\mathbf{J} = \frac{1}{\mu_0}\nabla\lambda\times\nabla r \tag{2.2.143}$$

$$\chi = \mu + \frac{\mu_0}{2\pi}\left(I_t\theta_f + I_p^d\zeta_f\right) \tag{2.2.144}$$

$$\lambda = -\frac{\mu_0}{2\pi}\left(2\pi G_0(r,\theta_f,\zeta_f) + \frac{\mathrm{d}I_t}{\mathrm{d}r}\theta_f + \frac{\mathrm{d}I_p^d}{\mathrm{d}r}\zeta_f\right) = -\mu_0 G(r,\theta_f,\zeta_f) \tag{2.2.145}$$

where I_p^d is the disk current. One can replace $I_p^d = -I_p^r$ if one wants to use ribbon currents. Boozer-Grad coordinates then change θ_f and ζ_f so that $\mu = 0$ everywhere.

The Jacobian determinant is given by

$$\mathcal{J} = \frac{\mu_0}{4\pi}\frac{\Phi_t I_p^d + \Phi_p^r I_t}{B^2 - \mathbf{B}\cdot\nabla\mu} = \frac{\mu_0}{4\pi}\frac{\Phi_t I_p^d + \Phi_p^r I_t}{B^2} \tag{2.2.146}$$

if we use $r = \Phi_t$ then

$$\mathcal{J} = \frac{\mu_0}{4\pi}\frac{\iota I_t + I_p^d}{B^2} \tag{2.2.147}$$

There actually is a good amount of information on Boozer Coordinates in D'haeseleer[9] and online resources, so I will defer to them.

2.3 Magnetic Field Terminology

Words are pegs to hang ideas on.

— Henry Ward Beecher

It is worth examining the terminology used for magentic fields in plasma physics in a bit more detail in order to understand why people talk of magnetic tension and pressure. One of the most common analogies is that of magnetic field lines as similar to strings in physical space. This model is not actually explicitly taught all that often, so it is worth examining it here. Back in the old days, magnetic field lines were often called magnetic lines of force, or just lines of force,[18] but the lines of force should be thought of as small tubes that represent properties of the magnetic field. When the lines of force are thought of as analogous to tubes or strings, we can use the same ideas that we use for tubes or strings to characterize the magnetic field lines. For example, a magnetic field line and a string can then both be under tension. And the tension increases if we pull them so that they are no longer a straight line.

That is, a string on a guitar has tension that tries to pull the string back towards its resting spot when it has been plucked. This tension acts such that it creates a force perpendicular to the direction of the cord itself. We call the tendency of magnetic field lines to want to be straight magnetic tension in analogy. In an actual string the tension is a force. For magnetic field lines it is a force density.

Another property with some other analogous situation is that for magnetic field strength $|\mathbf{B}|^2$ which can be thought of as analogous to regular fluid pressure. That is, the effects of a strong magnetic field strength have similar effects to having a high pressure. If you like the string analogy, think of an object that has many strings per volume. The more strings there are, the more difficult it is to move through that area because you have to push against all the strings. In the same way, we think of the density of magnetic field lines as increasing as $|\mathbf{B}|^2$ increases.[19] Because energy

[18] Leading to the memorable quip from Chen[6], "The magnetic field lines are often called 'lines of force.' They are not lines of force. The misnomer is perpetuated here to prepare the student for the treacheries of his profession."

[19] Do not take this situation too seriously, as it is simply helpful as an analogy. Of course in this analogy, the "pressure" is only increasing because there are more strings under tension, whereas in reality with magnetic field lines this increase in pressure is separate from the tension.

density and pressure have the same units, we can think of the energy density of a magnetic field as a pressure, which we will now explore mathematically.

This all comes together when we write $\mu_0\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{B}$ (ignore the Maxwell-Ampère correction) and so then the force density $\boldsymbol{\mathfrak{F}}$[20] is written (just for the magnetic contribution)

$$\boldsymbol{\mathfrak{F}} = \mathbf{J}\times\mathbf{B} = -\frac{\mathbf{B}}{\mu_0}\times(\boldsymbol{\nabla}\times\mathbf{B}) = -\nabla\overbrace{\left(\frac{B^2}{2\mu_0}\right)}^{\text{magnetic pressure}} + \overbrace{\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B}}{\mu_0}}^{\text{magnetic tension}} \tag{2.3.1}$$

Remember that in this way of thinking, the density of magnetic field lines represents a pressure, and magnetic tension is a pull on magnetic field lines to unbend them into straight lines. For an electrically conducting fluid, the force density has a component given by the fluid pressure $-\nabla p$ so that the magnetic pressure really does look like a regular fluid pressure since it is also $-\nabla[B^2/(2\mu_0)]$. One other useful decomposition is to recognize the magnetic curvature $\boldsymbol{\kappa}$ via

$$\begin{aligned}\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B}}{\mu_0} &= \frac{B}{\mu_0}\hat{\mathbf{b}}\cdot\nabla(B\hat{\mathbf{b}}) = \frac{B}{\mu_0}(\hat{\mathbf{b}}\cdot\nabla B)\hat{\mathbf{b}} + \frac{B^2}{\mu_0}\overbrace{\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}}^{\boldsymbol{\kappa}}\\ &= \hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\boldsymbol{\kappa}\\ &= \overbrace{\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla}^{\nabla_\parallel}\left(\frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\boldsymbol{\kappa}\end{aligned} \tag{2.3.2}$$

which means we could write

$$\boldsymbol{\mathfrak{F}} = \overbrace{-\nabla\left(\frac{B^2}{2\mu_0}\right) + \nabla_\parallel\left(\frac{B^2}{2\mu_0}\right)}^{\equiv-\nabla_\perp\left(\frac{B^2}{2\mu_0}\right)} + \frac{B^2}{\mu_0}\boldsymbol{\kappa} \tag{2.3.3}$$

In an Ideal MHD situation where the force density is given by

$$\boldsymbol{\mathfrak{F}} = \mathbf{J}\times\hat{\mathbf{B}} - \nabla p \tag{2.3.4}$$

we can define the total pressure $p_T = p + B^2/(2\mu_0)$ and write

$$\boldsymbol{\mathfrak{F}} = -\nabla p_T + \frac{\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B}}{\mu_0} \tag{2.3.5}$$

$$= -\nabla_\perp p_T - \nabla_\parallel p + \frac{B^2}{\mu_0}\boldsymbol{\kappa} \tag{2.3.6}$$

The last line can also help further explain the mathematical analogy. If you look at the Frenet-Serret formula in Section 1.11, especially Section 1.11.3, we can remember that the normal component to $\hat{\mathbf{b}}$ is given by $\boldsymbol{\kappa}$, the magnetic curvature, and so just as the tension in the string leads to a net force normal to the string, we can identify $\frac{B^2}{\mu_0}\boldsymbol{\kappa}$ as the magnetic tension normal to the magnetic field line $\hat{\mathbf{b}}$.

[20] In this section I will write a force density as $\boldsymbol{\mathfrak{F}}$, but generally we write $\mathbf{F}$ even though this could be confused with a force rather than a force density.

There is another interesting analogy here. In a string we have for distance transverse to the string y and distance along the string in equilibrium x that the tension T for a string of mass per unit length λ is given by Newton's Law as

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\lambda}\frac{\partial^2 y}{\partial x^2} \tag{2.3.7}$$

where $\sqrt{T/\lambda}$ is the velocity of the transverse wave along the string. As we will later find, a shear Alfvén wave has velocity $v_A = \sqrt{B^2/(\mu_0 m_i n)}$ and so if we said magnetic tension is given by $T = B^2/(\mu_0)$ (the coefficient on the curvature $\boldsymbol{\kappa}$) then we can recognize x as along a magnetic field line and y as perpendicular to field lines (in addition we have mass per volume $m_i n$ instead of mass per length λ) as a strong resemblance. That is our analgous equation is

$$\frac{\overbrace{F}^{\sim\frac{\partial^2 y}{\partial t^2}}}{nm_i} = \frac{\overbrace{B^2}^{\sim\frac{T}{\lambda}}}{\mu_0 nm_i}\overbrace{\boldsymbol{\kappa}}^{\sim\frac{\partial^2 y}{\partial x^2}} \tag{2.3.8}$$

Note how these are dimensionally consistent identifications, with the only somewhat questionable identification being $\boldsymbol{\kappa}$ and $\frac{\partial^2 y}{\partial x^2}$. This identification is more difficult to see because $\boldsymbol{\kappa}$ explicitly refers to the magnetic field line while $\frac{\partial^2 y}{\partial x^2}$ instead refers to the change in the transverse component along the string's equilibrium distance x. However, it is not hard to see that it is dimensionally consistent and that if we used a vector equation along the string, it is plausible it would lead to a more simlar equation.

We then see how shear Alfvén waves are considered "plucking" magnetic field lines, similar to a string on a guitar being plucked by the analogies between their equations.

This then relates to the idea of good and bad curvature regions. Remember that $\mathbf{B}\cdot\nabla\mathbf{B}/\mu_0$ tries to unbend field lines so that they are straight. If we have an equilibrium, we can place our origin at the center of the plasma and then there is also fluid pressure from the plasma which will always be pointing radially outward. If the magnetic field lines circle around the plasma, then the fluid (and magnetic) pressure is balancing the magnetic tension. If we push a small amount of fluid outward and the magnetic field can swap places with it,[21] then the magnetic tension in the outwardly moved fluid will decrease and the pressure gradient will decrease, while the fluid that was moved inwardly have more magnetic tension pushing it inward leading to a configuration that can continue to repeat itself; if this process repeats we have an unstable situation as plasma and magnetic field interchange, similar to the Rayleigh-Taylor instability. If the magnetic field lines bend the other way [see the good curvature in Figure 2.6] then if we put the fluid outward into the field line, it would require the magnetic tension to increase so that it opposes the movement and so we are in a stable situation. In an unstable situation this is called an interchange instability because if we view magnetic field lines as magnetic flux tubes, then moving a fluid parcel outward is like swapping or interchanging flux tubes.

A common way of explaining instabilities is to talk about energy and that it is energetically favorable to go to a low energy state. This has the potential to be a very confusing way of explaining the situation. This is because in an isolated system the total energy is constant, so a

[21] For example, one can consider a cylinder and then have the radial perturbation be a sine wave and then calculate if the sine perturbation increases or decreases.

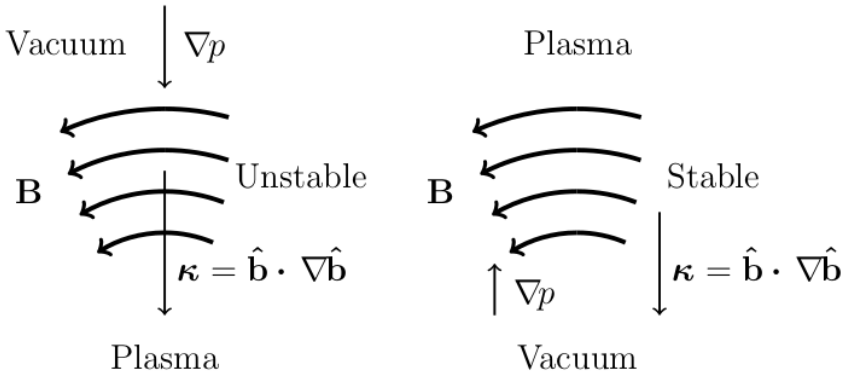

Figure 2.6: When the curvature and pressure gradient align, the interchange stability can occur, whereas when they point in opposite directions, interchange is not favorable. So the configuration on the right has "good curvature" (or favorable curvature) and the configuration on the left has "bad curvature" (or unfavorable curvature).

"low energy state" does not exist. What is actually meant is that it is entropically favorable, i.e., the entropy clearly increases when we have an unstable situation, whereas in a stable situation there is no easy way to increase entropy. When people talk about "lowest energy states", they are instead talking about the potential energy. If you think of kinetic energy as dissipating via friction or other mechanisms as something we no longer count then the lowest *potential* energy state is the most likely state (or the most entropically favorable). The use of "lowest energy" is ubiquitous, but always remember that it depends on what energy you are counting.

Sometimes you will hear an analogy that if it is not carefully explained as a memory device may make you look at this situation and get a misleading impression.[22] We tend to think of pushing against a shape bending towards us as more difficult then pushing on a shape bending away from us. So if we were in a room under pressure with a plastic sheet bulging towards another less pressurized room, we think it would be easier to push the plastic sheet from inside the high pressure room towards the low pressure room than from the low pressure room towards the high pressure room. Thus, if we think about pushing on the magnetic field lines from the plasma side (with $-\nabla p$, the fluid pressure), it is more stable to push on lines that are bending "against" us rather than bending with us (that is if we push on the magnetic field line and it is against us, we are working to make the line straight, whereas if it is bending with us, then us pushing on the magnetic field line will make it less straight). This works as a good device for remembering which curvature is good and bad. With "hard to bend" being good curvature and "easy to bend" being bad curvature. But remember that in fact magnetic field lines want to be unbent into straight lines so the analogy has a core problem with the idea that pushing against the bend should be easier.

We can see a good curvature and bad curvature configuration in Figure 2.6.

[22] If you think I am chiding others, I am guilty of using this analogy as an explanation in my own thesis.

2.4 Magnetic Confinement Devices

There have been a number of magnetic confinement devices designed since the idea of using nuclear fusion for energy was first entertained. These devices can be broadly divided into two types. Those that are "linear", often called "open", and those that close on themselves and so are "toroidal" or "closed". The most popular research devices are stellarators and tokamaks. I will discuss a number of different concepts. First we will look at the closed concepts, those being stellarators, tokamaks, and reversed-field pinches.

2.4.1 Tokamaks

Tokamaks have progressed farthest along the path to break even, and have "simpler" initial designs. The main defining property of a tokamak is the use of a net toroidal current to create the poloidal magnetic field. Another important property is axisymmetry (or near axisymmetry) around the geometric axis. Tokamaks also satisfy other important relations, such as $\frac{B_\theta}{B_\zeta} \sim \frac{B_p}{B_t} \sim \frac{r_0}{R_0}$ where B_t and B_p are toroidal and poloidal magnetic fields and r_0 is the minor radius and R_0 the major radius. The aspect ratio is given by $A = R_0/r_0$ and the inverse aspect ratio is usually denoted $\epsilon = r_0/R_0 = 1/A$ and is considered a small parameter. In reality, $\epsilon \sim 1/3$ is typical, but the approximation of high-aspect ratio does lead to insight into most tokamak physics. When $A \to \infty$ or $\epsilon \to 0$ we have a high-aspect ratio tokamak (again, in reality $A \sim 10$ would be considered very high aspect ratio, with values $A > 3$ being considered fairly large aspect ratio), because in this limit the torus becomes essentially a cylinder. What I mean by this, is that the torus major radius is so large that we can consider the curvature from bending the cylinder into a toroidal shape to be negligible, so we treat it like a cylinder. Thus, a straight tokamak is interpreted as taking the "major radius" to be $2\pi L$ where L is the length of a cylinder. Thus we treat the torus in a cylindrical geometry where $R_0 = 2\pi L$ and $\zeta = z/L$. This analysis will miss effects that do not take into affect curvature of the torus, but usually suffices for a basic analysis of a problem and has the advantage of being simple to analyze analytically.

Tokamaks have a couple of large drawbacks, however. They are susceptible to pressure and current instabilities. Stellarators avoid the current instabilities by not having net currents for the most part. Tokamaks are found to be unstable for number densities growing too large, a phenomenon called the Greenwald density limit.

There are also spherical tokamaks[23] where $A \sim 1$, so that they look like a cored apple (see Figure 2.8). Note that $A = 1$ would simply be a sphere, so it is strictly speaking impossible to achieve as a toroidal magnetic confinement device, but this "low" aspect ratio is the defining property of a spherical tokamak. Spherical tokamaks have a number of advantages and disadvantages relative to "regular" tokamaks. The key advantage is that the magnetic field lines generally twist around the geometric axis near the geometric axis where the stability properties of the plasma are the best.[24] The key disadvantage is that the "central stack", the center of the torus, is very small and so it is more difficult to access the inboard side (the area of the torus nearest to the geometric axis) of the device or to put diagnostics or plasma heating devices in this space.

The most important property of a tokamak is that the poloidal field is generated by a net current in the plasma, and that the plasma is axisymmetric.

[23] Do not confuse these with spheromaks which are a completely different type of magnetic confinement concept.
[24] Think about good and bad curvature regions, and you will see why.

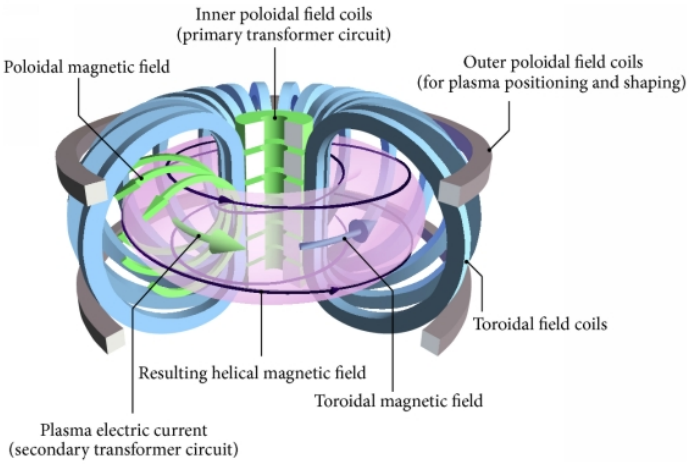

Figure 2.7: The image shows an exmaple of a generic tokamak. Note the axisymmetric toroidal magnetic field coils. The image is from S. Li, H. Jiang, Z. Ren, C. Xu - S. Li et. al. "Optimal Tracking for a Divergent-Type Parabolic PDE System in Current Profile Control" , Abstract and Applied Analysis doi:10.1155/2014/940965. It is available to the public under a CC 4.0 license.

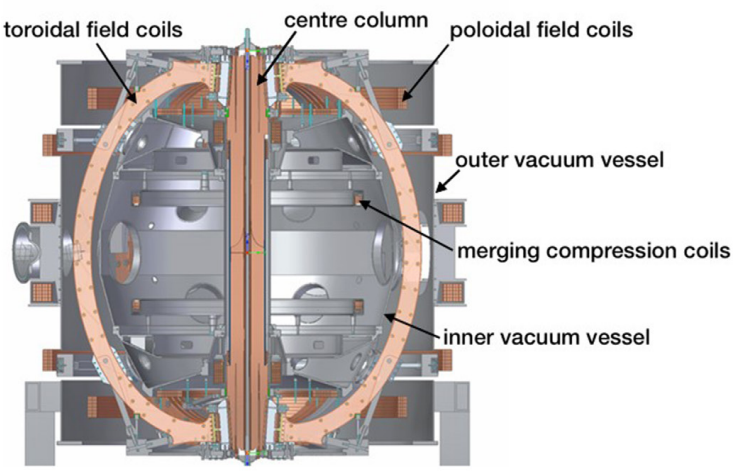

Figure 2.8: The image shows an example of a spherical tokamak. Note the very small central column (the central hole for the torus). The image is from A. Sykes et al. - (2018). "Compact fusion energy based on the spherical tokamak". Nuclear Fusion 58 (1): 016039. doi:10.1088/1741-4326/aa8c8d. ISSN 0029-5515. It is available to the public under a CC 3.0 license.

2.4.2 Stellarators and Heliotrons

Stellarators' and heliotrons' distinctive property is a lack of net toroidal current within the plasma, and so the external magnetic coils determine the magnetic flux surfaces. In fact, it is desired that the plasma not produce much net toroidal current in stellarators or heliotrons, because the toroidal currents deform or destroy the flux surfaces which are usually designed for optimal confinement; in addition, for a reactor this would be non-steady state.[25] One way of stating this is by using the rotational transform. The external coils or magnets are designed to create a rotational transform.

First, let's examine the difference between a stellarator and a heliotron. The difference is somewhat subtle and often ignored in practice. A stellarator device achieves the magnetic flux surfaces with external magnetic field coils in general. A heliotron uses magnetic coils that helically (that is both in the toroidal and poloidal directions) wind around the plasma. If one subscribes to such a definition then Wendelstein-7X (W7X) is a stellarator but not a heliotron, whereas the Large Helical Device (LHD) is both a stellarator and a heliotron.[26] I personally find this treatment somewhat hair-splitting, but it is good to be aware that there can be differences implied by this splitting of stellarator and heliotron. From now onward, I will just speak of stellarators, since they are the more general concept.

General principles establish that plasma confinement and stability require a magnetic field with both toroidal and a poloidal magnetic fields. One way to achieve the right combination is to use magnetic coils to produce the flux surfaces desired, and make sure that such an equilibrium is consistent. Designing coils to do this is still a subject of much investigation, and generally requires computational tools to produce coils with desired flux surfaces. As we will learn, plasma particles drift off flux surfaces, and so designing the flux surfaces so that the drift doesn't lead to loss of confinement is of major importance. This led the original stellarators to have figure-eight like confinement vessels with simpler coils around them. Current stellarators have toroidal confinement vessels with exotic-looking coils to create the required flux surfaces.

Because of the helical coils (or coils that create magnetic fields similar to a helical coil) there is generally not an angle over which we can say that the flux surfaces are invariant. That is there is no axisymmetric angles (where quantities are the same along an angle). In some cases, there is an angle (though it is not the simple geometric angle) that can satisfy a similar condition but only for the magnetic field modulus. In this case, it is $B = |\mathbf{B}|$ that is symmetric in an angle along flux surfaces. That is for $B(r, \alpha, \beta)$ for angles α and β, with β the quasisymmetric angle, then $B(r, \alpha, \beta = a) = B(r, \alpha, \beta = b)$ and so $B(r, \alpha, \beta) = B(r, \alpha)$ only. Then we say a "quasisymmetry" exists. There are three types. Quasihelical symmetry, where the quasisymmetric angle is a simple combination of the geometric θ and ζ in primitive toroidal coordinates. We could write $B(r, \theta, \zeta) = B(r, m\theta - n\zeta)$ for some values of m and n (integers so that the new angle closes on itself). Quasi-poloidal symmetry has $B(r, \theta, \zeta) = B(r, \zeta)$ so that the B is independent of the poloidal angle. Quasi-axisymmetry has $B(r, \theta, \zeta) = B(r, \theta)$ so that B is independent of the toroidal angle. Why we care about B is that in Boozer coordinates one can write the guiding center equations of motion such that they only depend on a position via r and B, so that these

[25] Of course, the magnetic coils may be designed such that they create fields that expect a net toroidal current in the stellarator plasma. The important thing is that the magnetic flux surfaces are generated mostly by external coils and magnets.

[26] The terminology is in fact a good bit more complicated. The geometric "figure eight" stellarator is sometimes called a spatial stellarator. Subsequent devices which were more toroidal, but had separate helical coils on each end are called classical stellarators. As far as I can tell, torsatron and heliotron are synonymous.

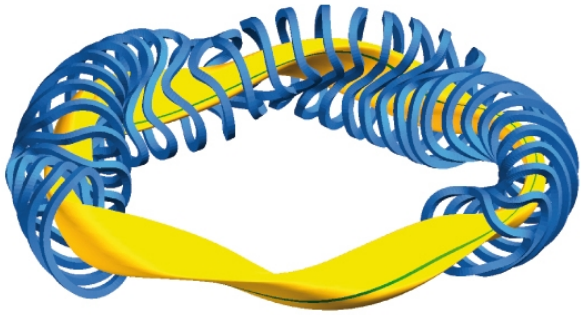

Figure 2.9: The image shows an example of a stellarator with the Wendelstein 7X design. Note how the coils are not simple. The yellow surface is a flux surface and the green line an example magnetic field line. The image is from Max-Planck Institut für Plasmaphysik (IPP) and IPP has the copyright. It is available to the public under a CC 3.0 license.

equations are similar to what happens in an axisymmetric field. Another way of saying this is that there is an angular momentum for some angle coordinate that is conserved.

Because stellarators keep their equilibrium flux and don't have a net current (or a very small net current) at steady-state, they could be easily operated in a steady-state. In fact, this is one large advantage of stellarators, as it is quite obvious how to operate one in steady state. A related disadvantage is that stellarators cannot really use ohmic heating to start and maintain a plasma since they require little net current.

In some sense, stellarators have more freedom than tokamak plasmas because they are not restricted by axisymmetry. This means that it is much more difficult to design, but you can also tailor your design much more easily. In a day of vast computational resources, making a tailored design for a stellarator is not as large of a burden as it once was.

The bottom line is that a stellarator does not impose axisymmetry, and tries to reduce net current to zero so that the flux surfaces are completely due to the magnetic field coils.

2.4.3 Reversed Field Pinch

A reversed field pinch, almost always referred to as an RFP, is yet another toroidal confinement concept. In this one, the most defining feature is that the toroidal magnetic field reverses direction as one goes from the magnetic axis ($r = 0$) outward radially towards the confinement vessel. This means at some radial point that $B_t = 0$, which is usually near the edge, that is for a the minor radius of the plasma, the field reversal is near $r \approx a$. This means that B_t/B_p is no longer always a large number as it is in a tokamak. Generally speaking, this means that $B_t \sim B_p$ and that $|B|$ is moderate in comparison to tokamaks (or stellarators). This means less energy is required to power the magnetic coils. The major disadvantage is that RFPs have terrible stability near the edge. The ways of dealing with this are putting a close-fitting electrically conductive shell (the shell generates currents that counteract the instabilities) or putting a series of magnetic coils that receive feedback from the plasma and act to prevent instabilities directly. Another disadvantage is that the current required to produce an RFP is rather large and so would require advances in current production. Ohmic current production is the only known way to get the required currents,

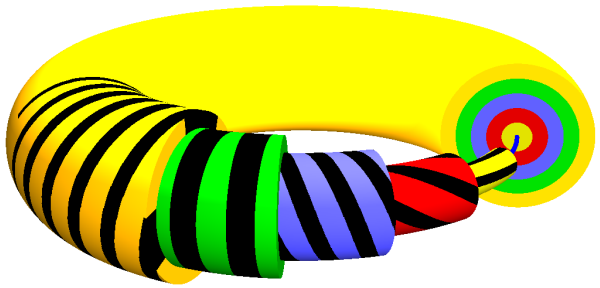

Figure 2.10: The image shows an example of an RFP, especially the q profile. Note how the black lines indicating the q profile reverse their slant as one moves out in minor radius. The image was made by user DaveBurke on Wikipedia. It is available to the public under a CC 2.5 license.

but ohmic current production becomes much less efficient as the plasma temperature rises.

The "pinch" part of RFP refers to the fact that the magnetic field produced by the currents in the RFP pinch the plasma radially inward away from the wall.

The most important property of an RFP is that the toroidal magnetic field reverses as one goes out in (minor) radius. Hence q goes through zero and changes sign. RFPs require less powerful magnets, but have plasma surface instabilities requiring some sort of intervention. For the close-fitting wall solution, the problems for a fusion reactor will be the heat and neutrons into the close-fitting wall.

2.4.4 Z Pinch

The Z pinch or, sometimes zeta pinch, uses the pinch effect previously described. That is, by driving a large axial current (axial meaning along the Z direction, which in toroidal Z pinches means in the ζ direction), the plasma is pinched radially inward. The Z pinches proved to be too unstable due to the large currents being driven, and so were abandoned in favor of tokamaks where larger magnetic fields were instead used. Currently, Z pinches tend to be linear devices (so they are cylinders), and are less studied as energy producing devices, than as devices capable of producing high energy x-rays, and for national defense adjacent concepts.[27] Z pinches as fusion devices are no longer a major area of study, though there are some ideas of using some sort of magnetic liner[28] that would aid in the pinch to produce energy.

2.4.5 Field Reversed Configurations and Spheromaks

Field Reversed Configurations or FRCs, are plasmas that look like smoke rings, and are, along with spheromaks, compact toroids. They are generated in linear devices, and are self-stable (for a while...). Self-stable actually means that magnetic coils are not (directly) causing the toroidal shape of the plasma. FRCs have a weak to no toroidal magnetic field on them, and so have only

[27] If you do not know what national defense adjacent is, then feel free to look at the uses of Z pinches online.
[28] By magnetic liner, I mean some material surface around the plasma that has inertia when pinched inward.

a confining poloidal magnetic field which allows larger $\beta = 2\mu_0 p/B^2$ which translates into more pressure for less magnetic field and so better power production capabilities. Spheromaks have a toroidal magnetic component along and in them.[29]

Compact toroids still do not have the performance of stellarators or tokamaks, but their linear design and high β have led to continued interest as fusion concepts.

2.4.6 Magnetic Mirrors

Magnetic mirrors are linear devices whose defining feature is that end losses are prevented by a steep gradient in $|\mathbf{B}|$. The particles then "bounce" when they meet the ends, as if they were hitting a mirror, and go the other direction. The problem that magnetic mirrors have faced is that they don't stop high energy particles and so the end losses have proved to be too large. With more powerful magnets and a larger gradient in $|\mathbf{B}|$, one can contain more particles, and so there is some renewed interest in magnetic mirrors, often called gas dynamic traps (GDTs) because they also use other strategies to prevent ion and electron losses at the ends.

2.5 Plasma Kinetics

> *Ludwig Boltzmann, who spent much of his life studying statistical mechanics, died in 1906, by his own hand. Paul Ehrenfest, carrying on the same work, died similarly in 1933. Now it is our turn to study statistical mechanics.*
>
> — David Goodstein in *States of Matter*

Now let's start with a very basic way of looking at a collection of particles.[30] This will just touch the surface, if you would like some good plasma kinetics references, such as notes I helped edit J. D. Callen's notes[5]. Montgomery[15] is an old, but quite readable reference. There are also many modern texts now available on the subject.

We can start with Newton's equations for each particle, considering each a δ-function in position. This will lead to the Klimontovich equation which, though exact, is also not continuous and not very useful.[31] We consider there to be distinct plasma species labeled by s. Then in each species there are N_s particles of the species. We could then write out the forces for each particle via

$$\sum_s \sum_{i=1}^{N_s} \frac{\mathrm{d}\mathbf{p}_{s,i}(\mathbf{x}_{s,i},t)}{\mathrm{d}t} = q_s \left[\mathbf{E}(\mathbf{x}_{s,i},t) + \mathbf{v}_{s,i} \times \mathbf{B}(\mathbf{x}_{s,i},t)\right] \tag{2.5.1}$$

where $\mathbf{p}_{s,i}(\mathbf{x}_{s,i},t)$ is the momentum of the ith particle of species s as a function of the position of that particle at time t. For our non-relativistic, non-quantum case $\mathbf{p}_{s,i} = m_s \mathbf{v}_{s,i}(\mathbf{x}_{s,i},t)$ the mass of the species times the velocity of the ith particle of species s. The $\mathbf{E}$ and $\mathbf{B}$ come from Maxwell's equations, and are in fact very ugly because we are still using point particles. Thus with $\mathbf{x}_{s,i}$

[29] If you take a spheromak and put a central core conductor through its hole and add external magnetic coils, you have basically created a spherical tokamak.

[30] Do not worry. Despite the quote by Goodstein, learning plasma kinetics, which encompasses statistical mechanics, is pretty harmless.

[31] I mean not practically useful to find particle trajectories. It is extremely useful for seeing how to derive equations!

pointing to the ith particle of species s we have

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\sum_s\sum_{i=1}^{N_s}\frac{q_s(\mathbf{x}_{s,i}-\mathbf{x})}{|\mathbf{x}_{s,i}-\mathbf{x}|^3} = \iiint_V \mathrm{d}^3x'\ \frac{\rho_q(\mathbf{x}')(\mathbf{x}'-\mathbf{x})}{4\pi\epsilon_0|\mathbf{x}'-\mathbf{x}|^3} \tag{2.5.2}$$

$$\mathbf{B}(\mathbf{x}) = \iiint_V \mathrm{d}^3x'\ \frac{\mu_0\mathbf{J}(\mathbf{x}')\times(\mathbf{x}'-\mathbf{x})}{4\pi|\mathbf{x}'-\mathbf{x}|^3} \tag{2.5.3}$$

$$\mathbf{J}(\mathbf{x}) = \sum_s\sum_{i=1}^{N_s}\rho_{q,s}\mathbf{v}_{s,i} \tag{2.5.4}$$

where $\mathbf{J}$ is the definition of the current density. These are "ugly" because

$$\rho_q(\mathbf{x}') = \sum_s\sum_{i=1}^{N_s}q_s\delta(\mathbf{x}_{s,i}) \tag{2.5.5}$$

$$\mathbf{J}(\mathbf{x}') = \sum_s\sum_{i=1}^{N_s}q_s\delta(\mathbf{x}_{s,i})\mathbf{v}_{s,i}(\mathbf{x}_{s,i},t) \tag{2.5.6}$$

Note that these definitions are if we care only about the plasma particles. If we include externally applied fields, then we can do the same thing, specifying all the particles of the total system. Alternatively, we could instead focus on a specific volume and make sure our $\mathbf{E}$ and $\mathbf{B}$ satisfy the proper boundary conditions consistent with the external fields and our distribution of particles.

This is a set of $N_0 = \sum_s N_s$ equations that are all coupled to each other. Solving N_0 equations when it is orders of magnitude larger than Avogadro's number[32] is an impossible proposition for complicated dynamics. We can make some progress by introducing a distribution function $f_s(\mathbf{x}_s, \mathbf{v}_s, t)$ for each species s. This distribution function will have units of inverse (volume times speed cubed) or in SI units $\mathrm{m}^{-6}\,\mathrm{s}^3$. Then f_s is a way of telling us the probability of a particle being in a specific volume of phase space,[33] where phase space is an abstract space of all possible velocities and positions. That is, the probability of a particle of species s having a position between $\mathbf{x}$ and $\mathbf{x}+\mathrm{d}\mathbf{x}$ and a velocity between $\mathbf{v}$ and $\mathbf{v}+\mathrm{d}\mathbf{v}$ at time t is given by $f_s(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}$

There is often confusion over using this phase space because one should not think of it as the actual position or velocity of a particle in the system. Instead, the trajectory of the particle is a specific curve in this generalized phase space and f_s is a way of giving us the proper information to determine what curve the particle is likely to be on. This also explains why $\mathbf{v}$ and $\mathbf{x}$ are considered independent of each other. For any position $\mathbf{x}$ in this phase space, any $\mathbf{v}$ is possible. When we move to actual trajectories of particles, then the actual $\mathbf{v}^a_{s,i}$ and $\mathbf{x}^a_{s,i}$ followed by particle i of species s are obviously related. Thus, if we are talking about a specific actual trajectory, we can use the relation $\frac{\mathrm{d}\mathbf{x}^a_{s,i}}{\mathrm{d}t} = \mathbf{v}^a_{s,i}$ but if we are talking about the phase space $\mathbf{x}$ and $\mathbf{v}$, $\frac{\mathrm{d}x_i}{\mathrm{d}v_j} = 0$ for all i and j because the positions and velocities are all independent of each other. That is, for a particle trajectory $\frac{\mathrm{d}\mathbf{x}^a_{s,i}}{\mathrm{d}t} = \mathbf{v}^a_{s,i}$, clearly $\mathbf{x}^a_{s,i}$ and $\mathbf{v}^a_{s,i}$ have a relationship of some form that is like $\mathbf{x}^a_{s,i} = \mathbf{x}^a_{s,i}(t(\mathbf{v}^a_{s,i}))$ because they are both parameterized by t. In the phase space, there is no time dependence and hence no connection.

[32] Avogadro's number N_A is the number of constituent particles in one mole (by definition) and is defined to be $6.022\times10^{23}\,\mathrm{mol}^{-1}$.

[33] When generalizing to quantum or relativistic situations, one should use momentum and position instead of velocity.

In our case, we will assume we do know the trajectories of the particles exactly as given above as the Dirac delta functions. Then clearly

$$f_s(\mathbf{x},\mathbf{v},t)=\sum_{i=1}^{N_s}\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)] \tag{2.5.7}$$

because it is guaranteed to be at a specific velocity and position but zero elsewhere. In general, it won't be possible to know this, but let's see what happens if we look at that change in the distribution over time. We will have

$$\begin{aligned}\frac{\partial f_s}{\partial t}&=\sum_{i=1}^{N_s}\left[\frac{\partial\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]}{\partial t}\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]+\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]\frac{\partial\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]}{\partial t}\right]\\&=\sum_{i=1}^{N_s}\left[\frac{\partial\mathbf{x}_{s,i}}{\partial t}\cdot\frac{\partial\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]}{\partial\mathbf{x}_{s,i}}\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]+\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]\frac{\partial\mathbf{v}_{s,i}}{\partial t}\cdot\frac{\partial\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]}{\partial\mathbf{v}_{s,i}}\right]\end{aligned} \tag{2.5.8}$$

Now we can switch from partial to full derivatives on the $\mathbf{x}_{s,i}$ and the $\mathbf{v}_{s,i}$ because they are variables of only t. Thus we have

$$\frac{\partial f_s}{\partial t}(\mathbf{x},\mathbf{v},t)=\sum_{i=1}^{N_s}\left[\frac{\mathrm{d}\mathbf{x}_{s,i}}{\mathrm{d}t}\cdot\frac{\partial\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]}{\partial\mathbf{x}_{s,i}}\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]+\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]\frac{\mathrm{d}\mathbf{v}_{s,i}}{\mathrm{d}t}\cdot\frac{\partial\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]}{\partial\mathbf{v}_{s,i}}\right] \tag{2.5.9}$$

We can use two things now. For the $\mathbf{v}_{s,i}$ and $\mathbf{x}_{s,i}$ we have the actual trajectories so

$$\frac{\mathrm{d}\mathbf{v}_{s,i}}{\mathrm{d}t}=\mathbf{a}_{s,i}(t)=\frac{q_s}{m_s}\left(\mathbf{E}+\mathbf{v}_{s,i}\times\mathbf{B}\right) \tag{2.5.10}$$

$$\frac{\mathrm{d}\mathbf{x}_{s,i}}{\mathrm{d}t}=\mathbf{v}_{s,i} \tag{2.5.11}$$

and that because of the delta functions we have for any $g(\mathbf{x}-\mathbf{x}_{s,i})$ that

$$\frac{\partial g(\mathbf{x}-\mathbf{x}_{s,i})}{\partial\mathbf{x}_{s,i}}=-\frac{\partial g(\mathbf{x}-\mathbf{x}_{s,i})}{\partial\mathbf{x}}\quad\text{and}\quad\mathbf{x}_{s,i}=\mathbf{x} \tag{2.5.12}$$

$$\frac{\partial g(\mathbf{x}-\mathbf{x}_{s,i})}{\partial\mathbf{v}_{s,i}}=-\frac{\partial g(\mathbf{x}-\mathbf{x}_{s,i})}{\partial\mathbf{v}}\quad\text{and}\quad\mathbf{v}_{s,i}=\mathbf{v} \tag{2.5.13}$$

Using these two identities we can then write

$$\begin{aligned}\frac{\partial f_s}{\partial t}(\mathbf{x},\mathbf{v},t)&=\sum_{i=1}^{N_s}\left[-\mathbf{v}_{s,i}\cdot\frac{\partial\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]}{\partial\mathbf{x}}\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]-\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]\mathbf{a}_{s,i}\cdot\frac{\partial\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]}{\partial\mathbf{v}}\right]\\&=\sum_{i=1}^{N_s}\left[-\mathbf{v}\cdot\frac{\partial\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]}{\partial\mathbf{x}}\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]-\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]\mathbf{a}\cdot\frac{\partial\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]}{\partial\mathbf{v}}\right]\end{aligned} \tag{2.5.14}$$

where

$$\mathbf{a}\equiv\frac{q_s}{m_s}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

We can use that $\mathbf{x}$ and $\mathbf{v}$ are independent of each other, so

$$\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]\frac{\partial\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]}{\partial\mathbf{v}} = \frac{\partial}{\partial\mathbf{v}}\big(\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]\big) \tag{2.5.15}$$

$$\frac{\partial\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]}{\partial\mathbf{x}}\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)] = \frac{\partial}{\partial\mathbf{x}}\big(\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]\big) \tag{2.5.16}$$

It is important to use this order if we are to conceptually keep the independence of $\mathbf{x}$ and $\mathbf{v}$ but not of the particular trajectories of the particles $\mathbf{x}_{s,i}$ and $\mathbf{v}_{s,i}$. By doing this, we can also take our $\mathbf{v}$ and $\mathbf{a}$ terms and derivatives outside of the sum to find

$$\begin{aligned}\frac{\partial f_s}{\partial t}(\mathbf{x},\mathbf{v},t) &= -\left[\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}+\mathbf{a}\cdot\frac{\partial}{\partial\mathbf{v}}\right]\sum_{i=1}^{N_s}\delta[\mathbf{x}-\mathbf{x}_{s,i}(t)]\delta[\mathbf{v}-\mathbf{v}_{s,i}(t)]\\ &= -\left[\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}+\mathbf{a}\cdot\frac{\partial}{\partial\mathbf{v}}\right]f_s\end{aligned} \tag{2.5.17}$$

We can then take the full time derivative of f_s and find

$$\begin{aligned}\frac{\mathrm{d}f_s}{\mathrm{d}t} &= \frac{\partial f_s}{\partial t}+\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\frac{\partial f_s}{\partial\mathbf{x}}+\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\frac{\partial f_s}{\partial\mathbf{v}} = \frac{\partial f_s}{\partial t}+\mathbf{v}\cdot\frac{\partial f_s}{\partial\mathbf{x}}+\mathbf{a}\cdot\frac{\partial f_s}{\partial\mathbf{v}}\\ &= \overbrace{-\left\{\cancel{\mathbf{v}\cdot\frac{\partial f_s}{\partial\mathbf{x}}}+\cancel{\mathbf{a}\cdot\frac{\partial f_s}{\partial\mathbf{v}}}\right\}}^{=\frac{\partial f_s}{\partial t}}+\cancel{\mathbf{v}\cdot\frac{\partial f_s}{\partial\mathbf{x}}}+\cancel{\mathbf{a}\cdot\frac{\partial f_s}{\partial\mathbf{v}}} = 0\end{aligned} \tag{2.5.18}$$

This is not as surprising as one might at first expect. It is a statement that the probability can only be moved around, but that no particles are disappearing.

We can sum over all species and we will find the same answers because the $\mathbf{E}$ and $\mathbf{B}$ are from the total and not just for a single species. Thus with $f=\sum_s f_s$ we write the total Klimontovich equation as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_s f_s = 0 \tag{2.5.19}$$

This is of course equivalent to simply writing out our N_0 equations above and solving for the particle trajectories using Newton's Second Law. We have simply rearranged the information by putting it into the distribution function f. Because $N_0 > N_A$ and sometimes $N_0 \gg N_A$, the actual calculation of these trajectories would be enormously time consuming for computers. You would also be left with the trajectories of N_0 particles, which may not give you any intuition for what is happening. Imagine I gave you the positions and velocities of all the particles in a tire. Would you be able to easily tell me what the pressure inside the tire is? We also have mathematical headaches because of those Dirac delta functions lurking in f_s. If we were to actually look at the $\mathbf{E}$ and $\mathbf{B}$ generated by the point particles we would have singularities at all the point particle positions. What we'd like to do is reduce the information we have, smooth the results, and have a way of getting information about the system as a whole rather than of specific particles. This is where plasma kinetics comes in.

2.5.1 Smoothing and Fluid Quantities

There are a couple of ways of imagining going from the spiky functions of the Klimontovich equation to smooth distributions for the plasma kinetic equation.

One way is inspired from thermodynamics, and thus has all of the intricacies and troubles of interpretation that comes from such thermodynamic thought experiments. Namely, that it seems what we're doing is sensible, but it requires us to consider an immense number of states that are "macroscopically" indistinguishable. This requires us to determine what is "macroscopic" and what is "distinguishable" which may not seem physically relevant. In any case, this is the ensemble approach. This approach says that we should average over an ensemble of systems that are consistent with the values we see macroscopically, but could differ in their microscopic arrangements. We are usually dealing with a 6D+1 (six-dimensional plus time) arrangement,[34] but the interpretational issues are no different if we restrict ourselves for discussion to 2D+1. The simplest method is to simply say that things are macroscopically indistinguishable for a small volume around a point in phase space. Deciding what is small enough is then always a question, and one has to really think about whether we can choose the same volume for every point in our phase space. You may also have trouble deciding if particular points in phase space really are indistinguishable for the macroscopic quantities you care about.

Another subtlety is in what does "average" mean. Is this an average over time? Or is it an average in the sense that we should appropriately weight each configuration in the ensemble. If that is the case, how should each weighting be chosen? In statistical mechanics the latter definition of weighting configurations is used. Then different ensembles are used depending on what macroscopic properties you wish to keep constant. See a statistical mechanics book on the microcanonical, the canonical, the grand canonical, the Gibbs, or the enthalpy ensembles to see all of the possibilities. One must now be very careful about counting all the ways that a system is represented correctly. If you accidentally overcount or overrepresent a state, you can get incorrect answers. Thus, this approach requires mathematical sophistication, perseverance, and accuracy to get results. There is nothing wrong with this approach, but an easier method presents itself.

The second way is to simply think of splitting our point particles in a manner consistent with charge and mass conservation. This is often called a mush limit, and involves changing our particles by spreading them into a continuous smear near their initial $\mathbf{x}_{s,i}$ and $\mathbf{v}_{s,i}$, and thus forming a continuous distribution. Operationally, we can think of this as choosing a box size in phase space and spreading the charge and mass throughout the box so that the same total charge and mass is in it, and it is consistent with surrounding box edges. Simplistically we could think of doing a process where we split $q_s \to q_s/2 + q_s/2$ and $m_s \to m_s/2 + m_s/2$ with each particle spread a little bit in space and repeating this on each of the subdivided particles until we have a sufficiently smoothed out function.

Then we would consider the variations in $\mathbf{v}$ and $\mathbf{x}$ to be negligible (small enough box sizes), and, of course, we are not messing with t.[35] Both ρ_q and $\mathbf{J}$ are negligibly changed for the scale we are looking at so $\mathbf{E}$ and $\mathbf{B}$ are the same (except very near particle positions, where they are now

[34] The 6D is often written 3D3V to emphasize three spatial and three velocity dimensions. If quantum or relativistic effects are included this is still written this way, though $\mathbf{p}$ will replace $\mathbf{v}$ in the phase space considered

[35] You might argue that this is essentially choosing small enough phase space volumes for an ensemble average, and it is mathematically equivalent, though I would say this mush limit process is easier for me to comprehend. Depending on how an ensemble average is done, it does not even necessarily yield a continuous distribution.

smooth and not singular). By design, q_s/m_s remains the same. Thus if we use $\langle\cdot\rangle$ to indicate the new mushed quantities we would find

$$\left\langle \frac{\mathrm{d}f}{\mathrm{d}t} \right\rangle = \sum_s \left\langle \frac{\mathrm{d}f_s}{\mathrm{d}t} \right\rangle = 0 \tag{2.5.20}$$

$$\left\langle \frac{\mathrm{d}f_s}{\mathrm{d}t} \right\rangle = \left\langle \left\{ \frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial}{\partial \mathbf{v}} \right\} f_s \right\rangle = 0 \tag{2.5.21}$$

$$\frac{\partial \langle f_s\rangle}{\partial t} + \mathbf{v}\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{x}} + \frac{q_s}{m_s}\left\langle (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_s}{\partial \mathbf{v}} \right\rangle = 0 \tag{2.5.22}$$

For the last term, we can consider a formal expansion around the true $\mathbf{E}$ such that $\mathbf{E} = \langle\mathbf{E}\rangle + \widetilde{\mathbf{E}}$ and $f_s = \langle f_s\rangle + \widetilde{f_s}$. We can then write

$$\mathbf{E}\cdot\frac{\partial f_s}{\partial \mathbf{v}} = \langle\mathbf{E}\rangle\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{v}} + \widetilde{\mathbf{E}}\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{v}} + \langle\mathbf{E}\rangle\cdot\frac{\partial \widetilde{f_s}}{\partial \mathbf{v}} + \widetilde{\mathbf{E}}\cdot\frac{\partial \widetilde{f_s}}{\partial \mathbf{v}} \tag{2.5.23}$$

By our definition of $\langle\widetilde{q}\rangle = 0$ for any quantity q. So we have

$$\left\langle \widetilde{\mathbf{E}}\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{v}} \right\rangle = \left\langle \langle\mathbf{E}\rangle\cdot\frac{\partial \widetilde{f_s}}{\partial \mathbf{v}} \right\rangle = 0 \tag{2.5.24}$$

Similarly for the $\mathbf{v}\times\mathbf{B}$ term where we can use $\mathbf{B} = \langle\mathbf{B}\rangle + \widetilde{\mathbf{B}}$. Then we can say

$$\left\langle \frac{\mathrm{d}f_s}{\mathrm{d}t} \right\rangle = \frac{\partial \langle f_s\rangle}{\partial t} + \mathbf{v}\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\langle\mathbf{E}\rangle + \mathbf{v}\times\langle\mathbf{B}\rangle)\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{v}} = -\left\langle (\widetilde{\mathbf{E}} + \mathbf{v}\times\widetilde{\mathbf{B}})\cdot\frac{\partial \widetilde{f_s}}{\partial \mathbf{v}} \right\rangle \tag{2.5.25}$$

If the change from the average is small, then we expect the terms on the right hand side (the "twiddle" terms) to be near zero and we get the Vlasov equation

$$\left\langle \frac{\mathrm{d}f_s}{\mathrm{d}t} \right\rangle = \frac{\partial \langle f_s\rangle}{\partial t} + \mathbf{v}\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\langle\mathbf{E}\rangle + \mathbf{v}\times\langle\mathbf{B}\rangle)\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{v}} = 0 \tag{2.5.26}$$

with the $\langle\mathbf{E}\rangle$ and $\langle\mathbf{B}\rangle$ being macroscopic averages of the fields. This also gives a heuristic explanation of collisions as being from the higher-order effects of the electric and magnetic field interactions where the plasma kinetic equation is written

$$\left\langle \frac{\mathrm{d}f_s}{\mathrm{d}t} \right\rangle = \frac{\partial \langle f_s\rangle}{\partial t} + \mathbf{v}\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\langle\mathbf{E}\rangle + \mathbf{v}\times\langle\mathbf{B}\rangle)\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{v}} - \sum_r C_{sr}(f_s, f_r) \tag{2.5.27}$$

$$\frac{\partial \langle f_s\rangle}{\partial t} + \mathbf{v}\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\langle\mathbf{E}\rangle + \mathbf{v}\times\langle\mathbf{B}\rangle)\cdot\frac{\partial \langle f_s\rangle}{\partial \mathbf{v}} = \sum_r C_{sr}(f_s, f_r) \tag{2.5.28}$$

with $C_{sr}(f_s, f_r)$ is the collision operator for collisions between species s and r. We can define the sum as $C(f_s) = \sum_r C_{sr}(f_s, f_r)$. They are coming from the ignored interactions between different charge species (by changing the $\mathbf{E}$ and $\mathbf{B}$ to smoothed quantities we miss what really happens when charged particles actually do get near each other) which will alter the electric and magnetic fields.

Because it is often assumed we're working in the mush limit or some ensemble average, we then omit the averaging brackets and the plasma kinetic equation is written

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_s}{\partial \mathbf{v}} = C(f_s) \tag{2.5.29}$$

2.5.2 Fluid Moments

Now to getting information about the system as a whole rather than to individual particle trajectories. We will follow a hierarchy of moments approach. This will allow us to understand fluid quantities (such as temperature, pressure, flow velocity, and number density) as coming from the plasma kinetics. Thus, we take our distribution function f_s and we take moments, integrating over the velocity space $\mathbf{v}$. We begin by defining some quantities using $\mathbf{v}'_s \equiv \mathbf{v} - \mathbf{V}_s$ so that

$$0 = \iiint \mathrm{d}^3v \; \mathbf{v}'_s f_s \tag{2.5.30}$$

$$n_s(\mathbf{x}) \equiv \iiint_{-\infty}^{\infty} \mathrm{d}^3v \; f_s(\mathbf{x}, \mathbf{v}) \tag{2.5.31}$$

$$n_s(\mathbf{x})\mathbf{V}_s(\mathbf{x}) \equiv \iiint_{-\infty}^{\infty} \mathrm{d}^3v \; \mathbf{v} f_s(\mathbf{x}, \mathbf{v}) \tag{2.5.32}$$

$$p_s(\mathbf{x}) \equiv \iiint_{-\infty}^{\infty} \mathrm{d}^3v \; \frac{m_s v_s'^2}{3} f_s \tag{2.5.33}$$

$$\overleftrightarrow{\mathbf{P}}_s(\mathbf{x}) \equiv \iiint_{-\infty}^{\infty} \mathrm{d}^3v \; m_s \mathbf{v}'_s \mathbf{v}'_s f_s = p_s(\mathbf{x})\mathbb{1} + \overleftrightarrow{\mathbf{\Pi}}_s(\mathbf{x}) \tag{2.5.34}$$

$$\overleftrightarrow{\mathbf{\Pi}}_s(\mathbf{x}) \equiv \iiint_{-\infty}^{\infty} \mathrm{d}^3v \; m_s \left(\mathbf{v}'_s \mathbf{v}'_s - \frac{v'^2}{3} \right) f_s \tag{2.5.35}$$

$$\mathbf{Q}_s(\mathbf{x}) \equiv \iiint_{-\infty}^{\infty} \mathrm{d}^3v \; \frac{m_s v^2}{2} \mathbf{v} f_s = \mathbf{q}_s + \frac{5}{2} n_s k_B T_s \mathbf{V}_s + \frac{1}{2} n_s m_s V_s^2 \mathbf{V}_s + \mathbf{V}_s \cdot \overleftrightarrow{\mathbf{\Pi}}_s \tag{2.5.36}$$

$$\mathbf{q}_s(\mathbf{x}) \equiv \iiint_{-\infty}^{\infty} \mathrm{d}^3v \; \mathbf{v}'_s \frac{m_s v_s'^2}{2} f_s \tag{2.5.37}$$

These quantities are given names that correspond to features of fluids that we measure. The n_s is the number density, $\mathbf{V}_s$ is the flow velocity, $p_s = n_s k_B T_s$ is the pressure, $\overleftrightarrow{\mathbf{P}}_s$ is the total pressure tensor, $\overleftrightarrow{\mathbf{\Pi}}_s$ is the stress tensor, $\mathbf{Q}_s$ is the total heat flow, and $\mathbf{q}_s$ is the conductive heat flow. Sometimes, $\mathbf{Q}_s$ is called the energy flux and $\mathbf{q}_s$ the heat flux, but these term names are not always very descriptive, anyway.

It is also convenient to define moments of the total collision operator.

$$\iiint_{-\infty}^{\infty} \mathrm{d}^3v \; C(f_s) = 0 \tag{2.5.38}$$

$$\iiint_{-\infty}^{\infty} \mathrm{d}^3v \; m\mathbf{v} C(f_s) = \iiint_{-\infty}^{\infty} m\mathbf{v}'_s C(f_s) = \mathbf{R}_s \tag{2.5.39}$$

$$\iiint_{-\infty}^{\infty} \mathrm{d}^3v \; \frac{m v_s'^2}{2} C(f_s) = Q_s \tag{2.5.40}$$

Then $\mathbf{R}_s$ is the collisional friction or the energy-weighted stress, while Q_s is, thankfully, just called the collisional heating.

One can continue to go up the moment chain by multiplying f_s by $(1, \mathbf{v}, v^2, \mathbf{v}v^2, v^4, \dots)$ with various factors to calculate an unending chain of quantities. These become ever more difficult to understand physically and more difficult to measure experimentally.

Now, we can take our plasma kinetic equation and take moments. We have

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = C(f_s) \tag{2.5.41}$$

and so we can simply take it one term at a time. We begin with just taking the moment with no factors.

$$\iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; \frac{\partial f_s}{\partial t} = \frac{\partial}{\partial t} \iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; f_s = \frac{\partial n_s}{\partial t} \tag{2.5.42}$$

where we have used that we can interchange the integration and differentiation operations. The next term is given by

$$\iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} = \iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; \left[\frac{\partial}{\partial \mathbf{x}} \cdot (f_s \mathbf{v}) - f_s \cancel{\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v}} \right] \tag{2.5.43}$$

where we use that $\mathbf{v}$ is independent of $\mathbf{x}$. The divergence operator can also be taken out of the integral and so we find

$$\frac{\partial}{\partial \mathbf{x}} \cdot \iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; \mathbf{v} f_s = \frac{\partial}{\partial \mathbf{x}} \cdot (n_s \mathbf{V}_s) \tag{2.5.44}$$

The next term is given by

$$\iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; \frac{q_s}{m_s} \mathbf{E} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \frac{q_s}{m_s} \iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; \left[\frac{\partial}{\partial \mathbf{v}} \cdot (f_s \mathbf{E}) - f_s \cancel{\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{E}} \right] \tag{2.5.45}$$

where we have used that $\mathbf{E}$ is independent of $\mathbf{v}$. We can then use the divergence theorem and we have

$$\frac{q_s}{m_s} \iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; \frac{\partial}{\partial \mathbf{v}} \cdot (f_s \mathbf{E}) = \frac{q_s}{m_s} \iint_{S_\infty} \mathrm{d}^2 v \; \hat{\mathbf{n}} \cdot \mathbf{E} f_s = 0 \tag{2.5.46}$$

Now we have used that at the surface at infinity S_∞, we must have $f_s \to 0$ and $\mathbf{E} \to \mathbf{0}$ because there are no particles far, far away.

The next term is given by

$$\iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; \frac{q_s}{m_s} (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} \tag{2.5.47}$$

We can use a similar trick. We use

$$\mathbf{v} \times \mathbf{B} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot [f_s \mathbf{v} \times \mathbf{B}] - f_s \frac{\partial}{\partial \mathbf{v}} \cdot [\mathbf{v} \times \mathbf{B}] \tag{2.5.48}$$

because $\mathbf{B}$ is independent of $\mathbf{v}$, then we have the final term as

$$f_s \frac{\partial}{\partial \mathbf{v}} \cdot [\mathbf{v} \times \mathbf{B}] = f_s \left[\mathbf{B} \cdot \frac{\partial}{\partial \mathbf{v}} \times \mathbf{v} - \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{v}} \times \mathbf{B} \right] = f_s \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{v}} \times \mathbf{v} \tag{2.5.49}$$

We then can use that $\frac{\partial}{\partial \mathbf{v}} \times \mathbf{v} = \mathbf{0}$ so that in fact this term identically vanishes.

We then have the moment of the collision operator which vanishes. Thus, we have

$$\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (n_s \mathbf{V}_s) = 0 \tag{2.5.50}$$

$$\frac{\partial n_s}{\partial t} + \boldsymbol{\nabla} \cdot (n_s \mathbf{V}_s) = \frac{\partial n_s}{\partial t} + n_s \boldsymbol{\nabla} \cdot \mathbf{V}_s + \nabla n_s \cdot \mathbf{V}_s = 0 \tag{2.5.51}$$

where I have switched to more conventional notation with $\boldsymbol{\nabla}$ and ∇ rather than $\frac{\partial}{\partial \mathbf{x}}$.

It is common to define a new type of time derivative[36] called alternatively the convective derivative, the advective derivative, or the total derivative for fluid equations.[37] This follows from assuming that all quantities q are functions of $\mathbf{x}$ and t. Thus

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial q}{\partial t} + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial q}{\partial \mathbf{x}} = \frac{\partial q}{\partial t} + \mathbf{V} \cdot \nabla q \tag{2.5.52}$$

If we consider only particles of a single species we use

$$\frac{\mathrm{d}_s q}{\mathrm{d}t} = \frac{\partial q}{\partial t} + \mathbf{V}_s \cdot \nabla q \tag{2.5.53}$$

With this definition of the total derivative the above first moment equation reads

$$\frac{\mathrm{d}_s n_s}{\mathrm{d}t} + n_s \boldsymbol{\nabla} \cdot \mathbf{V}_s = 0 \tag{2.5.54}$$

the number density continuity equation. This is a statement that particles are neither created or destroyed in our processes. Thus the total change is due only to compression or expansion of particles due to the flow velocity field.

We can then do the $m_s \mathbf{v}$ moment. Again, let's take these one term at a time.

$$\iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; m_s \mathbf{v} \frac{\partial f_s}{\partial t} = m_s \frac{\partial}{\partial t} \iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; \mathbf{v} f_s = m_s \frac{\partial n_s \mathbf{V}_s}{\partial t} \tag{2.5.55}$$

Then

$$\begin{aligned}
&\iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; m_s \mathbf{v}\mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} = m_s \iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; \left[\frac{\partial}{\partial \mathbf{x}} \cdot (f_s \mathbf{v}\mathbf{v}) - f_s \cancel{\frac{\partial}{\partial \mathbf{x}} \cdot [\mathbf{v}\mathbf{v}]} \right] \\
&= m_s \frac{\partial}{\partial \mathbf{x}} \cdot \iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; (f_s \mathbf{v}\mathbf{v}) \\
&= m_s \frac{\partial}{\partial \mathbf{x}} \cdot \iiint_{-\infty}^{\infty} \mathrm{d}^3 v \; [f_s (\mathbf{V}_s + \mathbf{v}_s')(\mathbf{V}_s + \mathbf{v}_s')] \\
&= m_s \frac{\partial}{\partial \mathbf{x}} \cdot \left[n_s \mathbf{V}_s \mathbf{V}_s + 0 + 0 + \overleftrightarrow{\mathbf{P}}_s \right] \\
&= m_s \boldsymbol{\nabla} \cdot \left[n_s \mathbf{V}_s \mathbf{V}_s + \overleftrightarrow{\mathbf{P}}_s \right]
\end{aligned} \tag{2.5.56}$$

[36] Remember this term has at least nine other names in the literature. This naming scheme should be considered a (horrific) model for creating a confusing mess of notation over a concept that already can can cause confusion.

[37] It is unfortunate that there is so much terminology. The advective derivative is sometimes also used to refer only to the $\mathbf{V} \cdot \nabla$ part of the total derivative $\frac{\mathrm{d}}{\mathrm{d}t}$. Caution when using these words should apply, as always, until you see an author's definition.

It is convenient to use our definition of $\mathbf{a}$ from before. We have then shown that

$$\mathbf{a}\cdot\frac{\partial f_s}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\cdot(f_s\mathbf{a}) \tag{2.5.57}$$

and so

$$\begin{aligned}
&\iiint_{-\infty}^{\infty} \mathrm{d}^3v\; m_s\mathbf{v}\mathbf{a}\cdot\frac{\partial f_s}{\partial \mathbf{v}} = m_s\frac{\partial}{\partial t}\iiint_{-\infty}^{\infty}\mathrm{d}^3v\;\mathbf{v}\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{a}f_s)\\
&= m_s\iiint_{-\infty}^{\infty}\mathrm{d}^3v\;\left[\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{a}\mathbf{v}f_s) - \mathbf{a}f_s\cdot\frac{\partial \mathbf{v}}{\partial \mathbf{v}}^{\mathbb{1}}\right]\\
&= m_s\cancel{\iint_{S_\infty}\mathrm{d}^2v\;\hat{\mathbf{n}}\cdot\mathbf{a}\mathbf{v}f_s} - \iiint_{-\infty}^{\infty}\mathrm{d}^3v\;[q_s\mathbf{E}f_s + q_s\mathbf{v}\times\mathbf{B}f_s]\\
&= -n_sq_s\mathbf{E} - n_sq_s\mathbf{V}_s\times\mathbf{B}
\end{aligned} \tag{2.5.58}$$

where we have been careful about dot product placement. Remember that $\frac{\partial}{\partial\mathbf{x}}\cdot(\mathbf{A}\mathbf{B})$ indicates that the derivative's components are matching onto $\mathbf{A}$, that is, for Cartesian coordinates we can write this in Einstein notation as

$$\frac{\partial}{\partial\mathbf{x}}\cdot(\mathbf{A}\mathbf{B}) = \partial_i(A_iB_j) \neq \frac{\partial}{\partial\mathbf{x}}\cdot(\mathbf{B}\mathbf{A}) = \partial_i(B_iA_j) \tag{2.5.59}$$

By definition we have from the collision integral that we get $\mathbf{R}_s$. Thus, we find all together that

$$m_s\frac{\partial n_s\mathbf{V}_s}{\partial t} + m_s\boldsymbol{\nabla}\cdot(n_s\mathbf{V}_s\mathbf{V}_s) + \boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{P}}_s - n_sq_s\mathbf{E} - n_s\mathbf{V}_s\times\mathbf{B} = \mathbf{R}_s \tag{2.5.60}$$

We can use that $\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{P}}_s = \nabla p_s + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_s$ and write

$$\begin{aligned}
&\cancel{m_s\mathbf{V}_s\frac{\partial n_s}{\partial t}} + m_sn_s\frac{\partial\mathbf{V}_s}{\partial t} + m_sn_s\mathbf{V}_s\cdot\boldsymbol{\nabla}\mathbf{V}_s + \cancel{m_s\mathbf{V}_s\boldsymbol{\nabla}\cdot(n_s\mathbf{V}_s)} = n_sq_s\left(\mathbf{E}+\mathbf{V}_s\times\mathbf{B}\right)\\
&\qquad\qquad - \nabla p - \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_s + \mathbf{R}_s
\end{aligned} \tag{2.5.61}$$

$$m_sn_s\frac{\mathrm{d}_s\mathbf{V}_s}{\mathrm{d}t} = n_sq_s\left(\mathbf{E}+\mathbf{V}_s\times\mathbf{B}\right) - \nabla p - \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_s + \mathbf{R}_s \tag{2.5.62}$$

Where we have used the total derivative definition. It is important to note at this point that the total derivative will differ for each species s for the $\mathbf{V}_s\cdot\nabla$ part of the total derivative. This is the momentum balance equation.

The energy/temperature equation is calculated similarly, though one must use the density and momentum balance equations to get it into this standard form.

$$\frac{3}{2}n_s\frac{\mathrm{d}_s(k_BT_s)}{\mathrm{d}t} + p_s\boldsymbol{\nabla}\cdot\mathbf{V}_s = -\boldsymbol{\nabla}\cdot\mathbf{q}_s - \overleftrightarrow{\boldsymbol{\Pi}}_s : \nabla\mathbf{V}_s + Q_s \tag{2.5.63}$$

Higher order equations can also be derived but they are seldom used in practice analytically.[38] We note that for every moment equation, we have terms that depend on higher order moments. The momentum equation has $\overleftrightarrow{\boldsymbol{\Pi}}_s$ and $\mathbf{q}_s$ terms that are not defined from within our fluid approximation.

[38] There are always exceptions, and higher moment methods are used quite a lot in numerical simulations.

We could get equations for them via doing higher order moments, but would find that they in turn had other even higher order moments that they needed. This problem is often called the moment hierarchy problem. There are a few ways to "solve" it.

The first way to "solve" the moment hierarchy problem is to not use moments. You just solve the plasma kinetic equation and then take moments later if you need insight. This means you need to solve a 3D3V+1 problem, which will scale very poorly computationally if you need to increase resolution in either the velocity space or the position space.

The second way is to truncate the equations. This just means we ignore any pieces we don't have an expression for, and just substitute them with zero. The pesky $\overleftrightarrow{\mathbf{\Pi}}_s$ and $\mathbf{q}_s$ are thus ignored in our equations. This has the advantage of being simple. It has the disadvantage of being ad hoc and non-rigorous. This can be seen as leading to the ideal MHD equations, however, and offers surprising insight into many plasma problems, so these terms turn out to be unimportant to many problems of interest.

The third way is to use an asymptotic approximation scheme. The most common method is often called a Chapman-Enskog-like closure. A closure here is just any way of "closing" the set of equations (truncation or supplying values for the higher order moments).[39] Chapman-Enskog applies for neutral gases, and so these closure schemes are Chapman-Enskog-like. These schemes are only good for certain parameter regimes, where a value for $\overleftrightarrow{\mathbf{\Pi}}_s$ or $\mathbf{q}_s$ [or higher order terms] is supplied via the closure scheme. The parameter of interest for validity is that the characteristic length of our system is much longer than a mean free path $L \gg \lambda_{\text{mfp}} = v_{\text{th}_s}\tau_s$ with v_{th_s} the thermal velocity and τ_s the collision time. The Knudsen number defined as $\text{Kn} = \frac{\lambda_{\text{mfp}}}{L}$ is sometimes referred to instead of L and λ_{mfp} directly. We see that a small Knudsen number is then what we desire, and, that this just means the plasma is very collisional. When we include magnetic fields, we must further use that ion Larmor radius satisfies $\rho_s = \frac{v_{\text{th}_s}}{\Omega_s} = \frac{\sqrt{2k_BT_s/m_s}}{q_sB/m_s} = \frac{\sqrt{2k_BT_sm_s}}{q_sB_s} \gg L$. The most famous of this type of closure scheme is called the Braginskii closure scheme[4]. Other closure schemes use a strong magnetic field rather than collisions as an asymptotic parameter.

The problem of closures is just as acute for some astrophysical plasmas as it is for fusion-relevant plasmas. The particles in astrophyiscal plasmas are collisionless over extreme distances because the number density is so low; there are just very few particles to collide with.

The ways of coming up with rigorously valid closures are few. There are Grad closure schemes rather than a Chapman-Enskog-like closure, which instead of assuming collisionality is necessary, assumes that the distribution function is only a small distortion from a background f_0 distribution. Then one can indeed find the requirements for the approximation scheme to be true. These will depend on the collisionality if they are a part of your kinetic equation, but they do not *a priori* assume large collisionality is necessary. Usually f_0 is an isotropic Maxwellian, and so the requirements then turn out to be that one has a highly collisional plasma. Indeed, the Grad schemes turn out to be difficult to prove they are valid in areas where they would be most useful. In addition, there is the CGL (Chew, Goldberger, Low) closure[7] where a strong magnetic field can take the place of high collisionality to a certain extent. This assumes a "double adiabaticity", which is a way of saying a pressure tensor of a certain form. Assuming such a form turns out to

[39] A closure (or closure equation or closure relation) should be viewed as an equation that we use instead of calculating the infinite moments necessary for a kinetic description. That is if the nth moment is $\mathbf{v}^n$ then a closure at order k is such that we do not need to consider moments of $\mathbf{v}^m$ for $m \geq k$. Instead we use the closure relation.

be very limiting on the actual physical situations, as you have to check if any physical plasmas actually have the requirement met. The double adiabaticity requirement turns out to be rather restrictive, in practice.

Thus, a closure scheme requires extreme care if you want it to be accurate. One needs to keep in mind the relevant symmetries that the closure should obey, and how simple or complicated it is to use. For example, the CGL laws are valid when the divergence of the heat flux tensor is zero (the double adiabaticity or adiabatic hypothesis), which means that time variations ΔT_c must be shorter than the time for particles to move along characteristic distances along magnetic field lines $1/k_\parallel$ at thermal speeds v_{th}. In other words, the frequencies $\omega \sim 1/T_c$ we are interested in must be larger than the "thermal frequency" along magnetic field lines, or $\omega \gg k_\parallel v_{\text{th}}$. Methods that use linear closures based off of a background can be constructed for a wide variety of situations, but they usually suffer from having restrictions like these. The idea of using strong magnetic fields,[40] however is a fruitful one, and can lead to fluid closures with well-defined regions of validity for practical situations. See Chust[8] for an especially cogent review of collisionless scenarios, and how to create closure schemes that are valid in some collisionless scenarios.

You may have noticed that two of the "solutions" to our moment hierarchy problem are not really even solutions: ignore doing moments altogether and ignore all higher order moments (that is truncate and hope).[41] Alas, the final solution of Chapman-Enskog-like closures turns out not to be applicable for the plasmas of interest in fusion. The mean free path turns out to be longer than the plasma devices. The strong magnetic field closures usually have severe restrictions on what the pressure can be like, and so their applicability to collisionless plasmas is also often questionable. Thus, in reality, none of our solutions is rigorous for turning our plasmas into fluids.

Do not worry, this will not stop us from doing it anyway. It turns out that with suitable kinetic corrections,[42] this is not even that bad of an approximation. Even without the kinetic corrections (or other empirical corrections), the magnetic fluid approximation turns out to yield an incredible amount of insight and information about plasma systems.

2.5.2.1 The Meaning of Temperature

One subtlety of plasma kinetics that is often underappreciated is that what is called temperature in a plasma kinetic setting is not a thermodynamic temperature unless the plasma distribution function is in fact Maxwellian in all directions. Remember that we defined

$$k_B T_s = \frac{\iiint_{-\infty}^{\infty} \mathrm{d}^3 v \ \frac{m_s v_s'^2}{3} f_s}{\int_{-\infty}^{\infty} \mathrm{d}^3 v \ f_s} \tag{2.5.64}$$

There was no assumption that f_s is Maxwellian. Using a Maxwellian with temperature Θ_s defined thermodynamically via $\Theta_s = (\partial U/\partial S)_{V,N}$ with U internal energy, S entropy, V volume, and N the

[40] You will see the term gyrotropy. This means that on the time and spatial scales we care about, the motion of particles is gyrations around magnetic field lines and movement along magnetic field lines. That is, the helical motion of particles around magnetic field lines.

[41] If you have done research long enough, you will recognize the wisdom in avoiding an unnecessary problem. You could spend your time making sure that the problem is handled correctly. But if you have another method that works without trouble, why wouldn't you use it? In this case, the 3D3V+1 method requires vast computational resources and sometimes more complicated computer programs.

[42] That is, we add terms coming from kinetic theory to our fluid problems that replicate behavior we desire. These can be justified rigorously in some cases. Kinetic corrections may also refer to using actual kinetic calculations (rather than approximations) in the fluid equations.

number of particles will lead to an expected result. Then direct substitution of the Maxwellian, now given by

$$f_s = \frac{n_s}{\pi^{3/2} v_{\text{th}_s}^3} \exp\left(-\frac{v_s^2}{v_{\text{th}_s}^2}\right) = n_s \left(\frac{m_s}{2\pi k_B \Theta_s}\right)^{3/2} \exp\left(-\frac{m_s v_s^2}{2k_B\Theta_s}\right) \tag{2.5.65}$$

will show you that for Maxwellian with temperature Θ_s, our definition is completely consistent and $\Theta_s = T_s$. Note that I did not prove that a Maxwellian is the proper distribution at thermal equilibrium, but if you accept that, then $T_s = \Theta_s$ must follow. However, Θ_s is not necessarily equal to T_s in general. Our temperature is different than the thermodynamic one and should be thought of as a measure of the random motion of a large number of particles and so it is sometimes called the kinetic temperature. This means we can define things like $T_{\parallel,s}$ or $T_{\perp,s}$ which tells us an effective temperature for particles along the parallel or perpendicular direction. If $\parallel$ and $\perp$ easily separate for temperature then we write a bi-Maxwellian given by

$$\begin{aligned} f_s &= n_s \left(\frac{m_s}{2\pi k_B T_{\perp,s}}\right) \exp\left(-\frac{m_s v_{s,\perp}^2}{2k_B T_{s,\perp}}\right) \left(\frac{m_s}{2\pi k_B T_{\parallel,s}}\right)^{1/2} \exp\left(-\frac{m_s v_{s,\parallel}^2}{2k_B T_{s,\parallel}}\right) \\ &= n_s \left(\frac{1}{\pi v_{\text{th},s,\perp}^2}\right) \exp\left(-\frac{v_{s,\perp}^2}{v_{\text{th},s,\perp}}\right) \left(\frac{1}{\pi^{1/2} v_{\text{th},s,\parallel}}\right) \exp\left(-\frac{v_{s,\parallel}^2}{v_{\text{th},s,\parallel}}\right) \end{aligned} \tag{2.5.66}$$

which we can define as $v_{\text{th},s,\perp}^2 = 2k_B T_{s,\perp}/m_s$ and $v_{\text{th},s,\parallel}^2 = 2k_B T_{s,\parallel}/m_s$. In this case the plasma has an anisotropic temperature, but can be "in equilibrium" in a relevant sense along different directions. Note in this anisotropic case we can still define a kinetic temperature T_s but it will not simply be given by $v_{\text{th}_s}^2 = 2k_B T_s/m_s$, and in general will have a complicated relationship to $T_\parallel$ and $T_\perp$ if they exist.

In general we write $f_s(v_\parallel, v_\perp) = n_s g(v_\parallel) h(v_\perp)$ with[43]

$$1 = \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \; g(v_\parallel) \tag{2.5.67}$$

$$1 = \int_0^{\infty} \mathrm{d}v_\perp \; 2\pi v_\perp h(v_\perp) \tag{2.5.68}$$

If we have a bi-Maxwellian we then have the simple relationship

$$k_B T_s = \frac{k_B T_{s,\parallel} + 2k_B T_{s,\perp}}{3} \tag{2.5.69}$$

2.6 Magnetohydrodynamics

> *Early on in my career I was told that the one of the simplest plasma models is a nonlinear system of 8 coupled differential equations. I thought, "That sounds like a field for me." [From the author's recollection of a statement in a class]*
>
> — C. C. Hegna

[43] It is not always possible to cleanly separate into perpendicular and parallel components. Note also that $v_\perp$ is essentially r in a polar coordinate system to see the form of the integrals.

Now that we have started from plasma kinetics, let's actually get to a fluid theory. For simplicity, let's just consider a positive ion and (negative) electron plasma. We then have from before the fluid equations

$$\frac{\mathrm{d}n_s}{\mathrm{d}t} + n_s \boldsymbol{\nabla}\cdot\mathbf{V}_s = 0 \tag{2.6.1}$$

$$m_s n_s \frac{\mathrm{d}\mathbf{V}_s}{\mathrm{d}t} = n_s q_s \left(\mathbf{E} + \mathbf{V}_s \times \mathbf{B}\right) - \nabla p_s - \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_s + \mathbf{R}_s \tag{2.6.2}$$

$$\frac{3}{2} n_s \frac{\mathrm{d}k_B T_s}{\mathrm{d}t} + p_s \boldsymbol{\nabla}\cdot\mathbf{V}_s = -\boldsymbol{\nabla}\cdot\mathbf{q}_s - \overleftrightarrow{\boldsymbol{\Pi}}_s : \nabla\mathbf{V}_s + Q_s \tag{2.6.3}$$

In the momentum balance equation we often omit the $\mathbf{R}_s$ friction term and we will soon see that many of the other terms are often omitted because they can be shown to be negligible under certain approximations. The temperature/energy evolution equation also was used with a definition of pressure for a monatomic gas. We can slightly generalize by using the adiabatic index γ so we get

$$\frac{n_s}{\gamma - 1}\frac{\mathrm{d}k_B T_s}{\mathrm{d}t} + p_s \boldsymbol{\nabla}\cdot\mathbf{V}_s = -\boldsymbol{\nabla}\cdot\mathbf{q}_s - \overleftrightarrow{\boldsymbol{\Pi}}_s : \nabla\mathbf{V}_s + Q_s \tag{2.6.4}$$

We also will use some of Maxwell's equations for electrodynamics

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{2.6.5}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{2.6.6}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} \tag{2.6.7}$$

$$\mathbf{J} = \sum_s q_s n_s \mathbf{V}_s \tag{2.6.8}$$

$$\mathbf{E} = \eta\mathbf{J} \tag{2.6.9}$$

We can then find $\mathbf{E}$ in multiple ways. One useful fact is that $m_e/m_i \lll 1$ so $\sqrt{m_e/m_i} \ll 1$ since $m_e/m_i \approx 1/1836$ and $\sqrt{m_e/m_i} \approx 1/43$. For now, consider the number density continuity equation for ions and electrons

$$\frac{\mathrm{d}_e n_e}{\mathrm{d}t} + n_e \boldsymbol{\nabla}\cdot\mathbf{V}_e = 0 \tag{2.6.10}$$

$$\frac{\mathrm{d}_i n_i}{\mathrm{d}t} + n_i \boldsymbol{\nabla}\cdot\mathbf{V}_i = 0 \tag{2.6.11}$$

remembering that

$$\frac{\mathrm{d}_s q}{\mathrm{d}t} = \frac{\partial q}{\partial t} + \mathbf{V}_s\cdot\nabla q \tag{2.6.12}$$

for any quantity q. We can define a center of mass velocity $\mathbf{V}$ via

$$\mathbf{V} = \frac{m_i n_i \mathbf{V}_i + n_e m_e \mathbf{V}_e}{n_i m_i + n_e m_e} = \frac{\mathbf{V}_i}{1 + \frac{m_e}{m_i}\frac{n_e}{n_i}} + \frac{\frac{m_e}{m_i}\frac{n_e}{n_i}\mathbf{V}_e}{1 + \frac{m_e}{m_i}\frac{n_e}{n_i}} = \mathbf{V}_i + \frac{n_e m_e}{n_i m_i}\mathbf{V}_e + \mathcal{O}\left(\left[\frac{m_e}{m_i}\right]^2\right) \tag{2.6.13}$$

where I have used $m_e/m_i \lll 1$ to get a good approximation. It is possible to write this slightly differently using quasineutrality so that $n_e \simeq n_i$ so $n_e = n_i = n$. Then we can note that $\mathbf{J} =$

$q_e n_e \mathbf{V}_e + q_i n_i \mathbf{V}_i$ so that

$$\mathbf{J} = ne\left(\mathbf{V}_i - \mathbf{V}_e\right) \tag{2.6.14}$$

$$\mathbf{V}_e = \mathbf{V}_i - \frac{\mathbf{J}}{ne} \tag{2.6.15}$$

We can then write

$$\mathbf{V} = \frac{\mathbf{V}_i}{1+\frac{m_e}{m_i}} + \frac{\frac{m_e}{m_i}(\mathbf{V}_i - \mathbf{J}/[ne])}{1+\frac{m_e}{m_i}} = \mathbf{V}_i - \frac{\frac{m_e}{m_i}\mathbf{J}}{ne(1+\frac{m_e}{m_i})} = \mathbf{V}_i - \lambda_i \frac{\mathbf{J}}{ne} \tag{2.6.16}$$

$$\mathbf{V} = \frac{\mathbf{V}_e + \mathbf{J}/(ne)}{1+\frac{m_e}{m_i}} + \frac{\frac{m_e}{m_i}\mathbf{V}_e}{1+\frac{m_e}{m_i}} = \mathbf{V}_e + \frac{\mathbf{J}}{ne(1+\frac{m_e}{m_i})} = \mathbf{V}_e + \lambda_e \frac{\mathbf{J}}{ne} \tag{2.6.17}$$

For simplicity we can define two useful parameters

$$\lambda_i = \frac{\frac{m_e}{m_i}}{1+\frac{m_e}{m_i}} \tag{2.6.18}$$

$$\lambda_e = \frac{1}{1+\frac{m_e}{m_i}} \tag{2.6.19}$$

and we see that $\lambda_e + \lambda_i = 1$. These can then be rearranged to

$$\mathbf{V}_i = \mathbf{V} + \frac{\lambda_i \mathbf{J}}{ne} = \mathbf{V} + \mathcal{O}\left(\frac{m_e}{m_i}\right) \tag{2.6.20}$$

$$\mathbf{V}_e = \mathbf{V} - \frac{\lambda_e \mathbf{J}}{ne} = \mathbf{V} - \frac{\mathbf{J}}{ne} + \mathcal{O}\left(\frac{m_e}{m_i}\right) \tag{2.6.21}$$

This means that $\mathbf{V}_i \approx \mathbf{V}$, which to zeroth order in mass ratio usually means that we can simply replace $\mathbf{V}_i$ with $\mathbf{V}$ and ignore any terms with $\mathbf{V}_e$. However, we shall look at this rigorously to make sure this makes sense. We have to remember that $\frac{\mathrm{d}_e n_e}{\mathrm{d}t} = \frac{\partial n_e}{\partial t} + \mathbf{V}_e \cdot \nabla n_e$ so

$$\frac{\partial n_e}{\partial t} + \boldsymbol{\nabla} \cdot (n_e \mathbf{V}_e) = 0 \tag{2.6.22}$$

$$\frac{\partial n_i}{\partial t} + \boldsymbol{\nabla} \cdot (n_i \mathbf{V}_i) = 0 \tag{2.6.23}$$

We can replace the $\mathbf{V}_s$ and find

$$\frac{\partial n_e}{\partial t} + \boldsymbol{\nabla} \cdot \left(n_e \mathbf{V} - \frac{\lambda_e \mathbf{J}}{e}\right) = 0 \tag{2.6.24}$$

$$\frac{\partial n_i}{\partial t} + \boldsymbol{\nabla} \cdot \left(n_i \mathbf{V} + \frac{\lambda_i \mathbf{J}}{e}\right) = 0 \tag{2.6.25}$$

So if we were to add the equations together with $n = n_e + n_i$ we'd find

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{V}) + \frac{\lambda_i - \lambda_e}{e} \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \tag{2.6.26}$$

Because of our definition of $\mathbf{J}$ then $\mu_0 \boldsymbol{\nabla} \cdot \mathbf{J} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{B} = 0$ identically so we find

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{V}) = 0 \tag{2.6.27}$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} + n\boldsymbol{\nabla} \cdot \mathbf{V} = 0 \tag{2.6.28}$$

$$\tag{2.6.29}$$

where the new $\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\partial n}{\partial t} + \mathbf{V} \cdot \nabla n$ has no subscript on the total derivative to denote the center-of-mass velocity is being used.

We can now consider momentum balance for electrons and ions. First let's consider ion momentum balance.

$$m_i n_i \frac{\mathrm{d}_i \mathbf{V}_i}{\mathrm{d}t} = ne\left(\mathbf{E} + \mathbf{V}_i \times \mathbf{B}\right) - \nabla p_i - \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\Pi}}_i \tag{2.6.30}$$

Now if we add the electron balance equation

$$m_e n_e \frac{\mathrm{d}_e \mathbf{V}_e}{\mathrm{d}t} = -ne\left(\mathbf{E} + \mathbf{V}_e \times \mathbf{B}\right) - \nabla p_e - \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\Pi}}_e \tag{2.6.31}$$

we find (using $ne\mathbf{V}_i - ne\mathbf{V}_e = \mathbf{J}$)

$$m_i n \frac{\mathrm{d}_i \mathbf{V}_i}{\mathrm{d}t} + m_e n \frac{\mathrm{d}_e \mathbf{V}_e}{\mathrm{d}t} = \mathbf{J} \times \mathbf{B} - \nabla p - \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{\Pi}} \tag{2.6.32}$$

We remember that

$$\frac{\mathrm{d}_i \mathbf{V}_i}{\mathrm{d}t} = \frac{\partial \mathbf{V}_i}{\partial t} + \mathbf{V}_i \cdot \nabla \mathbf{V}_i \tag{2.6.33}$$

$$\frac{\mathrm{d}_e \mathbf{V}_e}{\mathrm{d}t} = \frac{\partial \mathbf{V}_e}{\partial t} + \mathbf{V}_e \cdot \nabla \mathbf{V}_e \tag{2.6.34}$$

Using our relations for $\mathbf{V}$ we can find the correct quantities. Then we find

$$\mathbf{V}_i \cdot \nabla \mathbf{V}_i = \left(\mathbf{V} + \frac{\lambda_i \mathbf{J}}{ne}\right) \cdot \nabla \left(\mathbf{V} + \frac{\lambda_i \mathbf{J}}{ne}\right) \tag{2.6.35}$$

$$= \mathbf{V} \cdot \nabla \mathbf{V} + \frac{\lambda_i}{e} \mathbf{V} \cdot \nabla \left(\frac{\mathbf{J}}{n}\right) + \frac{\lambda_i \mathbf{J}}{ne} \cdot \nabla \mathbf{V} + \frac{\lambda_i^2}{ne^2} \mathbf{J} \cdot \nabla \left(\frac{\mathbf{J}}{n}\right) \tag{2.6.36}$$

Then we can use

$$\boldsymbol{\nabla} \cdot (\mathbf{AB}) = (\boldsymbol{\nabla} \cdot \mathbf{A})\,\mathbf{B} + \mathbf{A} \cdot \nabla \mathbf{B} \tag{2.6.37}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{BA}) = (\boldsymbol{\nabla} \cdot \mathbf{B})\,\mathbf{A} + \mathbf{B} \cdot \nabla \mathbf{A} \tag{2.6.38}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{AB} + \mathbf{BA}) = (\boldsymbol{\nabla} \cdot \mathbf{A})\mathbf{B} + (\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{A} + \mathbf{A} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{A} \tag{2.6.39}$$

So that

$$\mathbf{V} \cdot \nabla \left(\frac{\mathbf{J}}{n}\right) + \frac{\mathbf{J}}{n} \cdot \nabla \mathbf{V} = \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{VJ} + \mathbf{JV}}{n}\right) - (\boldsymbol{\nabla} \cdot \mathbf{V})\frac{\mathbf{J}}{n} - \left(\boldsymbol{\nabla} \cdot \frac{\mathbf{J}}{n}\right)\mathbf{V} \tag{2.6.40}$$

We also have $\boldsymbol{\nabla}\cdot(\mathbf{J}/n) = -\nabla n\cdot\mathbf{J}/n^2$ so that we find

$$\mathbf{V}\cdot\nabla\left(\frac{\mathbf{J}}{n}\right)+\frac{\mathbf{J}}{n}\cdot\nabla\mathbf{V}=\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{VJ}+\mathbf{JV}}{n}\right)-(\boldsymbol{\nabla}\cdot\mathbf{V})\frac{\mathbf{J}}{n}+\left(\frac{\nabla n}{n^2}\cdot\mathbf{J}\right)\mathbf{V} \tag{2.6.41}$$

We can then write

$$\mathbf{V}_i\cdot\nabla\mathbf{V}_i=\mathbf{V}\cdot\nabla\mathbf{V}+\boldsymbol{\nabla}\cdot\left(\frac{\lambda_i[\mathbf{VJ}+\mathbf{JV}]+\frac{\lambda_i^2}{e}\mathbf{JJ}}{ne}\right)-\frac{\lambda_i\boldsymbol{\nabla}\cdot\mathbf{V}}{n}\mathbf{J}+\frac{\lambda_i\nabla n}{n^2}\cdot\mathbf{JV} \tag{2.6.42}$$

Similarly for the electron terms we find (simply substitute $\lambda_i\to\lambda_e$)

$$\mathbf{V}_e\cdot\nabla\mathbf{V}_e=\mathbf{V}\cdot\nabla\mathbf{V}+\boldsymbol{\nabla}\cdot\left(\frac{-\lambda_e[\mathbf{VJ}+\mathbf{JV}]+\frac{\lambda_e^2}{e}\mathbf{JJ}}{ne}\right)+\frac{\lambda_e\boldsymbol{\nabla}\cdot\mathbf{V}}{n}\mathbf{J}-\frac{\lambda_e\nabla n}{n^2}\cdot\mathbf{JV} \tag{2.6.43}$$

$$\frac{m_e}{m_i}\mathbf{V}_e\cdot\nabla\mathbf{V}_e=\frac{m_e}{m_i}\mathbf{V}\cdot\nabla\mathbf{V}+\boldsymbol{\nabla}\cdot\left(\frac{-\lambda_i[\mathbf{VJ}+\mathbf{JV}]+\frac{\lambda_i\lambda_e}{e}\mathbf{JJ}}{ne}\right)+\frac{\lambda_i\boldsymbol{\nabla}\cdot\mathbf{V}}{n}\mathbf{J}-\frac{\lambda_i\nabla n}{n^2}\cdot\mathbf{JV} \tag{2.6.44}$$

noting that $\lambda_e m_e/m_i=\lambda_i$. So we find

$$\mathbf{V}_i\cdot\nabla\mathbf{V}_i+\frac{m_e}{m_i}\mathbf{V}_e\cdot\nabla\mathbf{V}_e=\left(1+\frac{m_e}{m_i}\right)\mathbf{V}\cdot\nabla\mathbf{V}+\boldsymbol{\nabla}\cdot\left(\lambda_i\frac{(\lambda_i+\lambda_e)\mathbf{JJ}}{ne^2}\right) \tag{2.6.45}$$

Remember $\lambda_i+\lambda_e=1$ so this actually says

$$\mathbf{V}_i\cdot\nabla\mathbf{V}_i+\frac{m_i}{m_e}\mathbf{V}_e\cdot\nabla\mathbf{V}_e=\left(1+\frac{m_e}{m_i}\right)\mathbf{V}\cdot\nabla\mathbf{V}+\boldsymbol{\nabla}\cdot\left(\lambda_i\frac{\mathbf{JJ}}{ne^2}\right) \tag{2.6.46}$$

It is also clear that through similar procedures we can find

$$\frac{\partial\mathbf{V}_i}{\partial t}+\frac{m_e}{m_i}\frac{\partial\mathbf{V}_e}{\partial t}=\left(1+\frac{m_e}{m_i}\right)\frac{\partial\mathbf{V}}{\partial t}+\frac{\partial}{\partial t}\left[\frac{\lambda_i\mathbf{J}}{n}-\cancel{\frac{\frac{m_e}{m_i}\lambda_e\mathbf{J}}{n}}\right] \tag{2.6.47}$$

Thus we find

$$\frac{\mathrm{d}_i\mathbf{V}_i}{\mathrm{d}t}+\frac{m_e}{m_i}\frac{\mathrm{d}_e\mathbf{V}_e}{\mathrm{d}t}=\left(1+\frac{m_e}{m_i}\right)\left[\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right]+\boldsymbol{\nabla}\cdot\left(\frac{\lambda_i\mathbf{JJ}}{ne^2}\right) \tag{2.6.48}$$

$$\frac{\mathrm{d}\mathbf{V}_i}{\mathrm{d}t}+\frac{m_e}{m_i}\frac{\mathrm{d}\mathbf{V}_e}{\mathrm{d}t}=\left(1+\frac{m_e}{m_i}\right)\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}+\boldsymbol{\nabla}\cdot\left(\frac{\lambda_i\mathbf{JJ}}{ne^2}\right) \tag{2.6.49}$$

Thus, the total equation would yield

$$nm_i\left(1+\frac{m_e}{m_i}\right)\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}+nm_i\boldsymbol{\nabla}\cdot\left(\frac{m_e}{m_i}\frac{\mathbf{JJ}}{ne^2(1+\frac{m_e}{m_i})}\right)=\mathbf{J}\times\mathbf{B}-\nabla p-\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}} \tag{2.6.50}$$

which is surprisingly simple considering it is not an approximation. If we only take zeroth order in mass ratio terms, we get an equation that looks like a single species equation

$$nm_i\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}=\mathbf{J}\times\mathbf{B}-\nabla p-\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}} \tag{2.6.51}$$

If we had simply said that $\mathcal{O}(m_e/m_i)$ should be ignored for each species, and used our $\mathbf{V}_i \to \mathbf{V}$ rule and then added our two-species equations we would arrive at the same equation. In either case, we can ask ourselves about $\mathbf{E}$.

We can use the electron equation anticipating that we can find equations in the parameter m_e/m_i. The $\mathbf{E}$ we'd find would then be

$$\mathbf{E} = -\frac{m_e}{e}\frac{\mathrm{d}\mathbf{V}_e}{\mathrm{d}t} - \mathbf{V}_e \times \mathbf{B} - \frac{\nabla p_e}{ne} - \frac{\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_e}{ne} \tag{2.6.52}$$

We then can write

$$\frac{\mathrm{d}\mathbf{V}_e}{\mathrm{d}t} = \left[\frac{\partial \mathbf{V}}{\partial t} - \frac{\partial}{\partial t}\left[\frac{\lambda_e \mathbf{J}}{n}\right] + \mathbf{V}\cdot\nabla\mathbf{V} + \boldsymbol{\nabla}\cdot\left(\frac{-\lambda_e[\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}] + \frac{\lambda_e^2}{e}\mathbf{J}\mathbf{J}}{ne}\right) + \frac{\lambda_e \boldsymbol{\nabla}\cdot\mathbf{V}}{n}\mathbf{J} - \frac{\lambda_e \nabla n\cdot\mathbf{J}}{n^2}\mathbf{V}\right] \tag{2.6.53}$$

$$\mathbf{V}_e \times \mathbf{B} = \mathbf{V}\times\mathbf{B} - \lambda_e \frac{\mathbf{J}\times\mathbf{B}}{ne} \tag{2.6.54}$$

We can factor an ion mass to find

$$\mathbf{E} = m_i\left[-\frac{m_e}{m_i e}\frac{\mathrm{d}\mathbf{V}_e}{\mathrm{d}t} - \frac{\mathbf{V}_e\times\mathbf{B}}{m_i} - \frac{\nabla p_e - \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_e}{m_i ne}\right] \tag{2.6.55}$$

Then the $\lambda_e \to \lambda_i$ which are on the order of the mass ratio. So

$$\begin{aligned}\mathbf{E} = &-\frac{m_i}{e}\left[\frac{m_e}{m_i}\frac{\partial\mathbf{V}}{\partial t} - \frac{\partial}{\partial t}\frac{\lambda_i\mathbf{J}}{n} + \frac{m_e}{m_i}\mathbf{V}\cdot\nabla\mathbf{V} + \boldsymbol{\nabla}\cdot\left(\frac{-\lambda_i[\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}] + \frac{\lambda_i\lambda_e}{e}\mathbf{J}\mathbf{J}}{ne}\right)\right] \\ &- \frac{m_i}{e}\left[\frac{\lambda_i\boldsymbol{\nabla}\cdot\mathbf{V}}{n}\mathbf{J} - \frac{\lambda_i\nabla n\cdot\mathbf{J}}{n^2}\mathbf{V}\right] - \mathbf{V}\times\mathbf{B} + \frac{\lambda_e\mathbf{J}\times\mathbf{B}}{ne} - \frac{\nabla p_e + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_e}{ne}\end{aligned} \tag{2.6.56}$$

We can of course, via symmetry, use our relations for the ions instead of the electrons for $\mathbf{E}$. Then we find

$$\mathbf{E} = \frac{m_i}{e}\frac{\mathrm{d}\mathbf{V}_i}{\mathrm{d}t} - \mathbf{V}_i\times\mathbf{B} + \frac{\nabla p_i + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_i}{ne} \tag{2.6.57}$$

$$\begin{aligned}\mathbf{E} = &\frac{m_i}{e}\left[\frac{\partial\mathbf{V}}{\partial t} + \frac{\partial}{\partial t}\frac{\lambda_i\mathbf{J}}{n} + \mathbf{V}\cdot\nabla\mathbf{V} + \boldsymbol{\nabla}\cdot\left(\frac{\lambda_i[\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}] + \frac{\lambda_i^2}{e}\mathbf{J}\mathbf{J}}{ne}\right)\right] \\ &+ \frac{m_i}{e}\left[-\frac{\lambda_i\boldsymbol{\nabla}\cdot\mathbf{V}}{n}\mathbf{J} + \frac{\lambda_i\nabla n\cdot\mathbf{J}}{n^2}\mathbf{V}\right] - \mathbf{V}\times\mathbf{B} - \frac{\lambda_i\mathbf{J}\times\mathbf{B}}{ne} - \frac{\nabla p_i + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_i}{ne}\end{aligned} \tag{2.6.58}$$

Thus we can find the electric field via addition

$$\begin{aligned}2\mathbf{E} = &\frac{m_i}{e}\left[\left(1-\frac{m_e}{m_i}\right)\left(\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) + 2\frac{\partial}{\partial t}\frac{\lambda_i\mathbf{J}}{n} + \boldsymbol{\nabla}\cdot\left(2\lambda_i\frac{\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}}{ne} + \frac{\lambda_i\mathbf{J}\mathbf{J}}{ne^2}\right)\right] \\ &+ \frac{2m_i\lambda_i}{e}\left[\frac{\nabla n\cdot\mathbf{J}}{n^2}\mathbf{V} - \frac{\boldsymbol{\nabla}\cdot\mathbf{V}}{n}\mathbf{J}\right] - 2\mathbf{V}\times\mathbf{B} + \frac{(\lambda_e-\lambda_i)\mathbf{J}\times\mathbf{B}}{ne} - \frac{\nabla p + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}}{ne}\end{aligned} \tag{2.6.59}$$

$$\begin{aligned}\mathbf{E} = &\frac{m_i}{e}\left[\frac{1}{2}\left(1-\frac{m_e}{m_i}\right)\left(\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) + \frac{\partial}{\partial t}\frac{\lambda_i\mathbf{J}}{n} + \boldsymbol{\nabla}\cdot\left(\lambda_i\frac{\mathbf{V}\mathbf{J}+\mathbf{J}\mathbf{V}}{ne} + \frac{\lambda_i\mathbf{J}\mathbf{J}}{2ne^2}\right)\right] \\ &+ \frac{m_i\lambda_i}{e}\left[\frac{\nabla n\cdot\mathbf{J}}{n^2}\mathbf{V} - \frac{\boldsymbol{\nabla}\cdot\mathbf{V}}{n}\mathbf{J}\right] - \mathbf{V}\times\mathbf{B} + \frac{(\lambda_e-\lambda_i)\mathbf{J}\times\mathbf{B}}{2ne} - \frac{\nabla p + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}}{2ne}\end{aligned} \tag{2.6.60}$$

We can then use any of the values for $\mathbf{E}$ that are convenient. Now if we take zeroth order in the mass ratio we get

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \frac{\mathbf{J}\times\mathbf{B}}{ne} - \frac{\nabla p_e + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_e}{ne} \tag{2.6.61}$$

$$\mathbf{E} = \frac{m_i}{e}\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} - \mathbf{V}\times\mathbf{B} - \frac{\nabla p_i + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_i}{ne} \tag{2.6.62}$$

$$\mathbf{E} = \frac{m_i}{2e}\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} - \mathbf{V}\times\mathbf{B} + \frac{\mathbf{J}\times\mathbf{B}}{2ne} - \frac{\nabla p + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}}{2ne} \tag{2.6.63}$$

The clear winner is (2.6.61) for zeroth order as there are no time derivatives necessary.

Finally, we should make some comments on the energy/temperature evolution equation. As previously mentioned, we know that the 3/2 factor comes from assuming an ideal monatomic gas, and so can be generalized to $1/(\gamma-1)$. Then we have

$$\frac{n_s}{\gamma-1}\frac{\mathrm{d}k_B T_s}{\mathrm{d}t} + p_s\boldsymbol{\nabla}\cdot\mathbf{V}_s = -\boldsymbol{\nabla}\cdot\mathbf{q}_s - \overleftrightarrow{\boldsymbol{\Pi}}_s : \nabla\mathbf{V}_s + Q_s \tag{2.6.64}$$

as our equations for T_s. Let's remove $\mathbf{V}_s$ from these equations. then

$$\mathbf{V}_e\cdot\nabla T_e = \mathbf{V}\cdot\nabla T_e - \frac{\lambda_e\mathbf{J}}{ne}\cdot\nabla T_e \tag{2.6.65}$$

$$p_e\boldsymbol{\nabla}\cdot\mathbf{V}_e = p_e\boldsymbol{\nabla}\cdot\mathbf{V} - \frac{p_e\lambda_e}{e}\boldsymbol{\nabla}\cdot\frac{\mathbf{J}}{n} = p_e\boldsymbol{\nabla}\cdot\mathbf{V} - \frac{\lambda_e p_e}{2e}\nabla\left(\frac{1}{n}\right)\cdot\mathbf{J} \tag{2.6.66}$$

There is a further trick using $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}$ so

$$\nabla a\cdot\mathbf{J} = \nabla a\cdot\boldsymbol{\nabla}\times\frac{\mathbf{B}}{\mu_0} = \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}}{\mu_0}\times\nabla a\right) + \mathbf{B}\cdot\cancel{\boldsymbol{\nabla}\times\nabla a} \tag{2.6.67}$$

So

$$\mathbf{V}_e\cdot\nabla T_e = \mathbf{V}\cdot\nabla T_e - \frac{\lambda_e}{\mu_0 ne}\boldsymbol{\nabla}\cdot(\mathbf{B}\times\nabla T_e) \tag{2.6.68}$$

$$p_e\boldsymbol{\nabla}\cdot\mathbf{V}_e = p_e\boldsymbol{\nabla}\cdot\mathbf{V} + \frac{\lambda_e p_e}{2\mu_0 e}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla n}{n^2}\right) \tag{2.6.69}$$

For ions, we thus find

$$\mathbf{V}_i\cdot\nabla T_i = \mathbf{V}\cdot\nabla T_i + \frac{\lambda_i}{\mu_0 ne}\boldsymbol{\nabla}\cdot(\mathbf{B}\times\nabla T_i) \tag{2.6.70}$$

$$p_i\boldsymbol{\nabla}\cdot\mathbf{V}_i = p_i\boldsymbol{\nabla}\cdot\mathbf{V} - \frac{\lambda_i p_i}{2\mu_0 e}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla n}{n^2}\right) \tag{2.6.71}$$

We see that

$$\overleftrightarrow{\boldsymbol{\Pi}}_i : \nabla\mathbf{V}_i = \overleftrightarrow{\boldsymbol{\Pi}}_i : \left[\nabla\mathbf{V} + \nabla\frac{\lambda_i\mathbf{J}}{ne}\right] \tag{2.6.72}$$

$$\overleftrightarrow{\boldsymbol{\Pi}}_e : \nabla\mathbf{V}_e = \overleftrightarrow{\boldsymbol{\Pi}}_e : \left[\nabla\mathbf{V} - \nabla\frac{\lambda_e\mathbf{J}}{ne}\right] \tag{2.6.73}$$

Then if we add add electron and ion temperature equations we find

$$\begin{aligned}\frac{n}{\gamma-1}\left[\frac{\partial(k_BT)}{\partial t}+\mathbf{V}\cdot\nabla(k_BT)+\frac{\lambda_i}{\mu_0 ne}\boldsymbol{\nabla}\cdot(\mathbf{B}\times\nabla(k_BT_i))-\frac{\lambda_e}{\mu_0 ne}\boldsymbol{\nabla}\cdot(\mathbf{B}\times\nabla(k_BT_e))\right]\\ +p\boldsymbol{\nabla}\cdot\mathbf{V}+\frac{\lambda_e p_e}{2\mu_0 e}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla n}{n^2}\right)-\frac{\lambda_i p_i}{2\mu_0 e}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla n}{n^2}\right)\\ =-\boldsymbol{\nabla}\cdot\mathbf{q}-\overleftrightarrow{\boldsymbol{\Pi}}:\nabla\mathbf{V}+\left(\lambda_e\overleftrightarrow{\boldsymbol{\Pi}}_e+\lambda_i\overleftrightarrow{\boldsymbol{\Pi}}_i\right):\nabla\frac{\mathbf{J}}{ne}+Q\end{aligned} \tag{2.6.74}$$

Now if we use zeroth order mass ratio we find

$$\begin{aligned}\frac{n}{\gamma-1}\left[\frac{\partial(k_BT)}{\partial t}+\mathbf{V}\cdot\nabla(k_BT)-\frac{1}{2\mu_0 ne}\boldsymbol{\nabla}\cdot(\mathbf{B}\times\nabla(k_BT_e))\right]\\ =-p\boldsymbol{\nabla}\cdot\mathbf{V}-\frac{p_e}{2\mu_0 e}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\nabla n}{n^2}\right)-\boldsymbol{\nabla}\cdot\mathbf{q}-\overleftrightarrow{\boldsymbol{\Pi}}:\nabla\mathbf{V}+\overleftrightarrow{\boldsymbol{\Pi}}_e:\nabla\frac{\mathbf{J}}{ne}+Q\end{aligned} \tag{2.6.75}$$

Note that I have not taken into account any mass ratio dependencies in pressure or the stress tensor, though, so $\overleftrightarrow{\boldsymbol{\Pi}}_e$ may still be quite small. This also shows that we still require T_e for the equation for T. we are losing information by only following T as we have to either find T_e explicitly or enforce a relation between T and T_e rather than calculating T_i and T_e separately.

2.6.1 MHD Validity

You may wonder about the actual requirements for MHD validity. There are four criteria that are typically used for determining the (formal) applicability of ideal MHD. These are given by low frequency behavior, quasineutrality, an isotropic distribution function,[44] and coupling between electrons and ions. It should be emphasized that these are not independent criteria. Often satisfying one will have implications for how the other criteria can be satisfied.

The low frequency behavior allows the displacement current and some polarization effects to be ignored. This means that the flow velocity is dominated by a parallel portion and the $\mathbf{E}\times\mathbf{B}$ drift. The frequency we then care about is the ion gyrofrequency. This is because the ion gyrofrequency is the smallest of relevant frequencies for most fusion parameter regimes and we do not want our frequency to be large enough that we have to deal with the gyromotion of any of the particles. Thus for a frequency f, we desire any frequency that we care about in our system has $f<\Omega_i$.

The equation $\boldsymbol{\nabla}\cdot\mathbf{E}=\rho_q/\epsilon_0$ is eliminated in MHD by assuming "quasineutrality". Quasineutrality is usually stated as $n_i\approx n_e$ so that $\rho_q=q(n_i-n_e)\approx 0$ and so $\boldsymbol{\nabla}\cdot\mathbf{E}\simeq 0$. It is important to realize that this is not equivalent to saying $\boldsymbol{\nabla}\cdot\mathbf{E}=0$. Instead we are saying that the charge difference is very small, but not necessarily zero and that over regions larger than a Debye sphere, charges are balanced. A clear way of thinking about this is to consider the contribution of $|\rho\mathbf{E}|$ to the force density equation. When quasineutrality holds $|\rho\mathbf{E}|$ is a small contribution to force balance and so is ignored in MHD. This also is consistent with $\boldsymbol{\nabla}\cdot\mathbf{J}=0$ so that there are no sources or sinks of current density in an MHD plasma. This last comment is elucidated by writing

$$\mu_0\mathbf{J}=\boldsymbol{\nabla}\times\mathbf{B}-\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{2.6.76}$$

[44] In reality isotropic is too restrictive. What is actually needed is that kinetic effects can be represented as a simple closure, although it is no longer necessarily "ideal MHD" then.

By taking $\boldsymbol{\nabla}\cdot\mathbf{E}\approx 0$ via quasineutrality then it seems reasonable to write

$$\mu_0\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{B} \tag{2.6.77}$$

$$\boldsymbol{\nabla}\cdot(\mu_0\mathbf{J}) = \cancel{\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{B})} \equiv \mathbf{0} \tag{2.6.78}$$

where the last equality comes from general vector identities (B.30). We will show that the displacement current $\epsilon_0\partial\mathbf{E}/\partial t$ is small via other means, but it is reassuring that quasineutrality seems to imply the divergence of $\mathbf{J}$ is zero, as well.

The isotropic distribution function is in fact where MHD has most of its problems formally, though it is often possible to retain important kinetic effects from simple closure schemes. An isotropic distribution function implies that all directions are treated equally, and this is clearly something that magnetic confinement does not usually satisfy. Fortunately, the perpendicular directions are indistinguishable because we are on time scales where the random phase of particles gyrating in gyroorbits means it is essentially random and no perpendicular direction is preferred. Along the parallel direction we have no such guarantee, and kinetic effects are usually important for getting parallel dynamics accurately. It is also possible for collisional plasmas to have enough collisions to isotropize the distribution, but this would be ruinous from a plasma confinement perspective, as the magnetic field is no longer confining the plasma. However, there are cases where one can get nearly isotropic distributions through fast pitch-angle scattering in the plasma due to collisions while still having a strong enough magnetic field to get magnetic confinement effects. This is just to say that the isotropic distribution function requirement is more flexible than it may at first seem. If we were to actually write out the isotropic condition, then we require the mean free path to be less than a device size, or $L_{\text{mfp}}/L \ll 1$ or $\frac{v_{\text{th}}}{\nu L} \ll 1$ with ν the collision frequency.

We also need electron and ions to be coupled. Quasineutrality and low frequency behavior are necessary but insufficient conditions for this. This is so that the plasma can be treated as a single fluid, and this requires that $\rho_{L_i}/\ell_\perp \lll 1$ with ρ_{L_i} the ion Larmor radius and $\ell_\perp$ is a characteristic length in a direction perpendicular to $\mathbf{B}$. For $\rho_{L_i}/\ell_\perp \ll 1$ two-fluid effects can be covered through the Hall terms. Another required coupling would be that the temperatures must be coupled, which essentially just means that $L_{\text{mfp}}/L \ll 1$ again.

As an example of looking at relevant terms, we can examine the displacement current. First, let's normalize our equations by first letting τ be a characteristic time, ℓ a characteristic length, and then E_0, B_0 and J_0 are characteristic electric fields, magnetic fields, and current densities, respectively. Then from Faraday's law we have the scaling

$$\frac{E_0}{\ell} \sim \frac{B_0}{\tau} \tag{2.6.79}$$

which yields $E_0/B_0 \sim \ell/\tau \equiv V_E$ where V_E is a characteristic velocity of the system. Thus, the Maxwell-Ampère equation when scaled reads ($\mu_0\epsilon_0 = 1/c^2$)

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{2.6.80}$$

$$\frac{B_0}{\ell} \sim \mu_0 J_0 + \frac{E_0}{c^2} \tag{2.6.81}$$

comparing terms, we see

$$\frac{\boldsymbol{\nabla}\times\mathbf{B}}{\mu_0\mathbf{J}} \sim \frac{B_0}{\ell\mu_0 J_0} \tag{2.6.82}$$

which has no clear scaling whereas

$$\frac{\frac{1}{c^2}\frac{\partial E}{\partial t}}{\boldsymbol{\nabla}\times\mathbf{B}} \sim \frac{\ell E_0}{c^2\tau B_0} \sim \frac{V_E^2}{c^2} \ll 1 \tag{2.6.83}$$

so that the displacement current term is negligible so long as $V_E \ll c$ which we require so that our equations are non-relativistic.

As a quick example showing how approximations lead to MHD equations, let's use the MHD ordering to get an idea of which terms are applicable. An MHD ordering means that, $V_E/v_{\text{th}} \sim 1$ and $\omega/\Omega_i \sim \rho_L/\ell$. So the $\mathbf{E}\times\mathbf{B}$ drift is approximately the thermal speed, and the frequency is (small) and on the same order as the FLR effects.

Then in the generalized Ohm's law we find for rough scaling that

$$\frac{\mathbf{E}}{\mathbf{V}\times\mathbf{B}} \sim \frac{E_0}{v_{\text{th}}B_0} \sim 1 \tag{2.6.84}$$

$$\frac{\mathbf{E}}{\eta\mathbf{J}} \sim \frac{\mu_0\ell v_{\text{th}}}{\eta} \tag{2.6.85}$$

$$\frac{\frac{\mathbf{J}\times\mathbf{B}}{n_e q}}{\mathbf{E}} \sim \frac{p}{n_e q L E_0} \sim \frac{\rho}{\ell} \tag{2.6.86}$$

$$\frac{\frac{\nabla p_e}{n_e q}}{\mathbf{E}} \sim \frac{p}{n_e q \ell E_0} \sim \frac{\rho}{\ell} \tag{2.6.87}$$

The $\mu_0\ell v_{\text{th}}/\eta$ measures how important resistive diffusion is for the problem, and is not included in ideal MHD. The two-fluid terms then are the two terms that become unimportant as the ion Larmor radius $\frac{v_{\text{th}}}{\Omega_i} = \frac{m_i v_{\text{th}}}{eB_0}$ becomes quite small. The electron inertia term, $\partial\mathbf{J}/\partial t$, drops out in single fluid MHD because we ignore electron inertia completely.

Note that this ordering would in fact show that the ion stress tensor should not be included, as it is a higher order effect, as well.

We can see how well these approximations hold by using a fairly typical modern tokamak's parameters

$$\begin{aligned} &B_0 \sim 1\,\text{T} \quad R_0 \sim 3\,\text{m} \quad a \sim 0.8\,\text{m} \\ &n_0 \sim 1\times10^{20}\,\text{m}^{-3} \quad T_0 \sim 10\,\text{keV} \quad m_i \sim 3.34\times10^{-27}\,\text{kg} \end{aligned} \tag{2.6.88}$$

Then we find

$$\frac{v_{\text{th}}}{c} \approx \frac{9.8\times10^5\,\text{m/s}}{3\times10^8\,\text{m/s}} \approx 0.003 \ll 1 \tag{2.6.89}$$

$$\frac{v_A}{c} = \frac{B}{c\sqrt{m_i n \mu_0}} \approx \frac{1.5\times10^6\,\text{m/s}}{3\times10^8\,\text{m/s}} \approx 0.005 \ll 1 \tag{2.6.90}$$

$$\rho_L \approx 2\times10^{-2}\,\text{m} \ll a \ll R_0 \tag{2.6.91}$$

$$\Omega_i = \frac{eB_0}{m_i} \approx 5\times10^7\,\text{s}^{-1} \gg \frac{v_{\text{th}}}{a} \approx 1.2\times10^6\,\text{s}^{-1} \tag{2.6.92}$$

$$\Omega_i = \frac{eB_0}{m_i} \approx 5\times10^7\,\text{s}^{-1} \gg \frac{v_A}{a} \approx 1.9\times10^6\,\text{s}^{-1} \tag{2.6.93}$$

Criterion	Approximation
Low Frequency	$f \ll \Omega_i$
Quasineutrality	$\lvert\rho_q \mathbf{E}\rvert \ll \mathbf{F}_{\text{MHD}}$
Isotropic Distribution Function	$L_{\text{mfp}}/L \ll 1$
Coupling of Ions and Electrons	$\rho_L/\ell_\perp \lll 1$

Table 2.2: The four ideal MHD criteria. Here f is a frequency, Ω_i is the ion gyrofrequency, $\rho_q\mathbf{E}$ is the electric force $\mathbf{F}_{\text{MHD}}$ are forces considered in MHD, ρ_L is the ion Larmor radius, and $\ell_\perp$ is a characteristic length perpendicular to $\mathbf{B}$.

So then we see that the displacement current and the Alfvénic activity can easily be dismissed in terms of frequency, as desired by our orderings. In addition, the length scales are definitely in the correct limit with the ion Larmor radius quite a bit smaller than length scales of interest for MHD.

The four criteria are summarized in Table 2.2.

2.6.2 Extended MHD

Extended MHD can be thought of as using our previous equations with $m_e/m_i = 0$. Two fluid MHD is another type of MHD,[45] but I will consider this essentially the same as extended MHD. In some sense, everything is an extended MHD model besides ideal MHD, and my nomenclature is mainly driven by computational distinctions. Using energy conservation arguments, one can show that if certain terms are retained in the definition of the electric field, then the temperature equation must be of a certain form, with extended MHD being one consistent formulation.

Again, we restrict ourselves to a two component plasma, singly ionized ions and electrons. It is possible to consider generalizations, but this is outside conventional MHD models. Usually a two-species plasma is used, and we shall continue that tradition here. Thus the two-fluid extended MHD equations can be written as

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{V}) = 0 \tag{2.6.94}$$

$$nm_i\frac{\partial \mathbf{V}}{\partial t} + nm_i\mathbf{V}\cdot\nabla\mathbf{V} = \mathbf{J}\times\mathbf{B} - \nabla p - \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}} \tag{2.6.95}$$

$$\frac{n_s}{\gamma-1}\left(\frac{\partial(k_BT_s)}{\partial t} + \mathbf{V}_s\cdot\nabla(k_BT_s)\right) + p_s\boldsymbol{\nabla}\cdot\mathbf{V}_s = -\boldsymbol{\nabla}\cdot\mathbf{q}_s - \overleftrightarrow{\boldsymbol{\Pi}}_s : \nabla\mathbf{V}_s + Q_s \tag{2.6.96}$$

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} = \eta\mathbf{J} + \frac{\mathbf{J}\times\mathbf{B}}{ne} - \frac{\nabla p_e + \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_e}{ne} \tag{2.6.97}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{2.6.98}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} \tag{2.6.99}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E} \tag{2.6.100}$$

$$\mathbf{V}_e = \mathbf{V} - \frac{\mathbf{J}}{ne} \tag{2.6.101}$$

Notice that I have left out what exactly $\overleftrightarrow{\boldsymbol{\Pi}}_s$, $\mathbf{q}_s$ and Q_s actually are. They are problem dependent, and can be changed depending on the specific model being used with extended MHD. For

[45] Besides Ideal and Resistive MHD, there are a zoo of names. Some do not have a standard form.

example, when using a Braginskii closure model (remember, this model is not rigorously justified at temperature ranges of fusion interest), then we can find the form of $\overleftrightarrow{\mathbf{\Pi}}_s$ and $\mathbf{q}_s$ using kinetics and an asymptotic closure.

These equations have proven to be very useful for extending our understanding of plasmas beyond the ideal MHD model which I will cover below. For example, the generalized Ohm's Law, which is just a way of saying what is the relationship of $\mathbf{E}$ to the other variables contains a lot of new physics. The $\eta\mathbf{J}$ term allows electrical resistivity and so allows magnetic field lines to "break" and "reconnect", thus allowing the process of magnetic reconnection. The electron pressure and stress terms tend not to have any exciting names, whereas the $\mathbf{J}\times\mathbf{B}$ is called the Hall term which lets the electron flow velocity to carry magnetic field lines rather than the center-of-mass velocity, and generally allows electron flow to affect important physical processes. Sometimes the electron pressure and stress terms are considered to be part of the Hall terms. Last, I completely ignored electron inertia. Sometimes people retain an extra term corresponding to electron inertia, $\frac{m_e}{e^2}\frac{\mathrm{d}(\mathbf{J}/n_e)}{\mathrm{d}t}$ which comes from the $\mathrm{d}\mathbf{V}_e/\mathrm{d}t$ term we ignored above. This term turns out to be useful computationally, even though it is small, because it helps make the resulting matrix to be solved more diagonally dominant with a smaller condition number.[46]

2.6.3 Resistive MHD

Another popular version of the MHD equations is resistive MHD. This is usually a single fluid, with a single temperature. Thus, we ignore the electron specific terms in the Ohm's law and get

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot(n\mathbf{V})=0 \tag{2.6.102}$$

$$nm_i\frac{\partial\mathbf{V}}{\partial t}+nm_i\mathbf{V}\cdot\nabla\mathbf{V}=\mathbf{J}\times\mathbf{B}-\nabla p-\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{\Pi}} \tag{2.6.103}$$

$$\frac{n}{\gamma-1}\left(\frac{\partial(k_BT)}{\partial t}+\mathbf{V}\cdot\nabla(k_BT)\right)+p\boldsymbol{\nabla}\cdot\mathbf{V}=-\boldsymbol{\nabla}\cdot\mathbf{q}-\overleftrightarrow{\mathbf{\Pi}}:\nabla\mathbf{V}+Q \tag{2.6.104}$$

$$\mathbf{E}=\eta\mathbf{J}-\mathbf{V}\times\mathbf{B} \tag{2.6.105}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0 \tag{2.6.106}$$

$$\boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{J} \tag{2.6.107}$$

$$\frac{\partial\mathbf{B}}{\partial t}=-\boldsymbol{\nabla}\times\mathbf{E} \tag{2.6.108}$$

This allows magnetic reconnection, and is a simplification upon extended MHD that is often useful. The separate temperatures and separate fluid treatment is more expensive to solve on computers and often not of interest, so this model is often used. The question of what T actually is in these cases is a bit unobvious, but it can be viewed as electrons and ions having similar enough temperatures to be considered the same. This is often not very true experimentally, but so long as the temperature does not vary differently between electrons and ions too much, it is not usually too terrible of an approximation for getting computational results.

[46] The smaller the condition number, the more accurately one can invert a matrix. This is essentially saying that we can accurately solve the problem even if we have a little bit of error in our calculations.

2.7 Ideal MHD

> *How are we to look for functions that will do what we want? In such cases, it is usually wise to take the simplest possible example and examine it carefully for hints of what happens in the more complicated cases. This is a rule of general value; if you cannot solve some problem, make up for yourself the simplest problem of the same kind that you can devise, and see if it suggests anything.*
>
> — W. W. Sawyer[17, p. 51]

This is perhaps one of the most lauded and useful set of equations for gaining insight into plasma phenomena with a fluid model. In this case, we ignore the resistivity of the plasma. Plasma resistivity scales as $T^{-3/2}$ and so for sufficiently hot plasmas the resistivity becomes essentially zero, giving some justification for the approximation. The equations are essentially identical from before but I will list them anyway.

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{V}) = 0 \tag{2.7.1}$$

$$nm_i\frac{\partial \mathbf{V}}{\partial t} + nm_i\mathbf{V}\cdot\nabla\mathbf{V} = \mathbf{J}\times\mathbf{B} - \nabla p - \boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}} \tag{2.7.2}$$

$$\frac{n}{\gamma-1}\left(\frac{\partial(k_BT)}{\partial t} + \mathbf{V}\cdot\nabla(k_BT)\right) + p\boldsymbol{\nabla}\cdot\mathbf{V} = -\boldsymbol{\nabla}\cdot\mathbf{q} - \overleftrightarrow{\boldsymbol{\Pi}}:\nabla\mathbf{V} + Q \tag{2.7.3}$$

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} \tag{2.7.4}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{2.7.5}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} \tag{2.7.6}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E} \tag{2.7.7}$$

The $\mathbf{E}$ is the electric field as seen from the "lab", that is if you are standing in place unaffected by the flow velocity from the plasma fluid. If you consider the electric field in the frame of reference moving with the fluid, then we will find that $\mathbf{E}' = \mathbf{0}$ and so our equation for $\mathbf{E}$ is in fact just a fancy way of saying that when moving with the flow velocity, you see no electric field (another reason for "ideal" as a moniker). We have an ideal perfect conductor,[47] which can be seen as another reason this is a good name.

2.7.1 Frozen Flux Theorem

Now let's prove the frozen flux theorem and hence that $\mathbf{E}' = \mathbf{0}$. This theorem says that magnetic flux is frozen into the fluid, so that as the fluid moves, the magnetic field in that fluid moves with it. We can begin by writing Faraday's Law in a lab frame

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E} \tag{2.7.8}$$

We start with the magnetic flux for a surface defined as

$$\psi_{\mathbf{B}} = \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{B} \tag{2.7.9}$$

[47] a perfect conductor is not the same as a superconductor, which expels magnetic fields when it is formed

Suppose we consider the change of $\psi_{\mathbf{B}}$ now for that surface through time

$$\frac{\mathrm{d}\psi_{\mathbf{B}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{B} \tag{2.7.10}$$

Note that many proofs of this principle omit an important part of the derivation. We need a generalization of the Leibniz rule for differentiation under the integral sign. For three dimensions and flux integrals this is usually called the Helmholtz transport theorem. The correct equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{B} = \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{V}\boldsymbol{\nabla}\cdot\mathbf{B} - \oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot(\mathbf{V}\times\mathbf{B}) + \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\frac{\partial\mathbf{B}}{\partial t} \tag{2.7.11}$$

where $C = \partial S$ is a bounding curve for the surface S. Many frozen flux derivations leave out the first term on the right hand side as it is not apparent when thinking about how the surface changing would affect the flux. Because $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ our form then becomes

$$\frac{\mathrm{d}\psi_{\mathbf{B}}}{\mathrm{d}t} = -\oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot(\mathbf{V}\times\mathbf{B}) + \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\frac{\partial\mathbf{B}}{\partial t} = \iint_S \hat{\mathbf{n}}\cdot\left[\frac{\partial\mathbf{B}}{\partial t} - \boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B})\right] \tag{2.7.12}$$

Most proofs do not explain the origin of this equation except through an argument that the change in the surface must be of the form given by $\int \mathrm{d}\boldsymbol{\ell}$ term. We will give a rigorous proof later.

We can substitute our form for $\partial\mathbf{B}/\partial t$ to find

$$\frac{\mathrm{d}\psi_{\mathbf{B}}}{\mathrm{d}t} = -\oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot(\mathbf{V}\times\mathbf{B}) + \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\mathbf{E} \tag{2.7.13}$$

$$\frac{\mathrm{d}\psi_{\mathbf{B}}}{\mathrm{d}t} = -\oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot[(\mathbf{V}\times\mathbf{B}) + \mathbf{E}] = -\oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot[\mathbf{E}'] = 0 \tag{2.7.14}$$

Thus, our flux does not change in time, and it is frozen into the plasma. If we wish to go with the plasma flow then we will find $\mathbf{E}' = \mathbf{0}$ since then there is no velocity changes to move magnetic field around.

To elaborate on the weaknesses of many proofs for the above relation (2.7.12), what I mean is that they do not actually explicitly mention the $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ requirement and so seem like proofs for any field instead of only divergenceless ones. For the frozen flux theorem, they are true, but misleading.

We will fix this with a completely general proof for any vector field $\mathbf{g}$.[48] This is most clearly accomplished by first looking at a vector field that has no explicit time dependence and then considering an explicitly time dependent vector field.

So first consider a general flux from an arbitrary vector field $\mathbf{g} = \mathbf{g}(\mathbf{x})$. When the velocity field moves the surface and changes it, we evalute $\mathbf{g}$ at a different position which is handled through the surface we are evaluating on. Because $\mathbf{g}$ does not change with time we do not need to worry about $\mathbf{g}(\mathbf{x})$ changing, just what it is integrated over. Then we can write

$$\Psi_{\mathbf{g}}(r,t) = \iint_{S_1(t)} \mathrm{d}S_1(t)\ \hat{\mathbf{n}}_1(t)\cdot\mathbf{g}(\mathbf{x}) \tag{2.7.15}$$

with r representing the remaining spatial variation and where $S_1(t)\hat{\mathbf{n}}_1(t)$ defines the time dependent surface and normal vector. We can consider a time Δt later, and the volume formed by forming

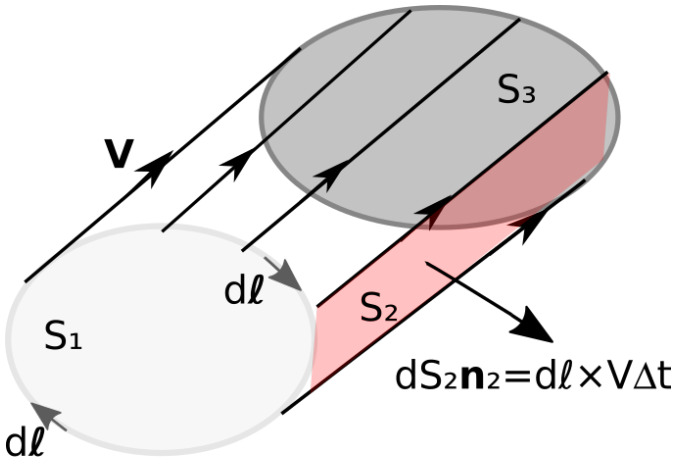

Figure 2.11: This shows the volume over which we apply Gauss's Law to derive the generalized Leibniz rule for integration. Note that we use simple ovals to show how the integration works, but the shape can be much more complicated.

a surface $S_2(\Delta t\,)$ that covers $S_1(t)$ and $S_3 = S_1(t+\Delta t\,)$ with straight lines (see Figure 2.11).[49] We will apply Gauss's law over the volume defined by the cylindrical-like volume defined by $S_1(t)$, $S_1(t+\Delta t\,)$ and the sides $S_2(\Delta t\,)$. Usually we have $\hat{\mathbf{n}}$ point normally outward, but we can use $\hat{\mathbf{n}}$ as pointing inward with a negative sign on the $S_1(t)$ surface and leave the $\hat{\mathbf{n}}$ on $S_1(t+\Delta t\,) = S_3(t)$ positive to write the integrals over the surfaces as

$$\begin{aligned}&\iint_{S_1(t)} \mathrm{d}S_1(t)\ (-\hat{\mathbf{n}}_1(t))\cdot\mathbf{g}(\mathbf{x}) + \iint_{S_1(t+\Delta t\,)} \mathrm{d}S_1(t+\Delta t\,)\ \hat{\mathbf{n}}_1(t+\Delta t\,)\cdot\mathbf{g}(\mathbf{x})\\ &+ \iint_{S_2(\Delta t\,)} \mathrm{d}S_2(\Delta t\,)\ \hat{\mathbf{n}}_2\cdot\mathbf{g}(\mathbf{x})\end{aligned} \tag{2.7.16}$$

because we have now formed a closed volume, we can use the divergence theorem for $\mathbf{g}(\mathbf{x}) = \mathbf{g}$. If you are disturbed by a volume extending through "time", remember we're using the velocity and time as a parameterization and that we are therefore still actually dealing with a 3D position space, but where one of the coordinate directions happens to correspond to time in a specific way.

$$\begin{aligned}&\iint_{S_1(t)} \mathrm{d}S_1(t)\ [-\hat{\mathbf{n}}_1(t)\cdot\mathbf{g}] + \iint_{S_1(t+\Delta t\,)} \mathrm{d}S_1(t+\Delta t\,)\ \hat{\mathbf{n}}_1(t+\Delta t\,)\cdot\mathbf{g}\\ &+ \iint_{S_2(\Delta t\,)} \mathrm{d}S_2(\Delta t\,)\ \hat{\mathbf{n}}_2\cdot\mathbf{g} = \iiint_V \mathrm{d}V\ \boldsymbol{\nabla}\cdot\mathbf{g} = \iint_{S_1(t)} \mathrm{d}S_1(t)\ \hat{\mathbf{n}}_1(t)\cdot\int \mathrm{d}\mathbf{s}\ \boldsymbol{\nabla}\cdot\mathbf{g}\end{aligned} \tag{2.7.17}$$

where $\mathrm{d}\mathbf{s}$ can point at parts of the volume by using for our short time we have $\int \mathrm{d}\mathbf{s} = \mathbf{V}\Delta t$ where

[48] This somewhat follows a proof from a recent paper[1].

[49] An eagle-eyed reader will notice that this image is not the most general case. It assumes $\hat{\mathbf{n}}_1(t)\cdot\mathbf{V}$ is positive over the entire surface $S_1(t)$. If $\hat{\mathbf{n}}_1(t)\cdot\mathbf{V}$ changes sign over $S_1(t)$, we would have an unphysical volume. In these cases, split $S_1(t)$ into subsurfaces $S_1^i(t)$ where $\hat{\mathbf{n}}_1(t)\cdot\mathbf{V}$ is either completely non-negative or completely negative. Each volume is now well-defined, and contributes either positive or negative correctly because of our use of $\mathrm{d}S_1(t)\cdot\mathrm{d}\mathbf{s}$ and $\mathrm{d}\boldsymbol{\ell}\times\mathrm{d}\mathbf{s}$. We can apply the exact same arguments given for each $S_1^i(t)$ as we do for $S_1(t)$ with $\mathbf{n}\cdot\mathbf{V}>0$. We will get the same equation as (2.7.24) for each $S_1^i(t)$ and so if we sum all the surfaces we will get (2.7.24) exactly.

we are considering the time so short we can ignore $\mathcal{O}([\Delta t\,]^2)$ corrections. Thus

$$\begin{aligned}&\iint_{S_1(t+\Delta t\,)} \mathrm{d}S_1(t+\Delta t\,)\ \hat{\mathbf{n}}_1(t+\Delta t\,)\cdot\mathbf{g} - \iint_{S_1(t)} \mathrm{d}S_1(t)\ (\hat{\mathbf{n}}_1(t))\cdot\mathbf{g}\\ &+\iint_{S_2(\Delta t\,)} \mathrm{d}S_2(\Delta t\,)\ \hat{\mathbf{n}}_2\cdot\mathbf{g} = \Delta t\iint_S \mathrm{d}S_1(t)\ \hat{\mathbf{n}}_1\cdot\mathbf{V}\boldsymbol{\nabla}\cdot\mathbf{g}\end{aligned} \tag{2.7.18}$$

The $\mathrm{d}S_2(\Delta t\,)$ term can be converted since we have a bounding curve $C=\partial S$ at time t with differential element $\mathrm{d}\boldsymbol{\ell}$ along the bounding curve so that

$$\iint_{S_2}\mathrm{d}S_2\hat{\mathbf{n}}_2\cdot\mathbf{g} = \oint_C \mathrm{d}\boldsymbol{\ell}\times\overbrace{\int}^{\approx\Delta t\,\mathbf{V}}\mathrm{d}\mathbf{s}\cdot[\mathbf{g}] = \oint_C \mathrm{d}\boldsymbol{\ell}\times\left[\mathbf{V}\Delta t\,+\mathcal{O}([\Delta t\,]^2)\cdot\mathbf{g}\right] \tag{2.7.19}$$

where we are using that $\mathrm{d}S_2\mathbf{n}_2 = \mathrm{d}\boldsymbol{\ell}\times\Delta t\,\mathbf{V}$. And so to accuracy $\mathcal{O}(\Delta t\,)$ we find

$$\iint_{S_2}\mathrm{d}S_2\hat{\mathbf{n}}_2\cdot\mathbf{g} = \oint_C \mathrm{d}\boldsymbol{\ell}\ \times\mathbf{V}\cdot\mathbf{g} = \oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot\mathbf{V}\times\mathbf{g} \tag{2.7.20}$$

Remember that we have by definition that

$$\begin{aligned}&\iint_{S_1(t+\Delta t\,)} \mathrm{d}S_1(t+\Delta t\,)\ \hat{\mathbf{n}}_1(t+\Delta t\,)\cdot\mathbf{g}(\mathbf{x}) - \iint_{S_1(t)} \mathrm{d}S_1(t)\ \hat{\mathbf{n}}_1(t)\cdot\mathbf{g}(\mathbf{x}(t))\\ &=\Psi_{\mathbf{g}}(r,t+\Delta t\,)-\Psi_{\mathbf{g}}(r,t)\end{aligned} \tag{2.7.21}$$

And so

$$\Psi(r,t+\Delta t\,)-\Psi(r,t) = \Delta t\,\left[-\oint_C \mathrm{d}\boldsymbol{\ell}\cdot\mathbf{V}\times\mathbf{g} + \iint_{S_1(t)}\mathrm{d}S_1\ \hat{\mathbf{n}}_1\cdot\mathbf{V}(\boldsymbol{\nabla}\cdot\mathbf{g})\right] + \mathcal{O}([\Delta t\,]^2) \tag{2.7.22}$$

We can then write

$$\overbrace{\lim_{\Delta t\to 0}\frac{\Psi(r,t+\Delta t\,)-\Psi(r,t)}{\Delta t}}^{\frac{\mathrm{d}\Psi_{\mathbf{g}}}{\mathrm{d}t}} = \lim_{\Delta t\to 0}\left[-\oint_C \mathrm{d}\boldsymbol{\ell}\cdot\mathbf{V}\times\mathbf{g} + \iint_{S_1(t)}\mathrm{d}S_1\ \hat{\mathbf{n}}_1\cdot\mathbf{V}(\boldsymbol{\nabla}\cdot\mathbf{g})\right] + \mathcal{O}([\Delta t\,]^2) \tag{2.7.23}$$

and so

$$\frac{\mathrm{d}\Psi_{\mathbf{g}}}{\mathrm{d}t} = -\oint_C \mathrm{d}\boldsymbol{\ell}\cdot\mathbf{V}\times\mathbf{g} + \iint_{S_1(t)}\mathrm{d}S_1\ \hat{\mathbf{n}}_1\cdot\mathbf{V}(\boldsymbol{\nabla}\cdot\mathbf{g}) \tag{2.7.24}$$

for $\mathbf{g}(\mathbf{x}(t))$ with no explicit time dependence. We could also use Stokes' theorem to rewrite this as

$$\frac{\mathrm{d}\Psi_{\mathbf{g}}}{\mathrm{d}t} = \iint_{S_1(t)}\mathrm{d}S_1\ \hat{\mathbf{n}}_1\cdot[\mathbf{V}(\boldsymbol{\nabla}\cdot\mathbf{g})-\boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{g})] \tag{2.7.25}$$

Now we consider $\mathbf{G}=\mathbf{G}(\mathbf{x},t)$ such that at exactly time τ it is the time invariant $\mathbf{g}(\mathbf{x})$ that we considered above. We use

$$\begin{aligned}\mathbf{G}(\mathbf{x},t) &= \overbrace{\mathbf{g}(\mathbf{x})}^{\mathbf{G}(\mathbf{x},\tau)} + \overbrace{\Delta t}^{=t-\tau}\left.\frac{\partial\mathbf{G}}{\partial t}\right|_{\text{fixed }\mathbf{x}} + \mathcal{O}([\Delta t\,]^2)\\ &= \mathbf{g}(\mathbf{x}(t)) + \Delta t\ \left.\frac{\partial\mathbf{G}}{\partial t}\right|_{\text{fixed }\mathbf{x}} + \mathcal{O}([\Delta t\,]^2)\end{aligned} \tag{2.7.26}$$

and that for $\mathbf{g}$ explicitly time independent we have already found the answer. So we find

$$\frac{\mathrm{d}\Psi_{\mathbf{G}}}{\mathrm{d}t} = \lim_{\Delta t\to 0}\frac{\Psi(r,\tau+\Delta t\,)-\Psi(r,\tau)}{\Delta t} = \tag{2.7.27}$$

$$= \lim_{\Delta t\to 0}\frac{\iint_{S_1(\tau+\Delta t\,)}\mathrm{d}S_1(\tau+\Delta t\,)\;\hat{\mathbf{n}}_1(\tau+\Delta t\,)\cdot\mathbf{G}(\mathbf{x},\tau+\Delta t) - \iint_{S_1(\tau)}\mathrm{d}S_1(\tau)\;\hat{\mathbf{n}}_1(\tau)\cdot\overbrace{\mathbf{G}(\mathbf{x},\tau)}^{=\mathbf{g}}}{\Delta t} \tag{2.7.28}$$

$$= \lim_{\Delta t\to 0}\frac{\overbrace{\iint_{S_1(\tau+\Delta t\,)}\mathrm{d}S_1(\tau+\Delta t\,)\;\hat{\mathbf{n}}_1(\tau+\Delta t\,)\cdot\left(\mathbf{g}+\Delta t\,\frac{\partial\mathbf{G}}{\partial t}\right)}^{(2.7.21)} - \overbrace{\iint_{S_1(\tau)}\mathrm{d}S_1(\tau)\;\hat{\mathbf{n}}_1(\tau)\cdot\mathbf{g}}^{(2.7.21)}}{\Delta t} \tag{2.7.29}$$
$$+\mathcal{O}([\Delta t\,])^2$$

$$= \overbrace{\frac{\mathrm{d}\Psi_{\mathbf{g}}}{\mathrm{d}t}}^{(2.7.21)} + \lim_{\Delta t\to 0}\iint_{S_1(\tau+\Delta t\,)}\mathrm{d}S_1(\tau+\Delta T\,)\;\hat{\mathbf{n}}_1(\tau+\Delta t\,)\cdot\frac{\partial\mathbf{G}}{\partial t} \tag{2.7.30}$$

This can then be rewritten as

$$\frac{\mathrm{d}\Psi_{\mathbf{G}}}{\mathrm{d}t} = \overbrace{-\oint_C \mathrm{d}\boldsymbol{\ell}\cdot\mathbf{V}\times\mathbf{g} + \iint_{S_1(t)}\hat{\mathbf{n}}_1\cdot\mathbf{V}(\boldsymbol{\nabla}\cdot\mathbf{g})}^{\frac{\mathrm{d}\Psi_{\mathbf{g}}}{\mathrm{d}t}} + \lim_{\Delta t\to 0}\iint_{S_1(\tau+\Delta t\,)}\mathrm{d}S_1(\tau+\Delta t\,)\;\hat{\mathbf{n}}_1(\tau+\Delta t\,)\cdot\frac{\partial\mathbf{G}}{\partial t} \tag{2.7.31}$$

with $\Delta t\to 0$, we know that $\mathbf{g}=\mathbf{G}$ and so

$$\frac{\mathrm{d}\Psi_{\mathbf{G}}}{\mathrm{d}t} = \oint_C \mathrm{d}\boldsymbol{\ell}\cdot\mathbf{V}\times\mathbf{G} + \iint_{S_1(t)}\hat{\mathbf{n}}_1\cdot\mathbf{V}(\boldsymbol{\nabla}\cdot\mathbf{G}) + \iint_{S_1(t)}\mathrm{d}S_1(t)\;\hat{\mathbf{n}}_1\cdot\frac{\partial\mathbf{G}}{\partial t} \tag{2.7.32}$$

$$= \iint_{S_1(t)}\mathrm{d}S_1(t)\;\hat{\mathbf{n}}_1\cdot\left[\frac{\partial\mathbf{G}}{\partial t} - \boldsymbol{\nabla}\cdot(\mathbf{V}\times\mathbf{G}) + \mathbf{V}\boldsymbol{\nabla}\cdot\mathbf{G})\right] \tag{2.7.33}$$

as it must be. A more general proof using differential forms (and for multiple dimensions) can be done[10].

2.7.2 Variational Methods

It is common to consider perturbations and then find the "minimum" energy situation. Once again, minimum energy really means maximum entropy and corresponds to a minimum potential energy situation. Still, this concept allows us to learn a great deal. The variational energy method has been used in plasmas for many years beginning with Hain[12] and Bernstein[2].

We begin with finding the fluid potential energy of the plasma in an Ideal MHD situation. The idea is simple. We first find the force density find the potential energy by looking at the work done over a small displacement and then extremize the potential energy to learn about stability. We begin by assuming that $\widetilde{\mathbf{V}}_1 = \frac{\partial\boldsymbol{\xi}}{\partial t}$, so that the perturbed velocity is completely due to the perturbation $\boldsymbol{\xi}$. This means that $\mathbf{V}_0 = \mathbf{0}$ (no background flow) which hugely simplifies our analysis. We can

then say (assuming $\mathbf{q} = \mathbf{q}_0 + \delta\mathbf{q}_1 + \mathcal{O}(\delta^2)$ for all quantities)

$$nm_i \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{F} = nm_i \left[\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right] \tag{2.7.34}$$

Then we find at zeroth order that

$$\mathbf{F}_0 = n_0 m_i \left(\frac{\partial \mathbf{V}_0}{\partial t} + \mathbf{V}_0 \cdot \nabla \mathbf{V}_0 \right) = \mathbf{0} \tag{2.7.35}$$

the $\mathcal{O}(\delta)$ terms are

$$\begin{aligned} \widetilde{\mathbf{F}}_1 &= n_1 m_i \cancel{\left[\frac{\partial \mathbf{V}_0}{\partial t} + \mathbf{V}_0 \cdot \nabla \mathbf{V}_0 \right]} + n_0 m_i \left(\frac{\partial \mathbf{V}_1}{\partial t} + \cancel{\mathbf{V}_0 \cdot \nabla \widetilde{\mathbf{V}}_1 + \widetilde{\mathbf{V}}_1 \cdot \nabla \mathbf{V}_0} \right) \\ &= n_0 m_i \frac{\partial \mathbf{V}_1}{\partial t} = n_0 m_i \frac{\partial}{\partial t} \frac{\partial \boldsymbol{\xi}}{\partial t} = n_0 m_i \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \end{aligned} \tag{2.7.36}$$

Now we remember that

$$\mathbf{F} = -\nabla p + \mathbf{J} \times \mathbf{B} \tag{2.7.37}$$

$$\mathbf{F}_0 = 0 = -\nabla p_0 + \mathbf{J}_0 \times \mathbf{B}_0 \tag{2.7.38}$$

$$\nabla p_0 = \mathbf{J}_0 \times \mathbf{B}_0 \tag{2.7.39}$$

$$\widetilde{\mathbf{F}}_1 = -\nabla \widetilde{p}_1 + \mathbf{J}_0 \times \widetilde{\mathbf{B}}_1 + \widetilde{\mathbf{J}}_1 \times \mathbf{B}_0 \tag{2.7.40}$$

We then use Ampère's Law

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2.7.41}$$

$$\boldsymbol{\nabla} \times \mathbf{B}_0 = \mu_0 \mathbf{J}_0 \tag{2.7.42}$$

$$\boldsymbol{\nabla} \times \mathbf{B}_1 = \mu_0 \mathbf{J}_1 \tag{2.7.43}$$

to write

$$\widetilde{\mathbf{F}}_1 = -\nabla \widetilde{p}_1 + \mathbf{J}_0 \times \widetilde{\mathbf{B}}_1 + \frac{\boldsymbol{\nabla} \times \mathbf{B}_1}{\mu_0} \times \mathbf{B}_0 \tag{2.7.44}$$

We now need a connection between $\widetilde{p}_1$, $\widetilde{\mathbf{B}}_1$ and $\boldsymbol{\xi}$ to write our force density's dependence on a perturbation. We use $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$ so

$$\mathbf{E}_0 = \mathbf{V}_0 \times \mathbf{B}_0 = \mathbf{0} \tag{2.7.45}$$

$$\widetilde{\mathbf{E}}_1 = \widetilde{\mathbf{V}}_1 \times \mathbf{B}_0 + \cancel{\mathbf{V}_0 \times \widetilde{\mathbf{B}}_1} = \frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B}_0 \tag{2.7.46}$$

$$\widetilde{\mathbf{E}}_1 = \widetilde{\mathbf{V}}_1 \times \mathbf{B}_0 + \cancel{\mathbf{V}_0 \times \widetilde{\mathbf{B}}_1} = \frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B}_0 \tag{2.7.47}$$

and so Faraday's law relays

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E} \tag{2.7.48}$$

$$\frac{\partial \mathbf{B}_0}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E}_0 = \mathbf{0} \tag{2.7.49}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\boldsymbol{\nabla} \times \widetilde{\mathbf{E}}_1 = \boldsymbol{\nabla} \times \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B}_0 \right) = \frac{\partial}{\partial t} \left[\boldsymbol{\xi} \times \mathbf{B}_0 \right] \tag{2.7.50}$$

This implies

$$\widetilde{\mathbf{B}}_1 = \boldsymbol{\xi} \times \mathbf{B}_0 + \mathbf{C}(\mathbf{x}) \tag{2.7.51}$$

where $\mathbf{C}$ is a time independent constant. We'd like $\widetilde{\mathbf{B}}_1 = \mathbf{0}$ when $\boldsymbol{\xi} = \mathbf{0}$ which then implies $\mathbf{C} = \mathbf{0}$ and we get

$$\widetilde{\mathbf{B}}_1 = \boldsymbol{\xi} \times \mathbf{B}_0 \tag{2.7.52}$$

We will use the continuity equation to find a relation for pressure.

$$\frac{\partial n_0}{\partial t} = -\boldsymbol{\nabla} \cdot (n_0 \mathbf{V}_0) = 0 \tag{2.7.53}$$

$$\frac{\partial \widetilde{n}_1}{\partial t} = -\boldsymbol{\nabla} \cdot (n_0 \widetilde{\mathbf{V}}_1) = -\nabla n_0 \cdot \widetilde{\mathbf{V}}_1 - n_0 \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} = -\frac{\partial}{\partial t} (\nabla n_0 \cdot \boldsymbol{\xi} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}) \tag{2.7.54}$$

$$\widetilde{n}_1 = -\nabla n_0 \cdot \boldsymbol{\xi} - n_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} + \mathbf{C}(\mathbf{x}) \tag{2.7.55}$$

where $\mathbf{C} = \mathbf{0}$ because we desire $\widetilde{n}_1 = 0$ when $\boldsymbol{\xi} = \mathbf{0}$. We have

$$n \frac{\mathrm{d}(k_B T)}{\mathrm{d}t} = -(\gamma - 1) p \boldsymbol{\nabla} \cdot \mathbf{V} \tag{2.7.56}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} - k_B T \frac{\mathrm{d}n}{\mathrm{d}t} = -(\gamma - 1) p \boldsymbol{\nabla} \cdot \mathbf{V} \tag{2.7.57}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} - k_B T \left(\frac{\partial n}{\partial t} + \mathbf{V} \cdot \nabla n \right) = -(\gamma - 1) p \boldsymbol{\nabla} \cdot \mathbf{V} \tag{2.7.58}$$

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p - k_B T \left(\frac{\partial n}{\partial t} + \mathbf{V} \cdot \nabla n \right) = -(\gamma - 1) p \boldsymbol{\nabla} \cdot \mathbf{V} \tag{2.7.59}$$

and so

$$\frac{\partial p_0}{\partial t} + \cancel{\mathbf{V}_0 \cdot \nabla p_0} - k_B T_0 \cancel{\left(\frac{\partial n_0}{\partial t} + \mathbf{V}_0 \cdot \nabla n_0 \right)} = \cancel{-(\gamma - 1) p_0 \boldsymbol{\nabla} \cdot \mathbf{V}_0} \tag{2.7.60}$$

$$\frac{\partial p_0}{\partial t} = 0 \tag{2.7.61}$$

and so

$$\begin{aligned} &\frac{\partial \widetilde{p}_1}{\partial t} + \cancel{\mathbf{V}_0 \cdot \nabla \widetilde{p}_1} + \widetilde{\mathbf{V}}_1 \cdot \nabla p_0 - k_B T_1 \cancel{\left(\frac{\partial n_0}{\partial t} + \mathbf{V}_0 \cdot \nabla n_0 \right)} - k_B T_0 \left(\frac{\partial \widetilde{n}_1}{\partial t} + \cancel{\mathbf{V}_0 \cdot \nabla \widetilde{n}_1} + \widetilde{\mathbf{V}}_1 \cdot \nabla n_0 \right) \\ &= -(\gamma - 1) p_0 \boldsymbol{\nabla} \cdot \widetilde{\mathbf{V}}_1 - \cancel{(\gamma - 1) \widetilde{p}_1 \boldsymbol{\nabla} \cdot \mathbf{V}_0} \end{aligned} \tag{2.7.62}$$

$$\frac{\partial \widetilde{p}_1}{\partial t} + \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \nabla p_0 - k_B T_0 \frac{\partial \widetilde{n}_1}{\partial t} - k_B T_0 \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \nabla n_0 = -(\gamma - 1) p_0 \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \tag{2.7.63}$$

Remember that

$$\frac{\partial p_0}{\partial t} = \frac{\partial \cancel{n_0}}{\partial t} k_B T_0 + n_0 k_B \frac{\partial T_0}{\partial t} = n_0 k_B \frac{\partial T_0}{\partial t} = 0 \tag{2.7.64}$$

so

$$\frac{\partial \widetilde{p}_1}{\partial t} + \frac{\partial}{\partial t}\left[\boldsymbol{\xi}\cdot\nabla p_0\right] + \frac{\partial}{\partial t}\left[k_B T_0 \nabla n_0 \cdot \boldsymbol{\xi} + p_0 \boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right] - \frac{\partial}{\partial t}\left[\boldsymbol{\xi}\cdot k_B T_0 \nabla n_0\right] = -\frac{\partial}{\partial t}\left[(\gamma-1)p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right] \tag{2.7.65}$$

$$\frac{\partial \widetilde{p}_1}{\partial t} + \frac{\partial}{\partial t}\left[\boldsymbol{\xi}\cdot\nabla p_0 + p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right] = -\frac{\partial}{\partial t}\left[(\gamma-1)p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right] + \frac{\partial}{\partial t}\left[p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right] \tag{2.7.66}$$

$$\widetilde{p}_1 = -\boldsymbol{\xi}\cdot\nabla p_0 - (\gamma-\cancel{1})p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi} - \cancel{p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}} + \mathbf{C}(\mathbf{x}) \tag{2.7.67}$$

$$\widetilde{p}_1 = -\boldsymbol{\xi}\cdot\nabla p_0 - \gamma p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \mathbf{C}(\mathbf{x}) \tag{2.7.68}$$

$$\tag{2.7.69}$$

which with $\widetilde{p}_1 = 0$ at $\boldsymbol{\xi} = 0$ yields

$$\widetilde{p}_1 = -\boldsymbol{\xi}\cdot\nabla p_0 - \gamma p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi} \tag{2.7.70}$$

We can then write

$$\widetilde{\mathbf{F}}_1 = -\nabla\widetilde{p}_1 + \mathbf{J}_0\times\widetilde{\mathbf{B}}_1 + \widetilde{\mathbf{J}}_1\times\mathbf{B}_0 \tag{2.7.71}$$

$$\widetilde{\mathbf{B}}_1 = \boldsymbol{\xi}\times\mathbf{B}_0 \tag{2.7.72}$$

$$\widetilde{\mathbf{J}}_1 = \frac{\boldsymbol{\nabla}\times\mathbf{B}_1}{\mu_0} = \frac{\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B}_0)}{\mu_0} \tag{2.7.73}$$

$$\widetilde{p}_1 = -\boldsymbol{\xi}\cdot\nabla p_0 - \gamma p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi} \tag{2.7.74}$$

This expands to

$$\widetilde{\mathbf{F}}_1 = \nabla(\boldsymbol{\xi}\cdot\nabla p_0) + \gamma\nabla(p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \frac{(\boldsymbol{\nabla}\times\mathbf{B}_0)\times(\boldsymbol{\xi}\times\mathbf{B}_0)}{\mu_0} + \frac{[\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B}_0)]\times\mathbf{B}_0}{\mu_0} \tag{2.7.75}$$

It is fairly typical to keep $\widetilde{\mathbf{B}}_1$ simply as $\widetilde{\mathbf{B}}_1$ in many applications, however.

Now because all of the above may be allowed to be complex valued, when we perform a displacement and wish to find the work done (by the linearized force), we need to work with either the real portion or imaginary portions carefully. This is because we have a quadratic now. Thus the work done D can be thought of as

$$D = \int_V \mathrm{d}^3x\ \Re[\boldsymbol{\xi}]\cdot\Re[\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})] = \int_V \mathrm{d}^3x\ \left(\frac{\boldsymbol{\xi}+\boldsymbol{\xi}^*}{2}\cdot\frac{\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})+\widetilde{\mathbf{F}}_1(\boldsymbol{\xi}^*)}{2}\right) \tag{2.7.76}$$

$$= \frac{1}{2}\int_V \mathrm{d}^3x\ \left[\Re[\boldsymbol{\xi}\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})] + \Re[\boldsymbol{\xi}^*\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})]\right] \tag{2.7.77}$$

The other solution should also work

$$D = \int_V \mathrm{d}^3x\ \Im[\boldsymbol{\xi}]\Im[\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})] = \int_V \mathrm{d}^3x\ \left(\frac{\boldsymbol{\xi}-\boldsymbol{\xi}^*}{2i}\cdot\frac{\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})-\widetilde{\mathbf{F}}_1(\boldsymbol{\xi}^*)}{2i}\right) \tag{2.7.78}$$

$$= -\frac{1}{2}\int_V \mathrm{d}^3x\ \left[\Re[\boldsymbol{\xi}\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})] - \Re[\boldsymbol{\xi}^*\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})]\right] \tag{2.7.79}$$

If we add these two solutions[50] we find

$$2D = \frac{1}{2}\int_V \mathrm{d}^3x \ \cancel{\left[\Re[\boldsymbol{\xi}\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})] + \Re[\boldsymbol{\xi}^*\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})]\right]} + -\frac{1}{2}\int_V \mathrm{d}^3x \ \cancel{\left[\Re[\boldsymbol{\xi}\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})] - \Re[\boldsymbol{\xi}^*\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})]\right]} \tag{2.7.80}$$

$$2D = \Re\left[\int_V \mathrm{d}^3x \ \boldsymbol{\xi}^*\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\xi})\right] \tag{2.7.81}$$

and so it is sensible to think of the D as related to the real part of the complex conjugate of the displacement multiplied by the force. That is, we write the work done for a general displacement is given by (reminding ourselves that we care only about the real part in the end)

$$D(\boldsymbol{\eta}) = \frac{1}{2}\int_V \mathrm{d}^3x \ \boldsymbol{\eta}^*\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\eta}) \tag{2.7.82}$$

where one must implicitly require that the real part is taken when we want the actual solution.

The negative of this can then be viewed as the potential energy, because this is the energy required by the linear forces to enact a displacement $\boldsymbol{\eta}$. In our case we let $\boldsymbol{\eta} = \boldsymbol{\xi}$ and so the potential energy W is given by[51]

$$W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = -\frac{1}{2}\int_V \mathrm{d}^3x \ \boldsymbol{\xi}^*\cdot\widetilde{\mathbf{F}}_1(\boldsymbol{\xi}) = -\frac{1}{2}D(\boldsymbol{\xi}) \tag{2.7.83}$$

The above is second order in δ because $\boldsymbol{\xi}$ is $\mathcal{O}(\delta)$. The kinetic energy T is given by

$$T(\frac{\partial\boldsymbol{\xi}^*}{\partial t}, \frac{\partial\boldsymbol{\xi}}{\partial t}) = \int_V \mathrm{d}^3x \ \frac{n_0 m_i}{2}\frac{\partial\boldsymbol{\xi}^*}{\partial t}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t} \tag{2.7.84}$$

We write both T and W as functions of two variables, because an important property is that of being self-adjoint. This means that the operator is Hermitian, so that given an inner product and operator A that can act on objects v and w, we have $\langle Av, w\rangle = \langle v, Aw\rangle$. For complex cases, it is usually true that $\langle v, w\rangle = v^*w$ where the object in the left of the inner product has used in its dual, that is complex conjugate, form. In our case, the operator is the integration and the objects are $\boldsymbol{\xi}$ and its complex conjugate. Thus to be self-adjoint we require (it is only self-adjoint with certain chocies of $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$)

$$W(\boldsymbol{\eta}^*, \boldsymbol{\xi}) = W(\boldsymbol{\xi}^*, \boldsymbol{\eta}) \tag{2.7.85}$$

Proving this is not so difficult for the $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ we are required to use. We have to remember that we are taking the real part at the end, though so that we can immediately see that $\Re[\boldsymbol{\eta}^*\cdot\boldsymbol{\xi}] = \Re[\boldsymbol{\xi}^*\cdot\boldsymbol{\eta}]$ and so we can switch the complex conjugate from one linear in $\boldsymbol{\xi}$ to $\boldsymbol{\eta}$ and vice versa without altering our actual final answer. When we take the $\boldsymbol{\xi} = \boldsymbol{\eta}$ case, then the above is trivial. We assume energy conservation so that

$$\frac{\partial T(\boldsymbol{\eta}^*, \boldsymbol{\xi})}{\partial t} + \frac{\partial W(\boldsymbol{\eta}^*, \boldsymbol{\xi})}{\partial t} = 0 \tag{2.7.86}$$

[50] This is not a mathematically rigorous proof unless we require that our solution be equally weighted by the two solutions. Can you think of why they must be equally weighted?

[51] Many texts call this potential energy from the linearized force δW . I avoid this use for two reasons. First, if we are going to extremize it then we would have to speak of $\delta\delta W$ which is very confusing notation. Second, the δW seems to indicate variation from the calculus of variations, which is also unnecessarily confusing. Just be warned that my W is often called δW .

and use

$$\frac{\partial T}{\partial t} = \frac{1}{2}\Re\left[\int_V \mathrm{d}^3x\ \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{\eta}^*}{\partial t}\cdot\frac{\partial \boldsymbol{\xi}}{\partial t}\right)\right] = \frac{1}{2}\Re\left[\int_V \mathrm{d}^3x\ \frac{\partial^2\boldsymbol{\eta}^*}{\partial t^2}\cdot\frac{\partial \boldsymbol{\xi}}{\partial t} + \frac{\partial \boldsymbol{\eta}^*}{\partial t}\cdot\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}\right] \tag{2.7.87}$$

Since T is self-adjoint, $\frac{\partial T}{\partial t}$ is as well. We can then note that

$$\frac{\partial T}{\partial t} = W(\frac{\partial \boldsymbol{\xi}}{\partial t}, \boldsymbol{\eta}^*) + W(\frac{\partial \boldsymbol{\eta}^*}{\partial t}, \boldsymbol{\xi}) = W(\frac{\partial \boldsymbol{\xi}^*}{\partial t}, \boldsymbol{\eta}) + W(\frac{\partial \boldsymbol{\eta}^*}{\partial t}, \boldsymbol{\xi}) \tag{2.7.88}$$

Then

$$\frac{\partial W}{\partial t} = \frac{1}{2}\Re\left[\frac{\partial}{\partial t}\int_V \mathrm{d}^3x\ \boldsymbol{\eta}^*\cdot\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}\right] \tag{2.7.89}$$

$$= \frac{1}{2}\Re\left[\int_V \mathrm{d}^3x\ \frac{\partial \boldsymbol{\eta}^*}{\partial t}\cdot\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}\right] + \frac{1}{2}\Re\left[\int_V \mathrm{d}^3x\ \boldsymbol{\eta}^*\cdot\frac{\partial^3\boldsymbol{\xi}}{\partial t^3}\right] \tag{2.7.90}$$

$$= W(\frac{\partial \boldsymbol{\eta}^*}{\partial t}, \boldsymbol{\xi}) + W(\boldsymbol{\eta}^*, \frac{\partial \boldsymbol{\xi}}{\partial t}) \tag{2.7.91}$$

And so we must have

$$W(\frac{\partial \boldsymbol{\xi}^*}{\partial t}, \boldsymbol{\eta}) + \cancel{W(\frac{\partial \boldsymbol{\eta}^*}{\partial t}, \boldsymbol{\xi})} = \cancel{W(\frac{\partial \boldsymbol{\eta}^*}{\partial t}, \boldsymbol{\xi})} + W(\boldsymbol{\eta}^*, \frac{\partial \boldsymbol{\xi}}{\partial t}) \tag{2.7.92}$$

which means

$$W(\frac{\partial \boldsymbol{\xi}^*}{\partial t}, \boldsymbol{\eta}) = W(\boldsymbol{\eta}^*, \frac{\partial \boldsymbol{\xi}}{\partial t}) \tag{2.7.93}$$

which is the exact property we desired. All that we need to recognize is that there is nothing special about $\frac{\partial \boldsymbol{\xi}}{\partial t}$ (we could have started our proof with $\int \mathrm{d}t\ \boldsymbol{\xi}$ instead, if this bothers you) so long as they satisfy the correct conditions imposed for the problem to be physically relevant [that is, boundary conditions and the function is a permissible function, in the space of functions we care about].

One should note that T is self-evidently self-adjoint. Once we have self-adjointness, then we know the eigenvalues of the operator are real, which makes the job much easier. In our case this means that the frequencies squared (ω^2) are real and so the frequencies are either purely real or purely imaginary. If we imagine that the total energy is unchanged during our displacement process, then we can state

$$E = W + T \tag{2.7.94}$$

for E some constant number. We can then see that if we allow $W < 0$, then T can become arbitrarily large. We view such a situation as likely to be unstable since this means that $\partial\boldsymbol{\xi}/\partial t$ is becoming enormous which means we can have explosive growth of the perturbation. On the other hand, if $W \geq 0$ this implies that T can only get so large and is thus bounded. This means that we are possibly in a stable configuration. To prove it is stable, would require a lot more analysis, but it is reasonable to suspect that $W \geq 0$ is necessary and sufficient for stability. Certainly in physical situations where we expect energy conservation, then we can argue it is sufficient because $T \geq 0$ so when $W \geq 0$ we have $E > 0$ and so T can only get so large, preventing unstable growth. To prove it is necessary, we can consider a displacement $W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) < 0$ exists. We can then solve

the problem where the initial displacement is $\boldsymbol{\xi}$ and $\partial\boldsymbol{\xi}/\partial t = 0$ at $t = 0$. In this case $E < 0$ initially, but still a constant. Then using that T is self-adjoint so that $T(\boldsymbol{\xi}^*, \boldsymbol{\eta}) = T(\boldsymbol{\eta}^*, \boldsymbol{\xi})$ we find (we found this above)

$$\frac{\partial T(\boldsymbol{\xi}^*, \boldsymbol{\xi})}{\partial t} = \Re\left\{\int_V \mathrm{d}^3x\ \frac{\partial}{\partial t}\left[\boldsymbol{\xi}^* \cdot \boldsymbol{\xi}\right]\right\} = T(\frac{\partial \boldsymbol{\xi}^*}{\partial t}, \boldsymbol{\xi}) + T(\boldsymbol{\xi}^*, \frac{\partial \boldsymbol{\xi}}{\partial t}) \tag{2.7.95}$$

We have forced T to be real in this instance. Since it is defined as the real part of an integral. Thus we notice that $\frac{\partial}{\partial t}$ distributes on the $\boldsymbol{\xi}$ for $T(\boldsymbol{\xi}^*, \boldsymbol{\xi})$. And so (let a prime denote $\frac{\partial}{\partial t}$ so $\frac{\partial \boldsymbol{\xi}}{\partial t} \equiv \boldsymbol{\xi}'$)

$$\frac{\partial^2 T(\boldsymbol{\xi}^*, \boldsymbol{\xi})}{\partial t^2} = T(\boldsymbol{\xi}^{*\prime\prime}, \boldsymbol{\xi}) + T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}') + T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}') + T(\boldsymbol{\xi}^*, \boldsymbol{\xi}'') \tag{2.7.96}$$

Using the self-adjointness of T with $\boldsymbol{\eta}^* \to \boldsymbol{\xi}^{*\prime}$ and $\boldsymbol{\xi} \to \boldsymbol{\xi}''$ we see that we get

$$\frac{\partial^2 T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}')}{\partial t^2} = 2T(\boldsymbol{\xi}^{*\prime\prime}, \boldsymbol{\xi}'') + 2T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}''') \tag{2.7.97}$$

Now we use

$$E = T\left(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}'\right) + W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) \tag{2.7.98}$$

We can then note that

$$T(\boldsymbol{\xi}^*, \boldsymbol{\xi}'') = \Re\left[\int_V \mathrm{d}^3x\ \boldsymbol{\xi}^* \cdot \widetilde{\mathbf{F}}_1(\boldsymbol{\xi})\right] = -W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) \tag{2.7.99}$$

Then we find

$$\begin{aligned}\frac{\partial^2 T(\boldsymbol{\xi}^*, \boldsymbol{\xi})}{\partial t^2} &= 2T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}') - 2\Re[W(\boldsymbol{\xi}^*, \boldsymbol{\xi})] \\ &= 2T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}') - 2\Re[H - T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}')] \\ &= 4T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}') - 2H\end{aligned} \tag{2.7.100}$$

Remember that $H < 0$ and so $T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}') \geq 0$ so

$$\frac{\partial^2 T(\boldsymbol{\xi}^*, \boldsymbol{\xi})}{\partial t^2} = 4T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}') - 2H \geq -2H \geq 0 \tag{2.7.101}$$

which implies that $T(\boldsymbol{\xi}^*, \boldsymbol{\xi})$ is growing in time even if initially we have $T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}') = 0$. Even if the $T(\boldsymbol{\xi}^{*\prime}, \boldsymbol{\xi}')$ contribution remains small in comparison to the H contribution, we'd initially have growth for a given W of at least $-2H = -2W > 0$. If we look at $\min(W)$ (over all possible W with permissible $\boldsymbol{\xi}$ in our configuration) then if $\min(W) < 0$ this implies we could have $-2W > 0$ and the growth. Thus we require $\min(W) \geq 0$ so that $-2W \leq 0$ to ensure that there is no such instability.

If we accept that $\min W \geq 0$ for stability then we are left with a variational calculus problem once we are given a configuration. All we need to do is calculate δW to find an extrema, check if $W > 0$ at the extrema if it is a minimum, and then if $W > 0$ and we are at a minimum we are stable. If $W < 0$, then we are unstable.

Finally, we calculate (using that W is self-adjoint)

$$\delta W = W(\boldsymbol{\xi}^* + \delta\boldsymbol{\xi}^*, \boldsymbol{\xi} + \delta\boldsymbol{\xi}) - W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) + \mathcal{O}(\delta^2) \tag{2.7.102}$$

$$= \Re\left[\int_V \mathrm{d}^3x \ \left(\boldsymbol{\xi}^* \cdot \widetilde{\mathbf{F}}_1(\delta\boldsymbol{\xi}) + \delta\boldsymbol{\xi}^* \cdot \widetilde{\mathbf{F}}_1(\boldsymbol{\xi})\right)\right] \tag{2.7.103}$$

$$= W(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi}) + W(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi}) = 2W(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi}) \tag{2.7.104}$$

And so at $\delta W = 0$ we need $W(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi}) = 0$. Thus if we can find a $\delta\boldsymbol{\xi}$ that gives $W = 0$, that $\delta\boldsymbol{\xi}$ is the maximizing or minimizing function. We can also use that $W = \Re\left[\int_V \mathrm{d}^3x \ L(\boldsymbol{\xi}^*, \boldsymbol{\xi})\right]$ to find the minimum via[52]

$$\delta\boldsymbol{\xi} \cdot \frac{\partial L}{\partial \boldsymbol{\xi}} + \delta\boldsymbol{\xi}^* \cdot \frac{\partial L}{\partial \boldsymbol{\xi}^*} \tag{2.7.105}$$

with $\frac{\partial \mathbf{L}}{\partial \boldsymbol{\xi}^*} = \mathbb{1} \cdot \widetilde{\mathbf{F}}_1(\boldsymbol{\xi}) = \widetilde{\mathbf{F}}_1(\boldsymbol{\xi})$. We can then use we are looking at real values to switch the complex conjugation off of the $\delta\boldsymbol{\xi}^*$. Then we have

$$\left(\frac{\partial L}{\partial \boldsymbol{\xi}} + \widetilde{\mathbf{F}}(\boldsymbol{\xi}^*)\right) \cdot \delta\boldsymbol{\xi} \tag{2.7.106}$$

as the integrand and so we must have $\frac{\partial L}{\partial \boldsymbol{\xi}} = -\widetilde{\mathbf{F}}(\boldsymbol{\xi}^*)$ to have a minimum. That is

$$\frac{\partial \widetilde{\mathbf{F}}_1(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \cdot \boldsymbol{\xi}^* = -\widetilde{\mathbf{F}}(\boldsymbol{\xi}^*) \tag{2.7.107}$$

In fact, we only require the real part to satisfy the above equation. One can then extend to other cases where the plasma is surrounded by a vacuum and conducting wall, but the machinery is similar. We simply have more integrals of energy to consider.

Finally, we come back to $\boldsymbol{\xi}^* \cdot \widetilde{\mathbf{F}}_1(\boldsymbol{\xi})$. We can write this as

$$\boldsymbol{\xi}^* \cdot \left[\nabla(\boldsymbol{\xi} \cdot \nabla p_0) + \nabla(\gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}) + \frac{\overbrace{\boldsymbol{\nabla} \times \mathbf{B}_0}^{\mathbf{J}_0}}{\mu_0} \times \widetilde{\mathbf{B}}_1 + \frac{\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1}{\mu_0} \times \mathbf{B}_0\right] \tag{2.7.108}$$

We can use $\boldsymbol{\xi}^* = \boldsymbol{\xi}^*_\perp + \boldsymbol{\xi}^*_\parallel$ with $\boldsymbol{\xi}^*_\parallel = \hat{\mathbf{b}}_0 \xi^*_\parallel$ where $\hat{\mathbf{b}}_0 = \mathbf{B}_0/|\mathbf{B}_0|$. Then

$$\xi^*_\parallel \hat{\mathbf{b}}_0 \cdot \frac{\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1}{\mu_0} \times \mathbf{B}_0 = 0 \tag{2.7.109}$$

via the vector identity $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$ (with $\mathbf{B}_0 = B_0 \hat{\mathbf{b}}_0$ being employed). Second we use $\boldsymbol{\nabla} \cdot \mathbf{B}_0 = 0$ and so

$$\mathbf{B}_0 \cdot \nabla(\boldsymbol{\xi} \cdot \nabla p_0) = \boldsymbol{\nabla} \cdot (\mathbf{B}_0[\boldsymbol{\xi} \cdot \nabla p_0]) - \boldsymbol{\xi} \cdot \nabla p_0 \cancel{\boldsymbol{\nabla} \cdot \mathbf{B}_0} \tag{2.7.110}$$

[52] We are formally regarding $\boldsymbol{\xi}^*$ and $\boldsymbol{\xi}$ as independent functions for this derivation. In reality, they are of course related, just like with derivatives.

with $\mathbf{J}_0 \times \mathbf{B}_0 = \nabla p_0$

$$\mathbf{B}_0 \cdot \mathbf{J}_0 \times [\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1] = -[\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1] \cdot \mathbf{J}_0 \times \mathbf{B}_0 = -\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1 \cdot \nabla p_0 \tag{2.7.111}$$

Then

$$\boldsymbol{\nabla} \cdot \Big[\nabla p_0 \times \widetilde{\mathbf{B}}_1 \Big] = \widetilde{\mathbf{B}}_1 \cdot \cancel{\boldsymbol{\nabla} \times \nabla p_0} - \nabla p_0 \cdot \boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1 \tag{2.7.112}$$

Then we combine

$$\mathbf{B}_0 \cdot \Big[\mathbf{J}_0 \times [\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1] + \nabla(\boldsymbol{\xi} \cdot \nabla p_0) \Big] = \boldsymbol{\nabla} \cdot \Big(\mathbf{B}_0 [\boldsymbol{\xi} \cdot \nabla p_0] + \nabla p_0 \times \widetilde{\mathbf{B}}_1 \Big) \tag{2.7.113}$$

$$= \boldsymbol{\nabla} \cdot (\mathbf{B}_0 [\boldsymbol{\xi} \cdot \nabla p_0] + \nabla p_0 \times (\boldsymbol{\xi} \times \mathbf{B}_0)) \tag{2.7.114}$$

$$= \boldsymbol{\nabla} \cdot \big(\cancel{\mathbf{B}_0 [\boldsymbol{\xi} \cdot \nabla p_0]} + \boldsymbol{\xi}(\nabla p_0 \cdot \mathbf{B}_0) - \cancel{\mathbf{B}_0 (\boldsymbol{\xi} \cdot \nabla p_0)} \big) = 0 \tag{2.7.115}$$

$$\tag{2.7.116}$$

where we used $\nabla p_0 \cdot \mathbf{B}_0 = 0$. The last step here is to recognize

$$\boldsymbol{\xi}^* \cdot \nabla(\gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}) = \boldsymbol{\nabla} \cdot (\boldsymbol{\xi}^* \gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}) - \gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \boldsymbol{\nabla} \cdot \boldsymbol{\xi}^* = \boldsymbol{\nabla} \cdot (\boldsymbol{\xi}^* \gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}) - \gamma p_0 |\boldsymbol{\nabla} \cdot \boldsymbol{\xi}|^2 \tag{2.7.117}$$

We then have for the integrand (ignoring divergences, which we will incorporate into a surface integral via Stokes' theorem)

$$\boldsymbol{\xi}_\perp^* \cdot \left(\nabla(\boldsymbol{\xi} \cdot \nabla p_0) + \mathbf{J}_0 \times \widetilde{\mathbf{B}}_1 + \frac{\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1}{\mu_0} \times \mathbf{B}_0 \right) - \gamma p_0 |\boldsymbol{\nabla} \cdot \boldsymbol{\xi}|^2 \tag{2.7.118}$$

We can perform the same ritual on $\boldsymbol{\xi}_\perp^* \cdot \nabla(\boldsymbol{\xi} \cdot \nabla p_0)$ to get

$$\boldsymbol{\xi}_\perp^* \cdot \nabla(\boldsymbol{\xi} \cdot \nabla p_0) = \boldsymbol{\nabla} \cdot (\boldsymbol{\xi}_\perp^* \boldsymbol{\xi} \cdot \nabla p_0) - (\boldsymbol{\xi} \cdot \nabla p_0) \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp^* \tag{2.7.119}$$

Then we have (ignoring the divergences again for now)

$$\boldsymbol{\xi}_\perp^* \cdot \left(\mathbf{J}_0 \times \widetilde{\mathbf{B}}_1 + \frac{\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1}{\mu_0} \times \mathbf{B}_0 \right) - \gamma p_0 |\boldsymbol{\nabla} \cdot \boldsymbol{\xi}|^2 - (\boldsymbol{\xi} \cdot \nabla p_0)(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp^*) \tag{2.7.120}$$

One last ritual is to convert the $\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1$ term.

$$\begin{aligned} \boldsymbol{\xi}_\perp^* \cdot (\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1) \times \mathbf{B}_0 &= (\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1) \cdot (\mathbf{B}_0 \times \boldsymbol{\xi}_\perp^*) \\ &= \boldsymbol{\nabla} \cdot (\widetilde{\mathbf{B}}_1 \times [\mathbf{B}_0 \times \boldsymbol{\xi}_\perp^*]) + \widetilde{\mathbf{B}}_1 \cdot \boldsymbol{\nabla} \times (\mathbf{B}_0 \times \boldsymbol{\xi}_\perp^*) \\ &= \boldsymbol{\nabla} \cdot (\mathbf{B}_0 (\widetilde{\mathbf{B}}_1 \cdot \boldsymbol{\xi}_\perp^*) - \boldsymbol{\xi}_\perp^* [\widetilde{\mathbf{B}}_1 \cdot \mathbf{B}_0]) + \widetilde{\mathbf{B}}_1 \cdot \boldsymbol{\nabla} \times (\mathbf{B}_0 \times \boldsymbol{\xi}_\perp^*) \end{aligned} \tag{2.7.121}$$

We then use $\mathbf{B}_0 \times \boldsymbol{\xi}_\perp^* = \mathbf{B}_0 \times \boldsymbol{\xi}^* = -\widetilde{\mathbf{B}}_1^*$. Thus

$$\boldsymbol{\xi}_\perp^* \cdot (\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1) \times \mathbf{B}_0 = \boldsymbol{\nabla} \cdot (\mathbf{B}_0 (\widetilde{\mathbf{B}}_1 \cdot \boldsymbol{\xi}_\perp^*) - \boldsymbol{\xi}_\perp^* [\widetilde{\mathbf{B}}_1 \cdot \mathbf{B}_0]) - \overbrace{\widetilde{\mathbf{B}}_1 \cdot \widetilde{\mathbf{B}}_1^*}^{=|\widetilde{\mathbf{B}}_1|^2} \tag{2.7.122}$$

Then if we gather all of the terms we find

$$\begin{aligned}
&\boldsymbol{\xi}^* \cdot \left(\nabla(\boldsymbol{\xi} \cdot \nabla p_0) + \nabla(\gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}) + \mathbf{J}_0 \times \widetilde{\mathbf{B}}_1 + \frac{\boldsymbol{\nabla} \times \widetilde{\mathbf{B}}_1}{\mu_0} \times \mathbf{B}_0 \right) \\
&= \boldsymbol{\nabla} \cdot \left(\boldsymbol{\xi}^* \gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} + \boldsymbol{\xi}_\perp^* \gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} + \mathbf{B}_0 (\widetilde{\mathbf{B}}_1 \cdot \boldsymbol{\xi}_\perp^*) - \boldsymbol{\xi}_\perp^* [\widetilde{\mathbf{B}}_1 \cdot \mathbf{B}_0] \right) \\
&\quad + \boldsymbol{\xi}_\perp^* \cdot \mathbf{J}_0 \times \widetilde{\mathbf{B}}_1 - \frac{|\widetilde{\mathbf{B}}_1|^2}{\mu_0} - \gamma p_0 |\boldsymbol{\nabla} \cdot \boldsymbol{\xi}|^2 - (\boldsymbol{\xi} \cdot \nabla p_0)(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp^*)
\end{aligned} \tag{2.7.123}$$

If we then integrate over a volume V with surface $S = \partial V$ and $\hat{\mathbf{n}} \cdot \mathbf{B}_0 = 0$ we see that $W = W(\boldsymbol{\xi}^*, \boldsymbol{\xi})$ is given by (remember that $\boldsymbol{\xi} \cdot \nabla p_0 = \boldsymbol{\xi}_\perp \cdot \nabla p_0$ since $\nabla p_0 \cdot \mathbf{B}_0 = 0$)

$$\begin{aligned}
W &= -\int_V \mathrm{d}^3 x \; \boldsymbol{\xi}^* \cdot \widetilde{\mathbf{F}}_1(\boldsymbol{\xi}) \\
&= \int_V \mathrm{d}^3 x \; \left[\frac{|\widetilde{\mathbf{B}}_1|^2}{\mu_0} - \boldsymbol{\xi}_\perp^* \cdot \mathbf{J}_0 \times \widetilde{\mathbf{B}}_1 + \gamma p_0 |\boldsymbol{\nabla} \cdot \boldsymbol{\xi}|^2 + (\boldsymbol{\xi} \cdot \nabla p_0)(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp^*) \right] \\
&\quad + \int_{\partial V} \mathrm{d}S \; \hat{\mathbf{n}} \cdot \left[\boldsymbol{\xi}_\perp^* [\widetilde{\mathbf{B}}_1 \cdot \mathbf{B}_0] - \boldsymbol{\xi}_\perp^* \boldsymbol{\xi} \cdot \nabla p_0 - \overbrace{\boldsymbol{\xi}^*}^{\hat{\mathbf{n}} \cdot \boldsymbol{\xi}^* = \hat{\mathbf{n}} \cdot \boldsymbol{\xi}_\perp^*} \gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} - \cancel{\mathbf{B}_0} (\widetilde{\mathbf{B}}_1 \cdot \boldsymbol{\xi}_\perp^*) \right] \\
&= \int_V \mathrm{d}^3 x \; \left[\frac{|\widetilde{\mathbf{B}}_1|^2}{\mu_0} - \boldsymbol{\xi}_\perp^* \cdot \mathbf{J}_0 \times \widetilde{\mathbf{B}}_1 + \gamma p_0 |\boldsymbol{\nabla} \cdot \boldsymbol{\xi}|^2 + (\boldsymbol{\xi}_\perp \cdot \nabla p_0)(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp^*) \right] \\
&\quad + \int_{\partial V} \mathrm{d}S \; \hat{\mathbf{n}} \cdot \boldsymbol{\xi}_\perp^* \left[\widetilde{\mathbf{B}}_1 \cdot \mathbf{B}_0 - \boldsymbol{\xi}_\perp \cdot \nabla p_0 - \gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right]
\end{aligned} \tag{2.7.124}$$

Remember that this W is really just the plasma or fluid portion of the energy. There can be energy from the plasma surface and surrounding vacuum as well. Finally, a more "intuitive" form is often used for the integrand. Instead of

$$\frac{|\widetilde{\mathbf{B}}_1|^2}{\mu_0} - \boldsymbol{\xi}_\perp^* \cdot \mathbf{J}_0 \times \widetilde{\mathbf{B}}_1 + \gamma p_0 |\boldsymbol{\nabla} \cdot \boldsymbol{\xi}|^2 + (\boldsymbol{\xi}_\perp \cdot \nabla p_0)(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp^*) \tag{2.7.125}$$

one uses[11]

$$\frac{|\widetilde{\mathbf{B}}_{1\perp}|^2}{\mu_0} + \frac{B_0^2}{\mu_0} |\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp|^2 + \gamma p_0 |\boldsymbol{\nabla} \cdot \boldsymbol{\xi}|^2 - 2(\boldsymbol{\xi}_\perp \cdot \nabla p_0)(\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp^*) - \lambda (\boldsymbol{\xi}_\perp^* \times \mathbf{B}_0) \cdot \widetilde{\mathbf{B}}_{1\perp} \tag{2.7.126}$$

with

$$\boldsymbol{\kappa} = \hat{\mathbf{b}}_0 \cdot \nabla \hat{\mathbf{b}}_0 \tag{2.7.127}$$

$$\lambda = \frac{\mathbf{J}_0 \cdot \mathbf{B}_0}{B_0^2} \tag{2.7.128}$$

This is more intuitive because many of the terms are squares and hence cannot contribute to instability. Only the last two terms can be negative, and they can be interpreted as representing curvature $(\boldsymbol{\xi}_\perp \cdot \nabla p_0)(\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp^*)$ and current instabilities $\lambda(\boldsymbol{\xi}_\perp^* \times \mathbf{B}_0) \cdot \widetilde{\mathbf{B}}_{1\perp}$.

Proving this is a job best done once in one's life. It convinces you that there is no magic going on, but exhausts you with getting the details correct.

2.7.3 Non-Ideal MHD

Our arguments were entirely general before. Yet, if one looks up information on the resistive MHD force operator, you will see people immediately saying that the resistive MHD force operator is not self-adjoint. Does such a thing come into conflict with our derivation above? (The answer is no, it does not.)

In fact, the matter is more subtle than one thinks. The first thing we used above was that the total energy is conserved. You might think that this solves the conundrum. In fact, it does not. It is true that resistive MHD has energy being "lost" via ohmic heating, but we can add such a term into our temperature equation, and so no energy is lost. Indeed, consider Problem 2.6.3. and the following problem in the problem set. We see we can have energy conservation if we deposit the ohmic heating into the temperature of the plasma.

The second thing that was important for our calculations was the property

$$\frac{\partial}{\partial t}W(\boldsymbol{\xi}^*,\boldsymbol{\xi}) = W(\frac{\partial \boldsymbol{\xi}^*}{\partial t},\boldsymbol{\xi}) + W(\boldsymbol{\xi}^*,\frac{\partial \boldsymbol{\xi}}{\partial t}) \tag{2.7.129}$$

This means that no other quantity but $\boldsymbol{\xi}$ is time dependent. That is all of our zeroth order terms are time independent. This is a statement that if our expansion is around a steady-state, then the only time dependent quantity will be $\boldsymbol{\xi}$. This is what turns out to be crucial. We saw before that in Ideal MHD such a statement is trivial. For resistive MHD, the steady-state zeroth order restricts us to only certain situations. Consider

$$\mathbf{E} = \eta\mathbf{J} - \mathbf{V}\times\mathbf{B} \tag{2.7.130}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E} \tag{2.7.131}$$

at zeroth order with no background flow $\mathbf{V}_0 = \mathbf{0}$. We then have

$$\frac{\partial \mathbf{B}_0}{\partial t} = -\boldsymbol{\nabla}\times(\eta_0\mathbf{J}_0) \tag{2.7.132}$$

This means that for $\frac{\partial \mathbf{B}_0}{\partial t} = \mathbf{0}$ we must have

$$\boldsymbol{\nabla}\times(\eta_0\mathbf{J}_0) = \nabla\eta_0\times\frac{\boldsymbol{\nabla}\times\mathbf{B}_0}{\mu_0} + \eta_0\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\nabla}\times\mathbf{B}_0}{\mu_0}\right) = \mathbf{0} \tag{2.7.133}$$

This highly restricts the possible η_0 and $\mathbf{B}_0$ that are possible. Otherwise, we lose the property that $\frac{\partial W}{\partial t}$ has the time derivative only distribute to the arguments of W.

This also brings up the importance of $\mathbf{V}_0 = \mathbf{0}$. With background flow, if we desire a steady-state in Ideal MHD, then only certain $\mathbf{V}_0$ are possible.

This is simply to say that if you find a steady-state zeroth order and construct the potential energy from displacements of the linearized force, then by construction you will have a self-adjoint potential energy. This as true in resistive MHD as it is in Ideal MHD. The real difference is in whether such a steady state is a useful state for physics analysis. In most resistive MHD situations, there is no steady-state near the state we are interested in, and so, were we to construct a linearized force, it would have time dependence outside of the displacements and so would not be self-adjoint.

2.8 Types of Plasma Instabilities

> *One must watch the convergence of a numerical code as carefully as a father watching his four-year-old play near a busy road.*
>
> — J. P. Boyd[3, p. 109]

Much of plasma physics theory and experiment for the past 70 years has been an increasing understanding of the various ways that plasmas display unstable behavior, and what can be done to mitigate or prevent these instabilities.[53] In this time, there have been numerous instabilities found, so that instability analysis often fills multiple chapters of textbooks.

We begin with the MHD model itself, such as in the set of equations in Section 2.6. In a perfect world, one could analytically analyze a set of equations for various initial or boundary conditions and determine their behavior over time. That is, the equations simply could be solved for all conditions that are consistent. Unfortunately, most equations of interest in the physical world cannot be so analyzed. The next hope would be that one could look at the equations and determine properties of the solution such as does it grow quickly, or come to a steady state solution. Again, most equations representing physical phenomena are nonlinear and so are difficult to analyze. So one is often left with looking at linear analysis, where the nonlinear equations are linearized by imagining conditions near an equilibrium and only keeping linear contributions to deviations from this equilibrium. Even with just a linear stability analysis, the full model is usually simplified to gain a better understanding of how different terms impact the overall stability of the system. Linear stability is usually investigated because linearly unstable configurations tend to be unstable non-linearly, or at the very least, unstable enough to prevent plasma confinement. A linearly unstable set of equations could be non-linearly stable and so linearly unstable does not automatically entail poor plasma confinement and performance, but, in practice, linear instabilities tend to arise and be destructive. The only way to know for sure, is to do some type of non-linear analysis (or experiment or computation).

Even with this caveat, linear analysis is useful in categorizing the types of instabilities available to plasmas. As an example we can consider the ideal MHD system, and construct an energy integral[12][2] to determine linear stability. The variational principles used in Section 2.7.2 for W are especially useful in this respect.

$$W = \frac{1}{2}\int \mathrm{d}^3x \left\{ |\mathbf{Q}_\perp|^2 + B_0^2|\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{\xi}}_\perp + 2\widetilde{\boldsymbol{\xi}}_\perp\cdot\boldsymbol{\kappa}|^2 + \gamma p_0|\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{\xi}}|^2 \right. \\ \left. - 2(\widetilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_0)(\boldsymbol{\kappa}\cdot\widetilde{\boldsymbol{\xi}}_\perp^*) - \frac{\mathbf{J}_0\cdot\mathbf{B}_0}{B_0^2}(\widetilde{\boldsymbol{\xi}}_\perp^*\times\mathbf{B}_0)\cdot\mathbf{Q}_\perp \right\} \tag{2.8.1}$$

with

$$\mathbf{Q} = \widetilde{\mathbf{B}}_1 = \boldsymbol{\nabla}\times(\widetilde{\boldsymbol{\xi}}\times\mathbf{B}_0) \tag{2.8.2}$$

$$\hat{\mathbf{b}} = \mathbf{B}_0/B_0 \tag{2.8.3}$$

$$\boldsymbol{\kappa} = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} \tag{2.8.4}$$

The first term represents energy for bending field lines (or it can be thought of as energy in shear Alfvén waves), the second term is energy in compressional Alvén waves, and the third term is the

[53] The Boyd quote applies to plasmas as well as it does to numerical codes. Think of the fun when you combine them!

energy in compressing the plasma. These three terms are always non-negative, and so the last two terms are the possibly destabilizing terms. The fourth term is $-(\widetilde{\boldsymbol{\xi}}_\perp \cdot \nabla p_0)(\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp)$. Thus, if $\boldsymbol{\kappa}$, the magnetic curvature vector,[54] and the pressure gradient, ∇p, point in the same direction, this term is destabilizing. Note this is a term where the pressure gradient (related to perpendicular current by $\mathbf{J}_0 \times \mathbf{B}_0 = \nabla p_0$) is causing instabilities. The final term is destabilizing and driven by parallel current $\mathbf{J}_0 \cdot \mathbf{B}_0$.

With these ideas in mind, MHD instabilities are usually divided into useful categories, some of which overlap. In addition, while instabilities are usually able to be differentiated from an experimental standpoint, they are usually not rigid categories with precise mathematical definitions, so that an instability can have multiple valid descriptions that emphasize different aspects of the instability. As hinted at before, one way of discriminating between instabilities (which are often called "modes", because the instabilities are usually linear eigenmodes of the equations) is looking at where the energy driving the instability is coming from. Thus, there are (parallel) current instabilities and pressure (gradient) instabilities. It is worth remembering that pressure instabilities can be viewed as perpendicular current instabilities, but that when the literature refers to current instabilities it almost always means parallel current instabilities.

A further classification is between internal and external instabilities. Here external instability simply means that the surface of the plasma is perturbed. If the plasma surface is unperturbed, then it is considered an internal instability.

In addition, there are a couple of instabilities that are common and physically apparent so that we can classify them as kink modes, interchange modes, and ballooning modes. A kink mode can be either internal or external, and is current driven. External kinks are well-named, as the plasma column appears to kink or twist into a braid-like or helical shape. Visualizations of the external perturbation for various mode numbers in a cylinder are shown in Figures 1.6 and 1.7. An intuitive reminder of the drive of this mode is that as the plasma kinks and forms an elbow (think of circular rings on a straight arm and then forming your arm into an elbow), the magnetic field on the outside of the elbow is rarefied (on the outside of your elbow the rings are now further apart) while the magnetic field on the inside of the elbow is compressed (on the inside of your arm the rings are bunched together). This compressed magnetic field tries to push itself apart (here the arm metaphor breaks down as your bones and tendons prevent your arm for spreading apart, though you should feel pressure on your elbow from the rings) and in so doing makes the elbow larger, reinforcing the kink. Internal kinks are not as clearly visible because they do not affect the external boundary, but are also current driven.

Interchange instabilities are an internal instability and pressure driven. They depend on the pressure gradient and magnetic curvature, and cause a fluted like structure to the plasma. They are "interchange" because if you imagine flux tubes (they can be thought of as the volume around a magnetic field line) next to each other, with the configuration being energetically favorable for the flux tubes to switch locations, you get interchange instabilities. This is analogous to the Rayleigh-Taylor instability[16][20] where under the force of gravity, a dense fluid above a less dense fluid is unstable and so the two fluids try to interchange locations to get to a more energetically favorable configuration. In plasmas, this often brings colder, less dense plasma nearer to the center of the plasma, affecting the main plasma properties. This can be seen in Figure 2.6. It is worth taking an

[54] Remember that the magnetic curvature vector points toward the center of the circle that would fit the magnetic field line locally. It is perpendicular to the local magnetic field, as can easily be seen via the identity $\boldsymbol{\kappa} = \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} = -\hat{\mathbf{b}} \times (\boldsymbol{\nabla} \times \hat{\mathbf{b}})$.

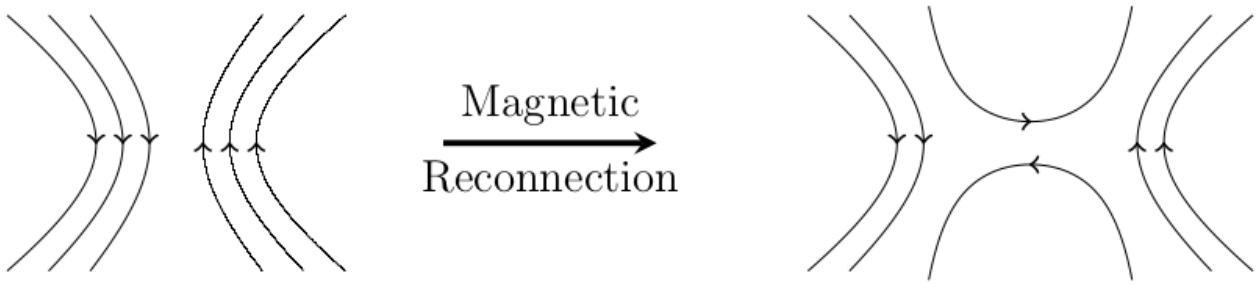

Figure 2.12: The magnetic field lines "break" and "reconnect" to form a new magnetic topology.

extended look at the fluted nature of interchange modes as it offers a geometric understanding of the importance of the safety factor and why having magnetic shear (when the helicity of field lines is changing) is useful for stabilizing interchange modes. Consider a plasma column in a cylindrical system, and then a perturbation of the cylindrical column $\boldsymbol{\xi}$. We can then consider a Fourier decomposition in the poloidal and toroidal angles of this perturbation of multiple wavenumbers defined in (r, θ, ζ) where ζ is the toroidal "angle" along the Z direction (for example, take $\zeta = Z/L$ for L some distance along the Z axis)

$$\mathbf{k}_{mn} = m\,\nabla\theta - n\,\nabla\zeta \tag{2.8.5}$$

Then m is the poloidal mode number and n is the toroidal mode number of the wavevector. In many cases, a specific (m, n) combination is dominant and so we can speak of an (m, n) mode, which will have a helicity of m/n. Also recall that the safety factor q is related to the helicity of magnetic field lines. So when q and m/n are the same, that is, the perturbations $\boldsymbol{\xi}$ align with the magnetic field, then interchange instabilities can occur because the field lines can go past each other without getting entangled with each other because they are of the same "shape". When there is shear, this means that q is changing and so the field lines cannot easily slip past each other because they are of different helicities and are more easily entangled. These interchange modes are often called flute modes because they often create a fluted structure to the plasma column, like the classical fluted Roman pillars, but with helicity like the stripes on a barber's pole.

There are also ballooning instabilities which are also pressure driven and can be internal or external instabilities. They are manifest on the bad curvature region (the outboard side of the tokamak) and have balloon-like protrusions on the outboard side. They are driven in a similar way as an interchange instability, but because of the tokamak toroidal geometry, no single helicity is dominant leading to distinct behavior from regular interchange modes.

In addition, tearing modes should be mentioned as a type of instability that does not arise in ideal MHD. These instabilities occur when magnetic diffusion (due to electrical resistivity) causes the magnetic field topology to change on small scales by breaking and reconnecting magnetic field lines. On large scales, this process is typically called magnetic diffusion. The topology change can turn closed field lines into open field lines, and thus leads to worse confinement. The breaking of field lines (which is at a current sheet via $\mu_0\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B}$) can be visualized as in Figure 2.12.

There are of course far more instabilities than this. There is a whole zoo of instabilities that occur from plasma wave interactions, but the above analysis is a good introduction to the majority of MHD related instabilities.

2.9 Plasma Waves and Instabilities

> *In view of the bewildering number of parameters and the confusion that could result, a remarkable amount of organization and sanity is regained by using the dielectric tensor. The difficult problems of plasma waves in a homogeneous medium become, for the most part, just difficult problems in algebra. The algebra is not trivial... The human qualities that are required to carry out these remaining calculations are, in small proportion, insight, and in large proportion, stamina.*
>
> — T. H. Stix[18, pp. 265–266]

It is worth going over the procedures used to find linear waves and instabilities from equations because this is a time-honored tradition in plasma theory. The usefulness of linearization and analyzing plasmas with these methods has been crucial to designing and understanding experiments. If you would like to get more into the details of a variety of plasmas, you should consult a textbook on the subject such as Stix[18] or Swanson[19].

One of the most important things to derive for any wave, is the dispersion relation, which tells us how the frequency (ν) of a wave depends on the wavelength (λ). As physicists we usually deal with the angular frequency $\omega = 2\pi\nu$ and the wavenumber $k = 2\pi/\lambda$ instead. So the dispersion relation is $\omega = f(k)$ for some function of k. This relation determines many properties of the wave, such as the phase velocity given by ω/k, and the group velocity $\mathrm{d}\omega/\mathrm{d}k$. This is then generalized to $\mathrm{d}\omega/\mathrm{d}\mathbf{k}$ for three dimensions with $\mathbf{k}$ a wavenumber vector. Once you have a dispersion relation, you can calculate how a wave will propagate in a system, hence its importance.

Let's first deal with ideal MHD waves. We begin by assuming that there is a dimensionless parameter $\delta \ll 1$ that exists that separates "equilibrium" or the steady-state[55] from a small perturbation from that state. We write this as for all quantities q they go to $q = q_0 + \delta q_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = q_0 + \delta\widetilde{q}_1$. The signs on $\mathbf{k}\cdot\mathbf{x}$ and ω terms is conventional for plasmas and means ω/k goes in the positive spatial direction. This could be called the cheap Fourier transform, as we simply assume the form of a plane wave. Then, we begin with constant in space and time $\mathbf{B}_0$, no steady state flow $\mathbf{V}_0 = \mathbf{0}$, a spatially and temporally constant T_0. This means that $\mathbf{J}_0 = \mathbf{0}$. As a reminder, our ideal MHD equations are at (2.7.7) to (2.7.1). Also, note that I am making stronger assumptions than simple linearization by assuming a plane wave like dependence in addition to a simple steady-state. This formulation also avoids the problems of saying $q_1 \ll q_0$ when $q_0 = 0$ because we invoke an ordering parameter instead.

We begin with the number density equation

$$\frac{\partial(\cancel{n_0} + \delta\widetilde{n}_1)}{\partial t} + \boldsymbol{\nabla}\cdot\left([n_0 + \widetilde{n}_1][\cancel{\mathbf{V}_0} + \widetilde{\mathbf{V}}_1]\right) = 0 \tag{2.9.1}$$

$$\delta\left(\frac{\partial\widetilde{n}_1}{\partial t} + n_0\boldsymbol{\nabla}\cdot\widetilde{\mathbf{V}}_1\right) + \delta^2\left(\boldsymbol{\nabla}\cdot(\widetilde{n}_1\widetilde{\mathbf{V}}_1)\right) = 0 \tag{2.9.2}$$

Now we ignore $\mathcal{O}(\delta^2)$ terms because we are looking only at the first order terms in a linearization.

[55] The word equilibrium in plasma should really be understood as a steady state. It does not mean thermodynamic equilibrium, just that if we wait long enough without changing sources and sinks, what state do we get. Steady state is then, perhaps, a better term.

Thus, our continuity equation states

$$-i\omega n_1 + in_0\mathbf{k}\cdot\mathbf{V}_1 = 0 \tag{2.9.3}$$

$$\frac{n_1}{n_0} = \frac{\mathbf{k}\cdot\mathbf{V}_1}{\omega} \tag{2.9.4}$$

Next we look at the momentum balance equation

$$nm_i\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = nm_i\frac{\partial\mathbf{V}}{\partial t} + nm_i\mathbf{V}\cdot\mathbf{V} = \mathbf{J}\times\mathbf{B} - \nabla p \tag{2.9.5}$$

We will replace $p = nT$. Thus $p = p_0 + \delta p_1 = n_0k_BT_0 + \delta\widetilde{n}_1k_BT_0 + \delta n_0k_B\widetilde{T}_1 + \delta^2\widetilde{n}_1k_B\widetilde{T}_1$. Expanding, this becomes

$$\begin{aligned}(n_0+\delta n_1)\left[\frac{\partial}{\partial t} + (\cancel{\mathbf{V}_0} + \delta\widetilde{\mathbf{V}}_1)\cdot\nabla\right]\left(\cancel{\mathbf{V}_0} + \delta\widetilde{\mathbf{V}}_1\right) &= (\cancel{\mathbf{J}_0} + \delta\widetilde{\mathbf{J}}_1)\times(\mathbf{B}_0 + \delta\widetilde{\mathbf{B}}_1)\\ &- \nabla(n_0k_BT_0 + \delta[\widetilde{n}_1k_BT_0 + n_0k_B\widetilde{T}_1] + \delta^2\widetilde{n}_1k_B\widetilde{T}_1)\end{aligned} \tag{2.9.6}$$

$$\begin{aligned}m_i\left[\delta\left(n_0\frac{\partial\widetilde{\mathbf{V}}_1}{\partial t}\right) + \delta^2\left(n_1\frac{\partial\widetilde{\mathbf{V}}_1}{\partial t} + n_0\widetilde{\mathbf{V}}_1\cdot\nabla\widetilde{\mathbf{V}}_1\right) + \delta^3\left(n_1\widetilde{\mathbf{V}}_1\cdot\nabla\widetilde{\mathbf{V}}_1\right)\right]\\ = -k_B\nabla(n_0T_0) + \delta\left(-k_B\nabla(\widetilde{n}_1T_0 + n_0\widetilde{T}_1) + \widetilde{\mathbf{J}}_1\times\mathbf{B}_0\right)\\ + \delta^2\left(\widetilde{\mathbf{J}}_1\times\widetilde{\mathbf{B}}_1 - k_B\nabla(\widetilde{n}_1\widetilde{T}_1)\right)\end{aligned} \tag{2.9.7}$$

These can then be grouped in orders of δ. Again, for linearization we can ignore $\mathcal{O}(\delta^2)$.

$$0 = k_B\nabla(n_0T_0) \tag{2.9.8}$$

$$n_0m_i\frac{\partial\widetilde{\mathbf{V}}_1}{\partial t} = -k_B\nabla(\widetilde{n}_1T_0 + n_0\widetilde{T}_1) + \widetilde{\mathbf{J}}_1\times\mathbf{B}_0 \tag{2.9.9}$$

We have

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} \tag{2.9.10}$$

$$\boldsymbol{\nabla}\times(\mathbf{B}_0 + \widetilde{\mathbf{B}}_1) = \mu_0\left(\mathbf{J}_0 + \widetilde{\mathbf{J}}_1\right) \tag{2.9.11}$$

$$\boldsymbol{\nabla}\times(\mathbf{B}_0 + \delta\widetilde{\mathbf{B}}_1) = \mu_0\left(\mathbf{J}_0 + \delta\widetilde{\mathbf{J}}_1\right) \tag{2.9.12}$$

$$\mathbf{0} + \delta\boldsymbol{\nabla}\times\widetilde{\mathbf{B}}_1 = \mathbf{0} + \delta\mu_0\widetilde{\mathbf{J}}_1 \tag{2.9.13}$$

Thus (2.9.9) will say

$$n_0m_i\frac{\partial\widetilde{\mathbf{V}}_1}{\partial t} = -k_B\nabla(\widetilde{n}_1T_0 + n_0\widetilde{T}_1) + \frac{\boldsymbol{\nabla}\times\widetilde{\mathbf{B}}_1}{\mu_0}\times\mathbf{B}_0 \tag{2.9.14}$$

$$-i\omega n_0m_i\mathbf{V}_1 = -ik_B\mathbf{k}(T_0n_1 - n_0T_1) + \frac{i\mathbf{k}\times\mathbf{B}_1}{\mu_0}\times\mathbf{B}_0 \tag{2.9.15}$$

$$-i\omega n_0m_i\mathbf{V}_1 = -ik_B\mathbf{k}(T_0n_1 + n_0T_1) + i\frac{\mathbf{B}_1}{\mu_0}(\mathbf{B}_0\cdot\mathbf{k}) - i\mathbf{k}\left(\mathbf{B}_0\cdot\frac{\mathbf{B}_1}{\mu_0}\right) \tag{2.9.16}$$

$$\omega n_0m_i\mathbf{V}_1 = \mathbf{k}\left(k_BT_0n_1 + n_0k_BT_1 + \mathbf{B}_0\cdot\frac{\mathbf{B}_1}{\mu_0}\right) - \frac{\mathbf{B}_1}{\mu_0}(\mathbf{B}_0\cdot\mathbf{k}) \tag{2.9.17}$$

It is important to note that $\mathbf{k}\cdot\mathbf{B}_0 \neq 0$. $\mathbf{k}\cdot\mathbf{B}_1 = 0$ due to $\boldsymbol{\nabla}\cdot\mathbf{B}_1 = 0$, but there is no such relation for $\mathbf{B}_0$ and $\mathbf{k}$. For the temperature equation we have

$$\frac{n}{\gamma-1}\left[\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right]T = -nT\boldsymbol{\nabla}\cdot\mathbf{V} \tag{2.9.18}$$

$$\frac{n_0+\delta\widetilde{n}_1}{\gamma-1}\left[\frac{\partial}{\partial t} + [\cancel{\mathbf{V}_0} + \delta\widetilde{\mathbf{V}}_1]\cdot\nabla\right][k_BT_0 + \delta k_B\widetilde{T}_1] = -(n_0+\delta\widetilde{n}_1)[k_BT_0+\delta k_B\widetilde{T}_1]\boldsymbol{\nabla}\cdot[\cancel{\mathbf{V}_0} + \delta\widetilde{\mathbf{V}}_1] \tag{2.9.19}$$

Which translates into (we can ignore $\mathcal{O}(\delta^2)$ terms)

$$\begin{aligned}&\delta\left(\frac{n_0}{\gamma-1}\frac{\partial k_B\widetilde{T}_1}{\partial t}\right) + \delta^2\left(\widetilde{n}_1\frac{\partial k_B\widetilde{T}_1}{\partial t} + n_0\widetilde{\mathbf{V}}_1\cdot\nabla k_B\widetilde{T}_1\right) + \delta^3\left(\frac{\widetilde{n}_1}{\gamma-1}\widetilde{\mathbf{V}}_1\cdot\nabla k_B\widetilde{T}_1\right)\\ &= -\delta\left(n_0k_BT_0\boldsymbol{\nabla}\cdot\widetilde{\mathbf{V}}_1\right) - \delta^2\left([\widetilde{n}_1k_BT_0 + n_0k_B\widetilde{T}_1]\boldsymbol{\nabla}\cdot\widetilde{\mathbf{V}}_1\right) - \delta^3\left(\widetilde{n}_1k_B\widetilde{T}_1\boldsymbol{\nabla}\cdot\widetilde{\mathbf{V}}_1\right)\end{aligned} \tag{2.9.20}$$

So that we find up at $\mathcal{O}(\delta)$ that

$$\frac{n_0}{\gamma-1}\frac{\partial k_B\widetilde{T}_1}{\partial t} = -n_0k_BT_0\boldsymbol{\nabla}\cdot\widetilde{\mathbf{V}}_1 \tag{2.9.21}$$

$$\frac{-i\omega k_BT_1}{\gamma-1} = -ik_BT_0\mathbf{k}\cdot\mathbf{V}_1 \tag{2.9.22}$$

$$\frac{T_1}{T_0} = \frac{(\gamma-1)\mathbf{k}\cdot\mathbf{V}_1}{\omega} \tag{2.9.23}$$

We can find

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} \tag{2.9.24}$$

$$\mathbf{E}_0 + \delta\widetilde{\mathbf{E}}_1 = -(\cancel{\mathbf{V}_0} + \delta\widetilde{\mathbf{V}}_1)\times(\mathbf{B}_0 + \delta\widetilde{\mathbf{B}}_1) \tag{2.9.25}$$

$$\mathbf{E}_0 + \delta\widetilde{\mathbf{E}}_1 = -\delta(\widetilde{\mathbf{V}}_1\times\mathbf{B}_0) - \delta^2(\widetilde{\mathbf{V}}_1\times\widetilde{\mathbf{B}}_1) \tag{2.9.26}$$

so

$$\mathbf{E}_0 = \mathbf{0} \tag{2.9.27}$$

$$\widetilde{\mathbf{E}}_1 = -\widetilde{\mathbf{V}}_1\times\mathbf{B}_0 \tag{2.9.28}$$

and then

$$\frac{\partial\mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E} \tag{2.9.29}$$

$$\frac{\partial}{\partial t}\left[\mathbf{B}_0 + \delta\widetilde{\mathbf{B}}_1\right] = -\boldsymbol{\nabla}\times\left[\cancel{\mathbf{E}_0} + \delta\widetilde{\mathbf{E}}_1\right] \tag{2.9.30}$$

$$\delta\frac{\partial\widetilde{\mathbf{B}}_1}{\partial t} = \delta\boldsymbol{\nabla}\times\left[\widetilde{\mathbf{V}}_1\times\mathbf{B}_0\right] \tag{2.9.31}$$

$$-i\omega\mathbf{B}_1 = i\mathbf{k}\times(\mathbf{V}_1\times\mathbf{B}_0) = i\left[\mathbf{V}_1(\mathbf{k}\cdot\mathbf{B}_0) - \mathbf{B}_0(\mathbf{k}\cdot\mathbf{V}_1)\right] \tag{2.9.32}$$

$$\mathbf{B}_1 = \frac{\mathbf{B}_0(\mathbf{k}\cdot\mathbf{V}_1) - \mathbf{V}_1(\mathbf{k}\cdot\mathbf{B}_0)}{\omega} \tag{2.9.33}$$

So we now have

$$\frac{n_1}{n_0} = \frac{\mathbf{k}\cdot\mathbf{V}_1}{\omega} \tag{2.9.34}$$

$$\frac{T_1}{T_0} = \frac{(\gamma-1)\mathbf{k}\cdot\mathbf{V}_1}{\omega} \tag{2.9.35}$$

$$\mathbf{B}_1 = \frac{\mathbf{B}_0(\mathbf{k}\cdot\mathbf{V}_1) - \mathbf{V}_1(\mathbf{k}\cdot\mathbf{B}_0)}{\omega} \tag{2.9.36}$$

$$\omega n_0 m_i \mathbf{V}_1 = \mathbf{k}\left(k_B T_0 n_1 + n_0 k_B T_1 + \mathbf{B}_0\cdot\frac{\mathbf{B}_1}{\mu_0}\right) - \frac{\mathbf{B}_1}{\mu_0}(\mathbf{B}_0\cdot\mathbf{k}) \qquad ((2.9.17))$$

We can substitute everything into the momentum equation

$$\begin{aligned}\omega n_0 m_i \mathbf{V}_1 = \mathbf{k}\Bigg(k_B T_0 \frac{n_0\mathbf{k}\cdot\mathbf{V}_1}{\omega} + n_0\frac{(\gamma-1)k_B T_0\mathbf{k}\cdot\mathbf{V}_1}{\omega} + \frac{\mathbf{B}_0}{\mu_0}\cdot\left[\frac{\mathbf{B}_0(\mathbf{k}\cdot\mathbf{V}_1) - \mathbf{V}_1(\mathbf{k}\cdot\mathbf{B}_0)}{\omega}\right]\Bigg) \\ - \left[\frac{\mathbf{B}_0(\mathbf{k}\cdot\mathbf{V}_1) - \mathbf{V}_1(\mathbf{k}\cdot\mathbf{B}_0)}{\mu_0\omega}\right](\mathbf{B}_0\cdot\mathbf{k})\end{aligned} \tag{2.9.37}$$

We can factor this into three vector components, $\mathbf{V}_1$, $\mathbf{k}$ and $\mathbf{B}_0$. We find

$$\begin{aligned}\left[\omega^2 - \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{n_0 m_i\mu_0}\right]\mathbf{V}_1 = \mathbf{k}\left[\frac{\gamma k_B T_0(\mathbf{k}\cdot\mathbf{V}_1)}{m_i} + \frac{|\mathbf{B}_0|^2(\mathbf{k}\cdot\mathbf{V}_1) - (\mathbf{B}_0\cdot\mathbf{V}_1)(\mathbf{k}\cdot\mathbf{B}_0)}{n_0 m_i\mu_0}\right] \\ - \mathbf{B}_0\frac{(\mathbf{k}\cdot\mathbf{B}_0)(\mathbf{k}\cdot\mathbf{V}_1)}{n_0 m_i\mu_0}\end{aligned} \tag{2.9.38}$$

We can make another simplification with $v_A^2 = B_0^2/(nm_i\mu_0)$ the Alfvén speed and $\hat{\mathbf{b}} = \mathbf{B}_0/|\mathbf{B}_0|$ as well as $v_S^2 = \gamma k_B T_0/m_i$ the sound speed[56] so that

$$\left[\omega^2 - (\mathbf{k}\cdot\hat{\mathbf{b}})^2 v_A^2\right]\mathbf{V}_1 = \mathbf{k}\left[v_S^2(\mathbf{k}\cdot\mathbf{V}_1) + v_A^2[(\mathbf{k}\cdot\mathbf{V}_1) - (\hat{\mathbf{b}}\cdot\mathbf{V}_1)(\hat{\mathbf{k}}\cdot\hat{\mathbf{b}})]\right] - \hat{\mathbf{b}}v_A^2(\mathbf{k}\cdot\hat{\mathbf{b}})(\mathbf{k}\cdot\mathbf{V}_1) \tag{2.9.39}$$

Every wave in ideal MHD is summarized in this equation. When we assume a certain form for any of the vectors $\mathbf{B}_0$, $\mathbf{V}_1$, and $\mathbf{k}$ then we get the conventional waves. For example, for sound waves, we take $\mathbf{k} = k_\parallel\hat{\mathbf{b}}$ and $\mathbf{V}_1 = V_\parallel\hat{\mathbf{b}}$ we get an extreme simplification. Thus we find

$$\left[\omega^2 - k_\parallel^2 v_A^2\right]V_\parallel\hat{\mathbf{b}} = k_\parallel\hat{\mathbf{b}}\left[v_S^2 k_\parallel V_\parallel + v_A^2[\cancel{k_\parallel V_\parallel} - \cancel{k_\parallel V_\parallel}]\right] - \hat{\mathbf{b}}v_A^2 k_\parallel^2 V_\parallel \tag{2.9.40}$$

$$\omega^2 - \cancel{k_\parallel^2 v_A^2} = k_\parallel^2 v_S^2 - \cancel{k_\parallel^2 v_A^2} \tag{2.9.41}$$

$$\omega^2 = k_\parallel^2 v_S^2 \tag{2.9.42}$$

which are sound waves since v_S is a sound speed.

One can also investigate shear Alfvén waves with $k = k_\parallel$ and $V_\parallel = 0$

$$\omega^2 - k_\parallel^2 v_A^2 = 0 \tag{2.9.43}$$

and compressional Alfvén waves with $k_\parallel = 0$ and $v_\parallel = 0$

$$\omega^2(\omega^2 - k_\perp^2 v_A^2) = 0 \tag{2.9.44}$$

[56] or ion acoustic speed

2.9.1 Discontinuities

In order to look at an interesting instability case we need to consider interface relations across surfaces. This will require us to derive relations using discontinuities across surfaces.

We should derive the Leibniz integral rule for a 3D integral and vector, as we will use it to find interface relations for a small control volume[57] for our boundary conditions in general.

2.9.1.1 Generalized Leibniz Rule

We begin with wanting to know how to calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \mathrm{d}^3x\ \mathbf{F} \tag{2.9.45}$$

We will derive what is usually called the Reynolds transport theorem. We can begin by translating into a coordinate system where the volume $V(t)$ goes to a fixed volume in time $\mathcal{V}$. The Jacobian determinant for this transformation is given by $\frac{\partial \mathbf{x}}{\partial \xi^1}\cdot\frac{\partial \mathbf{x}}{\partial \xi^2}\times\frac{\partial \mathbf{x}}{\partial \xi^3}$. Then we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \mathrm{d}^3x\ \mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathcal{V}} \mathrm{d}^3\xi\ \frac{\partial \mathbf{x}}{\partial \xi^1}\cdot\frac{\partial \mathbf{x}}{\partial \xi^2}\times\frac{\partial \mathbf{x}}{\partial \xi^3}\mathbf{F} \tag{2.9.46}$$

Now it is obvious that the $\frac{\mathrm{d}}{\mathrm{d}t}$ can pass through the integral, and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \mathrm{d}^3x\ \mathbf{F} = \iiint_{\mathcal{V}} \mathrm{d}^3\xi\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathbf{x}}{\partial \xi^1}\cdot\frac{\partial \mathbf{x}}{\partial \xi^2}\times\frac{\partial \mathbf{x}}{\partial \xi^3}\mathbf{F}\right] \tag{2.9.47}$$

Now we can transform back into the original coordinate system to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \mathrm{d}^3x\ \mathbf{F} = \iiint_{V(t)} \mathrm{d}^3x\ \frac{\partial \xi^1}{\partial \mathbf{x}}\cdot\frac{\partial \xi^2}{\partial \mathbf{x}}\times\frac{\partial \xi^3}{\partial \mathbf{x}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathbf{x}}{\partial \xi^1}\cdot\frac{\partial \mathbf{x}}{\partial \xi^2}\times\frac{\partial \mathbf{x}}{\partial \xi^3}\mathbf{F}\right] \tag{2.9.48}$$

We can begin with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathbf{x}}{\partial \xi^1}\cdot\frac{\partial \mathbf{x}}{\partial \xi^2}\times\frac{\partial \mathbf{x}}{\partial \xi^3}\right] &= \frac{\partial \mathbf{V}}{\partial \xi^1}\cdot\frac{\partial \mathbf{x}}{\partial \xi^2}\times\frac{\partial \mathbf{x}}{\partial \xi^3} + \frac{\partial \mathbf{x}}{\partial \xi^1}\cdot\frac{\partial \mathbf{V}}{\partial \xi^2}\times\frac{\partial \mathbf{x}}{\partial \xi^3} + \frac{\partial \mathbf{x}}{\partial \xi^1}\cdot\frac{\partial \mathbf{x}}{\partial \xi^2}\times\frac{\partial \mathbf{V}}{\partial \xi^3}\\ &= \frac{\partial \mathbf{V}}{\partial \xi^1}\cdot\frac{\partial \mathbf{x}}{\partial \xi^2}\times\frac{\partial \mathbf{x}}{\partial \xi^3} + \frac{\partial \mathbf{V}}{\partial \xi^2}\cdot\frac{\partial \mathbf{x}}{\partial \xi^3}\times\frac{\partial \mathbf{x}}{\partial \xi^1} + \frac{\partial \mathbf{V}}{\partial \xi^3}\cdot\frac{\partial \mathbf{x}}{\partial \xi^1}\times\frac{\partial \mathbf{x}}{\partial \xi^2}\end{aligned} \tag{2.9.49}$$

Now we can use that [remember (i', j', k') are even cyclic permutation of $(1, 2, 3)$]

$$\frac{\partial \xi^1}{\partial \mathbf{x}}\cdot\frac{\partial \xi^2}{\partial \mathbf{x}}\times\frac{\partial \xi^3}{\partial \mathbf{x}} = \frac{1}{\frac{\partial \mathbf{x}}{\partial \xi^1}\cdot\frac{\partial \mathbf{x}}{\partial \xi^2}\times\frac{\partial \mathbf{x}}{\partial \xi^3}} = \frac{1}{\mathcal{J}} \tag{2.9.50}$$

$$\frac{\partial \mathbf{x}}{\partial \xi^{j'}}\times\frac{\partial \mathbf{x}}{\partial \xi^{k'}} = \mathcal{J}\,\nabla\xi^{i'} \tag{2.9.51}$$

So for example

$$\left(\frac{\partial \xi^1}{\partial \mathbf{x}}\cdot\frac{\partial \xi^2}{\partial \mathbf{x}}\times\frac{\partial \xi^3}{\partial \mathbf{x}}\right)\left(\frac{\partial \mathbf{V}}{\partial \xi^{i'}}\cdot\frac{\partial \mathbf{x}}{\partial \xi^{j'}}\times\frac{\partial \mathbf{x}}{\partial \xi^{k'}}\right) = \frac{\frac{\partial \mathbf{V}}{\partial \xi^{i'}}\cdot\mathcal{J}\,\nabla\xi^{i'}}{\mathcal{J}} = \frac{\partial \mathbf{V}}{\partial \xi^{i'}}\cdot\frac{\partial \xi^{i'}}{\partial \mathbf{x}} \tag{2.9.52}$$

[57] If you have not heard of the term control volume, it is simply a set volume where we consider things going in and out and the differential equations within this specified volume.

so that we get

$$\left(\frac{\partial\xi^1}{\partial\mathbf{x}}\cdot\frac{\partial\xi^2}{\partial\mathbf{x}}\times\frac{\partial\xi^3}{\partial\mathbf{x}}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathbf{x}}{\partial\xi^1}\cdot\frac{\partial\mathbf{x}}{\partial\xi^2}\times\frac{\partial\mathbf{x}}{\partial\xi^3}\right]=\sum_{i=1}^{3}\frac{\partial\mathbf{V}}{\partial\xi^i}\cdot\frac{\partial\xi^i}{\partial\mathbf{x}}=\mathbb{1}:\nabla\mathbf{V}=\boldsymbol{\nabla}\cdot\mathbf{V} \tag{2.9.53}$$

where I used

$$\nabla\mathbf{V}=\frac{\partial\mathbf{V}}{\partial\mathbf{x}}=\sum_{i=1}^{3}\frac{\partial\xi^i}{\partial\mathbf{x}}\frac{\partial\mathbf{V}}{\partial\xi^i} \tag{2.9.54}$$

$$\mathbb{1}:\nabla\mathbf{V}=\mathbb{1}:\frac{\partial\mathbf{V}}{\partial\mathbf{x}}=\sum_{i=1}^{3}\frac{\partial\xi^i}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{V}}{\partial\xi^i} \tag{2.9.55}$$

and the identity $\mathbb{1}:\nabla\mathbf{V}=\boldsymbol{\nabla}\cdot\mathbf{V}$ from (1.2.235).

This means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)}\mathrm{d}^3x\;\mathbf{F}=\iiint_{V(t)}\mathrm{d}^3x\;\left[(\boldsymbol{\nabla}\cdot\mathbf{V})\mathbf{F}+\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t}\right]=\iiint_{V(t)}\mathrm{d}^3x\;\left[(\boldsymbol{\nabla}\cdot\mathbf{V})\mathbf{F}+\frac{\partial\mathbf{F}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{F}\right] \tag{2.9.56}$$

We can then use that

$$\boldsymbol{\nabla}\cdot(\mathbf{V}\mathbf{F})=(\boldsymbol{\nabla}\cdot\mathbf{V})\mathbf{F}+\mathbf{V}\cdot\nabla\mathbf{F} \tag{2.9.57}$$

and so we recover the usual Reynolds transport theorem form where we can use Gauss's Law

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)}\mathrm{d}^3x\;\mathbf{F}=\iiint_{V(t)}\mathrm{d}^3x\;\left[\boldsymbol{\nabla}\cdot(\mathbf{V}\mathbf{F})+\frac{\partial\mathbf{F}}{\partial t}\right]=\iiint_{V(t)}\mathrm{d}^3x\;\frac{\partial\mathbf{F}}{\partial t}+\oiint_{\partial V(t)}\mathrm{d}S\;\hat{\mathbf{n}}\cdot\mathbf{V}\mathbf{F} \tag{2.9.58}$$

with $\partial V(t)$ the surface bounding $V(t)$ and outward unit normal $\mathbf{n}$.

It is instructive to consider why this method encounters difficulties when we are doing a time derivative of a flux surface integral.

2.9.1.2 Interface Relations

The general form of a conservation law is given by

$$\frac{\partial\mathbf{u}}{\partial t}+\boldsymbol{\nabla}\cdot(\overleftrightarrow{\mathbf{F}})=\mathbf{S} \tag{2.9.59}$$

which is usually called the conservative form, where $\mathbf{u}$ is a vector (or scalar) and $\overleftrightarrow{\mathbf{F}}$ is a second rank tensor (or vector) and $\mathbf{S}$ is a volumetric vector (or scalar) source that is non-singular.

We can consider a control volume across the interface between the two regions we are interested in with width across the interface of h (see Figure 2.13). We can also consider a small enough surface in the other directions such that this becomes a rectangular prism. Let's assume the interface is moving at velocity $\mathbf{V}_i$ at the time of interest, which may vary in time and space. We have for our volume for the total change in $\mathbf{u}$ over time that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)}\mathrm{d}^3x\;\mathbf{u}=\iiint_{V(t)}\mathrm{d}^3x\;\frac{\partial\mathbf{u}}{\partial t}+\oiint_{\partial V(t)}\mathrm{d}S\;\hat{\mathbf{n}}\cdot\mathbf{V}_i\mathbf{u} \tag{2.9.60}$$

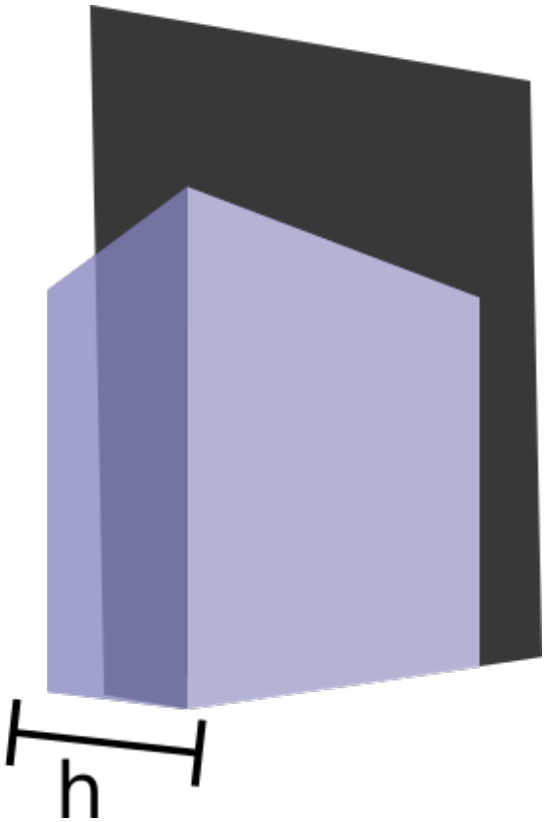

Figure 2.13: This shows the volume encompassing the interface in black. If we consider a small enough part of the interface surface we can make our volume a rectangular prism with vanishing width h.

We then can substitute in our form for $\partial\mathbf{u}/\partial t$ and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)}\mathrm{d}^3x\ \mathbf{u} = \iiint_{V(t)}\mathrm{d}^3x\ [-\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{F}}+\mathbf{S}] + \oiint_{\partial V(t)}\mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{V}_i\mathbf{u} \tag{2.9.61}$$

$$= \iiint_{V(t)}\mathrm{d}^3x\ \mathbf{S} + \oiint_{\partial V(t)}\mathrm{d}S\ \hat{\mathbf{n}}\cdot[\mathbf{V}_i\mathbf{u}-\overleftrightarrow{\mathbf{F}}] \tag{2.9.62}$$

If we now consider that h, the width normally, going to zero, then any volume terms with non-singular integrands (both $\mathbf{u}$ and $\mathbf{S}$ are non-singular because they don't blow up [go to $\pm\infty$]at our interface) must vanish. Thus, the two volume integrals must go to zero, and we must have

$$\oiint_{\partial V(t)}\mathrm{d}S\ \hat{\mathbf{n}}\cdot[\mathbf{V}_i\mathbf{u}-\overleftrightarrow{\mathbf{F}}] = \mathbf{0} \tag{2.9.63}$$

Because this is for any surface across the interface, then the integrand itself must satisfy this relation. Call one side of the volume + and the other side −. Then we must have

$$\left(\hat{\mathbf{n}}\cdot[\mathbf{V}_i\mathbf{u}-\overleftrightarrow{\mathbf{F}}]\right)_+ + \left(\hat{\mathbf{n}}\cdot[\mathbf{V}_i\mathbf{u}-\overleftrightarrow{\mathbf{F}}]\right)_- = \mathbf{0} \tag{2.9.64}$$

Because $\hat{\mathbf{n}}_+ = -\hat{\mathbf{n}}_-$ we can define $\hat{\mathbf{n}}_- = \hat{\mathbf{n}}$ and then write this as

$$\hat{\mathbf{n}}\cdot\left[\!\!\left[\overleftrightarrow{\mathbf{F}}-\mathbf{V}_i\mathbf{u}\right]\!\!\right] \equiv \hat{\mathbf{n}}\cdot\left\{\left[\overleftrightarrow{\mathbf{F}}_+-\mathbf{V}_i\mathbf{u}_+\right]-\left[\overleftrightarrow{\mathbf{F}}_--\mathbf{V}_i\mathbf{u}_-\right]\right\} = \mathbf{0} \tag{2.9.65}$$

Note because this equals zero we could use $\hat{\mathbf{n}}$ in either $\hat{\mathbf{n}}_-$ or $\hat{\mathbf{n}}_+$ as long as we are consistent, we could also write

$$\hat{\mathbf{n}}\cdot\left[\!\!\left[\mathbf{V}_i\mathbf{u}-\overleftrightarrow{\mathbf{F}}\right]\!\!\right] = \mathbf{0} \tag{2.9.66}$$

This relation is sometimes called the Rankine-Hugoniot condition.

Thus we have that for quantities across an interface (possibly discontinuous) we find

$$\frac{\partial\mathbf{u}}{\partial t}+\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{F}} = \mathbf{S} \tag{2.9.67}$$

$$\Rightarrow \hat{\mathbf{n}}\cdot\left[\!\!\left[\overleftrightarrow{\mathbf{F}}-\mathbf{V}_i\mathbf{u}\right]\!\!\right] = \mathbf{0} \tag{2.9.68}$$

2.9.2 Lagrangian and Eulerian Specifications

Leonhard Euler and Joseph-Louis Lagrange are two quite justifiably famous mathematician-physicists, and so they have an enormous number of things named after them. In fact, they are the ones for which the Euler-Lagrange equations are named. It is therefore somewhat amusing to note that Lagrangian and Eulerian were chosen for two different perspectives on observing fluid dynamics.

Consider the water in a river. Suppose you put a cork in the water. One can then consider the motion of the cork relative to the river water or relative to the banks of the river. When relative to the river water, it is as if you were in a boat on the river and dropped the cork into the water.

This is the Lagrangian specification of a flow field.[58] The other perspective, where the cork's movement is compared to a fixed location, such as if you stood on the bank and dropped the cork in, is called the Eulerian specification of a flow field[59] These two perspectives are equivalent, of course, as the cork's motion does not actually change depending on these perspectives. The cork here simply represents a small fluid parcel. Some calculations are much easier in one specification versus another, or a combination of both. Thus these two perspectives are often combined or the simpler perspective is used to do a calculation.

The importance of choosing a perspective can easily be illustrated. Consider the famous problem of the bottle and boat in the river.[60] Suppose a river has a constant velocity with respect to the land. A motorboat is going upstream at a constant velocity with respect to the water and when passing under a bridge a bottle falls out of it. Thirty minutes pass and the person in the motorboat realizes the bottle has fallen out. The motorboat is turned around (ignore the time it takes to do this) and goes downstream at the same constant velocity with respect to the water. The motorboat meets the bottle one kilometer downstream from the bridge. What is the velocity of the river with respect to the ground?

The difficult method of solution is to try and consider the motorboat's velocity solely relative to the land or solely in the frame of the river. Instead consider the motion relative to the bottle in the river. The motorboat moves away from it for 30 minutes, and so it will take it a further 30 minutes for it to return. We can then use that the bottle has moved 1 kilometer in that time. Therefore the river speed must be 1 km/h.

Now let's consider a specific coordinate system where $\mathbf{R}(t)$ specifies a location in space of a fluid parcel at time t. Then for any quantity $\mathbf{q}$ we can consider it at later time as at $\mathbf{q}_E(\mathbf{R}(t),t)$ which is the Eulerian specification. Instead we could label the fluid parcel at $t=0$ with $\mathbf{R}_0=\mathbf{R}(0)$, and the change since then given by $\mathbf{X}(\mathbf{R}_0,t)$ so that $\mathbf{q}_L(\mathbf{R}_0,t)$. We can use that $\mathbf{R}=\mathbf{R}_0+\mathbf{X}(\mathbf{R}_0,t)$. Because these are equivalent we know that

$$\mathbf{q}_L(\mathbf{R}_0,t)=\mathbf{q}_E(\mathbf{R}(t),t)=\mathbf{q}_E(\mathbf{R}_0+\mathbf{X}(\mathbf{R}_0,t),t) \tag{2.9.69}$$

This may seem confusing but remember in the Lagrangian specification we just need to know the initial location of a fluid parcel and then we can find its location at a later time by following the fluid flow. In the Eulerian perspective, we are not following any fluid parcels, but just says a parcel is at a particular location at a particular time. We can then consider over a small amount of time what the change would be when fixing the original location $\mathbf{R}_0$. This is easy in the Lagrangian specification, and fairly simple in the Eulerian perspective as well. It would be given by a Taylor Series approximation.

$$\left.\frac{\mathrm{d}\mathbf{q}_L}{\mathrm{d}t}\right|_{\mathbf{R}_0}=\overbrace{\left.\frac{\partial\mathbf{q}_L}{\partial t}\right|_{\mathbf{R}_0}}^{=\frac{\partial\mathbf{q}_E}{\partial t}|_{\mathbf{R}_0}}+\cancel{\frac{\partial\mathbf{R}_0}{\partial t}}\cdot\nabla\mathbf{q}_L+\cdots \tag{2.9.70}$$

$$\left.\frac{\mathrm{d}\mathbf{q}_E}{\mathrm{d}t}\right|_{\mathbf{R}_0}=\overbrace{\left.\frac{\partial\mathbf{q}_E}{\partial t}\right|_{\mathbf{R}_0}}^{=\frac{\partial\mathbf{q}_L}{\partial t}|_{\mathbf{R}_0}}+\frac{\partial\mathbf{X}}{\partial t}\cdot\nabla\mathbf{q}_E+\cdots \tag{2.9.71}$$

[58] Often shortened to Lagrangian frame, though this is somewhat imprecise as frame of reference usually refers to the coordinate system for an observer, as well.

[59] Often shortened to Eulerian frame.

[60] I first heard of it in George Gamow's book.

Here I have emphasized that the first terms are interchangeable as $\left(\frac{\partial \mathbf{q}_L}{\partial t}\right)_{\mathbf{R}_0} = \left(\frac{\partial \mathbf{q}_E}{\partial t}\right)_{\mathbf{R}_0}$ if these two specifications are to match at the initial time. The fluid velocity must be given by $\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}_E$.[61] Now if we consider small changes we can use that $\mathbf{q} = \mathbf{q}_0 + \delta \mathbf{q}_1$ with $\delta \ll 1$. Then we can write

$$\left.\frac{\mathrm{d}\mathbf{q}_L}{\mathrm{d}t}\right|_{\mathbf{R}_0} = \overbrace{\left.\frac{\partial \mathbf{q}_{0L}}{\partial t}\right|_{\mathbf{R}_0}}^{=\frac{\partial \mathbf{q}_{0E}}{\partial t}|_{\mathbf{R}_0}} + \delta \overbrace{\left.\frac{\partial \mathbf{q}_{1L}}{\partial t}\right|_{\mathbf{R}_0}}^{=\frac{\partial \mathbf{q}_{1E}}{\partial t}|_{\mathbf{R}_0}} + \mathcal{O}(\delta^2) \tag{2.9.72}$$

$$\begin{aligned}\left.\frac{\mathrm{d}\mathbf{q}_E}{\mathrm{d}t}\right|_{\mathbf{R}_0} &= \overbrace{\left.\frac{\partial \mathbf{q}_{0E}}{\partial t}\right|_{\mathbf{R}_0}}^{=\frac{\partial \mathbf{q}_{0L}}{\partial t}|_{\mathbf{R}_0}} + \delta \overbrace{\left.\frac{\partial \mathbf{q}_{1E}}{\partial t}\right|_{\mathbf{R}_0}}^{=\frac{\partial \mathbf{q}_{0L}}{\partial t}|_{\mathbf{R}_0}} + \frac{\partial \mathbf{X}}{\partial t} \cdot \nabla(\mathbf{q}_{0E} + \delta \mathbf{q}_{1E}) + \mathcal{O}(\delta^2) \\ &= \frac{\partial \mathbf{q}_{0L}}{\partial t} + \delta \frac{\partial \mathbf{q}_{1L}}{\partial t} + \mathbf{V}_{0E} \cdot \nabla \mathbf{q}_{0E} + \delta \mathbf{V}_{0E} \cdot \nabla \mathbf{q}_{1E} + \delta \mathbf{V}_{E1} \cdot \nabla \mathbf{q}_{0E} + \mathcal{O}(\delta^2) \\ &= \frac{\partial \mathbf{q}_{0L}}{\partial t} + \mathbf{V}_{0E} \cdot \nabla \mathbf{q}_{0E} + \delta \left[\frac{\partial \mathbf{q}_{1L}}{\partial t} + \mathbf{V}_{0E} \cdot \nabla \mathbf{q}_{1E} + \mathbf{V}_{E1} \cdot \nabla \mathbf{q}_{0E}\right] + \mathcal{O}(\delta^2)\end{aligned} \tag{2.9.73}$$

Thus, we find that when solving in a Lagrangian specification we consider $\frac{\mathrm{d}\mathbf{q}_L}{\mathrm{d}t} = \frac{\partial \mathbf{q}_L}{\partial t}$ and so we can write

$$\left.\frac{\mathrm{d}\mathbf{q}_{0E}}{\mathrm{d}t}\right|_{\mathbf{R}_0} = \left.\frac{\partial \mathbf{q}_{0L}}{\partial t}\right|_{\mathbf{R}_0} + \mathbf{V}_{0E} \cdot \nabla \mathbf{q}_{0E} \tag{2.9.74}$$

$$\left.\frac{\mathrm{d}\mathbf{q}_{1E}}{\mathrm{d}t}\right|_{\mathbf{R}_0} = \left.\frac{\partial \mathbf{q}_{1L}}{\partial t}\right|_{\mathbf{R}_0} + \mathbf{V}_{0E} \cdot \nabla \mathbf{q}_{1E} + \mathbf{V}_{1E} \cdot \nabla \mathbf{q}_{E0} \tag{2.9.75}$$

Now, we can use that $\mathbf{V}_{E1} = \frac{\partial \boldsymbol{\xi}}{\partial t} = \omega \boldsymbol{\xi}$ for a linearization with $\frac{\mathrm{d}\mathbf{q}_1}{\mathrm{d}t} = \omega \mathbf{q}_1$ so that

$$\omega \mathbf{q}_{1E} = \omega \mathbf{q}_{1L} + \mathbf{V}_{0E} \cdot \nabla \mathbf{q}_{1E} + \omega \boldsymbol{\xi} \cdot \nabla \mathbf{q}_{E0} \tag{2.9.76}$$

$$\mathbf{q}_{1E} = \mathbf{q}_{1L} + \frac{\mathbf{V}_{0E}}{\omega} \cdot \nabla \mathbf{q}_{1E} + \boldsymbol{\xi} \cdot \nabla \mathbf{q}_{E0} \tag{2.9.77}$$

When there is no background flow, this is a simple prescription and we find

$$\mathbf{q}_{1E} = \mathbf{q}_{1L} + \boldsymbol{\xi} \cdot \nabla \mathbf{q}_{0E} \tag{2.9.78}$$

2.9.3 Kruskal-Schwarzchild Problem

Let's assume that we make a small perturbation to a surface separating a more dense fluid ρ_+ on top and smaller density ρ_- at equilibrium. Let's assume the perturbation is of the form $\boldsymbol{\xi}$ and then we say that $\partial \boldsymbol{\xi}/\partial t = \mathbf{V}_1$. We can further assume that the $\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0$, that is the perturbation is divergenceless and so $\boldsymbol{\nabla} \cdot \mathbf{V}_1 = 0$, as well. Compression is usually stabilizing, so ignoring it gives us a worst-case situation. We will also consider no background flow $\mathbf{V}_0 = \mathbf{0}$. Let's put the interface at $y = 0$ and assume

$$\rho_0 = \begin{cases} \rho_- & y < 0 \\ \rho_+ & y \geq 0 \end{cases} \tag{2.9.79}$$

$$\mathbf{B}_0 = \begin{cases} B\left[\sin\alpha \hat{\mathbf{x}} + \cos\alpha \hat{\mathbf{z}}\right] & y < 0 \\ B\hat{\mathbf{z}} & y \geq 0 \end{cases} \tag{2.9.80}$$

[61] This is the fluid velocity measured in an Eulerian specification, so $\mathbf{V}_E$.

where $\rho = nm$ with a constant gravitational force $\mathbf{g} = -g\hat{\mathbf{y}}$.

Thus, the steady-state momentum balance equation states ($\mathbf{V}_0 = \mathbf{0}$)

$$\mathbf{0} = -\nabla p_0 + \rho\mathbf{g} \tag{2.9.81}$$

$$\nabla p_0 = -\rho g\hat{\mathbf{y}} \tag{2.9.82}$$

$$\frac{\partial p_0}{\partial y} = -\rho g \tag{2.9.83}$$

$$\frac{\partial p_0}{\partial x} = \frac{\partial p_0}{\partial y} = 0 \tag{2.9.84}$$

Now, let's solve in the Lagrangian specification, which means that the interface velocity will not need to be specified yet. We will use the ideal MHD equations. The linearized momentum equation says

$$\rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\nabla p_1 + \frac{\boldsymbol{\nabla}\times\mathbf{B}_1}{\mu_0}\times\mathbf{B}_0 \tag{2.9.85}$$

We also have for the magnetic equation that

$$\frac{\partial\mathbf{B}_1}{\partial t} = -\boldsymbol{\nabla}\times\left[\frac{\partial\boldsymbol{\xi}}{\partial t}\times\mathbf{B}_0\right] \tag{2.9.86}$$

Because $\partial\mathbf{B}_0/\partial t = \mathbf{0}$ we know that we can then write

$$\frac{\partial\mathbf{B}_1}{\partial t} = \frac{\partial\boldsymbol{\nabla}\times[\boldsymbol{\xi}\times\mathbf{B}_0]}{\partial t} \tag{2.9.87}$$

$$\mathbf{B}_1 = \boldsymbol{\nabla}\times[\boldsymbol{\xi}\times\mathbf{B}_0] + \mathbf{C}_0 \tag{2.9.88}$$

where $\mathbf{C}_0$ is time-independent. It must be zero since we desire $\mathbf{B}_1 = \mathbf{0}$ when $\boldsymbol{\xi} = \mathbf{0}$. We note that we can use the vector calculus identity with our assumptions of divergenceless $\mathbf{B}_0$ and $\boldsymbol{\xi}$ so that

$$\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B}_0) = \boldsymbol{\xi}\cancel{(\boldsymbol{\nabla}\cdot\mathbf{B}_0)} - \mathbf{B}_0\cancel{(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})} + \mathbf{B}_0\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{B}_0 = \mathbf{B}_0\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{B}_0 \tag{2.9.89}$$

Because $\mathbf{B}_0$ has no spatial dependence $\nabla\mathbf{B}_0 = \overset{\leftrightarrow}{\mathbf{0}}$ everywhere except across $y = 0$. We will consider each side separately, so

$$\mathbf{B}_1 = \mathbf{B}_0\cdot\nabla\boldsymbol{\xi} \tag{2.9.90}$$

We can then use

$$(\boldsymbol{\nabla}\times\mathbf{A})\times\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{A} - \nabla(\mathbf{A}\cdot\mathbf{B}) + \nabla\mathbf{B}\cdot\mathbf{A} = \mathbf{B}\cdot\nabla\mathbf{A} - \nabla\mathbf{A}\cdot\mathbf{B} \tag{2.9.91}$$

Thus

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{B}_1)\times\mathbf{B}_0 &= \mathbf{B}_0\cdot\nabla\mathbf{B}_1 - \nabla\mathbf{B}_1\cdot\mathbf{B}_0 = \mathbf{B}_0\cdot\nabla[\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}] - \nabla[\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}]\cdot\mathbf{B}_0\\ &= (\mathbf{B}_0\cdot\nabla)^2\boldsymbol{\xi} - \nabla[\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}]\cdot\mathbf{B}_0\end{aligned} \tag{2.9.92}$$

and so we find

$$\rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\nabla p_1 + \frac{(\mathbf{B}_0\cdot\nabla)^2}{\mu_0}\boldsymbol{\xi} - \nabla\overbrace{[\mathbf{B}_1]}^{\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}}\cdot\frac{\mathbf{B}_0}{\mu_0} \tag{2.9.93}$$

It is worth noting that the total pressure can now be employed

$$p_T = p + \frac{B^2}{2\mu_0} \tag{2.9.94}$$

$$p_{T1} = p_1 + \frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{\mu_0} \tag{2.9.95}$$

$$\nabla p_{T1} = \nabla p_1 + \frac{\nabla \mathbf{B}_1}{\mu_0} \cdot \mathbf{B}_0 \tag{2.9.96}$$

Thus we can rewrite this as

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla p_{1T} + \frac{(\mathbf{B}_0 \cdot \nabla)^2}{\mu_0} \boldsymbol{\xi} \tag{2.9.97}$$

Because there is no background gradients in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ direction, we can assume a form of $e^{ik_x x + ik_z z}$ for $\boldsymbol{\xi}$ (equivalent to taking Fourier transforms). We will also assume $e^{-i\omega t}$ time dependence on $\boldsymbol{\xi}$. Define $\mathbf{k} = k_x \hat{\mathbf{x}} + k_z \hat{\mathbf{z}}$ for ease of notation. Then

$$-\omega^2 \rho_0 \boldsymbol{\xi} = -i\mathbf{k} p_{1T} - \frac{\partial p_{1T}}{\partial y} \hat{\mathbf{y}} + \frac{(\mathbf{B}_0 \cdot i\mathbf{k})^2}{\mu_0} \boldsymbol{\xi} \tag{2.9.98}$$

where we have used that $\hat{\mathbf{y}} \cdot \mathbf{B}_0 = 0$. Then we can rewrite this as

$$([\mathbf{B}_0 \cdot \mathbf{k}]^2 - \mu_0 \rho_0 \omega^2) \boldsymbol{\xi} = -i\mathbf{k} \mu_0 p_{T_1} - \mu_0 \frac{\partial p_{T1}}{\partial y} \hat{\mathbf{y}} \tag{2.9.99}$$

This shows that if $\mathbf{B}_0 \cdot \mathbf{k} = \sqrt{\mu_0 \rho_0}\omega$ or

$$\omega = \frac{\mathbf{B}_0 \cdot \mathbf{k}}{\mu_0 \rho_0} \tag{2.9.100}$$

are solutions. This would be when shear Alfvén waves are excited. This leads simply to wave propagation and so is stable.

We can then multiply by $\mathbf{k}\cdot$ removing the y component and look at the final y component separately

$$([\mathbf{B}_0 \cdot \mathbf{k}]^2 - \mu_0 \rho_0 \omega^2) \mathbf{k} \cdot \boldsymbol{\xi} = -ik^2 \mu_0 p_{1T} \tag{2.9.101}$$

$$([\mathbf{B}_0 \cdot \mathbf{k}]^2 - \mu_0 \omega^2 \rho_0) \xi_y = -\mu_0 \frac{\partial p_{1T}}{\partial y} \tag{2.9.102}$$

Remember we used $\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0$ and with our assumptions this says

$$i\mathbf{k} \cdot \boldsymbol{\xi} + \frac{\partial \xi_y}{\partial y} = 0 \tag{2.9.103}$$

Thus, we can replace $\mathbf{k} \cdot \boldsymbol{\xi}$ and find

$$([\mathbf{B}_0 \cdot \mathbf{k}]^2 - \mu_0 \rho_0 \omega^2) i \frac{\partial \xi_y}{\partial y} = -ik^2 \mu_0 p_{1T} \tag{2.9.104}$$

$$([\mathbf{B}_0 \cdot \mathbf{k}]^2 - \mu_0 \rho_0 \omega^2) \frac{\partial \xi_y}{\partial y} = -k^2 \mu_0 p_{1T} \tag{2.9.105}$$

$$([\mathbf{B}_0 \cdot \mathbf{k}]^2 - \mu_0 \omega^2 \rho_0) \xi_y = -\mu_0 \frac{\partial p_{1T}}{\partial y} \tag{2.9.106}$$

We can take $\frac{\partial}{\partial y}$ on (2.9.105) to find (using pre-factors are constants)

$$([\mathbf{B}_0 \cdot \mathbf{k}]^2 - \mu_0\rho_0\omega^2)\frac{\partial^2 \xi_y}{\partial y^2} = -k^2\mu_0\frac{\partial p_{1T}}{\partial y} = k^2([\mathbf{B}_0 \cdot \mathbf{k}]^2 - \mu_0\omega^2\rho_0)\xi_y \tag{2.9.107}$$

$$\frac{\partial^2 \xi_y}{\partial y} = k^2\frac{\cancel{([\mathbf{B}_0 \cdot \mathbf{k}]^2 - \mu_0\omega^2\rho_0)}}{\cancel{[\mathbf{B}_0 \cdot \mathbf{k}]^2 - \mu_0\rho_0\omega^2}}\xi_y = k^2\xi_y \tag{2.9.108}$$

Thus

$$\xi_y = A_+ e^{ky} + A_- e^{-ky} \tag{2.9.109}$$

for constants A_+ and A_-. At $y = 0$ we need the value to just be ξ_{y0}. We desire $\xi_y \to 0$ as $y \to \pm\infty$ and thus we can write that the y dependence of ξ_y must be

$$\xi_y = \xi_{y0}e^{-k|y|} \tag{2.9.110}$$

Thus, by our assumption of $\boldsymbol{\xi} \sim e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}$ we have

$$\xi_y = \xi_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x}) - k|y|} \tag{2.9.111}$$

We can now consider our momentum balance interface relation. We will consider it in our original Eulerian specification, so that the interface now is no longer simply a plane. We need to use that momentum balance can be written as

$$\frac{\partial(\rho\mathbf{V})}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho\mathbf{V}\mathbf{V} + \left[p + \frac{B^2}{2\mu_0}\right]\mathbb{1} - \frac{\mathbf{B}\mathbf{B}}{\mu_0}\right) = \mathbf{0} \tag{2.9.112}$$

Thus using our interface relation (2.9.58) yields

$$\hat{\mathbf{n}}\cdot\left[\!\left[\rho\mathbf{V}\mathbf{V} + \left[p + \frac{B^2}{2\mu_0}\right]\mathbb{1} - \frac{\mathbf{B}\mathbf{B}}{\mu_0} - \rho\mathbf{V}_i\mathbf{V}\right]\!\right] = \mathbf{0} \tag{2.9.113}$$

For our case $\mathbf{V}\mathbf{V}$ and $\mathbf{V}_i\mathbf{V}$ terms go to zero to first order. We also have $\hat{\mathbf{n}}\cdot\mathbf{B} = 0$ on either side due to no divergence of the magnetic field. Thus this simplifies to

$$\hat{\mathbf{n}}\cdot\left[\!\left[\cancel{\rho\mathbf{V}\mathbf{V}} + \left[p + \frac{B^2}{2\mu_0}\right]\mathbb{1} - \cancel{\frac{\mathbf{B}\mathbf{B}}{\mu_0}} - \cancel{\rho\mathbf{V}_i\mathbf{V}}\right]\!\right] = \mathbf{0} \tag{2.9.114}$$

$$\left[\!\left[\left[p + \frac{B^2}{2\mu_0}\right]\hat{\mathbf{n}}\right]\!\right] = \mathbf{0} \tag{2.9.115}$$

$$\tag{2.9.116}$$

which must be true for each component of $\hat{\mathbf{n}}$ thus

$$\left[\!\left[p + \frac{B^2}{2\mu_0}\right]\!\right] = 0 \tag{2.9.117}$$

The crossed out term is zero via

$$\hat{\mathbf{n}}\cdot[\![\mathbf{B}\mathbf{B}]\!] = [\![(\hat{\mathbf{n}}\cdot\mathbf{B})\mathbf{B}]\!] \tag{2.9.118}$$

For the steady-state quantities $\hat{\mathbf{n}} = \hat{\mathbf{y}}$ and so $\hat{\mathbf{n}}_0 \cdot \mathbf{B}_0 = 0$. Thus we need only compute $[\![\mathbf{n}_1 \cdot \mathbf{B}_0 \mathbf{B}_0 + \hat{\mathbf{n}}_0 \cdot \mathbf{B}_1 \mathbf{B}_0]\!]$. We will find that

$$\mathbf{n}_1 \cdot \mathbf{B}_0 + \hat{\mathbf{n}}_0 \cdot \mathbf{B}_1 = 0 \tag{2.9.119}$$

and so this term does not contribute in our case.

We need a relation for $\mathbf{n}_1$ the perturbed normal. The way to find this is by thinking about the change of the $y = 0$ surface. It will get pushed by $\boldsymbol{\xi}$ and so we need to find the perturbation to $\hat{\mathbf{n}}_0$ that leads to $\hat{\mathbf{n}}_0 + \mathbf{n}_1 = \hat{\mathbf{n}}$. As a vector relation, we then have

$$\mathbf{n}_1 = \hat{\mathbf{n}} - \hat{\mathbf{n}}_0 \tag{2.9.120}$$

For n_0 we can use that we fix $y = 0$ and let x and z vary. Thus ∇y is clearly the correct $\hat{\mathbf{n}}_0$.

We have that the contour in space defining our new interface is given by adding the $\boldsymbol{\xi}(y = 0)$ to our previous surface. We can find $\hat{\mathbf{n}}$ by fixing $y = 0$ and using x and z as parameterizations of the surface through $\mathbf{r} = x\hat{\mathbf{x}} + \xi_y(x, y = 0, z)\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. We don't care about the ξ_x and ξ_z components because they are not changing the normal direction from the steady state value in any way. The normal is then given by

$$\hat{\mathbf{n}} = \frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial x} = \left[\frac{\partial \xi_y}{\partial z}\hat{\mathbf{y}} + \hat{\mathbf{z}}\right] \times \left[\hat{\mathbf{x}} + \frac{\partial \xi_y}{\partial x}\hat{\mathbf{y}}\right] = -\frac{\partial \xi_y}{\partial z}\hat{\mathbf{z}} + \hat{\mathbf{y}} - \frac{\partial \xi_y}{\partial x}\hat{\mathbf{x}} \tag{2.9.121}$$

So

$$\mathbf{n}_1 = \left(-\frac{\partial \xi_y}{\partial z}\hat{\mathbf{z}} + \hat{\mathbf{y}} - \frac{\partial \xi_y}{\partial x}\hat{\mathbf{x}}\right) - \hat{\mathbf{y}} = -\frac{\partial \xi_y}{\partial x}\hat{\mathbf{x}} - \frac{\partial \xi_y}{\partial z}\hat{\mathbf{z}} \tag{2.9.122}$$

In general, if we have a surface parameterized by $\mathbf{r} = x(s,t)\hat{\mathbf{x}} + y(s,t)\hat{\mathbf{y}} + z(s,t)\hat{\mathbf{z}}$ and perturb it with vector $\boldsymbol{\xi} = \xi_x(s,t)\hat{\mathbf{x}} + \xi_y(s,t)\hat{\mathbf{y}} + \xi_z(s,t)\hat{\mathbf{z}}$ then the normal is given by

$$\hat{\mathbf{n}}_0 = \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \tag{2.9.123}$$

$$\hat{\mathbf{n}} = \frac{\partial \mathbf{r} + \boldsymbol{\xi}}{\partial s} \times \frac{\partial \mathbf{r} + \boldsymbol{\xi}}{\partial s} = \overbrace{\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial s}}^{\hat{\mathbf{n}}_0} + \frac{\partial \boldsymbol{\xi}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} + \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \boldsymbol{\xi}}{\partial t} + \overbrace{\frac{\partial \boldsymbol{\xi}}{\partial s} \times \frac{\partial \boldsymbol{\xi}}{\partial t}}^{\mathcal{O}(|\xi|^2) \to 0} \tag{2.9.124}$$

One then must remove from $\boldsymbol{\xi}$ any parts that simply translate along the original surface if the problem has a symmetry on the original surface.

The general relation usually given for a perturbation of a surface by $\boldsymbol{\xi}$ is[62]

$$\mathbf{n}_1 = \overbrace{-\hat{\mathbf{n}}_0 \times (\boldsymbol{\nabla} \times \boldsymbol{\xi})}^{\substack{\text{rotation tangent} \\ \text{to surface}}} \overbrace{-\hat{\mathbf{n}}_0 \cdot \nabla\boldsymbol{\xi} + \hat{\mathbf{n}}_0[\hat{\mathbf{n}}_0\hat{\mathbf{n}}_0 : \nabla\boldsymbol{\xi}]}^{\substack{\text{removes tangential} \\ \text{displacement along surface}}} = (\hat{\mathbf{n}}_0\hat{\mathbf{n}}_0 - \mathbb{1}) \cdot \nabla\boldsymbol{\xi} \cdot \hat{\mathbf{n}}_0 \tag{2.9.125}$$

One can find this via assuming we have the equation for the surface $f(\mathbf{x}) = 0$ and so the normal is $\mathbf{n} = \pm\nabla f$. We can always redefine f (multiply by -1) so that it points in the direction we desire,

[62] I have never found a formal proof of this statement. There almost certainly is one somewhere, but not in any textbooks I have seen. So the following is my reconstruction of what I believe the proof must be. I actually find a more general relation.

so $\hat{\mathbf{n}} = \nabla f/|\nabla f|$. We can also imagine we redefine f such that $|\nabla f| = 1$. Then we consider the expansion $f(\mathbf{x}-\boldsymbol{\xi})$[63] and Taylor expand

$$f(\mathbf{x}-\boldsymbol{\xi}) = f(\mathbf{x}) - \boldsymbol{\xi}\cdot\nabla f\big|_{\boldsymbol{\xi}=\mathbf{0}} + \mathcal{O}(\xi^2) \tag{2.9.126}$$

$$\nabla f(\mathbf{x}-\boldsymbol{\xi}) = \nabla f(\mathbf{x}) - \nabla\left[\boldsymbol{\xi}\cdot\nabla f\big|_{\boldsymbol{\xi}=\mathbf{0}}\right] + \mathcal{O}(\xi^2) \tag{2.9.127}$$

$$\hat{\mathbf{n}} = \mathbf{n}_0 - \nabla\left[\boldsymbol{\xi}\cdot\hat{\mathbf{n}}\big|_{\boldsymbol{\xi}=\mathbf{0}}\right] + \mathcal{O}(\xi^2) \tag{2.9.128}$$

Now $\mathbf{n}_0$ is not necessarily a unit vector in this notation. We can replace it with $\mathbf{n}_0 = \hat{\mathbf{n}}_0 + C(\boldsymbol{\xi})\hat{\mathbf{n}}_0$ with a term $C(\boldsymbol{\xi}) = \mathcal{O}(\xi)$.[64]

$$\begin{aligned}
\hat{\mathbf{n}} &= \mathbf{n}_0 + C(\boldsymbol{\xi})\hat{\mathbf{n}}_0 - \nabla\left[\boldsymbol{\xi}\cdot\hat{\mathbf{n}}\big|_{\boldsymbol{\xi}=\mathbf{0}}\right] + \mathcal{O}(\xi^2)\\
&= \mathbf{n}_0 + C(\boldsymbol{\xi})\hat{\mathbf{n}}_0 - \nabla[\boldsymbol{\xi}\cdot\mathbf{n}_0] + \mathcal{O}(\xi^2)\\
&= \mathbf{n}_0 + C(\boldsymbol{\xi})\hat{\mathbf{n}}_0 - \boldsymbol{\xi}\cdot\nabla\mathbf{n}_0 - \mathbf{n}_0\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\times\cancel{\nabla\times\mathbf{n}_0} - \mathbf{n}_0\times\nabla\times\boldsymbol{\xi} + \mathcal{O}(\xi^2)\\
&= \mathbf{n}_0 + C(\boldsymbol{\xi})\hat{\mathbf{n}}_0 - \nabla\boldsymbol{\xi}\cdot\hat{\mathbf{n}}_0 - \nabla\hat{\mathbf{n}}_0\cdot\boldsymbol{\xi} + \mathcal{O}(\xi^2)
\end{aligned} \tag{2.9.129}$$

where I have used $\nabla\times\mathbf{n}_0 = \nabla\times\hat{\mathbf{n}}_0 + \mathcal{O}(\xi) = \nabla\times\nabla f + \mathcal{O}(\xi) = \mathbf{0} + \mathcal{O}(\xi)$. All of the terms coming form $\nabla[\boldsymbol{\xi}\cdot\mathbf{n}_0]$ can replace $\mathbf{n}_0$ with $\hat{\mathbf{n}}_0$ because these terms are already $\mathcal{O}(\xi)$ and so the $C(\boldsymbol{\xi})$ term goes into the $\mathcal{O}(\xi^2)$ terms. Because $\boldsymbol{\xi}\times\nabla\times\hat{\mathbf{n}}_0 = \mathbf{0}$, we also have $\nabla\hat{\mathbf{n}}_0\cdot\boldsymbol{\xi} = \boldsymbol{\xi}\cdot\nabla\hat{\mathbf{n}}_0$ from $\mathbf{A}\times\nabla\times\mathbf{B} = \nabla\mathbf{B}\cdot\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{B}$. We see we have basically already derived the previous relation but with an extra term $\boldsymbol{\xi}\cdot\nabla\hat{\mathbf{n}}_0$ and a missing term with the double dot product. To find the double dot product, remember we have $\mathbf{n}_0 = \hat{\mathbf{n}}_0 + C(\boldsymbol{\xi})\hat{\mathbf{n}}_0$ and so we need to find $C(\boldsymbol{\xi})$.

For convenience define the term $\mathbf{N}$ which contains all $\mathcal{O}(\xi)$ terms

$$\mathbf{N} = C(\boldsymbol{\xi})\hat{\mathbf{n}}_0 - \nabla\boldsymbol{\xi}\cdot\hat{\mathbf{n}}_0 - \nabla\hat{\mathbf{n}}_0\cdot\boldsymbol{\xi} \tag{2.9.130}$$

$$\hat{\mathbf{n}} = \hat{\mathbf{n}}_0 + \mathbf{N} \tag{2.9.131}$$

Now, remember we have $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1$, full stop. This means we need our approximation to satisfy this to all orders of ξ. So we write

$$\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = (\hat{\mathbf{n}}_0 + \mathbf{N})\cdot(\hat{\mathbf{n}}_0 + \mathbf{N}) + \mathcal{O}(\xi^2) = \hat{\mathbf{n}}_0\cdot\hat{\mathbf{n}}_0 + 2\mathbf{N}\cdot\hat{\mathbf{n}}_0 + \mathcal{O}(\xi^2) = 1 + 2\mathbf{N}\cdot\hat{\mathbf{n}}_0 + \mathcal{O}(\xi^2) \tag{2.9.132}$$

Thus we require $2\hat{\mathbf{n}}_0\cdot\mathbf{N} = 0$ or more simply ignore the factor of 2, and we require $\hat{\mathbf{n}}_0\cdot\mathbf{N} = 0$. So

$$\hat{\mathbf{n}}_0\cdot\mathbf{N} = C(\boldsymbol{\xi}) - \hat{\mathbf{n}}_0\cdot\nabla\boldsymbol{\xi}\cdot\hat{\mathbf{n}}_0 - \hat{\mathbf{n}}_0\cdot\nabla\hat{\mathbf{n}}_0\cdot\boldsymbol{\xi} = 0 \tag{2.9.133}$$

We use

$$\hat{\mathbf{n}}_0\cdot[\nabla\hat{\mathbf{n}}_0\cdot\boldsymbol{\xi}] = \hat{\mathbf{n}}_0\cdot[\boldsymbol{\xi}\cdot\nabla\hat{\mathbf{n}}_0] = \boldsymbol{\xi}\cdot[\nabla\hat{\mathbf{n}}_0\cdot\hat{\mathbf{n}}_0] = \boldsymbol{\xi}\cdot\left[\frac{1}{2}\nabla(\hat{\mathbf{n}}_0\cdot\hat{\mathbf{n}}_0)\right] = 0 \tag{2.9.134}$$

using $\hat{\mathbf{n}}_0\cdot\hat{\mathbf{n}}_0 = 1$ so $\nabla 1 = \mathbf{0}$. Thus, we have

$$0 = C(\boldsymbol{\xi}) - \hat{\mathbf{n}}_0\cdot\nabla\boldsymbol{\xi}\cdot\hat{\mathbf{n}}_0 \tag{2.9.135}$$

$$C(\boldsymbol{\xi}) = \hat{\mathbf{n}}_0\cdot\nabla\boldsymbol{\xi}\cdot\hat{\mathbf{n}}_0 \tag{2.9.136}$$

[63] It should be a minus sign. Consider the plane $y = 0$. If we perturb to $y = \xi_y$ then the new equation is $y - \xi_y = 0$.
[64] A priori, we do not know that it is proportional to ξ, so we are being conservative.

So if we now define $\mathbf{n}_1$ such that

$$\hat{\mathbf{n}} = \hat{\mathbf{n}}_0 + \mathbf{n}_1 + \mathcal{O}(\xi^2) \tag{2.9.137}$$

we have found that

$$\begin{aligned} \mathbf{n}_1 &= C(\boldsymbol{\xi})\hat{\mathbf{n}}_0 - \nabla\boldsymbol{\xi}\cdot\hat{\mathbf{n}}_0 - \nabla\hat{\mathbf{n}}_0\cdot\boldsymbol{\xi} \\ &= [\hat{\mathbf{n}}_0\cdot\nabla\boldsymbol{\xi}\cdot\hat{\mathbf{n}}_0]\hat{\mathbf{n}}_0 - \nabla\boldsymbol{\xi}\cdot\hat{\mathbf{n}}_0 - \nabla\hat{\mathbf{n}}_0\cdot\boldsymbol{\xi} \end{aligned} \tag{2.9.138}$$

or we can write

$$\mathbf{n}_1 = \hat{\mathbf{n}}_0[\hat{\mathbf{n}}_0\hat{\mathbf{n}}_0 : \nabla\boldsymbol{\xi}] - \nabla\boldsymbol{\xi}\cdot\hat{\mathbf{n}}_0 - \boldsymbol{\xi}\cdot\nabla\hat{\mathbf{n}}_0 \tag{2.9.139}$$

which using $\nabla\boldsymbol{\xi} = \mathbb{1}\cdot\nabla\boldsymbol{\xi}$ can be rewritten as

$$\hat{\mathbf{n}}_1 = (\hat{\mathbf{n}}_0\hat{\mathbf{n}}_0 - \mathbb{1})\cdot\nabla\boldsymbol{\xi}\cdot\hat{\mathbf{n}}_0 - \boldsymbol{\xi}\cdot\nabla\hat{\mathbf{n}}_0 \tag{2.9.140}$$

which is the correct formula for the new normal up to $\mathcal{O}(\xi^2)$ terms. When $\nabla\hat{\mathbf{n}}_0 = \overset{\leftrightarrow}{\mathbf{0}}$ or more generally $\boldsymbol{\xi}\cdot\nabla\hat{\mathbf{n}}_0 = \mathbf{0}$ then we recover the previous formula.

Using this formula we find

$$\hat{\mathbf{n}}_1 = (\hat{\mathbf{y}}\hat{\mathbf{y}} - \mathbb{1})\cdot\nabla\boldsymbol{\xi}\cdot\hat{\mathbf{y}} = (-\hat{\mathbf{x}}\hat{\mathbf{x}} - \hat{\mathbf{z}}\hat{\mathbf{z}})\,\nabla\xi_y = -\frac{\partial\xi_y}{\partial x}\hat{\mathbf{x}} - \frac{\partial\xi_y}{\partial z}\hat{\mathbf{z}} \tag{2.9.141}$$

agreeing with our previous result.

Thus we find for our jump condition that

$$\mathbf{n}_1\cdot\mathbf{B}_0 + \hat{\mathbf{n}}_0\cdot\mathbf{B}_1 = -i\mathbf{k}\cdot\mathbf{B}_0\xi_y + i(\mathbf{B}_0\cdot\mathbf{k})\xi_y = 0 \tag{2.9.142}$$

and $\hat{\mathbf{n}}\cdot[\![\mathbf{BB}]\!] = \mathbf{0}$.

Thus we need only look at the "total pressure" being continuous across the surface. But we have been using a Lagrangian specification. So to convert back into our lab Eulerian frame p_E we note that $p_{1E} = p_{1L} + \boldsymbol{\xi}\cdot\nabla p_{0E}$. We have previously calculated the background pressure p_{0E}. We can note that for $\mathbf{B}_{1L} = \mathbf{B}_{1E} + \cancel{\boldsymbol{\xi}\cdot\nabla\mathbf{B}_{0E}}$. Then

$$\left[\!\left[p + \frac{B^2}{2\mu_0}\right]\!\right] = 0 \tag{2.9.143}$$

$$p_+ + \frac{B_+^2}{2\mu_0} = p_- + \frac{B_-^2}{2\mu_0} \tag{2.9.144}$$

$$\cancel{p_{0+}} + p_{1+} + \frac{\cancel{B_{0+}^2} + 2\mathbf{B}_{0+}\cdot\mathbf{B}_{1+}}{2\mu_0} = \cancel{p_{0-}} + p_{1-} + \frac{\cancel{B_{0-}^2} + 2\mathbf{B}_{0-}\cdot\mathbf{B}_{1-}}{2\mu_0} \tag{2.9.145}$$

$$p_{1E+} + \boldsymbol{\xi}\cdot\nabla p_{0E+} + \frac{\mathbf{B}_{0+}\cdot\mathbf{B}_{1+}}{\mu_0} = p_{1E-} + \boldsymbol{\xi}\cdot\nabla p_{0E-} + \frac{\mathbf{B}_{0-}\cdot\mathbf{B}_{1-}}{\mu_0} \tag{2.9.146}$$

Here I have canceled steady-state terms which match since $P_0 = C_0 - \rho_\pm gy$ with $P_0 = C_0$ at $y = 0$, the interface we care about. We can rewrite this as

$$\underbrace{p_{1E+} + \frac{\mathbf{B}_{0+}\cdot\mathbf{B}_{1+}}{\mu_0}}_{p_{1T+}}\underbrace{-\rho_+ g\xi_y}_{\boldsymbol{\xi}\cdot\nabla p_{0E+}} = \underbrace{p_{1E-} + \frac{\mathbf{B}_{0-}\cdot\mathbf{B}_{1-}}{\mu_0}}_{p_{1T-}}\underbrace{-\rho_- g\xi_y}_{\boldsymbol{\xi}\cdot\nabla p_{0E-}} \tag{2.9.147}$$

We note that

$$p_{1T\pm} = \frac{[\mathbf{B}_{0\pm}\cdot\mathbf{k}]^2 - \mu_0\rho_\pm\omega^2}{-k^2\mu_0}(\mp k)\xi_y \tag{2.9.148}$$

$$p_{1T+} = \frac{B^2k_z^2 - \mu_0\rho_+\omega^2}{k\mu_0}\xi_y \tag{2.9.149}$$

$$p_{1T-} = -\frac{B^2[k_z\cos\alpha + k_x\sin\alpha]^2 - \mu_0\rho_-\omega^2}{k\mu_0}\xi_y \tag{2.9.150}$$

Thus

$$\frac{-(\rho_+ + \rho_-)\omega^2}{k}\xi_y - (\rho_+ - \rho_-)g\xi_y + \frac{B^2[k_z^2 + (k_z\cos\alpha + k_x\sin\alpha)^2]}{k\mu_0}\xi_y = 0 \tag{2.9.151}$$

$$\frac{(\rho_+ + \rho_-)\omega^2}{k} + (\rho_+ - \rho_-)g = \frac{B^2[k_z^2 + (k_z\cos\alpha + k_x\sin\alpha)^2]}{k\mu_0} \tag{2.9.152}$$

which we can write as a dispersion relation as

$$\omega^2 = \frac{B^2}{\mu_0(\rho_+ + \rho_-)}[k_z^2 + (k_z\cos\alpha + k_x\sin\alpha)^2] - \frac{\rho_+ - \rho_-}{\rho_+ + \rho_-}kg \tag{2.9.153}$$

It is clear that we have an instability if the kg term is larger than the B^2/μ_0 term.

2.10 Drifts

> *To study, and when the occasion arises to put what one has learned into practice — is that not deeply satisfying?*
>
> — Confucius in the *Analects*

This section goes over various types of drifts for particle or particle species due to forces. Single particle drifts are really only useful when considering guiding center approaches, where motion is divided into the motion of the center of a gyroorbit and the motion of the gyroorbit. One can find fluid analogues if you are considering an additional test particle being injected into a predetermined $\mathbf{E}$ and $\mathbf{B}$ to see how that particle's species acts. Thus, for gyrokinetics and similar other approaches, the particle drifts are required to determine what the $\mathbf{V}$ is for the guiding centers. We have "drifts" that appear as the guiding center perpendicular to the magnetic field. From an MHD point of view, where the gyroradius is ignored, many of these particle drifts are unimportant. Their inclusion in textbooks without giving their major motivation is somewhat curious. Single particle motions do not tell us much about plasma dynamics unless we include the magnetic and electric fields (and all forces) due to all the particles in the configuration already. Thus, given all the forces and assuming that a test particle doesn't alter things, single particle drifts can be useful for seeing what any particle of the test particle's species would be doing. But you must be careful not to infer that if you add a test particle to a configuration, that the test particle trajectory would be the same as if you injected large amounts of test particles into the same configuration. In addition, if you only consider a single test particle, you may be misled if you care about an entire species' behavior (fluid drifts would appear quite strange) because in a fluid you care about the net particle motion rather than the motion of any single particle. So long as one deals with time-independent forces, it is fairly easy to find the drifts that would occur. There are often comments or excuses of the trajectories becoming complicated beyond spatially uniform forces, but we will find that calculating time-independent particle trajectories is fairly simple.

2.10.1 Particle Drifts

We begin with the motion of a particle in a steady state magnetic field. This just means we choose $\mathbf{B}(\mathbf{x},t)=\mathbf{B}_0(\mathbf{x})$ to be independent of time. Often, this is chosen in analysis to be spatially constant even though such a magnetic field is of course unphysical. If we think simply of a large region of space, it is often true that the magnetic field is constant over the region of interest, and so this approximation is sometimes helpful. We will see when we talk of fluid drifts that a single particle drift with uniform fields is the same as a single particle drift, which appears to be the motivation in some texts for teaching single particle drifts.

In any case, we will allow $\mathbf{B}_0$ to vary spatially. We use the second law of Newton with the Lorentz force law to write (for a particle of species s, of constant mass m_s)

$$m_s\frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}t}=\mathbf{F}_{\text{other}}+q_s\mathbf{E}+q_s\mathbf{v}_s\times\mathbf{B}_0 \tag{2.10.1}$$

It is worth dwelling on what this equation actually says. This is for a single test particle trajectory. Thus, when I write the total time derivative, it does not imply there is an advective part. This can be explained physically and mathematically. Physically, if we were to adopt an Eulerian specification, we ask what is the change at a particular point $\mathbf{x}$. But, we are solving for the trajectory of a single particle, so if $\mathbf{x}$ isn't on the trajectory, nothing will change. In other words, when we talk of what $\mathbf{v}_s$ even is, we are talking about the velocity along the trajectory following the particle (a Lagrangian specification).[65] Mathematically, we are saying that $\mathbf{v}_s$ has no spatial dependence, but only depends on time, so $\nabla\mathbf{v}_s$ is simply zero. This says that changing $\mathbf{x}$ doesn't change the trajectory, which is true. If we change where we look in $\mathbf{x}$ it has no effect on what the trajectory does.

Now, let's resume an analysis for single particle drifts. For now, we will say there is neither an electric force, nor any other force on the particle. Thus

$$m_s\frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}t}=q_s\mathbf{v}_s\times\mathbf{B}_0 \tag{2.10.2}$$

We can note that no work is done via $m_s\mathbf{v}_s\cdot\frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}t}=\frac{m_s}{2}\frac{\mathrm{d}v_s^2}{\mathrm{d}t}=\frac{\mathrm{d}K}{\mathrm{d}t}$ where K is the kinetic energy. Thus,

$$\frac{\mathrm{d}K}{\mathrm{d}t}=q_s\mathbf{v}_s\cdot(\mathbf{v}_s\times\mathbf{B}_0)=0 \tag{2.10.3}$$

via the vector identity $\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B})=0$. This is a general instantiation of the rule "magnetic forces perform no work". Let's now solve the equation for $\mathbf{v}_s$. Because $\mathbf{B}_0\equiv B_0\hat{\mathbf{b}}$ is constant in time we can use the trick of projecting into perpendicular and parallel components with respect to the magnetic field at every point in space. Thus for the parallel portion we find

$$m_s\frac{\mathrm{d}(\mathbf{v}_s\cdot\hat{\mathbf{b}})}{\mathrm{d}t}=\hat{\mathbf{b}}\cdot q_s\mathbf{v}_s\times\mathbf{B}_0=0 \tag{2.10.4}$$

$$m_s\frac{\mathrm{d}v_{s\|}}{\mathrm{d}t}=0\Rightarrow v_{s\|}=v_{s\|0} \tag{2.10.5}$$

$$x_{s\|}=v_{s\|0}t+x_{s\|0} \tag{2.10.6}$$

[65] Note that particle physicists often use the phrase "on shell" to indicate that quantities are along physical trajectories, rather than being evaluated in a general phase space.

which simply says the particle continues at its original velocity in the parallel (to the magnetic field) direction for all time.

The perpendicular projection implies

$$m_s \frac{\mathrm{d}}{\mathrm{d}t}\left[-\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times\mathbf{v}_s)\right] = -q_s\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times(\mathbf{v}_s\times\mathbf{B})) \tag{2.10.7}$$

$$m_s\frac{\mathrm{d}\mathbf{v}_{s\perp}}{\mathrm{d}t} = -B_0 q_s\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times(\mathbf{v}_s\times\hat{\mathbf{b}})) = -q_sB_0\hat{\mathbf{b}}\times\mathbf{v}_{s\perp} \tag{2.10.8}$$

We also have

$$m_s\frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{\mathbf{b}}\times\mathbf{v}_s\right] = \hat{\mathbf{b}}\times(q_s\mathbf{v}_s\times\mathbf{B}) \tag{2.10.9}$$

$$m_s\frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{\mathbf{b}}\times\mathbf{v}_s\right] = q_sB_0\mathbf{v}_{s\perp} \tag{2.10.10}$$

This means we can summarize our equations as

$$m_s\frac{\mathrm{d}^2\mathbf{v}_{s\perp}}{\mathrm{d}t^2} = -q_sB_0\frac{\mathrm{d}[\hat{\mathbf{b}}\times\mathbf{v}_{s\perp}]}{\mathrm{d}t} = -q_sB_0\left[\frac{q_sB_0}{m_s}\mathbf{v}_{s\perp}\right] = -\frac{q_s^2B_0^2}{m_s}\mathbf{v}_{s\perp} \tag{2.10.11}$$

$$\frac{\mathrm{d}^2\mathbf{v}_{s\perp}}{\mathrm{d}t^2} = -\frac{q_s^2B_0^2}{m_s^2}\mathbf{v}_{s\perp} \equiv -\Omega^2\mathbf{v}_{s\perp} \tag{2.10.12}$$

$$\frac{\mathrm{d}\mathbf{v}_{s\perp}}{\mathrm{d}t} = -\Omega\hat{\mathbf{b}}\times\mathbf{v}_{s\perp} \tag{2.10.13}$$

with the definition $\Omega^2 = q_s^2B_0^2/m_s^2$ and $\Omega = |q_s|B_0/m_s$ being the gyrofrequency. In some textbooks, and in some situations it is convenient to use the definition $\Omega = q_sB_0/m_s$ so that Ω can be negative for electrons, but I find this definition more often leads to confusion. It is usually better to keep frequencies positive and just write out separate equations for electrons and ions.

We can easily find the solution to (2.10.12) as it is for simple harmonic motion. Then (2.10.13) provides the second initial condition relation for finding $\mathbf{v}_{s\perp}$, in general. It is worth mentioning that it is convenient that $|\mathbf{v}_{s\perp}| = |\mathbf{v}_{s\perp 0}| = v_{s\perp 0}$. That is, the magnitude of the velocity vector never changes. The general solution can be written in many ways such as

$$\mathbf{v}_{s\perp} = \mathbf{v}_+e^{i\Omega t} + \mathbf{v}_-e^{-i\Omega t} = \mathbf{v}_C\cos(\Omega t) + \mathbf{v}_S\sin(\Omega t) = \mathbf{v}_{C0}\cos(\Omega t+\delta) \tag{2.10.14}$$

where the relations between the various coefficients can easily be derived. The initial conditions determine which form is most useful. Thus, if we are given $\mathbf{v}_{s\perp 0}$ we can use that with our generic solution and then use the generic solution with relation (2.10.13) at $t = 0$ to fully specify $\mathbf{v}_{s\perp}$. Thus, given $\mathbf{v}_{s\perp 0}$ at $t = 0$ we can write

$$\mathbf{v}_{s\perp 0} = \mathbf{v}_+ + \mathbf{v}_- = \mathbf{v}_C = \mathbf{v}_{C0}\cos\delta \tag{2.10.15}$$

$$\begin{aligned}
\left.\frac{\mathrm{d}\mathbf{v}_{s\perp}}{\mathrm{d}t}\right|_{t=0} &= -\Omega\hat{\mathbf{b}}\times\mathbf{v}_{s\perp 0}\\
i\mathbf{v}_+ - i\mathbf{v}_- &= -\hat{\mathbf{b}}\times(\mathbf{v}_+ + \mathbf{v}_-)\\
\mathbf{v}_S &= -\hat{\mathbf{b}}\times\mathbf{v}_C\\
-\mathbf{v}_{C0}\sin\delta &= -\hat{\mathbf{b}}\times\mathbf{v}_{C0}\cos\delta
\end{aligned} \tag{2.10.16}$$

For example, if we assume $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ with initial velocity vector $\mathbf{v}_{s\perp 0} = v_{sm}(\hat{\mathbf{x}} + \hat{\mathbf{y}})$ at $t = 0$ then the above equations state

$$v_z = v_{z0} \quad z = v_{z0}t + z_0 \tag{2.10.17}$$
$$\mathbf{v}_C = v_{sm}(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \tag{2.10.18}$$
$$\mathbf{v}_S = -\hat{\mathbf{b}} \times \mathbf{v}_C = -\hat{\mathbf{z}} \times v_{sm}(\hat{\mathbf{x}} + \hat{\mathbf{y}}) = v_{sm}(\hat{\mathbf{x}} - \hat{\mathbf{y}}) \tag{2.10.19}$$

Thus, we find

$$\begin{aligned}\mathbf{v}_{s\perp} &= v_{sm}\left[(\hat{\mathbf{x}} + \hat{\mathbf{y}})\cos(\Omega t) + (\hat{\mathbf{x}} - \hat{\mathbf{y}})\sin(\Omega t)\right] \\ &= v_{sm}\left[\hat{\mathbf{x}}\left(\cos(\Omega t) + \sin(\Omega t)\right) + \hat{\mathbf{y}}\left(\cos(\Omega t) - \sin(\Omega t)\right)\right]\end{aligned} \tag{2.10.20}$$

In an even simpler case where we consider $\mathbf{v}_{s\perp 0} = v_{sm}\hat{\mathbf{x}}$ with $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ then $\mathbf{v}_C = v_{sm}\hat{\mathbf{x}}$ and $\mathbf{v}_s = \hat{\mathbf{b}} \times \mathbf{v}_C = \hat{\mathbf{z}} \times v_{sm}\hat{\mathbf{x}} = -v_{sm}\hat{\mathbf{y}}$ and we write

$$v_x = v_{sm}\cos(\Omega t)\,, \quad x = -\frac{v_{sm}}{\Omega}\sin(\Omega t) + x_0 \tag{2.10.21}$$
$$v_y = -v_{sm}\sin(\Omega t)\,, \quad y = \frac{v_{sm}}{\Omega}\cos(\Omega t) + y_0 \tag{2.10.22}$$
$$v_z = v_{z0}\,, \quad z = v_{z0}t + z_0 \tag{2.10.23}$$

Note that the only important assumption was a time independent magnetic field for this calculation. When $\mathbf{B}$ is time dependent, the examples are far more complicated because the projection operation can no longer pass through the time derivatives leading to much messier expressions.

We can now consider the presence of some other time-independent force (we'll just call it $\mathbf{F}$) in addition to the time independent magnetic field. Then our initial vector equation becomes

$$m_s\frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}t} = \mathbf{F} + q_s\mathbf{v}_s \times \mathbf{B}_0 \tag{2.10.24}$$

We again project parallel and perpendicular to $\mathbf{B}_0$ and write

$$\frac{\mathrm{d}v_{s\parallel}}{\mathrm{d}t} = \frac{F_\parallel}{m_s} + \mathbf{0} \tag{2.10.25}$$
$$\frac{\mathrm{d}\mathbf{v}_{s\perp}}{\mathrm{d}t} = \frac{\mathbf{F}_\perp}{m_s} - \Omega\hat{\mathbf{b}} \times \mathbf{v}_{s\perp} \tag{2.10.26}$$

We see that once again this was a good choice as the parallel equation shows no influence from the magnetic field. Of course $F_\parallel$ may be fairly complicated depending on the magnetic field structure, but it is constant in time. Thus

$$v_{s\parallel} = \frac{F_\parallel}{m_s}t + v_{s\parallel 0} \tag{2.10.27}$$

The perpendicular component is where there is a new twist. However, we can use tricks again to simplify. First, let's write (2.10.26) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{v}_{s\perp} - \frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{\Omega m_s}\right] = \frac{\mathbf{F}_\perp}{m_s} - \Omega\hat{\mathbf{b}} \times \mathbf{v}_{s\perp} \tag{2.10.28}$$

where we use the time independence of $\mathbf{F}$ and $\hat{\mathbf{b}}$. We define $\mathbf{v}_T = \mathbf{v}_{s\perp} - \frac{\mathbf{F}_\perp}{\Omega m_s} \times \hat{\mathbf{b}} = \mathbf{v}_{s\perp} + \frac{\hat{\mathbf{b}}\times\mathbf{F}_\perp}{\Omega m_s}$ which means that this is

$$\frac{\mathrm{d}\mathbf{v}_T}{\mathrm{d}t} = \frac{\mathbf{F}_\perp}{m_s} - \Omega\hat{\mathbf{b}} \times \left[\mathbf{v}_T - \frac{\hat{\mathbf{b}} \times \mathbf{F}_\perp}{\Omega m_s}\right] = \cancel{\frac{\mathbf{F}_\perp}{m_s}} - \Omega\hat{\mathbf{b}} \times \mathbf{v}_T - \cancel{\frac{\mathbf{F}_\perp}{m_s}} \tag{2.10.29}$$

$$\frac{\mathrm{d}\mathbf{v}_T}{\mathrm{d}t} = -\Omega\hat{\mathbf{b}} \times \mathbf{v}_T \tag{2.10.30}$$

It's fairly clear then that (2.10.30) for $\mathbf{v}_T$ is of the same form as (2.10.8) for $\mathbf{v}_{s\perp}$. This means that we have the same solutions for $\mathbf{v}_T$ as we did for $\mathbf{v}_{s\perp}$. Thus

$$\mathbf{v}_T = \mathbf{v}_+ e^{i\Omega t} + \mathbf{v}_- e^{-i\Omega t} = \mathbf{v}_C \cos(\Omega t) + \mathbf{v}_S \sin(\Omega t) = \mathbf{v}_{C0} \cos(\Omega t + \delta) \tag{2.10.31}$$

with a given $\mathbf{v}_{T0}$ at $t = 0$ we can write

$$\mathbf{v}_{T0} = \mathbf{v}_+ + \mathbf{v}_- = \mathbf{v}_C = \mathbf{v}_{C0} \cos\delta \tag{2.10.32}$$

$$\begin{aligned} \left.\frac{\mathrm{d}\mathbf{v}_T}{\mathrm{d}t}\right|_{t=0} &= -\Omega\hat{\mathbf{b}} \times \mathbf{v}_{D0} \\ i\mathbf{v}_+ - i\mathbf{v}_- &= -\hat{\mathbf{b}} \times (\mathbf{v}_+ + \mathbf{v}_-) \\ \mathbf{v}_S &= -\hat{\mathbf{b}} \times \mathbf{v}_C \\ -\mathbf{v}_{C0} \sin\delta &= -\hat{\mathbf{b}} \times \mathbf{v}_{C0} \cos\delta \end{aligned} \tag{2.10.33}$$

which we translate into our current solution

$$\begin{aligned} \mathbf{v}_{s\perp} - \frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{\Omega m_s} = \mathbf{v}_{s\perp} - \frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{q_s B_0} &= \\ = \mathbf{v}_+ e^{i\Omega t} + \mathbf{v}_- e^{-i\Omega t} &\\ = \mathbf{v}_C \cos(\Omega t) + \mathbf{v}_S \sin(\Omega t) &\\ = \mathbf{v}_{C0} \cos(\Omega t + \delta) & \end{aligned} \tag{2.10.34}$$

so

$$\mathbf{v}_{s\perp 0} = \mathbf{v}_+ + \mathbf{v}_- + \frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{q_s B_0} = \mathbf{v}_C + \frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{q_s B_0} = \mathbf{v}_{C0} \cos\delta + \frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{q_s B_0} \tag{2.10.35}$$

$$\begin{aligned} \left.\frac{\mathrm{d}\mathbf{v}_D}{\mathrm{d}t}\right|_{t=0} &= -\Omega\hat{\mathbf{b}} \times \mathbf{v}_{s\perp 0} \\ i\mathbf{v}_+ - i\mathbf{v}_- &= -\hat{\mathbf{b}} \times (\mathbf{v}_+ + \mathbf{v}_- + \frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{q_s B_0}) \\ \mathbf{v}_S &= -\hat{\mathbf{b}} \times (\mathbf{v}_C + \frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{q_s B_0}) \\ -\mathbf{v}_{C0} \sin\delta &= -\hat{\mathbf{b}} \times (\mathbf{v}_{C0} \cos\delta + \frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{q_s B_0}) \end{aligned} \tag{2.10.36}$$

which along with (2.10.27) gives the general solution. Again, note we made no assumptions on the

spatial structure of $\mathbf{F}$ or $\mathbf{B}_0$. We see that we can write

$$\mathbf{v}_{s\perp} = \mathbf{v}_{s\perp}^{\text{no }\mathbf{F}} + \overbrace{\frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{q_s B_0}}^{-\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times\mathbf{v}_D)=\mathbf{v}_{D\perp}} \tag{2.10.37}$$

$$v_{s\parallel} = v_{s\parallel}^{\text{no }\mathbf{F}} + \overbrace{\frac{F_\parallel}{m_s}t}^{v_{D\parallel}} \tag{2.10.38}$$

which is simply the solution with no other forces but translated via the "drift" term $\mathbf{v}_D$. Note also that we can write

$$\mathbf{v}_{D\perp} = \frac{\mathbf{F}_\perp \times \hat{\mathbf{b}}}{q_s B_0} = \frac{\mathbf{F} \times \hat{\mathbf{b}}}{q_s B_0} \tag{2.10.39}$$

via $\mathbf{F}_\parallel \times \hat{\mathbf{b}} = \mathbf{0}$. Many times the parallel drift is not considered because it is small or zero for many of the most common drifts.

Thus, we get the names of various drifts. The most famous is perhaps the $\mathbf{E} \times \mathbf{B}$ drift given by

$$\mathbf{v}_{\mathbf{E}\times\mathbf{B}} = \frac{q_s \mathbf{E} \times \hat{\mathbf{b}}}{q_s B_0} = \frac{\mathbf{E} \times \mathbf{B}_0}{B_0^2} \tag{2.10.40}$$

which is unique in being independent of the charge of the species. Thus it is in the same direction for both ions and electrons.

Another common drift is the gravity one (at least near Earth) where $\mathbf{F} = -m_s\mathbf{g}$ and so

$$\mathbf{v}_{\mathbf{g}} = \frac{-m_s \mathbf{g} \times \hat{\mathbf{b}}}{q_s B_0} \tag{2.10.41}$$

I would like to emphasize again that the time-independent forces are not difficult, but some textbooks make claims like non-constant (spatially) fields are too complicated. I believe what the textbooks are trying to say is that the trajectory is no longer a simple helical trajectory (or circular if there is no parallel velocity component).

If we use a helical trajectory as our base, then we can find what a spatially varying magnetic field does to alter the trajectory approximately. We can, of course, calculate such a trajectory explicitly, but it is often useful to find drift terms that approximates our more complicated trajectory if the magnetic field is not rapidly changing. We will look at this in Section 2.10.3.

Finally, it is worth exploring the results of a time varying force and magnetic field very briefly (and incompletely). We then have (I will drop the s species so the notation is less cluttered)

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F} + q\mathbf{v} \times \mathbf{B} \tag{2.10.42}$$

$$m\frac{\mathrm{d}(v_\parallel \hat{\mathbf{b}})}{\mathrm{d}t} + m\frac{\mathrm{d}\mathbf{v}_\perp}{\mathrm{d}t} = F_\parallel \hat{\mathbf{b}} + \mathbf{F}_\perp + q\mathbf{v}_\perp \times \mathbf{B} \tag{2.10.43}$$

Now we use

$$\frac{\mathrm{d}(v_\parallel \hat{\mathbf{b}})}{\mathrm{d}t} = \frac{\mathrm{d}v_\parallel}{\mathrm{d}t}\hat{\mathbf{b}} + v_\parallel \frac{\partial \hat{\mathbf{b}}}{\partial t} \tag{2.10.44}$$

$$\hat{\mathbf{b}} \cdot \frac{\mathrm{d}\mathbf{v}_\perp}{\mathrm{d}t} = \frac{\mathrm{d}\cancel{(\hat{\mathbf{b}} \cdot \mathbf{v}_\perp)}}{\mathrm{d}t} - \frac{\partial \hat{\mathbf{b}}}{\partial t} \cdot \mathbf{v}_\perp \tag{2.10.45}$$

where I have used that along the particle trajectory we have $\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}t} = \frac{\partial \hat{\mathbf{b}}}{\partial t}$. We note

$$\hat{\mathbf{b}} \cdot \frac{\partial \hat{\mathbf{b}}}{\partial t} = \frac{1}{2}\frac{\partial \hat{\mathbf{b}} \cdot \hat{\mathbf{b}}}{\partial t} = \frac{1}{2}\frac{\partial 1}{\partial t} = 0 \tag{2.10.46}$$

Thus $v_\parallel \frac{\partial \hat{\mathbf{b}}}{\partial t}$ is purely in the perpendicular direction. We subtract off the parallel components from the full equation to look at purely perpendicular portions and find

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{F_\parallel}{m} - \frac{\partial \hat{\mathbf{b}}}{\partial t} \cdot \mathbf{v}_\perp \tag{2.10.47}$$

$$\frac{\mathrm{d}\mathbf{v}_\perp}{\mathrm{d}t} + \hat{\mathbf{b}}\frac{\partial \hat{\mathbf{b}}}{\partial t} \cdot \mathbf{v}_\perp = \frac{\mathbf{F}_\perp}{m} + \Omega \mathbf{v}_\perp \times \hat{\mathbf{b}} - v_\parallel \frac{\partial \hat{\mathbf{b}}}{\partial t} \tag{2.10.48}$$

Further solution is difficult, and the above equations illustrate the difficulty of time-dependent forces and magnetic fields.

2.10.2 Fluid Drifts

We can now get to the more applicable drifts in most plasma applications (MHD). Now we consider a flow velocity for a species of particles s and write the equation of motion with negligible friction force between species and ignorable viscous stress.

$$n_s m_s \frac{\mathrm{d}\mathbf{V}_s}{\mathrm{d}t} = q_s n_s \mathbf{E} + q_s n_s \mathbf{V}_s \times \mathbf{B} - \nabla p_s \tag{2.10.49}$$

Now this is a nonlinear equation because $\mathbf{V}_s$ is a flow velocity and so this is equivalent to

$$n_s m_s \left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s \right) = q_s \mathbf{E} + q_s \mathbf{V}_s \times \mathbf{B} - \nabla p_s \tag{2.10.50}$$

If $\mathbf{V}_s \cdot \nabla \mathbf{V}_s = \mathbf{0}$ then our equations reduce to the single particle motion. This means that $\mathbf{V}_s$ must not change in the direction along $\mathbf{V}_s$ for all space. The same amount must leave a region of space as enters, and along the same route. If we think of our solution before

$$V_{s\parallel 0} = \frac{q_s E_\parallel}{m_s} - \frac{\nabla p_s}{n_s m_s} \tag{2.10.51}$$

$$\mathbf{V}_{s\perp 0} = \mathbf{V}_C \cos(\Omega t) + \hat{\mathbf{b}} \times \mathbf{V}_C \sin(\Omega t) + \frac{\mathbf{E} \times \hat{\mathbf{b}}}{B} + \frac{-\nabla p_s \times \hat{\mathbf{b}}}{q_s n_s m_s B} \tag{2.10.52}$$

The simplest way for this to be true is if all the terms are no longer spatially dependent. That is $\mathbf{V}_C$, $\mathbf{E}$, $\mathbf{B}$, ∇p_s and n_s are all constants in space. Thus, for spatially uniform cases the drifts are again given by

$$\mathbf{V}_{DF} = \frac{\mathbf{F} \times \hat{\mathbf{b}}}{q_s B} \tag{2.10.53}$$

but remember $\mathbf{F}$ must be spatially uniform.

It is sometimes commented upon that the diamagnetic drift (the $\nabla p_s \times \hat{\mathbf{b}}$ term) is a purely fluid phenomenon. This is true, but if we are talking about the fluid velocity, then essentially all drifts are fluid. It's true there is no analog of the diamagnetic drift for single particles, however. This is not all that surprising since plasmas require collective behavior, so that you cannot just analyze the particles one at a time in general. Note if we consider the motion of a single particle making up the fluid, our trajectories become simple to calculate again, but we no longer know if a single particle trajectory will translate into a fluid drift, or if we will miss some important fluid drifts by only considering particles one at a time.

We can also use that for a fluid equation of species s in general and find that the perpendicular fluid velocity must be small in MHD. We have

$$n_s m_s \left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s \right) = q_s n_s \mathbf{E} + q_s n_s \mathbf{V}_s \times \mathbf{B} - \nabla p_s \tag{2.10.54}$$

If we take the parallel component we find

$$m_s n_s \left(\hat{\mathbf{b}} \cdot \frac{\partial \mathbf{V}_s}{\partial t} + \hat{\mathbf{b}} \mathbf{V}_s : \nabla \mathbf{V}_s \right) = q_s n_s E_\parallel - \nabla_\parallel p_s \tag{2.10.55}$$

and taking $\hat{\mathbf{b}}\times$ the same equation we can find $\mathbf{V}_{s\perp}$ as

$$m_s n_s \hat{\mathbf{b}} \times \left(\frac{\partial \mathbf{V}_s}{\partial t} \mathbf{V}_s \cdot \nabla \mathbf{V}_s \right) = q_s n_s \hat{\mathbf{b}} \times \mathbf{E} + q_s n_s B \mathbf{V}_{s\perp} - \hat{\mathbf{b}} \times \nabla p_s \tag{2.10.56}$$

$$\mathbf{V}_{s\perp} = \frac{\hat{\mathbf{b}}}{B} \times \left(\frac{m_s}{q_s} \frac{\partial \mathbf{V}_s}{\partial t} + \frac{m_s}{q_s} \mathbf{V}_s \cdot \nabla \mathbf{V}_s - \mathbf{E} + \frac{\nabla p_s}{q_s n_s} \right) \tag{2.10.57}$$

$$\mathbf{V}_{s\perp} = \frac{\hat{\mathbf{b}}}{\Omega} \times \left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s - \frac{q_s}{m_s} \mathbf{E} + \frac{\nabla p_s}{m_s n_s} \right) \tag{2.10.58}$$

$$\tag{2.10.59}$$

We use $\epsilon = \frac{\omega}{\Omega}$ where ω is the angular frequency of calculations we care about (for MHD, $\omega \ll \Omega$) as an ordering parameter with an expansion $\mathbf{q} = \sum_n \epsilon^n \mathbf{q}_n$. First we rewrite the above as

$$\mathbf{V}_{s\perp} = \epsilon \frac{\hat{\mathbf{b}}}{\omega} \times \left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s - \frac{q_s}{m_s} \mathbf{E} + \frac{\nabla p_s}{m_s n_s} \right) \tag{2.10.60}$$

This means that $\mathbf{V}_{s\perp 0} = \mathbf{0}$ in MHD unless some of the terms are proportional to ϵ^{-1}. We can find $\mathbf{V}_{s\perp 1}$ via taking the zeroth order for all of those terms inside the parentheses. One other thing to consider is what order $\mathbf{V}_s$ is in terms of ϵ. Naively, we might expect $\mathbf{V}_{s0} \neq \mathbf{0}$, but we must first consider what that actually means. The pressure is related to the thermal speeds of the particles making up the fluid, and so if $\mathbf{V}_{s0}$ is to be of the same order, it must be approximately the thermal speed as well. However, in conventional MHD we need the flow speed to be smaller than the thermal particle speed, and usually have $\mathbf{V}_s = \mathcal{O}(\epsilon v_{\text{th}})$ for v_{th} the thermal speed. Thus we assume that the background flow velocity is essentially zero in comparison to the thermal speed, $\mathbf{V}_{s0} = \mathbf{0}$. This means that

$$\begin{aligned} \mathbf{V}_{s\perp 1} &= \frac{\hat{\mathbf{b}}_0}{\Omega} \times \left(-\frac{q_s}{m_s} \mathbf{E}_0 + \frac{\nabla p_{s0}}{m_s n_{s0}} \right) \\ &= \frac{\mathbf{E}_0 \times \hat{\mathbf{b}}_0}{B_0} + \frac{\nabla p_{s0} \times \hat{\mathbf{b}}_0}{q_s n_{s0} B_0} \end{aligned} \tag{2.10.61}$$

We recover our two fluid drifts and see that they must be small in the sense of our ordering parameter scheme.

2.10.3 Drifts for Guiding Centers

You will undoubtedly hear of curvature and grad-B drifts. These drifts really only make sense if you average out the gyromotion of particles, but also desire that the Larmor radius not be zero. For most fluid approximations, the Larmor radius must be very small, or else you get poor results, and so it does not always make sense to look at these drifts for a fluid approximation.[66]

The gyrokinetic equation is really where these drifts make the most sense. They are typically derived for students as single particle drifts, but this is a poor explanation for why they are important. Especially because the general statement is only for forces that are uniform, and so the derivation only tells you the form of the drifts in a particular situation that isn't guaranteed to tell you what you want.[67] I will now explain how to get the other single particle drifts, but as a means to get a good approximation for the gyrokinetic equation. It relies on the variation of $\mathbf{B}$ not being large, so that we can use approximations that will yield the grad-B and curvature drifts. I will not derive the gyrokinetic equation, instead it is worth just explaining where the drifts come from. We first use a guiding center variable $\mathbf{X}$ defined by

$$\mathbf{X}(\mathbf{x},\mathbf{v}) = \mathbf{x} - \boldsymbol{\rho}(\mathbf{x},\mathbf{v}) \tag{2.10.62}$$

$$\boldsymbol{\rho} = \frac{\hat{\mathbf{b}}(\mathbf{x})\times\mathbf{v}}{\Omega(\mathbf{x})} \tag{2.10.63}$$

This looks much simpler than it is. We need to find $\boldsymbol{\rho}(\mathbf{X},\mathbf{v})$ which is in general not easy since we defined ρ as a function of $\mathbf{x}$ rather than $\mathbf{X}$. Fix in your mind very clearly the difference between $\mathbf{X}$, the gyrocenter position, and $\mathbf{x}$ a generic position coordinate. For us, with only first order corrections in $\rho/L \ll 1$ where L is a characteristic lenght of our system much longer than the Larmor radius ρ, we can use that $\mathbf{x} = \mathbf{X} + \mathcal{O}(\rho/L)$. Thus $\boldsymbol{\rho} = \hat{\mathbf{b}}\times\mathbf{v}_\perp/\Omega$ evaluated at $\mathbf{X}$. Then we just need to calculate

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\frac{\partial\mathbf{X}}{\partial\mathbf{x}} + \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\frac{\partial\mathbf{X}}{\partial\mathbf{v}} + \frac{\partial\mathbf{X}}{\partial t} \tag{2.10.64}$$

We then use $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}$ and use the total $\mathbf{E}$ and $\mathbf{B}$ (so I write $\mathbf{E}_T$ to emphasize it is not linearized). We find (note how carefully we must put the order of things so that all operators act immediately to their right)[68]

$$\frac{\partial\mathbf{X}}{\partial\mathbf{v}} = \cancel{\frac{\partial\mathbf{x}}{\partial\mathbf{v}}} - \frac{\partial}{\partial\mathbf{v}}\left[\frac{\hat{\mathbf{b}}}{\Omega}\times\mathbf{v}_\perp\right] = \frac{\partial}{\partial\mathbf{v}}\left[\mathbf{v}_\perp\times\frac{\hat{\mathbf{b}}}{\Omega}\right] = \frac{\partial\mathbf{v}}{\partial\mathbf{v}}\times\frac{\hat{\mathbf{b}}}{\Omega} = \mathbb{1}\times\frac{\hat{\mathbf{b}}}{\Omega} \tag{2.10.65}$$

[66] There are gyrofluids, so it is not impossible to retain these effects in a fluid model. For these one would need to think of curvature and grad-B drifts.

[67] Especially because in a fluid perspective you do not want to know the single particle trajectories, but the net movement of particles as a fluid.

[68] Also note I used $\hat{\mathbf{b}}\times\mathbf{v}_\perp = \hat{\mathbf{b}}\times\mathbf{v}$ to make the $\frac{\partial}{\partial\mathbf{v}}$ derivatives easier to evaluate. I also kept the $\frac{\partial}{\partial\mathbf{v}}$ components to the farthest left so $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\mathbf{v}}$ dots into the correct components.

$$\begin{aligned}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} &= \cancel{\frac{\partial \mathbf{x}}{\partial \mathbf{x}}}^{\mathbb{1}} - \frac{1}{\Omega}\frac{\partial \hat{\mathbf{b}}}{\partial \mathbf{x}} \times \mathbf{v}_\perp + \frac{\partial \Omega}{\partial \mathbf{x}}\frac{\hat{\mathbf{b}} \times \mathbf{v}_\perp}{\Omega^2} \\ &= \mathbb{1} - \frac{1}{\Omega^2}\left(\Omega \frac{\partial \hat{\mathbf{b}}}{\partial \mathbf{x}} - \frac{\partial \Omega}{\partial \mathbf{x}}\hat{\mathbf{b}}\right) \times \mathbf{v}_\perp = \mathbb{1} - \frac{1}{\Omega^2}\left(\Omega \frac{\partial \hat{\mathbf{b}}}{\partial \mathbf{x}} - \frac{\partial \Omega}{\partial \mathbf{x}}\hat{\mathbf{b}}\right) \times \mathbf{v}\end{aligned} \tag{2.10.66}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q}{m}\mathbf{E}_T - \Omega^2 \boldsymbol{\rho} \tag{2.10.67}$$

Where (2.10.67) is simply the Lorentz equation in our new variables. Then

$$\begin{aligned}\mathbf{v} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{x}} &= \mathbf{v} - \frac{\mathbf{v}}{\Omega^2} \cdot \left(\Omega \frac{\partial \hat{\mathbf{b}}}{\partial \mathbf{x}} - \frac{\partial \Omega}{\partial \mathbf{x}}\hat{\mathbf{b}}\right) \times \mathbf{v} = \mathbf{v} - \frac{1}{\Omega}(\mathbf{v} \cdot \nabla \hat{\mathbf{b}}) \times \mathbf{v} + \frac{1}{\Omega^2}(\mathbf{v} \cdot \nabla \Omega)(\hat{\mathbf{b}} \times \mathbf{v}) \\ &= \mathbf{v} - \frac{1}{\Omega}(\mathbf{v} \cdot \nabla \hat{\mathbf{b}}) \times \mathbf{v} + \frac{m^2}{q^2 B^2}\frac{q}{m}(\mathbf{v} \cdot \nabla B)(\hat{\mathbf{b}} \times \mathbf{v}) \\ &= \mathbf{v} - \frac{1}{\Omega}\left[(\mathbf{v} \cdot \nabla \hat{\mathbf{b}}) \times \mathbf{v} - \frac{1}{B}(\mathbf{v} \cdot \nabla B)(\hat{\mathbf{b}} \times \mathbf{v})\right] \\ &= \mathbf{v} - \frac{1}{\Omega}(\mathbf{v} \cdot \nabla \hat{\mathbf{b}}) \times \mathbf{v} + \frac{\boldsymbol{\rho}}{B}(\mathbf{v} \cdot \nabla B)\end{aligned} \tag{2.10.68}$$

We then use $\mathbf{A} \cdot (\mathbb{1} \times \mathbf{B}) = \mathbf{A} \times \mathbf{B}$ to find

$$\begin{aligned}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{v}} &= \left(\frac{q}{m}\mathbf{E}_T - \Omega^2 \boldsymbol{\rho}\right) \cdot \left(\mathbb{1} \times \frac{\hat{\mathbf{b}}}{\Omega}\right) \\ &= \frac{q}{m}\mathbf{E}_T \times \frac{\hat{\mathbf{b}}}{\Omega} - \Omega \boldsymbol{\rho} \times \hat{\mathbf{b}} \\ &= \frac{1}{B}\mathbf{E}_T \times \hat{\mathbf{b}} - \mathbf{v}_\perp \\ &= \mathbf{v}_E - \mathbf{v}_\perp\end{aligned} \tag{2.10.69}$$

Now we can consider (remember $\frac{\partial}{\partial t}$ holds $\mathbf{x}$ and $\mathbf{v}$ constant)

$$\frac{\partial \mathbf{X}}{\partial t} = \cancel{\frac{\partial \mathbf{x}}{\partial t}} - \frac{\partial \boldsymbol{\rho}}{\partial t} = -\frac{1}{\Omega}\frac{\partial \hat{\mathbf{b}}}{\partial t} \times \mathbf{v} - \hat{\mathbf{b}} \times \mathbf{v}\frac{\partial}{\partial t}\frac{1}{\Omega} - \cancel{\frac{\hat{\mathbf{b}}}{\Omega} \times \frac{\partial \mathbf{v}}{\partial t}} \tag{2.10.70}$$

Thus, the only new term is $\frac{1}{\Omega}\frac{\partial \hat{\mathbf{b}}}{\partial t} \times \mathbf{v}_\perp$.

We care only about the zeroth order (in ρ/L) quantities, which are given by

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{v} - \mathbf{v}_\perp + \mathbf{v}_E - \frac{1}{\Omega}(\mathbf{v} \cdot \nabla \hat{\mathbf{b}}) \times \mathbf{v} + \frac{\boldsymbol{\rho}}{B}\mathbf{v} \cdot \nabla B - \frac{1}{\Omega}\frac{\partial \hat{\mathbf{b}}}{\partial t} \times \mathbf{v} - \hat{\mathbf{b}} \times \mathbf{v}\frac{\partial}{\partial t}\frac{1}{\Omega} \tag{2.10.71}$$

Now, we use an average of the gyrophase $\gamma(t)$ which to lowest order is just Ωt, with $\mathbf{v}_\perp$ given by

$$\mathbf{v}_\perp = v_\perp[\hat{\mathbf{e}}_2 \sin(\gamma) + \hat{\mathbf{e}}_3 \cos(\gamma)] \tag{2.10.72}$$

with $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ forming the local right-handed coordinate system $(\hat{\mathbf{b}}, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$. This can be constructed by choosing $\hat{\mathbf{e}}_3$ as the direction of the perpendicular velocity at $\gamma = 0$ and $\hat{\mathbf{e}}_2$ at $\gamma = \pi/2$.

In general, it is always possible to find a local right-handed coordinate system based on the magnetic field (other than where the field goes to zero) by using the magnetic field lines and Frenet-Serret formulas [see Section 1.11].

We define a gyrophase average as

$$\langle \mathbf{A} \rangle_\gamma = \oint \frac{\mathrm{d}\gamma}{2\pi} \mathbf{A}(\mathbf{x}, \ldots, \gamma) \tag{2.10.73}$$

where all dependencies but γ are fixed in $\mathbf{A}$. That is we go over a single period in γ. This is equivalent to assuming a period $T = 2\pi$ via

$$\langle \mathbf{A} \rangle_\gamma = \frac{\int_0^T \mathrm{d}\gamma\ \mathbf{A}(\gamma)}{\int_0^T \mathrm{d}\gamma} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\gamma\ \mathbf{A}(\gamma) \tag{2.10.74}$$

We then take

$$\left\langle \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} \right\rangle_\gamma = \left\langle \mathbf{v} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \right\rangle_\gamma + \left\langle \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{v}} \right\rangle_\gamma \tag{2.10.75}$$

$$\begin{aligned} &= \langle \mathbf{v} - \mathbf{v}_\perp \rangle_\gamma + \langle \mathbf{v}_E \rangle_\gamma - \frac{1}{\Omega} \left\langle \mathbf{v} \cdot \nabla \hat{\mathbf{b}} \times \mathbf{v} \right\rangle_\gamma + \frac{1}{B} \left\langle (\mathbf{v} \cdot \nabla \hat{\mathbf{b}}) \boldsymbol{\rho} \right\rangle_\gamma \\ &\quad - \frac{1}{\Omega} \left\langle \frac{\partial \hat{\mathbf{b}}}{\partial t} \times \mathbf{v} \right\rangle_\gamma - \hat{\mathbf{b}} \times \cancel{\langle \mathbf{v} \rangle_\gamma} \frac{\partial}{\partial t} \frac{1}{\Omega} \end{aligned} \tag{2.10.76}$$

Note that

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp = v_\parallel \hat{\mathbf{b}} + v_\perp [\hat{\mathbf{e}}_2 \sin\gamma + \hat{\mathbf{e}}_3 \cos\gamma] \tag{2.10.77}$$

so then we need to calculate gyrophase averages.

It is convenient to use the cross-dot product

$$\mathbf{AB} \overset{\times}{\cdot} \mathbf{CD} = (\mathbf{A} \times [\mathbf{B} \cdot \mathbf{C}] \mathbf{D}) \tag{2.10.78}$$

$$\mathbf{AB} \overset{\times}{\cdot} \nabla \mathbf{C} = \mathbf{A} \times (\mathbf{B} \cdot \nabla \mathbf{C}) = -\mathbf{B} \cdot \nabla \mathbf{C} \times \mathbf{A} \tag{2.10.79}$$

and we then see

$$\langle \mathbf{v} - \mathbf{v}_\perp \rangle_\gamma = \mathbf{v}_\parallel = v_\parallel \hat{\mathbf{b}} \tag{2.10.80}$$

We can find

$$\langle \mathbf{v}_\perp \mathbf{v}_\perp \rangle_\gamma = v_\perp^2 \left\langle \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 \sin^2\gamma + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_3 \sin\gamma\cos\gamma + \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_2 \cos\gamma \sin\gamma + \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 \cos^2\gamma \right\rangle_\gamma \tag{2.10.81}$$

$$= \frac{v_\perp^2}{2} (\hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3) = \frac{\mathbf{v}_\perp^2}{2} (\mathbb{1} - \hat{\mathbf{b}} \hat{\mathbf{b}}) \tag{2.10.82}$$

where the last is simply using that $\mathbb{1} = \hat{\mathbf{b}}\hat{\mathbf{b}} + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3$ for our right-handed system. Thus, we find

$$\begin{aligned} \left\langle -\mathbf{v} \cdot \nabla \hat{\mathbf{b}} \times \mathbf{v} \right\rangle_\gamma &= \langle \mathbf{vv} \rangle_\gamma \overset{\times}{\cdot} \nabla \hat{\mathbf{b}} = v_\parallel^2 \hat{\mathbf{b}} \hat{\mathbf{b}} \overset{\times}{\cdot} \nabla \hat{\mathbf{b}} + \frac{v_\perp^2}{2} (\mathbb{1} - \hat{\mathbf{b}} \hat{\mathbf{b}}) \overset{\times}{\cdot} \nabla \hat{\mathbf{b}} \\ &= v_\parallel^2 \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) + \frac{v_\perp^2}{2} \mathbb{1} \overset{\times}{\cdot} \nabla \hat{\mathbf{b}} - \frac{v_\perp^2}{2} \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \end{aligned} \tag{2.10.83}$$

We find

$$\mathbb{1} \overset{\times}{\cdot} \nabla\hat{\mathbf{b}} = \epsilon_{ijl}\delta_{jk}\partial_k b_l = \epsilon_{ikl}\partial_k b_l = \boldsymbol{\nabla} \times \hat{\mathbf{b}} \tag{2.10.84}$$

In addition we use

$$0 = \hat{\mathbf{b}} \cdot \nabla(1)/2 = \hat{\mathbf{b}} \cdot \nabla(\hat{\mathbf{b}} \cdot \hat{\mathbf{b}})/2 = \hat{\mathbf{b}} \cdot (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}) \tag{2.10.85}$$

and so we can write

$$\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} = -\hat{\mathbf{b}} \times (\boldsymbol{\nabla} \times \hat{\mathbf{b}}) \tag{2.10.86}$$

via the identity

$$\nabla(\mathbf{A} \cdot \mathbf{A})/2 = \mathbf{A} \cdot \nabla\mathbf{A} + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A}) \tag{2.10.87}$$

Thus we can write

$$\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}) = \hat{\mathbf{b}} \times (-\hat{\mathbf{b}} \times (\boldsymbol{\nabla} \times \hat{\mathbf{b}})) = -\hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{b}}) + \boldsymbol{\nabla} \times \hat{\mathbf{b}} \tag{2.10.88}$$

We can use $\boldsymbol{\kappa} = \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}$ as a definition of the magnetic curvature vector and so

$$\left\langle -\mathbf{v} \cdot \nabla\hat{\mathbf{b}} \times \mathbf{v} \right\rangle_\gamma = v_\parallel^2 \hat{\mathbf{b}} \times \boldsymbol{\kappa} + \frac{v_\perp^2}{2}\hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{b}}) \tag{2.10.89}$$

We then find (using $\Omega\boldsymbol{\rho} = \hat{\mathbf{b}} \times \mathbf{v}$)

$$\langle \Omega\boldsymbol{\rho}\mathbf{v} \rangle_\gamma = \hat{\mathbf{b}} \times \langle \mathbf{v}\mathbf{v} \rangle_\gamma = \hat{\mathbf{b}} \times \left[\cancel{v_\parallel^2 \hat{\mathbf{b}}\hat{\mathbf{b}}} + \frac{v_\perp^2}{2}(\mathbb{1} - \cancel{\hat{\mathbf{b}}\hat{\mathbf{b}}}) \right] = \frac{v_\perp^2}{2}\hat{\mathbf{b}} \times \mathbb{1} \tag{2.10.90}$$

So that

$$\frac{1}{B}\langle \boldsymbol{\rho}\mathbf{v} \rangle_\gamma \cdot \nabla B = \frac{v_\perp^2}{\Omega B}\hat{\mathbf{b}} \times \left(\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right) \cdot \nabla B = \frac{v_\perp^2}{\Omega B}\hat{\mathbf{b}} \times \mathbb{1} \cdot \nabla B - \cancel{v_\perp^2 \hat{\mathbf{b}} \times \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla B} \tag{2.10.91}$$

$$= \frac{v_\perp^2}{\Omega B}\hat{\mathbf{b}} \times \nabla B = \frac{\mu}{m\Omega}\hat{\mathbf{b}} \times \nabla B \tag{2.10.92}$$

with $\mu = mv_\perp^2/(2B)$ defined as the magnetic moment. Thus

$$\left\langle \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} \right\rangle_\gamma = \mathbf{v}_\parallel + \mathbf{v}_E + \frac{v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \boldsymbol{\kappa} + \frac{v_\perp^2}{2\Omega}\hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{b}}) + \frac{\mu}{m\Omega}\hat{\mathbf{b}} \times \nabla B - \frac{v_\parallel}{\Omega}\frac{\partial\hat{\mathbf{b}}}{\partial t} \times \hat{\mathbf{b}} \tag{2.10.93}$$

Thus, the guiding center drifts are given by

$$\mathbf{v}_D = \mathbf{v}_E + \mathbf{b} \times \left(\frac{v_\parallel^2}{\Omega}\boldsymbol{\kappa} + \frac{\mu}{m\Omega}\nabla B + \frac{v_\parallel}{\Omega}\frac{\partial\hat{\mathbf{b}}}{\partial t} \right) \tag{2.10.94}$$

and we note that we have a "drift" along the parallel direction given by

$$v_{\parallel D} = \frac{v_\perp^2}{2\Omega}\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{b}} = \frac{\mu B}{\Omega}\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{b}} \tag{2.10.95}$$

However, it is important to note that all of the above quantities are evaluated for $\mathbf{x}$ and not for the guiding center position $\mathbf{X}$ unless we use the approximation $\rho(\mathbf{x}) = \rho(\mathbf{X}) + \mathcal{O}(\rho/L)$ and ignore $\mathcal{O}(\rho/L)$ contributions.

If we desired, we could solve for $\boldsymbol{\rho}$ via

$$\boldsymbol{\rho}(\mathbf{x}, \mathbf{v}) = \frac{\hat{\mathbf{b}}}{\Omega} \times \mathbf{v} \tag{2.10.96}$$

and define $\mathbf{b}_\Omega = \hat{\mathbf{b}}(\mathbf{x})/\Omega(\mathbf{x})$. Then we write (ignoring the $\mathbf{v}$ dependence when writing $\boldsymbol{\rho}$)

$$\begin{aligned}
\boldsymbol{\rho}(\mathbf{x}) &= \boldsymbol{\rho}(\mathbf{X} + \boldsymbol{\rho}) = \boldsymbol{\rho}(\mathbf{X}) + \boldsymbol{\rho}(\mathbf{x}) \cdot \nabla\boldsymbol{\rho}|_{\mathbf{x}=\mathbf{X}} + \mathcal{O}(\rho^3) && (2.10.97)\\
&= \boldsymbol{\rho}(\mathbf{X}) + (\boldsymbol{\rho}(\mathbf{X}) + \boldsymbol{\rho}(\mathbf{x}) \cdot \nabla\boldsymbol{\rho}|_{\mathbf{x}=\mathbf{X}}) \cdot \nabla\boldsymbol{\rho}|_{\mathbf{x}=\mathbf{X}} + \mathcal{O}(\rho^3) && (2.10.98)\\
&= \boldsymbol{\rho}(\mathbf{X}) + \boldsymbol{\rho}(\mathbf{X}) \cdot \nabla\boldsymbol{\rho}|_{\mathbf{x}=\mathbf{X}} + \mathcal{O}(\rho^3) && (2.10.99)
\end{aligned}$$

where we can see the iterative way of solving must be done somewhat carefully beyond first order as we must retain all the Taylor order series to the correct order. However we are still left with the unfortuante result of needing to calculate $\nabla\boldsymbol{\rho}(\mathbf{x})$ with respect to $\mathbf{x}$ rather than $\mathbf{X}$. That is

$$\nabla\boldsymbol{\rho} = \frac{\partial\boldsymbol{\rho}}{\partial\mathbf{x}} \tag{2.10.100}$$

whereas it would be nice if we could find

$$\left.\frac{\partial\boldsymbol{\rho}}{\partial\mathbf{x}}\right|_{\mathbf{x}=\mathbf{X}} \tag{2.10.101}$$

directly in the form of $\frac{\partial\boldsymbol{\rho}(\mathbf{X})}{\partial\mathbf{X}}$. We see that we need $\frac{\partial\boldsymbol{\rho}}{\partial\mathbf{x}}$ only to first order in ρ to get our required accuracy. So

$$\frac{\partial\boldsymbol{\rho}}{\partial\mathbf{x}} = \frac{\partial\mathbf{X}}{\partial\mathbf{x}} \cdot \frac{\partial\boldsymbol{\rho}(\mathbf{X} + \boldsymbol{\rho})}{\partial\mathbf{X}} \tag{2.10.102}$$

Now we need this term only to first order in ρ for our procedure so we can use

$$\begin{aligned}
\frac{\partial\boldsymbol{\rho}(\mathbf{X} + \boldsymbol{\rho})}{\partial\mathbf{X}} &= \frac{\partial\boldsymbol{\rho}(\mathbf{X})}{\partial\mathbf{X}} + \frac{\partial}{\partial\mathbf{X}}\left[\boldsymbol{\rho}(\mathbf{x}) \cdot \left.\frac{\partial\boldsymbol{\rho}(\mathbf{x})}{\partial\mathbf{x}}\right|_{\mathbf{x}=\mathbf{X}}\right] + \mathcal{O}(\rho^2) \\
&= \frac{\partial\boldsymbol{\rho}(\mathbf{X})}{\partial\mathbf{X}} + \mathcal{O}(\rho^2)
\end{aligned} \tag{2.10.103}$$

$$\begin{aligned}
\frac{\partial\mathbf{X}}{\partial\mathbf{x}} &= \left(\mathbb{1} - \frac{\partial\boldsymbol{\rho}}{\partial\mathbf{x}}\right) = \left(\mathbb{1} - \left[\mathbb{1} - \frac{\partial\boldsymbol{\rho}}{\partial\mathbf{x}}\right] \cdot \frac{\partial\boldsymbol{\rho}(\mathbf{X})}{\partial\mathbf{X}}\right) + \mathcal{O}(\rho^2) \\
&= \mathbb{1} - \frac{\partial\boldsymbol{\rho}(\mathbf{X})}{\partial\mathbf{X}} + \mathcal{O}(\rho^2)
\end{aligned} \tag{2.10.104}$$

and so

$$\begin{aligned}
\frac{\partial\boldsymbol{\rho}}{\partial\mathbf{x}} &= \left(\mathbb{1} - \frac{\partial\boldsymbol{\rho}(\mathbf{X})}{\partial\mathbf{X}}\right) \cdot \frac{\partial\boldsymbol{\rho}(\mathbf{X})}{\partial\mathbf{X}} + \mathcal{O}(\rho^2) \\
&= \frac{\partial\boldsymbol{\rho}(\mathbf{X})}{\partial\mathbf{X}} + \mathcal{O}(\rho^2)
\end{aligned} \tag{2.10.105}$$

which means that

$$\frac{\partial \boldsymbol{\rho}}{\partial \mathbf{x}} = \frac{\partial \boldsymbol{\rho}(\mathbf{X})}{\partial \mathbf{X}} + \mathcal{O}(\rho^2) \tag{2.10.106}$$

and we therefore can write

$$\boldsymbol{\rho}(\mathbf{x}) = \boldsymbol{\rho}(\mathbf{X}) + \boldsymbol{\rho}(\mathbf{X}) \cdot \frac{\partial \boldsymbol{\rho}(\mathbf{X})}{\partial \mathbf{X}} + \mathcal{O}(\rho^3) \tag{2.10.107}$$

$$\boldsymbol{\rho}(\mathbf{x}) = \mathbf{b}_\Omega(\mathbf{X}) \times \mathbf{v} + (\mathbf{b}_\Omega(\mathbf{X}) \times \mathbf{v}) \cdot \frac{\partial \mathbf{b}_\Omega(\mathbf{X})}{\partial \mathbf{X}} \times \mathbf{v} + \mathcal{O}(\rho^3) \tag{2.10.108}$$

which is what one might naively expect anyway.

2.11 Further Reading

There is a large number of good plasma textbooks. Chen[6] is a popular introductory text. I have personally found D'haeseleer[9] great for flux coordinates, Hazeltine[13] as a more advanced introduction to plasma physics in general. It is good for plasma kinetics, deriving fluid equations, and even turbulence modeling. For a more comprehensive treatment of transport phenomena (so good plasma kinetics) Helander[14] was excellent. Braginskii's original paper[4] is a very readable text on asymptotically closing the MHD equations. And for plasma kinetics in general I recommend a text I helped edit based on Callen's notes[5], in addition Montgomery[15] while older is a great reference. Wesson[22] is a great reference in general, especially for tokamak physics, and Wakatani[21] is great for stellarators and heliotrons. There is such an abundance of reading materials, that consulting the bibliographies of any of these books will also provide you with a wealth of resources. If you are interested in waves in plasmas, a classic is Stix[18]. I would also recommend Swanson[19]. They cover the vast amount of physics of just understanding all the possible waves possible in plasmas.

2.12 Problem Set

2.1. For Section 2.1.

2.1.1. Consider the 1D Poisson problem

$$-\epsilon_0 \frac{\partial^2 \phi}{\partial x^2} = \rho_q + q_T \delta(x)$$

Perform the same calculations as we did for a Debye length. Do you find a screening length? Do you find the same Debye length?

2.1.2. Find $\nabla^2 \phi$ via our formula for a divergence (1.2.234) with spherical coordinates. Does it match our derivation in the text?

2.1.3. In our solution we used a solution for $r\phi$. Why do we care about $r\phi$? What happens to this quantity as $r \to 0$? Do you think we should approach $\phi r = q_T/(4\pi\epsilon_0)$, as $r \to 0$?

2.1.4. What is a typical Debye length for a fusion reactor and fusion device? Consider electrons and a single ion species with $n = 10^{20}\,\mathrm{m}^{-3}$ and $k_B T = 10\,\mathrm{keV}$. Is it larger than a Bohr radius?

2.1.5. What is the typical number of particles in a Debye sphere for the same fusion parameters as the previous problem?

2.1.6. With atmospheric pressure at sea level is given by the definition $1\,\text{atm} = 1.013\,25\,\text{bar} = 101\,325\,\text{Pa} = 101.325\,\text{kPa}$, what is the number density of an ideal gas at sea level at a temperature of 20 °C? This is sometimes called Loschmidt's constant or Loschmidt's constant.

2.1.7. Vacuum pressures are almost always quoted in Torr, $760\,\text{Torr} = 1\,\text{atm}$. Suppose a given experiment is at around a 1 mTorr and 20 °C. Now what is the number density?

2.1.8. The plasma species neutral collision frequency can be calculated via $\nu_{sn} = n_n \sigma_n \langle v \rangle$ with n_n the neutral number density, σ_n the neutral cross section, and $\langle v \rangle$ the average relative velocity between the plasma species and a neutral. We can take $\langle v \rangle = v_{\text{th}_s}$ for thermal motions of the plasma species and use $\sigma_n = \pi R_n^2$ where R_n is the size of the atomic radius of the neutral particle. What is its value when the neutral is approximately a Bohr radius, with number density at Locschmidt's constant and at room temperature, 20 °C?

2.1.9. Consider a fusion device with $n = 10^{20}\,\text{m}^{-3}$ and $k_B T = 10\,\text{keV}$. Estimate the collision frequency between neutrals and ions from the previous problem. Assume neutral number density are 0.1, 0.01 and 0.001times n.

2.1.10. Consider fusion experiments of today. Now $n = 10^{19}\,\text{m}^{-3}$ with $k_B T = 100\,\text{eV}$. Try the various neutral number density assumptions from the previous problem.

2.1.11. Consider interplanetary space. $n = 10^{6}\,\text{m}^{-3}$ and $k_B T = 200\,\text{meV}$. Try the various neutral number density fractions again.

2.1.12. Consider a flame. What sort of number density do you think seems reasonable? What temperatures? It is probably simpler to consider an extreme range of values and try both extremes. Are flames clearly a plasma or not?

2.2. For Section 2.2.

2.2.1. Consider a magnetic field of $10^{-3}\,\text{T}$ to 10 T. What is the gyroradius and gyrofrequency associated with these values for fusion parameters: $n = 10^{20}\,\text{m}^{-3}$ and $k_B T = 10\,\text{keV}$? The ion collision frequency is given by

$$\nu_i = 4.80 \times 10^{-14} \frac{Z^4}{\sqrt{\mu}} \ln \Lambda n_i (k_B T_i)^{-3/2} \text{Hz}$$

for n in SI units, $k_B T$ in eV, Z the ion charge and μ the factor the ion mass is larger than a proton mass, $\mu m_p = m_i$. Are δ_m and δ_c small?

2.2.2. Consider the relationship between the Boozer coordinate Jacobian determinant and other flux coordinates Jacobian determinants. This should explain the connection in general. Given θ_f and ζ_f, what $G_0 = G_B$ can be chosen to change the flux coordinates

into the Boozer representation? Use

$$\begin{aligned}\theta_B &= \theta_f + \frac{\mathrm{d}\Phi_p^r}{\mathrm{d}r} G_B(r,\theta_f,\zeta_f)\\ \phi_B &= \phi_f + \frac{\mathrm{d}\Phi_t}{\mathrm{d}r} G_B(r,\theta_f,\zeta_f)\\ \mathcal{J}_B &= \nabla r \cdot \nabla\theta_B \times \nabla\phi_B\\ \mathcal{J}_f &= \nabla r \cdot \nabla\theta_f \times \nabla\phi_f\\ 2\pi\mathbf{B} &= \frac{\mathrm{d}\Phi_t}{\mathrm{d}r}\nabla r \times \nabla\theta_f - \frac{\mathrm{d}\Phi_p^r}{\mathrm{d}r}\nabla r \times \nabla\phi_f\end{aligned}$$

2.2.3. Can one write $\mathcal{J}_f$ in terms of $\mathbf{B}$ in general? Use

$$2\pi\mathbf{B} = \frac{\mathrm{d}\Phi_t}{\mathrm{d}r}\nabla r \times \nabla\theta_f - \frac{\mathrm{d}\Phi_p^r}{\mathrm{d}r}\nabla r \times \nabla\phi_f$$

and compute $\mathbf{B}\cdot\mathbf{B}$. Find the contravariant and covariant components of this equation to do so.

2.3. For Section 2.4.

2.3.1. Summarize the differences between the major approaches in terms of linear or closed, and the topology of the magnetic field.

2.4. For Section 2.5.

2.4.1. Show that taking the energy moment $m_s v^2/2$ yields the equation

$$\frac{3}{2}n_s\frac{\mathrm{d}_s(k_B T_s)}{\mathrm{d}t} + p_s\boldsymbol{\nabla}\cdot\mathbf{V}_s = -\boldsymbol{\nabla}\cdot\mathbf{q}_s - \overleftrightarrow{\boldsymbol{\Pi}}_s : \nabla\mathbf{V}_s + Q_s$$

This is much more difficult than it looks. First just take the moments and find

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{3}{2}n_s k_B T_s + \frac{m_s n_s}{2}V_s^2\right] + \boldsymbol{\nabla}\cdot\left[\mathbf{q}_s + \frac{5}{2}n_s k_B T_s\mathbf{V}_s + \frac{n_s m_s}{2}V_s^2 + \mathbf{V}_s\cdot\overleftrightarrow{\boldsymbol{\Pi}}_s\right]\\ &\quad - n_s q_s\mathbf{V}_s\cdot\mathbf{E} = Q_s + \mathbf{V}_s\cdot\mathbf{R}_s\end{aligned}$$

Then plug in values for $\frac{\partial n}{\partial t}$ and $mn\frac{\partial\mathbf{V}_s}{\partial t}$ from number density continuity and momentum density equations we previously derived. Use the identities

$$\begin{aligned}\mathbf{V}_s\cdot\nabla\mathbf{V}_s &= \frac{1}{2}\nabla V_s^2 - \mathbf{V}_s\times(\boldsymbol{\nabla}\times\mathbf{V}_s)\\ \boldsymbol{\nabla}\cdot(\mathbf{V}_s\cdot\overleftrightarrow{\boldsymbol{\Pi}}_s) &= \nabla V_s : \overleftrightarrow{\boldsymbol{\Pi}}_s + \mathbf{V}\cdot\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}_s^{\mathsf{T}}\end{aligned}$$

and use that $\overleftrightarrow{\boldsymbol{\Pi}}_s$ is symmetric. Finally combine like terms and change $\frac{\partial}{\partial t}$ into $\frac{\mathrm{d}_s}{\mathrm{d}t}$.

2.5. For Section 2.6.

2.5.1. Explain why there is no simple form for the center-of-mass momentum density equation like (2.6.50) when there are three species: two ions and an electron. Consider

$$\mathbf{V} = \frac{\sum_s n_s \mathbf{V}_s}{\sum_s n_s m_s}$$
$$\mathbf{J} = \sum_s q_s n_s \mathbf{V}_s$$
$$0 = \sum_s q_s n_s$$

With two species there are two vector equations and a scalar equation relating $\mathbf{V}$, $\mathbf{V}_i$, $\mathbf{V}_e$, and $\mathbf{J}$. When there are three species we now have $\mathbf{V}_{i1}$, $\mathbf{V}_{i2}$, and $\mathbf{V}_e$, but the same two vector equations and a scalar relating with $\mathbf{V}$ and $\mathbf{J}$.

2.5.2. Explain why even with (2.6.50) for an arbitrary mass ratio, that a system of equations would still be complicated by the need to calculate $\mathbf{E}$ [when using an arbitrary mass ratio].

2.5.3. Consider the MHD criteria for a fusion experiment with $B_0 = 0.1\,\mathrm{T}$, $R_0 = 1\,\mathrm{m}$, $a = 0.2\,\mathrm{m}$, $n = 1 \times 10^{18}\,\mathrm{m}^{-3}$, $T_0 = 100\,\mathrm{eV}$ and $m_i = 3.34 \times 10^{-27}\,\mathrm{kg}$.

2.6. For Section 2.9.

2.6.1. Derive the dispersion relation for sound waves, shear Alfvén waves and compressional Alfvén waves directly by using the appropriate approximations before solving for the dispersion relation.

2.6.2. Show that (2.9.39) reproduces the correct dispersion relations for shear Alfvén waves and compressional Alfvén waves.

2.6.3. Consider energy-density conservation in Ideal MHD. The energy density is given by

$$\epsilon = \epsilon_f + \epsilon_B + \epsilon_p = \frac{nm}{2}|\mathbf{V}|^2 + \frac{|\mathbf{B}|^2}{2\mu_0} + \frac{p}{\gamma - 1}$$

where ϵ_f is the fluid energy density, ϵ_B is the magnetic energy density, and ϵ_p is the internal energy. Using $\frac{1}{2}\frac{\partial |\mathbf{q}|^2}{\partial t} = \mathbf{q} \cdot \frac{\partial \mathbf{q}}{\partial t}$ find the conservative form for ϵ as

$$\frac{\partial \epsilon}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\Gamma} = \mathbf{S}$$

where $\mathbf{S}$ is an energy-density source. For Ideal MHD, you should find

$$\boldsymbol{\Gamma} = \left[\frac{nm|\mathbf{V}|^2}{2} + \frac{\gamma p}{\gamma - 1}\right]\mathbf{V} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$$
$$\mathbf{S} = \mathbf{0}$$

2.6.4. Consider energy-density conservation in general. To do so, put δ tags onto terms in

only certain types of MHD. Thus,

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \delta_\eta\eta\mathbf{J} + \frac{\delta_h}{ne}\left[\mathbf{J}\times\mathbf{B} - \nabla p_e\right] + \delta_e\frac{m_e}{ne^2}\frac{\partial\mathbf{J}}{\partial t}$$

$$\epsilon_p = \sum_s \frac{p_s}{\gamma-1}$$

$$\frac{k_B n}{\gamma-1}\left(\frac{\partial T_s}{\partial t} + \mathbf{V}_s\cdot\nabla T_s\right) = -nk_BT_s\boldsymbol{\nabla}\cdot\mathbf{V}_s - \boldsymbol{\nabla}\cdot\mathbf{q}_s - Q_s$$

$$\mathbf{V}_e = \mathbf{V} - \delta_{se}\frac{\mathbf{J}}{ne(1+\frac{m_e}{m_i})}$$

$$\mathbf{V}_i = \mathbf{V} + \delta_{si}\frac{\frac{m_e}{m_i}\mathbf{J}}{ne(1+\frac{m_e}{m_i})}$$

$$nm_i\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = -\nabla p + \mathbf{J}\times\mathbf{B} - \delta_\Pi\boldsymbol{\nabla}\cdot\overleftrightarrow{\boldsymbol{\Pi}}$$

The conservative form with $m_e/m_i = 0$ should now have

$$\boldsymbol{\Gamma} = \left[\frac{nm|\mathbf{V}|^2}{2} + \frac{\gamma p}{\gamma-1}\right]\mathbf{V} + \delta_\Pi\overleftrightarrow{\boldsymbol{\Pi}}\cdot\mathbf{V} + \mathbf{q} + \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} + \frac{\delta_s\gamma\nabla T_e\times\mathbf{B}}{\mu_0 e(\gamma-1)}$$

$$\mathbf{S} = (\delta_h - \delta_s)\frac{\nabla p_e}{ne}\cdot\mathbf{J} - \delta_e\frac{m_e}{2ne^2}\frac{\partial|\mathbf{J}|^2}{\partial t} - Q + \delta_\Pi\overleftrightarrow{\boldsymbol{\Pi}}:\nabla\mathbf{V} - \delta_\eta\eta|\mathbf{J}|^2$$

2.6.5. Consider the last problem's conservative form. What must Q balance to not have energy injected into the system? Note that $\eta|\mathbf{J}|^2$ is often called Ohmic heating and $\overleftrightarrow{\boldsymbol{\Pi}}:\nabla\mathbf{V}$ is called viscous heating. If we include Hall terms δ_h, must we include separate temperatures δ_s? Are there any balancing terms for the electron inertia (δ_e)? If you were using a computational scheme, do you think it would be good to have $\mathbf{S} = \mathbf{0}$? If so, why? If not, why not?

2.6.6. Finally, consider deriving the Helmholtz transport theorem [see (2.7.11)] in the same manner as we did for the Reynolds transport theorem [see (2.9.58)]. What difficulties do you encounter? Do you prefer my derivation method?

2.7. For Section 2.9.1.

2.7.1. Find the dispersion relation for the Rayleigh-Taylor instability. That is, consider an interface at $y = 0$ with fluid $\rho_+ > \rho_-$ where ρ_+ is above the interface ($y \geq 0$) and ρ_- is below the interface $y < 0$. We have gravity as $\mathbf{g} = -g\hat{\mathbf{y}}$. Consider the fluid incompressible. Then our equations for either side of the interface are given by

$$\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = \mathbf{g} - \nabla p$$

Linearize this assuming that the interface is perturbed with $\boldsymbol{\xi} = \xi_y\hat{\mathbf{y}} + \xi_{x0}\exp(-ik_xx)\hat{\mathbf{x}} + \xi_{z0}\exp(-ik_zz)\hat{\mathbf{z}}$ where $\mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}$ and $\frac{\partial\boldsymbol{\xi}}{\partial t} = \widetilde{\mathbf{V}}$ with no background flow. Use our interface relations with no magnetic field to match across the interface. Solve the entire system and find

$$\omega^2 = \frac{\rho_- - \rho_+}{\rho_+ + \rho_-}kg$$

2.7.2. Now consider the Kelvin-Helmholtz equation. Use the same analysis as for the Rayleigh-Taylor analysis but with

$$\mathbf{V}_0 = \begin{cases} V_+ x & y \geq 0 \\ V_- x & y < 0 \end{cases}$$

Find the dispersion relation.

2.7.3. The dispersion relation we found for the Kruskal-Schwarzchild problem was

$$\omega^2 = \frac{B^2}{(\rho_+ + \rho_-)}\left[k_z^2 + (k_z \cos\alpha + k_x \sin\alpha)^2\right] - \frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} kg$$

For $0 < \alpha < \pi$, which orientation of k leads to the fastest growing mode? How much faster is it than when $\alpha = 0$ with the same k value?

2.7.4. Consider the Kruskal-Schwarzchild problem within a bounded box. This time let the top be a fluid with ρ_+ and the bottom a vacuum with $\rho_- = 0$. Let $\mathbf{B}_0 = \mathbf{0}$ on top and $\mathbf{B} = B_0\hat{\mathbf{z}}$ with the interface again at $y = 0$ and $\mathbf{g} = -g\hat{\mathbf{y}}$. Consider the box to have bounds at $y = \pm h$. Use $\boldsymbol{\xi} = \boldsymbol{\xi}_0(y)\exp(ik_x x + ik_z z - i\omega t)$ and $p = p_0(y)\exp(ik_x x + ik_z z - i\omega t)$ with $\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$. Use that the bottom is a vacuum solution so $\nabla\Phi = B_0\hat{\mathbf{z}}$ and $\nabla^2\Phi = 0$ in a vacuum. Use interface relations and sinh and cosh to simplify the expressions of your solutions. Find the dispersion relation

$$\omega^2 = \frac{B_0^2}{\mu_0\rho_0}k_z^2 - kg\tanh(kz)$$

What do you find when $kz \ll 1$?

2.7.5. Use the Rankine-Hugoniot relation

$$\hat{\mathbf{n}} \cdot \left[\!\!\left[\mathbf{V}_i\mathbf{u} - \overleftrightarrow{\mathbf{F}} \right]\!\!\right] = \mathbf{0}$$
$$\hat{\mathbf{n}} \cdot [\![\mathbf{V}_i u - \mathbf{F}]\!] = \mathbf{0}$$

for conservative forms

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{F}} = \mathbf{S}$$
$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{F} = S$$

considering $\mathbf{S}$ or S singular with all of Maxwell's equations. Singular $\mathbf{S}$ or S means that we have to include a contribution on the right from $\mathbf{S}$ or S that is a surface quantity that doesn't disappear as the volume goes to zero. That is the right hand side is not $\mathbf{0}$ or 0. Use a stationary interface, $\mathbf{V}_i = \mathbf{0}$.

$$\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} = 0$$
$$\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla} \times \mathbf{B} = -\mu_0\mathbf{J}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$
$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho_q}{\epsilon}$$

You should recover the normal interface relations for electromagnetic theory.

$$\begin{aligned}\hat{\mathbf{n}}\cdot[\![\mathbf{B}]\!] &= 0\\ \hat{\mathbf{n}}\times[\![\mathbf{B}]\!] &= \mu_0\mathbf{K}\\ \hat{\mathbf{n}}\cdot[\![\mathbf{E}]\!] &= \frac{\sigma_q}{\epsilon_0}\\ \hat{\mathbf{n}}\times[\![\mathbf{E}]\!] &= \mathbf{0}\end{aligned}$$

where $\hat{\mathbf{n}}$ points from the $-$ to $+$ side and $[\![q]\!] = q_+ - q_-$, σ_q is a surface charge and $\mathbf{K}$ is a surface current. You may find $\boldsymbol{\nabla}\cdot(\mathbb{1}\times\mathbf{A}) = \boldsymbol{\nabla}\times\mathbf{A}$ useful. You can try the non-singular forms first, so that $\sigma_q = 0$ and $\mathbf{K} = \mathbf{0}$ if you have trouble with the singular $\mathbf{S}$ or S forms.

2.8. For Section 2.10.

2.8.1. We found $\mathbf{v}_D = \frac{\mathbf{F}\times\hat{\mathbf{b}}}{q_s B_0}$. How large would the $\mathbf{E}_\perp$ field have to be to compete with the Alfvén velocity generically (find v_A/v_D)?

2.8.2. For a $1\,\mathrm{kV\,m^{-1}}$ electric field and a $100\,\mathrm{mT}$ magnetic field, what is the velocity of the $\mathbf{E}\times\mathbf{B}$ drift? What if the magnetic field is $1\,\mathrm{T}$?

2.8.3. For a $100\,\mathrm{mT}$ magnetic field, what is the gravitational drift near Earth's surface where $\mathbf{F} = -m_s\mathbf{g}$. Consider the worst case where the magnetic field is completely perpendicular to the direction of gravity.

2.8.4. What would a diamagnetic drift be for $\nabla p \approx p/L$ with $n = 10^{20}\,\mathrm{m^{-3}}$, $B_0 = 1\,\mathrm{T}$, and $k_B T = 10\,\mathrm{keV}$?

2.8.5. Calculate the grad B drift and curvature drift for representative parameters as given above.

2.8.6. Suppose you wanted to find $\mathcal{O}(\rho^3)$ corrections to finding $\rho(\mathbf{x})$. What would the new terms look like?

References

[1] Elmo Benedetto. "Some remarks about flux time derivative". In: *Afrika Matematika* 28.1-2 (2017), pp. 23–27.

[2] Ira B Bernstein et al. "An energy principle for hydromagnetic stability problems". In: *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 244.1236 (1958), pp. 17–40.

[3] J.P. Boyd. *Chebyshev and Fourier Spectral Methods: Second Revised Edition*. Dover Books on Mathematics. Dover Publications, 2013. ISBN: 9780486141923. URL: https://books.google.com/books?id=b4TCAgAAQBAJ.

[4] S. I. Braginskii. "Transport Processes in a Plasma". In: *Reviews of Plasma Physics* 1 (1965), p. 205.

[5] James D. Callen, Kyle J. Bunkers, and Jeffrey S. Kollasch. *Plasma Kinetic Theory*. 2018. URL: http://homepages.cae.wisc.edu/~callen/plasmas.html.

[6] F.F. Chen. *Introduction to Plasma Physics and Controlled Fusion: Volume 1: Plasma Physics*. Springer US, 2013. ISBN: 9781475755954. URL: https://books.google.com/books?id=WGbaBwAAQBAJ.

[7] GF Chew, ML Goldberger, and FE Low. "The Boltzmann equation an d the one-fluid hydromagnetic equations in the absence of particle collisions". In: *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 236.1204 (1956), pp. 112–118.

[8] T Chust and G Belmont. "Closure of fluid equations in collisionless magnetoplasmas". In: *Physics of plasmas* 13.1 (2006), p. 012506.

[9] W.D. D'haeseleer. *Flux coordinates and magnetic field structure: a guide to a fundamental tool of plasma structure*. Springer series in computational physics. Springer-Verlag, 1991. ISBN: 9783540524199. URL: https://books.google.com/books?id=X3x5AAAAIAAJ.

[10] Harley Flanders. "Differentiation Under the Integral Sign". In: *The American Mathematical Monthly* 80.6 (1973), pp. 615–627. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2319163.

[11] John M Greene and John L Johnson. "Interchange instabilities in ideal hydromagnetic theory". In: *Plasma Physics* 10.8 (1968), p. 729.

[12] K. Hain, R. Lust, and A. Schluter. "STABILITY OF A PLASMA". In: *Zeitschrift fuer Naturforschung (West Germany) Divided into Z. Nautrforsch., A, and Z. Naturforsch., B: Anorg. Chem., Org. Chem., Biochem., Biophys.*, Vol: 12a (Oct. 1957).

[13] R.D. Hazeltine and J.D. Meiss. *Plasma Confinement*. Dover Books on Physics. Dover Publications, 2013. ISBN: 9780486151038. URL: https://books.google.com/books?id=LDPDAgAAQBAJ.

[14] P. Helander and D.J. Sigmar. *Collisional Transport in Magnetized Plasmas*. Cambridge Monographs on Plasma Physics. Cambridge University Press, 2005. ISBN: 9780521020985. URL: https://books.google.com/books?id=nm-V01E0H2MC.

[15] D.C. Montgomery. *Theory of the Unmagnetized Plasma*. Gordon and Breach, 1971. ISBN: 9780677033501. URL: https://books.google.com/books?id=8w-IodNEv7gC.

[16] Rayleigh. "Investigation of the Character of the Equilibrium of an Incompressible Heavy Fluid of Variable Density". In: *Proceedings of the London Mathematical Society* s1-14.1 (1882), pp. 170–177. DOI: 10.1112/plms/s1-14.1.170. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/plms/s1-14.1.170. URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/plms/s1-14.1.170.

[17] W.W. Sawyer. *Vision in Elementary Mathematics*. Dover Books on Mathematics. Dover Publications, 2012. ISBN: 9780486143620. URL: https://books.google.com/books?id=LaK8AQAAQBAJ.

[18] T.H. Stix. *Waves in Plasmas*. American Inst. of Physics, 1992. ISBN: 9780883188590. URL: https://books.google.com/books?id=OsOWJ8iHpmMC.

[19] D.G. Swanson. *Plasma Waves*. Elsevier Science, 2012. ISBN: 9780323141635. URL: https://books.google.com/books?id=0GMwFeDDZ9UC.

[20] Geoffrey Taylor. "The Instability of Liquid Surfaces when Accelerated in a Direction Perpendicular to their Planes. I". In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 201.1065 (1950), pp. 192–196. ISSN: 00804630. URL: http://www.jstor.org/stable/98398.

[21] M. Wakatani. *Stellarator and Heliotron Devices*. International series of monographs on physics. Oxford University Press, 1998. ISBN: 9780195078312. URL: https://books.google.com/books?id=MxttViyhaaEC.

[22] John Wesson and David J Campbell. *Tokamaks*. Vol. 149. Oxford university press, 2011.

Chapter 3

Nuclear Fusion

> *It is no good trying to stop knowledge from going forward. Whatever nature has in store for mankind, unpleasant as it may be, men must accept, for ignorance is never better than knowledge.*
>
> — Enrico Fermi

Few plasma textbooks actually get into the details of how nuclear fusion works. This is because, strictly speaking, plasma physics doesn't really matter for nuclear fusion. That is, nuclear fusion is a nuclear physics problem, and not a plasma physics problem. Plasma physics yields very little information about the processes of nuclear decay or fission. When we view plasma physics through the lens of magnetic confinement for fusion, then ignorance of nuclear processes no longer makes sense. The plasma physics in this case is actually for promoting nuclear fusion. Thus, it is worth going through what we know in this arena and explaining how it could be used to produce electrical energy.

This chapter first goes over nuclear physics terminology, explains nuclear decay, and cross sections.[1] We then get what fusion reactions are important. We first study stellar fusion. This will show that we cannot do anything like stellar fusion on Earth, and so we will then study the reactions of interest for terrestrial fusion. It is called terrestrial fusion simply to make it clear that it is feasible on Earth. Then I will explain why beam fusion and muon catalyzed fusion cannot be done efficiently.

3.1 Nuclear Terminology

> *Many of the units of nuclear science have a whimsical sound, in direct proportion to their incomprehensibility.*
>
> — Herbert Arthur Klein[11, p. 691]

We may as well begin with what a nucleus is. It is where the word nuclear comes from. A nucleus is the center of an atom, made up of neutrons (neutral particles) and protons (positive particles) which are often called nucleons so that we do not have to say protons and neutrons. Under our current theories, protons and neutrons are further made up of quarks. Quarks are unimportant for

[1] To be clear, I mean nuclear cross sections, and not generic cross sections.

our level of analysis, so I will not mention them again.[2] The important thing about the nucleus is that it is held together very tightly by the strong nuclear force. This means that we can exploit the energy of nuclei[3] in various ways. If some configurations of protons and neutrons are at a lower (potential) energy level[4] than other configurations, then we can release energy by rearranging the higher energy configuration to the lower (potential) energy configuration. However, it may sometimes take some initial energy to allow this transition to a lower (potential) energy state to occur. It is sometimes helpful to think of this in analogy to normal chemistry, which is mostly due to the electrons orbiting[5] the nucleus. For some types of elements, it is energetically (in reality entropically) favorable to combine multiple atoms (including sharing their electrons), while for some molecules it is more energetically (again, technically entropically) favorable to separate the constituents.

For nuclear physics, splitting a nucleus is called nuclear fission (usually shortened to fission), named after the biological fissioning of cells. In this process, a nucleus splits into two or more new nuclei. Combining two nuclei[6] into one nucleus is called nuclear fusion (again, usually shortened to fusion). The determination of whether two nuclei fusing or a single nuclei fissioning will release energy[7] is determined by the type of nuclei, that is the configuration of protons and neutrons. In general, heavier nuclei split and release potential energy while the lightest nuclei combine and release potential energy. The "most stable"[8] element is at ${}^{56}_{26}\mathrm{Fe}_{30}$, with 56 the mass number (sometimes called nucleon mass number or atomic mass number and usually denoted A), 26 the number of protons, or atomic number (often denoted Z), Fe the chemical symbol, and 30 the number of neutrons (equal to $A - Z$). The neutron number (sometimes called N) is often omitted in this notation because it is determined by A and Z and so one sees ${}^{56}_{26}\mathrm{Fe}$. Because Z is technically encoded in the chemical element symbol, this often gets further shortened to ${}^{56}\mathrm{Fe}$. That is for chemical symbol X with proton/atomic number Z, and mass number A, the representation is given by ${}^{A}_{Z}\mathrm{X}_{A-Z}$ or ${}^{A}_{Z}\mathrm{X}$ or ${}^{A}\mathrm{X}$.

The theory of whether a specific combination of protons and neutrons is stable is not complete in the sense that most equations have both theoretical and empirical aspects. This is simply to say that if you were given a proton and neutron configuration, determining whether that configuration is stable through purely theoretical techniques is not currently possible in general. Instead, theoretical techniques offer clues, and empirical observations allow one to fine tune parameters to make the guesses more accurate.

Chemical elements with a different number of neutrons are called isotopes. It is often claimed that different isotopes of elements are chemically identical, but this is actually false. Generally

[2] Consult a particle physics book for more on quarks.

[3] plural of nucleus

[4] Once again, we usually hear "lower energy state", but realize that we are either considering only the atom so that the energy of the atom can actually change, as the energy that leaves, completely leaves the system. If we consider the entire system (atom and elsewhere) then the energy remains constant, but the lowest potential energy state of the atom becomes the most entropically favorable state.

[5] Of course, orbits is not really the right concept. This is classical usage in a quantum world. Electrons do not orbit nuclei in the way planets orbit the sun, but it is useful to use this language to at least assert that the electrons are associated with a certain nucleus.

[6] Technically more than two nuclei is possible, but it is very unlikely since it requires three particles to get near each other at the right time.

[7] Again, this is typical (somewhat sloppy) wording. What we mean is that *potential* energy from the nucleus is released.

[8] We will define this as most binding energy per nucleon (proton or neutron) later.

speaking, the chemistry is similar, but chemical differences can be important for processes that separate desired isotopes from the rest. For example, heavy water is not safe for drinking because it acts differently than regular water.[9] Another terminology is to use nuclides instead of isotopes. A nuclide is a nucleus with a specific number of protons and neutrons. If you think this sounds pretty much the same as an isotope, you are fairly correct. The difference in use is that nuclides usually refer to nuclides for every chemical element, whereas isotopes are usually isotopes of a specific element.[10] Thus, when we are speaking from a nuclear perspective, nuclides is usually preferred because we are not necessarily limiting ourselves to any particular element. This terminology difference is not universally held, so do not be surprised if someone talks of isotopes and means isotopes of several elements (what I have called nuclides).

The table of nuclides is like the periodic table, but for nuclear properties. It is a gigantic chart of all the nuclides with proton number as the y-axis (vertically) and neutron (rarely mass number) on the x-axis (horizontally). It also usually includes stability information and half-life times if applicable for each nuclide type. Many charts online include a great deal more information, with exact atomic masses, binding energies, and the nuclear decays that are possible. I suggest looking at one online, as the amount of information is a bit overwhelming unless you zoom in.[11]

Finally, nuclear reactions use a notation similar to chemistry for denoting reactions. For simple reactions the style is usually

$$\text{target} + \text{projectile} \rightarrow \text{final nucleus} + \text{ejected particles} + (\text{Energy}) \tag{3.1.1}$$

where the target is a nucleus, the "projectile" is whatever hits the target nucleus inducing a nuclear reaction, the final particle is the nucleus that results, the ejected particles are smaller nuclei (than the "final nucleus"), and the energy is the excess or needed energy for the particles. Technically, the reaction could go either way so that the arrow should be double-headed, but usually one process is much more likely and so it is usually written with a single arrow. In addition, it is not strictly necessary to have all the components of the equation. For example, some elements spontaneously decay and so no projectile is needed. Because this notation is not as compact as physicists desire, you will sometimes see a reaction written as

$$A(b,c)D \longleftrightarrow A + b \rightarrow c + D \tag{3.1.2}$$

for target A, projectile b, ejected particle c and final nucleus D. The lower versus upper case letters is conventional for some objects, but it should not be taken too seriously. In addition, you should be aware that A and b and c and D are exchangeable without changing the reaction since we can always change our frame of reference such that the target becomes the projectile and vice versa. I won't use that notation here because we won't have to deal with that many nuclear reactions and I think that the notation can often be a bit too terse. If there is no need for a projectile or no ejected particles, these are just omitted in the above equations. Remember that the prescription of labeling things on both sides of the equation should not be taken too seriously. We can always just think of the left side as the beginning particles with whatever energies, and the right side as the resulting particles and energy after a nuclear reaction.

[9] The more neutrons and protons for an isotope, however, the less it matters for chemistry generically speaking.

[10] That is you have isotopes of carbon. If you had only two pure elements such as ${}^{14}_{6}\text{C}$ and ${}^{4}_{2}\text{He}$ you generally would not say we have two isotopes, but would say we have two nuclides.

[11] Some good charts are available from the IAEA and the NNDC and Brookhaven National Labs.

Some other notation that is usually employed is for commonly seen particles during radioactive reactions. They will be further explained later in the text. Alpha particles [which are nuclei of ^{4}He] are often abbreviated to α, beta particles to β, gamma particles (i.e., gamma photons or gamma rays) to γ, protons to p, neutrons to n, electrons to e^- (often just e), and positrons to e^+. Because beta particles are just electrons or positrons, you hear β^+ (beta plus or beta plus particle) for positron emission and β^- (beta minus or beta minus particle) for electron emission. In fusion it is typical to refer to deuterons as $D = {}^2_1\mathrm{H}$ (sometimes d) and tritons as $T = {}^3_1\mathrm{H}$ (sometimes t). The excess or needed energy is often given as Q. This is defined by

$$\begin{aligned} Q &= \text{Kinetic Energy}_{\text{after reaction}} - \text{Kinetic Energy}_{\text{before reaction}} \\ &= (m_i - m_f)c^2 \end{aligned} \tag{3.1.3}$$

with m_i the initial rest mass energy of the particles and m_f is the final rest mass energy of the particles. The reason these two expressions are equal is that any excess kinetic energy must come from mass-energy equivalence. Positive Q values indicate the reaction is "exothermic" or releases energy whereas negative Q means that the reaction is "endothermic" or requires energy. Thus, when giving a Q value it matters which side of the equation it is put on, and what way the arrow points. If the arrow points toward Q then $Q > 0$ means an exothermic reaction. If the arrow points away from Q and $Q > 0$ then an endothermic reaction. Similarly if the arrow points toward Q and $Q < 0$ then it is an endothermic reaction and if the arrow points away from Q and $Q < 0$ then it is an exothermic reaction. Conventionally, Q has the arrow pointing at it so that $Q > 0$ is exothermic and $Q < 0$ is endothermic.[12]

If we have two initial particles W and X and two resultant particles Y and Z with $Q = K_F - K_I$ (the conventional nuclear Q discussed above) and with quantities with a subscript F for final (all resultant particles) and I for all initial, then we can derive what the distribution of energies among Y and Z are. The reaction is

$$W + X \to Y + Z + Q \tag{3.1.4}$$

We will assume $Q > 0$ for this derivation, because we are interested in exothermic reactions. Say E_I and $\mathbf{p}_I$ are the initial energy and momentum and E_F and $\mathbf{p}_F$ are the final energy and momentum of the system with E_i for $i \in \{W, X, Y, Z\}$ representing the total energy for particle i. For this we'll use the special relativistic forms so that we see it is correct in general. Then $\mathbf{p}_i = m_i\gamma_i\mathbf{v}_i$ with $\gamma_i = 1/\sqrt{1 - |\mathbf{v}_i|^2/c^2}$. Any masses m_i refer to the rest mass and $K_i = (\gamma_i - 1)m_ic^2 = E_i - m_ic^2$ is the kinetic energy. We will simply use $E_i = \sqrt{p_i^2c^2 + m_i^2c^4}$. We will consider the calculation in the center of momentum frame, as is conventional. This simplifies the motion to be along one dimension and we don't need to worry about the vector nature of the velocities or momenta. We use conservation of momentum and energy. We can square the energies to find

$$(E_W + E_X)^2 = E_I^2 = (E_Y + E_Z)^2 = E_Y^2 + E_Z^2 + 2E_YE_Z \tag{3.1.5}$$

$$E_I^2 = E_Y^2 + E_Z^2 + 2E_Y(E_I - E_Y) = E_Y^2 + E_Z^2 + 2E_YE_I - 2E_Y^2 \tag{3.1.6}$$

$$E_Y = \frac{E_I^2 + E_Y^2 - E_Z^2}{2E_I} = \frac{E_I^2 + p_Y^2c^2 + m_Y^2c^4 - p_Z^2c^2 - m_Z^2c^4}{2E_I} \tag{3.1.7}$$

We can then use $p_Y = -p_Z$ so $p_Y^2c^2 - p_Z^2c^2 = 0$ and we can also note that we could perform the

[12] Do not assume that this is the same convention as used in chemistry.

same action with $Y \leftrightarrow Z$ so that

$$E_Y = \frac{E_I^2 + m_Y^2c^4 - m_Z^2c^4}{2E_I} \tag{3.1.8}$$

$$E_Z = \frac{E_I^2 + m_Z^2c^4 - m_Y^2c^4}{2E_I} \tag{3.1.9}$$

We can use $K_F - K_I = Q = E_F - m_Y^2c^4 - m_Z^2c^4 - K_I = E_I - K_I - m_Y^2c^4 - m_Z^2c^4$ to find

$$\begin{aligned} K_Y = E_Y - m_Yc^2 &= \frac{E_I^2 + m_Y^2c^4 - m_Z^2c^4 - 2m_YE_Ic^2}{2E_I} = \frac{(E_I - m_Yc^2)^2 - m_Z^2c^4}{2E_i} \\ &= \frac{(m_Zc^2 + Q + K_I)^2 - m_Z^2c^4}{2E_I} = \frac{2(Q+K_I)m_Zc^2 + (Q+K_I)^2}{2E_I} \end{aligned} \tag{3.1.10}$$

$$K_Z = \frac{2(Q+K_I)m_Yc^2 + (Q+K_I)^2}{2E_I} \tag{3.1.11}$$

This is probably the most useful fully relativistic result. However, if we are in a limit where $K_I \ll m_ic^2$ and $K_I \ll Q$, it is useful to write $E_I = E_F = K_F + m_Yc^2 + m_Zc^2$ so that

$$K_Y = \frac{2(Q+K_I)m_Zc^2 + (Q+K_I)^2}{2(K_F + m_Yc^2 + m_Zc^2)} \tag{3.1.12}$$

$$K_Z = \frac{2(Q+K_I)m_Yc^2 + (Q+K_I)^2}{2(K_F + m_Yc^2 + m_Zc^2)} \tag{3.1.13}$$

We can then use that $K_F \ll m_ic^2$ with $K_F \gg K_I$ (consistent with $K_F - K_I = Q \gg K_I$) so $Q = K_F - K_I \ll m_ic^2$ and we then find

$$K_Y \approx \frac{2Qm_Zc^2}{2(m_Yc^2 + m_Zc^2)} = Q\frac{m_Z}{m_Y + m_Z} \tag{3.1.14}$$

$$K_Z \approx \frac{2Qm_Yc^2}{2(m_Yc^2 + m_Zc^2)} = Q\frac{m_Y}{m_Y + m_Z} \tag{3.1.15}$$

As we are in this limit for most of our nuclear reactions, then this is why one finds the given quoted kinetic energies for DT (deuterium and tritium) or DD (deuterium and deuterium) reactions. These reactions are often just shortened to *DT* and *DD* and mean processes using only the two named constituents. We find that it is simply Q weighted by the other reactant's mass in comparison to the total mass of the reactants.

It is worth mentioning that we could have derived our previous results quickly with momentum four-vectors because for a single particle's momentum four-vector we have in our sign convention that $\mathbb{P} \cdot \mathbb{P} = -m_0c^2$ with m_0 the particle's rest mass.

3.2 Nuclear Decay

> *In some sense, nuclear fission is not one of those developments in physics which arose logically and systematically in the course of progress. There was a great deal of accident and surprise in the process.*
>
> — John von Neumann

One of the achievements of physics in the late 1800's to early 1900's is the discovery of transmutation of elements through nuclear decay. There are three main processes through which an element can undergo (nuclear) decay. These processes can be generically divided into three categories. Alpha decay emits an alpha particle (simply a helium nucleus, so two protons and two neutrons). It is governed by the strong force and electromagnetic force. Beta decay releases an electron or positron and is governed by the weak force. Gamma decay releases a photon (usually assumed to be of high energy) and is governed mostly by the electromagnetic force. Alpha particles do not penetrate shielding well, but cause immense damage if they do interact with biological tissue. Paper thick materials are typically enough to ameliorate alpha particles. Beta particles also do not penetrate shielding well either, but better than alpha particles with aluminum foil (or a couple layers of aluminum foil) usually cited as being thick enough to prevent most beta particle exposure. Gamma rays usually require on the order of 300 mm of lead. Thus, skin is enough for α's, heavy clothing for β's, and bunkers for γ's. When I say protected, I am saying that we reduce the incoming radiation by a factor of about a billion or so, and the details always matter (what kinetic energy the α, β, or γ has and at what angle it hits the shielding).

In retrospect, better naming could have been used had they known what alpha, beta, and gamma radiation was when they were first discovered. Instead, we are left with the first three letters of the Greek alphabet, which work well enough. Later, it was learned that neutrons could be used to break up nuclei, and so are, in effect, another type of nuclear reaction process. By sending neutrons (which are not affected by electromagnetic fields[13]) at nuclei, the nuclei would sometimes break up, and so large nuclei would become multiple smaller ones. Once again, the kinetic energy of the neutrons is important in determining what occurs, and so a distinction between slow neutrons (low kinetic energy) and fast neutrons (high kinetic energy) developed. Fast neutrons are those with kinetic energies greater than about 0.1 MeV.[14] Slow neutrons are generally those with kinetic energies closer to thermal temperatures (usually 1 meV to 1 eV levels).[15]

There are in fact two other common types of spontaneous radiation events, but they are sensibly named. The first is neutron emission, where, you guessed it, a nucleus emits a neutron or neutrons. The second is electron capture. In this case, (you might have guessed) a nucleus absorbs an electron. In electron capture, the electron combines with a proton to create a neutron in the nucleus and so can be written as

$$p + e^- \rightarrow n + \nu_e \tag{3.2.1}$$

where ν_e is an electron neutrino. You do not need to worry about neutrinos if you are not curious, because neutrinos barely ever interact with other particles.

Let's look at typical processes for each type of nuclear decay. We can remember that on the nuclide table, an alpha decay removes two protons and two neutrons. Therefore, from the original location on the nuclide table you move left two places, and down two places to find the new nuclide created. For example, a typical alpha decay is one like radon.

$$^{226}_{88}\mathrm{Ra}_{138} \rightarrow {}^{222}_{86}\mathrm{Rn}_{136} + \alpha + 4.87\,\mathrm{MeV} \tag{3.2.2}$$

[13]Technically there are interactions since a neutron has a magnetic moment, but for us, they are basically immune to electromagnetic fields.

[14]I will not use SI units in favor of the ubiquitous electron volt, because it is how most nuclear physics references state results. Remember $|e| \approx 1.60 \times 10^{-19}$ is the conversion factor from 1 eV to 1 J.

[15]Remember 1 eV $\sim$ 11 605 K = 11 300 °C so 9 meV $\approx$ 100 °C.

For beta decay, we need to know if it is a minus or plus decay. A minus decay results in a neutron decaying into a proton, an electron and an electron antineutrino.[16] A plus decay result in the conversion of a proton into a neutron, a positron, and an electron neutrino. Note that the total number of protons and neutrons remains the same in a beta decay process.

So a beta minus decay decreases the neutron number by one and increases the proton number by one. On the nuclide table, we find the resulting nuclide by moving up one and to the left one from the original nuclide. A typical beta (minus) decay would be that of tritium

$$\mathrm{T} \rightarrow {}^{3}_{2}\mathrm{He}_1 + e^- + \bar{\nu}_e + 18.6\,\mathrm{keV} \tag{3.2.3}$$

where I have included neutrinos as ν and the bar indicates an antiparticle.[17] In nuclear terminology, this process changes a nuclide into an isobar (because the total number of protons and neutrons, the mass number, is conserved).

A beta plus decay occurs when a proton transforms into a neutron and positron (and neutrino). A typical process is given by

$${}^{23}_{12}\mathrm{Mg}_{11} \rightarrow {}^{23}_{11}\mathrm{Na}_{12} + e^+ + \nu_e \tag{3.2.4}$$

And so in a beta plus, we find the resultant nuclide by moving down one and to the right one from the original nuclide.

The way to think about this is that beta plus decay occurs in proton-rich nuclei and beta minus decay occurs in neutron-rich nuclei. In turns out that in general, a stable nucleus requires a balance of neutrons and protons. It is also important to realize that beta plus decay is not the same as proton decay, because beta plus decay is a nuclear process in a bound nucleus (whereas proton decay is for a bare proton).

Gamma decay occurs because there can be metastable (excited) nuclei. That is, the nucleus is not in its ground state. This typically occurs after an alpha or beta decay, and the excited state is often indicated by a superscript asterisk next to the nucleus. Cobalt to nickel is an example. This occurs in three steps. First cobalt changes to nickel via beta minus decay, and then goes from one metastable state to a lower metastable state by gamma decay. Then it goes from this lower metastable state to an unexcited state by another gamma decay. This is shown as

$${}^{60}_{27}\mathrm{Co}_{33} \rightarrow {}^{60}_{28}\mathrm{Ni}^* + e^- + \bar{\nu}_e + \gamma + 1.17\,\mathrm{MeV} \tag{3.2.5}$$

$${}^{60}_{28}\mathrm{Ni}^* \rightarrow {}^{60}_{28}\mathrm{Ni} + \gamma + 1.33\,\mathrm{MeV} \tag{3.2.6}$$

For neutron emission, the process is simple. You have a nuclide that is neutron-rich, and so to become more stable it emits a neutron or neutrons to get there. On the nuclide table, for a single neutron emission you simply move to the left one, as one neutron is removed from the nuclide. Some examples are for beryllium and helium

$${}^{13}_{4}\mathrm{Be}_9 \rightarrow {}^{12}_{4}\mathrm{Be}_8 + n \tag{3.2.7}$$

$${}^{5}_{2}\mathrm{He}_3 \rightarrow {}^{4}_{2}\mathrm{He}_2 + n \tag{3.2.8}$$

[16] Do not worry about neutrinos and antineutrinos. Consult a particle physics reference if you wish to understand them better.

[17] The neutrinos aren't important for the processes we are concerned with, but are important for balancing nuclear reactions.

For electron capture, the process is for proton-rich nuclides. On the table of nuclides you move down one and one to the right. An example of electron capture is given by aluminium[18]

$$^{26}_{13}\mathrm{Al}_{13} + e^- \rightarrow {}^{26}_{12}\mathrm{Mg}_{14} + \nu_e \tag{3.2.9}$$

As you can see, decay is actually not as simple as it may at first seem. Real radioactive decay at the nuclide level is usually a series of different radioactive processes. In addition, there are quite a few exotic possibilities such as double decays, three-body collisions, etc. These are usually exceedingly rare processes, but nuclear physicists have often characterized them nevertheless. We won't worry about the exotic processes, since they are not of much consequence for producing electrical energy.

All of the previous processes are spontaneous, in the sense that they naturally occur in nature all of the time. If we just have pieces of the element and do not do anything special, the above decays will occur naturally. However, as humans, we can alter the conditions so that processes become more likely to occur. When we do this with radioactive processes, we call this induced radioactivity. This just means what was a radioactively stable nuclide was changed into an unstable nuclide by some human intervention.

One way of doing this is bombarding nuclei with neutrons. These can lead to nuclear reactions as they can physically break up the nucleus. If, in the process of doing so, the reaction releases more neutrons, it can also lead to a chain reaction and an explosive or steady release of energy through nuclear fission. Since we are focused on fusion, I will not explore this process. If you'd like an overview with a good amount of history, consult Rhodes[15] or equivalent texts[17][1].

3.3 Cross Sections

> *It was quite the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you.*
>
> — Ernest Rutherford, [This refers to the discovery of the nucleus.]

I mentioned exotic processes, but how do we know if a process is exotic? Unlike in common parlance, exotic here means rare rather than occurring in a foreign land or culture. If some process is very unlikely to occur, then it is an exotic process. This sounds very good in words, but it would be better to have something quantitative to point at. What we want to characterize now is how easily a reaction can occur in a collision. For fusion, it is important that two nuclei "hit"[19] each other, and so we will look at the probability of such an event occurring.

The way of describing this is with cross sections. The reason for calling it a cross section comes from examining a simple case, two hard spheres of radius R_a and R_b. We then want to know what is the total area of the region where the spheres will hit each other. The spheres will only hit if their radii overlap. Thus the total area will be $\pi(R_a + R_b)^2$. If the particles are greater than $R_a + R_b$ apart, then they will clearly miss. Thus, the cross section $\sigma = \pi(R_a + R_b)^2$. Unfortunately

[18] More commonly called aluminum in the US. Aluminum is completely acceptable, but I think it's better to support the international standard.

[19] The terminology is that of a classical collision, though as always, the quantum shroud makes such classical depictions somewhat incorrect. The collision terminology does not mislead very much, though, and so it serves as a better way of remembering what is happening.

cross section is often used for differential cross section, which we will explore shortly. The cross section σ is often called the integral cross section or total cross section to make it clear that it is not per angle. However, the term total cross section should be avoided, because total cross section is usually associated with the cross section for all of the different decay/collision processes. Luckily, context usually makes it very clear whether we are talking of the integral cross section or differential cross section when speaking of a cross section.

The main idea is that the transverse area[20] that a particle must be in, in order to "hit" or collide with a target gives us an idea of how likely such a process is of occurring. The larger the cross section, the more probable the collision is.

Another cross section is the differential cross section. This yields the transverse area for a specific energy, angle, impact parameter,[21] or some other variable(s). The integral cross section is then arrived at via integrating over the variables. If the cross section is over a solid angle, then the differential cross section is typically written as $\mathrm{d}\sigma/\mathrm{d}\Omega$ with $\mathrm{d}\Omega = 2\pi \sin\theta \,\mathrm{d}\theta$ the solid angle for azimuthally symmetric situations.[22] By convention, the differential impact parameter is forced to be a positive number. In general we could write

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta, \varphi) \tag{3.3.1}$$

as the differential cross section with Ω a solid angle variable such that

$$\oiint \mathrm{d}\Omega \; \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \; \sin\theta \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \sigma \tag{3.3.2}$$

The Rutherford differential cross section for the Coulomb interaction is one of the most famous results in classical collisional physics. It is typically written with the impact parameter b as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \left(\frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 m v_0^2}\right)^2 \csc^4\frac{\theta}{2} \tag{3.3.3}$$

where v_0 is the speed at infinite distances and m is the reduced mass of the two particles. We can see the trajectory and definition of the angle θ and impact parameter b in Figure 3.1.

Note that the Rutherford integral cross section[23] can be written such that it only depends on the relative velocity between the target and the projectile.

Now, in reality, we do not figure out the exact potential from the nucleus and the neutron (this would require writing down what the strong force interaction is between the projectile and all of the nucleons, a quite difficult task). Instead, cross sections are calculated empirically by bombarding known materials (of a known number density). Because the radii of nuclei are on the order of 10 femtometers, 1×10^{14} m, then one would guess cross sections for nuclear reactions are often

[20] That is the area in the plane perpendicular to the line joining the two particles.

[21] Impact parameter is usually represented as b and is the perpendicular distance between the target and projectile.

[22] The azimuthal angle φ usually has symmetry and so can be ignored as a dependent variable. This just means we can rotate our system such that the change in φ does not affect any of our predictions.

[23] This is also called the microscopic cross section.

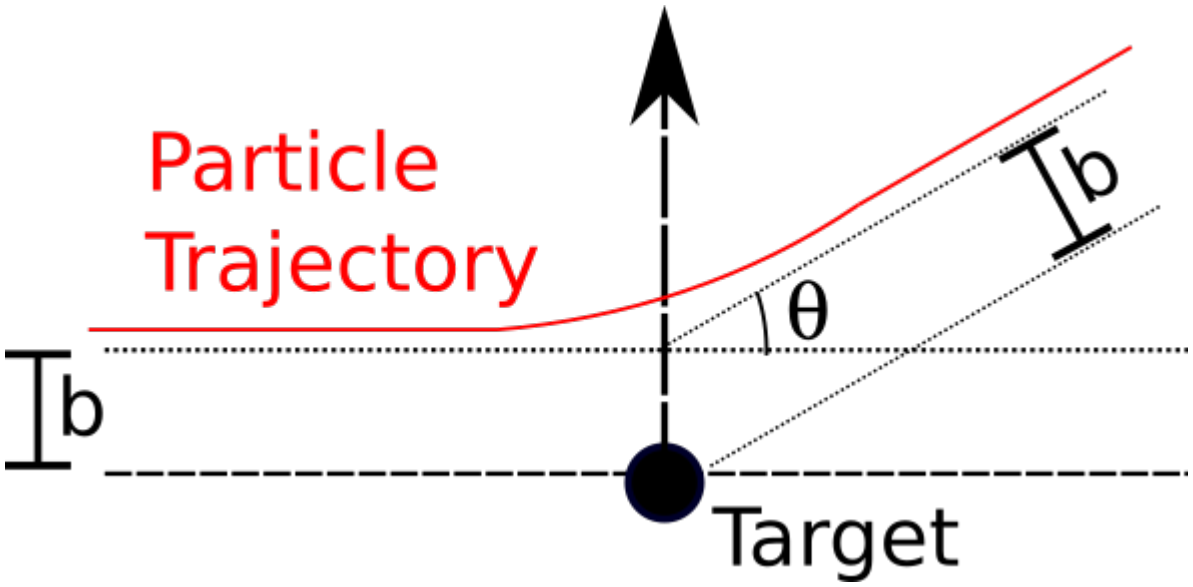

Figure 3.1: This shows the particle trajectory for Rutherford scattering of a particle off of a target with defined angle θ and impact parameter b.

about $\sim (1 \times 10^{14}\,\mathrm{m})^2 = 1 \times 10^{28}\,\mathrm{m}^2$. This is the reason that this area is given the special name of "barn".[24]

Now, to get a cross section empirically, it is convenient to think of a frame of reference where you create a beam of particles (the projectiles) that hit a stationary target material. Suppose for simplicity of the picture that we consider only a single target. Then if we have a beam of particles traveling at velocity v_b and the cross section for the target (for the reaction r we desire) is σ_r, then there will be a number of reactions in time Δt corresponding to every beam particle in the cylinder of volume $\sigma_r(v_b \Delta t)$. If there are n_b beam particles in a volume then the total number of reactions $r_r \Delta t$ in this time (with r_r the reaction rate) is then given by

$$r_r \Delta t = n_b \sigma_r v_b \Delta t \tag{3.3.4}$$

$$r_r = n_b v_b \sigma_r = \Phi_b \sigma_r \tag{3.3.5}$$

where $\Phi_b = n_b v_b$ is the flux of the beam particles. Note that this was for one target. If there are n_t targets in a unit volume (and so target number density is n_t), then we have roughly n_t of the cylinders per unit volume[25] and so the total reaction rate per volume R_r is in these cases

$$R_r = n_t \Phi_b \sigma_r \tag{3.3.6}$$

In reality, the cross section is a measured quantity by setting up a beam into a target and counting the number of reactions per time. When we go to the more realistic empirical case, we simply

[24] Back during the Manhattan project, the cross section was estimated as above and physicists wanted a secretive way of referring to cross sections that would not raise suspicions in conversations. Whether or not talking about "barns" in this manner would keep the conversation non-suspicious, I am a bit skeptical. In any case, despite the secretive reason for naming the unit "barn", it is now widespread in particle physics. It is also said that "barn" was apt as it connects to the idiom "couldn't hit the broad side of a barn".

[25] This assumption is that $\sqrt{\sigma} \ll n_t^{-1/3}$ so that the targets are spaced out such that cross sections for each target don't overlap.

define σ_r via the above relation, and so the cross section can theoretically depend on n_t, n_b, or v_b. In reality, σ_r is highly dependent on the initial beam particle energy, but not as sensitive to n_t and n_b and so we assume $\sigma_r = \sigma_r(v_b)$. Obviously this approximation could break down if $\sqrt{\sigma} \simeq n_t^{-1/3}$ or if we go to extreme number densities far beyond what is normally encountered for nuclear reaction measurements. In addition, we can consider the reaction rate for a distribution of beam velocities. That is, instead of a beam velocity, we consider a distribution of velocities. In this case, the reactivity is usually written as $\langle \sigma v \rangle$ and is related to the reaction rate per volume R_r for process r with target velocity $\mathbf{v}_t$ and beam velocity $\mathbf{v}_b$ via the velocity $\mathbf{v}' = \mathbf{v}_b - \mathbf{v}_t$ ($v' = |\mathbf{v}'|$) with velocity distributions for the target $f_t(\mathbf{v}_t)$ and beam $f_b(\mathbf{v}_b)$ as

$$R_r = n_t n_b \langle \sigma_r |v_b - v_t| \rangle = n_t n_b \langle \sigma_r v' \rangle \equiv \iiint_{-\infty}^{\infty} \mathrm{d}^3 v_t \iiint_{-\infty}^{\infty} \mathrm{d}^3 v_b \; \sigma(v') v' f_t(\mathbf{v}_t) f_b(\mathbf{v}_b) \tag{3.3.7}$$

This form is not particularly illuminating yet, however. We will consider the Maxwellian case later on. If the target and beam are of the same material, then the above is actually an overcount[26] and we must use

$$R_r = \frac{n_t n_t}{2} \langle \sigma_r v' \rangle \tag{3.3.8}$$

The reaction rate is usually what we actually care about, as it yields the number of reactions occurring per volume per time and so determines how much energy is being released.

One last word of caution, again concerning the cross section σ_r. It is also called the microscopic cross section. Sometimes a macroscopic cross section is used, defined by $\Sigma_t = n_t \sigma_t$. This means that Σ_t has units of inverse length, and so is even more divorced from a cross section than the microscopic cross section. The macroscopic cross section is useful because it can be easier to measure.

3.4 Fusion Reactions

> *When we look up at night and view the stars, everything we see is shining because of distant nuclear fusion.*
>
> — Carl Sagan

Now let us consider the processes most important for us from a magnetic confinement perspective. The first thing to consider is how hard it is for a two positively charged particles to hit each other. Clearly the Coulomb interaction, which is repulsive, will be dominant over distances greater than the rough size of a nucleus. Since a nucleus is on the order of femtometers, we can use this as a cutoff, and below that the strong force's attraction overcomes the Coulomb barrier. Suppose we consider the Coulomb barrier height near where it switches. Then we can approximate it as being about E_C in comparison to the strong force's potential valley. Classically with a Boltzmann distribution of particles with energy E we would expect that the probability of a particle getting

[26] We are in essence counting each reaction as happening twice because we cannot distinguish between the beam and target, so instead we pretend we can and divide by two. Think about the case where we have two indistinguishable balls and N distinguishable containers. We can count the number of configurations by first assuming the balls are distinguishable. Then there are $N(N-1)$ ways. However, for each of these configurations we could switch the location of the balls so we have overcounted. There are 2 ways of rearranging the balls for each configuration so the final answer is $N(N-1)/2$.

over the barrier E_C to be $\exp(-E_C/E)$, the proportion of particles with energy $E > E_C$. In fact this is too restrictive as it omits the possibility of quantum tunneling through the potential. Remember that this probability is for a given collision being in a regime where a fusion event is possible, and not the probability of a fusion event actually occurring.[27] Quantum mechanical scattering can yield a value for E_C given by

$$E_C = \frac{Z_1^2 Z_2^2 e^4 \mu}{8\epsilon_0^2 \hbar^2} \approx \frac{Z_1 Z_2 \mu}{m_p} 0.99\,\text{MeV} \tag{3.4.1}$$

where Z_1 and Z_2 are charges for the two particles and μ is the reduced mass of the system. The probability of penetrating the barrier quantum mechanically (that is, given the Coulomb ptotential, what is the probability of a collision, not necessarily of a fusion event) is then given by[12]

$$P \sim \exp\left(-\sqrt{\frac{E_C}{E}}\right) \tag{3.4.2}$$

with the square root coming from the use of a JWKB estimate.[28] E_C is sometimes denoted E_G and is called the Gamow energy.[29]

A cartoon drawing, but a good enough approximation of the potential shape for us, is shown in Figure 3.2.

3.4.1 Stellar Fusion Reactions

It is illuminating to consider the one (so far) functioning fusion reactor in our solar system, our sun. One can consider the powerhouse reactions for stellar fusion. This turns out to not be a trivial process, but a chain of reactions that leads from protons to helium. We begin with

$$p + p \to D + e^+ + \nu_e \overset{\sim}{\to} D + \nu_e + 2\gamma + 1.44\,\text{MeV} \tag{3.4.3}$$

Because we live in a matter universe, the positron will almost always end up annihilating with an electron and forming two gamma rays hence the $\overset{\sim}{\to}$. This process is not very feasible anywhere but in stars since a single proton waits on average almost 10 billion years before it reacts with another proton. Luckily for us, the sun has much more than 10 billion protons, and keeps them together for extended periods of time.

It is instructive to consider the quantum and classical predictions for the probability of a collision. The actual (quantum influenced) probability of a collision was given before as $P \sim \exp\left(\sqrt{E_C/E}\right)$ with the Gamow energy (sometimes called the Coulomb energy[30]) defined as in (3.4.1). Then for the sun with a core temperature of about $1.4\,\text{keV}{=}1.6 \times 10^7\,\text{K}{=}16\,\text{MK}$, and $\mu = m_p/2$ we'd find

$$P \sim \exp\left(-\sqrt{500\,\text{keV}/1.4\,\text{keV}}\right) \approx \exp(-18.9) \approx 10^{-9} \tag{3.4.4}$$

[27] Even if the collision occurs, there may be other circumstances that drastically change the probability of a fusion event actually occurring.

[28] Effectively, this comes from the fact that the JWKB estimate has $\exp(\int dx' \sqrt{U(x')})$ and $U(x') \sim 1/x'$.

[29] Alas, sometimes the Gamow energy is defined differently depending on whether you are caring about a volume rate or a rate through a surface.

[30] I wouldn't recommend calling it this, since it E_C comes from a quantum mechanical estimate, but you still see it called this in some physics or astronomy literature.

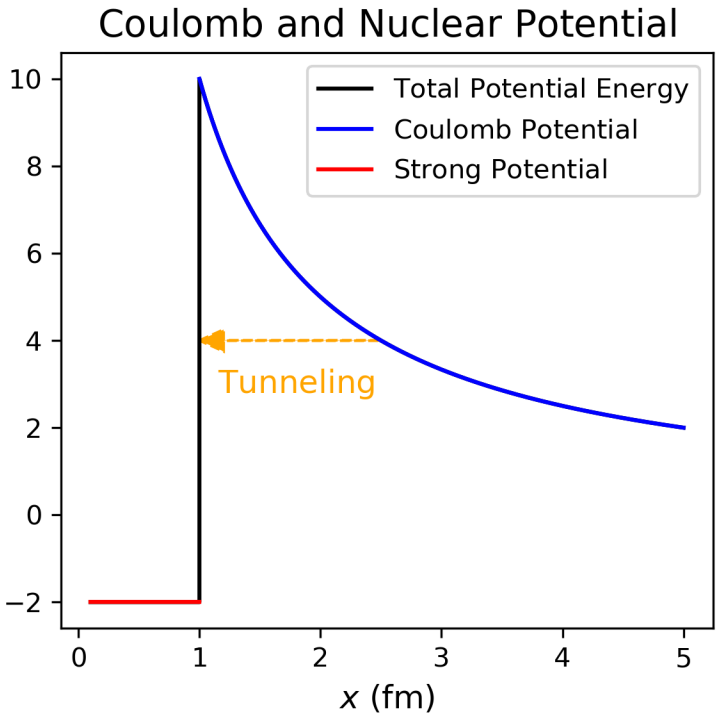

Figure 3.2: This shows a cartoon-figure for the potential a positive particle sees as it approaches a nucleus. As it gets close to about 1 fm the nuclear potential dominates and brings it down to a low level. Quantum tunneling allowing one to access the nuclear potential in classically forbidden regions, is also represented by the dashed arrow.

Classically, we would simply consider the Boltzmann fraction of particles with energy greater than 500 keV, which are the only ones that have enough energy to penetrate the Coulomb barrier, because there is no quantum tunnelling. This would lead to an estimate

$$P_{\text{classical}} \sim \exp(-E_C/E) \approx \exp\left(500\,\text{keV}/1.4\,\text{keV}\right) \approx 10^{-155} \tag{3.4.5}$$

which would mean it would be essentially impossible in stellar cores like our sun. I want you to stop and think about this difference because it is mind-boggling. The difference is like if you classically expected \$0.01 or one (US) cent, but were given $\$10^{144}$. If you gave each particle in the observable universe (all 10^{80} of them) the world's total GDP per year of about $\$10^{14}$ or 100 trillion US dollars every attosecond (10^{-18} seconds) of the year, you would have to wait nearly 3×10^{24} yr to exhaust the $\$10^{144}$. The universe has only lasted about 13×10^9 yr, so you could wait 100 trillion times the age of the universe before you ran out of money!

Note that a cross section for proton-proton fusion is actually lower than the probability of two protons getting close enough by penetrating the Coulomb barrier because the protons need a weak interaction beta decay in addition to getting close enough. Most weak interactions take a long time in comparison to electromagnetic or strong interactions, and so the necessity of a weak interaction actually imposes quite a penalty on the likelihood of a successful fusion event. The energy dependent fusion cross section is usually written in the form

$$\sigma(E) = \frac{S(E)}{E}\exp(-\sqrt{E_C/E}) \tag{3.4.6}$$

with $S(E)$ a factor that takes into account the probability of the fusion once the nuclei come into contact. One can estimate the fusion reaction rate in our sun via the fusion rate being $6 \times 10^{11}\,\text{kg}\,\text{s}^{-1}$ of hydrogen which translates into a number of protons produced of about $4 \times 10^{38}\,\text{s}^{-1}$. We will use a central sun density (n_0) of $10^{31}\,\text{m}^{-3}$ with exponential number density given by $n_0 \exp(-\frac{x \ln 2}{0.1 R_\odot})$ and $R_\odot$ being a solar radius (6.96×10^8 m). The solar core then has about 1×10^{56} protons. So the fusion reaction rate per proton is about $4 \times 10^{-18}\,\text{s}^{-1}$. Thus, on average it takes a proton $1/(4 \times 10^{-18}\,\text{s}^{-1})$ or about 8 billion years for a proton to fuse, and justifies our earlier claims of about 10 billion years. Because the central number density of the sun is so large at $10^{31}\,\text{m}^{-3}$ this allows substantial energy production with a process that is so rare. At the very core then, we have a volume reaction rate of $4 \times 10^{13}\,\text{m}^{-3}\,\text{s}^{-1}$. One final thing to consider is what the likelihood of a collision is. We use the collision frequency at the core to be estimated by (we use the reduced de Broglie wavelength of the hydrogen as an estimate for the cross section $\lambda\!\!\!^{-}_{dB} = \frac{\hbar}{m_p v_{\text{th}}}$ with $v_{\text{th}} = \sqrt{2k_B T/m_p} \approx 5.2 \times 10^5\,\text{m}\,\text{s}^{-1}$ this implies $\lambda\!\!\!^{-}_{dB} \approx 1.2 \times 10^{-13}\,\text{m}$)[31]

$$\nu = n\pi\lambda\!\!\!^{-2}_{dB} v_{\text{th}} \approx \pi(10^{31}\,\text{m}^{-3})(1.2 \times 10^{-13}\,\text{m})^2(5.2 \times 10^5\,\text{m}\,\text{s}^{-1}) \approx 2 \times 10^{11}\,\text{s}^{-1} \tag{3.4.7}$$

Which would imply a probability of fusion upon collision of $4 \times 10^{-18}\,\text{s}^{-1}/2 \times 10^{11}\,\text{s}^{-1} \approx 2 \times 10^{-29}$ which is much smaller than our previous estimate based on the Gamow energy of 10^{-9}. This is because we ignored the $S(E)/E$ portion of the cross section, which for a weak interaction dependent process can be quite small.

[31] We use the de Broglie wavelength because this serves as an effective quantum cutoff for us. We use the reduced value for convenience as it is better for cross section estimates. See Section 3.5.1.1 for the details.

The process of producing energy requires a chain, and the deuteron then produced by the proton-proton reaction can interact with the surrounding solar material to produce much more energy per reaction.

The deuteron just produced can undergo a reaction

$$D + p \to {}^{3}_{2}\mathrm{He}_1 + \gamma + 5.49\,\mathrm{MeV} \tag{3.4.8}$$

This process is mediated by the strong force and usually occurs on a scale of seconds for a deuteron (this is mostly saying that the $S(E)$ is much more favorable and so the Gamow energy estimate is more accurate).[32]

There are three main ways for the resulting ${}^{3}_{2}\mathrm{He}_1$ to be converted into the more stable ${}^{4}_{2}\mathrm{He}_2$, with the first two dominating the probability. They are usually labeled *p-p* I-III.

$$2\,{}^{3}_{2}\mathrm{He}_1 \to {}^{4}_{2}\mathrm{He}_2 + 2\,{}^{1}_{1}\mathrm{H} + 12.9\,\mathrm{MeV} \tag{*p-p* I}$$

The total process releases about 26.8 MeV and about 83% of reactions take this route in our sun. It releases 26.7 MeV because if we start only with protons then we have to have both (3.4.3) and (3.4.8) occur twice in order to finish (*p-p* I). It is dominant for temperatures greater than 10 MK (about 860 eV) but less than 14 MK (about 1200 eV).

The *p-p* II route involves lithium and so is often called lithium burning. It is a chain

$$\begin{aligned} {}^{3}_{2}\mathrm{He}_1 + {}^{4}_{2}\mathrm{He}_2 &\to {}^{7}_{4}\mathrm{Be}_3 + 1.59\,\mathrm{MeV} \\ {}^{7}_{4}\mathrm{Be}_3 + e^- &\to {}^{7}_{3}\mathrm{Li}_4 + \nu_e + 1.37\,\mathrm{MeV} \\ {}^{7}_{3}\mathrm{Li}_4 + p &\to 2\,{}^{4}_{2}\mathrm{He}_2 + 16.8\,\mathrm{MeV} \end{aligned} \tag{*p-p* II}$$

This chain occurs about 17% of the time in our sun and is dominant for 14 MK (about 1.2 keV) to 23 MK (about 2.0 keV).

The *p-p* III route also involves beryllium and boron

$$\begin{aligned} {}^{3}_{2}\mathrm{He}_1 + {}^{4}_{2}\mathrm{He}_2 &\to {}^{7}_{4}\mathrm{Be}_3 + \gamma + 1.59\,\mathrm{MeV} \\ {}^{7}_{4}\mathrm{Be}_3 + p &\to {}^{8}_{5}\mathrm{B}_3 + \gamma - 0.374\,\mathrm{MeV} \\ {}^{8}_{5}\mathrm{B}_3 &\to {}^{8}_{4}\mathrm{Be}_4 + e^+ + \nu_e + 17.5\,\mathrm{MeV} \\ {}^{8}_{4}\mathrm{Be}_4 &\to 2\,{}^{4}_{2}\mathrm{He}_2 + 0.0918\,\mathrm{MeV} \end{aligned} \tag{*p-p* III}$$

This chain only occurs 0.02% of the time in our sun but is believed to be important because it generates neutrinos with high energy and is dominant for 23 MK (about 2.0 keV) or higher.

There is one more twist to the story. For stars with larger masses than our sun (about 1.3 times), a different process dominates. This is called the CNO process. This stands for carbon-nitrogen-oxygen. In these cases, carbon, nitrogen, and oxygen catalyze fusion reactions. There are again detailed paths for how this can occur, but the main reaction is given by

$$4p + 2e^- \to \alpha + 2e^+ + 2e^- + 2\nu_e + 3\gamma + 24.7\,\mathrm{MeV} \to \alpha + 2\nu_e + 7\gamma + 26.7\,\mathrm{MeV} \tag{3.4.9}$$

[32] You can already begin to see why terrestrial fusion eyes deuterons over protons. If we use that the Gamow energy estimate of probability of fusion per collision of about 10^{-10} with a similar collision frequency, as computed above, then the reaction rate is about $10^{-10}(2 \times 10^{11}\,\mathrm{s}^{-1}) = 2\,\mathrm{s}^{-1}$. Such a rough estimate shows the average lifetime of a deuteron is on the order of seconds.

The step in between the arrows is where carbon, nitrogen, and oxygen nuclides make the end result reactions by capturing protons (that is a nucleus gains one proton via fusing a proton with some other nuclide) and then beta decaying.

The reason that larger mass stars are dominated by CNO is because of the temperature dependence of all these reactions. A self-sustaining proton-proton chain can start in stars near 4 MK (about 350 eV) but the CNO process only becomes self-sustaining near 15 MK (about 1.3 keV) and rapidly becomes more dominant (that is, it releases more energy more quickly as temperature rises) such that it is the dominant process for temperatures above 17 MK (about 1.5 keV). You can see that our sun just barely misses this with a core temperature of 15.7 MK (about 1.4 keV).

3.4.2 Terrestrial Fusion

It is now time to look at reactions that are of interest for fusion reactions on Earth (or terrestrial fusion). We previously mentioned that proton-proton fusion requires nearly 10 billion years for a single proton in the sun. This immediately tells us a couple of things. We either need a lot of protons in one place or this process is not going to release energy on a timescale useful to humans. For the sun, there is a massive number of protons held together by gravitation. We do not want to create a literal star on Earth,[33] so this method will not be what we desire. Later we will see how this can be quantified in the Lawson criterion, but for now let's focus on the nuclear reactions we will consider. This will be looking at the problem from a magnetic confinement perspective. Thus, we are using electromagnetic fields to contain particles and hoping they collide through random thermal motion.

There are a number of things we can consider for reactions that are potentially good for this type of energy production. Obviously we want reactions that release energy, but it would be better to use lower atomic number nuclei because then the Coulomb interaction is not as strong of a barrier. It is also ideal to have two reactants because getting more than two things to hit each other is difficult. In addition, it is better to have only two products so that energy and momentum conservation are simple (which also means that we do not get momentum or energy taken from the rest of the plasma possibly altering the equilibrium and causing stresses on the confining device). Finally, we don't want to rely on (weak interaction) radioactive decay within the device; these are reactions that do not conserve proton and neutron number separately. This is mostly because (weak interaction) radioactive decay doesn't happen often enough.[34] We see that this step is the main reason we cannot rely on stellar processes because the proton-proton reaction relied on a beta decay which does not conserve the number of protons or neutrons (separately). Stellar processes usually rely on beta decays which are too slow for us, because we cannot hold the particles together as long as a star can (nor can we hold a gigantic number of them together for a shorter period of time).

Some reactions that satisfy these requirements include

$$D + T \rightarrow {}^{4}_{2}\mathrm{He}_2 + n + 17.6\,\mathrm{MeV} \tag{3.4.10}$$

$$D + D \rightarrow T + p + 4.03\,\mathrm{MeV} \tag{3.4.11}$$

[33] If you think you do, rethink what having a literal star "on" Earth would mean. Here's a hint. There would be no "Earth" left for the star to be on. There would just be a star.

[34] If the radioactive decay were the main source of energy in the process, you would also wonder if the fusion reaction part is even necessary.

$$D + D \rightarrow {}^{3}_{2}\mathrm{He} + n + 3.27\,\mathrm{MeV} \tag{3.4.12}$$
$$D + {}^{3}_{2}\mathrm{He} \rightarrow {}^{4}_{2}\mathrm{He} + p + 18.3\,\mathrm{MeV} \tag{3.4.13}$$

We can use (3.1.15) in these reactions to find what proportion of the kinetic energy is distributed to the reactants. We simply use

$$\frac{m_{{}^{4}_{2}\mathrm{He}}}{m_n + m_{{}^{4}_{2}\mathrm{He}}} \approx 0.799 \Rightarrow K_n \approx 14.1\,\mathrm{MeV} \tag{3.4.14}$$
$$\frac{m_n}{m_n + m_{{}^{4}_{2}\mathrm{He}}} \approx 0.201 \Rightarrow K_{{}^{4}_{2}\mathrm{He}} \approx 3.54\,\mathrm{MeV} \tag{3.4.15}$$
$$\frac{m_T}{m_T + m_p} \approx 0.750 \Rightarrow K_p \approx 3.02\,\mathrm{MeV} \tag{3.4.16}$$
$$\frac{m_p}{m_T + m_p} \approx 0.250 \Rightarrow K_T \approx 1.01\,\mathrm{MeV} \tag{3.4.17}$$
$$\frac{m_{{}^{3}_{2}\mathrm{He}}}{m_n + m_{{}^{3}_{2}\mathrm{He}}} \approx 0.749 \Rightarrow K_n \approx 2.45\,\mathrm{MeV} \tag{3.4.18}$$
$$\frac{m_n}{m_n + m_{{}^{3}_{2}\mathrm{He}}} \approx 0.251 \Rightarrow K_{{}^{3}_{2}\mathrm{He}} \approx 0.821\,\mathrm{MeV} \tag{3.4.19}$$
$$\frac{m_{{}^{4}_{2}\mathrm{He}}}{m_p + m_{{}^{4}_{2}\mathrm{He}}} \approx 0.799 \Rightarrow K_p \approx 14.6\,\mathrm{MeV} \tag{3.4.20}$$
$$\frac{m_p}{m_p + m_{{}^{4}_{2}\mathrm{He}}} \approx 0.201 \Rightarrow K_{{}^{4}_{2}\mathrm{He}} \approx 3.68\,\mathrm{MeV} \tag{3.4.21}$$

The DD reactions occur with about 50% probability for each (all branching ratios are given at cross section peaks). If we relax our restrictions, we can go for some more reactions (some of these happen if we primarily use the previous reactions) with the percentage of the time a particular reaction branch is used indicated in square brackets

$$T + T \rightarrow {}^{4}_{2}\mathrm{He} + 2n + 11.3\,\mathrm{MeV} \tag{3.4.22}$$
$${}^{3}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} \rightarrow {}^{4}_{2}\mathrm{He} + 2p + 12.9\,\mathrm{MeV} \tag{3.4.23}$$
$${}^{3}_{2}\mathrm{He} + T \rightarrow {}^{4}_{2}\mathrm{He} + p + n + 12.1\,\mathrm{MeV} \quad [57\%] \tag{3.4.24}$$
$${}^{3}_{2}\mathrm{He} + T \rightarrow {}^{4}_{2}\mathrm{He} + D + 14.3\,\mathrm{MeV} \quad [43\%] \tag{3.4.25}$$
$$p + {}^{6}_{3}\mathrm{Li} \rightarrow {}^{4}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} + 4.0\,\mathrm{MeV} \tag{3.4.26}$$
$$p + {}^{11}_{5}\mathrm{B} \rightarrow 3\,{}^{4}_{2}\mathrm{He} + 8.7\,\mathrm{MeV} \tag{3.4.27}$$

Another consideration is that we would like to avoid neutrons because they can activate [i.e., make radioactive] the confining materials and cause damage to the confinement device and magnets.

The cross sections and reactivities are most advantageous for DD and DT fusion, as we will soon learn, with DT the most advantageous. One of the other problems is the relative abundances of the elements. Tritium is not stable with a 12.3 year half life and so if this reaction is to be primary then a method of producing more tritium must be introduced. Fortunately for fusion, neutrons bombarding lithium leads to significant tritium production

$$n + {}^{6}_{3}\mathrm{Li} \rightarrow T + {}^{4}_{2}\mathrm{He} + 4.78\,\mathrm{MeV} \tag{3.4.28}$$
$$n + {}^{7}_{3}\mathrm{Li} \rightarrow T + {}^{4}_{2}\mathrm{He} + n - 2.76\,\mathrm{MeV} \tag{3.4.29}$$

The lithium-7 reaction has unfavorable cross sections unless the impinging neutrons are high energy.[35] For fusion, it is imagined that a lithium-6 enriched "blanket" will be needed to produce the required tritium. This blanket will also require some other element(s) that will improve the likelihood of neutron and lithium-6 reactions.

3.4.2.1 Cross Section and Reactivity Values

First let's consider our reaction rate per volume

$$R_r \equiv \iiint_{-\infty}^{\infty} \mathrm{d}^3 v_t \iiint_{-\infty}^{\infty} \mathrm{d}^3 v_b \; \sigma(v') v' f_t(\mathbf{v}_t) f_b(\mathbf{v}_b) \tag{3.4.30}$$

in the case of Maxwellian distributions

$$f_i = \frac{n_i}{\pi^{3/2} v_{\mathrm{th}_i}^3} \exp\left(-\frac{v_i^2}{v_{\mathrm{th}_i}^2}\right) \tag{3.4.31}$$

$$v_{\mathrm{th}_i}^2 = \frac{2k_B T_i}{m_i} \tag{3.4.32}$$

And we can introduce the total velocity $\mathbf{V} = \mathbf{v}_t + \mathbf{v}_b$. The argument of the exponentials can then be rewritten

$$\frac{v_t^2}{v_{\mathrm{th}_t}^2} + \frac{v_b^2}{v_{\mathrm{th}_b}^2} = \frac{v_{\mathrm{th}_b}^2 v_t^2 + v_{\mathrm{th}_t}^2 v_b^2}{v_{\mathrm{th}_b}^2 v_{\mathrm{th}_t}^2} \tag{3.4.33}$$

$$|\mathbf{V}|^2 = v_t^2 + v_b^2 + 2\mathbf{v}_t \cdot \mathbf{v}_b \tag{3.4.34}$$

If we assume that both are at the same temperature, then we can define $v_{\mathrm{th}}^2 = 2k_B T/\mu$ with μ the reduced mass $\mu = m_t m_b/(m_t + m_b)$. We'd like to get everything in terms of $\mathbf{v}'$ and $\mathbf{V}$ and we assume that we can factor out v'^2/v_{th}^2 out. We note that this is

$$\frac{|\mathbf{v}'|^2}{v_{\mathrm{th}}^2} = \mu \frac{v_t^2 + v_b^2 - 2\mathbf{v}_t \mathbf{v}_b}{2k_B T} \tag{3.4.35}$$

and so

$$\begin{aligned} 2k_B T \left(\frac{v_t^2}{v_{\mathrm{th}_t}^2} + \frac{v_b^2}{v_{\mathrm{th}_b}^2} - \frac{|\mathbf{v}'|^2}{v_{\mathrm{th}}^2}\right) &= m_t v_t^2 + m_b v_b^2 - \frac{m_t m_b}{m_t + m_b}(v_t^2 + v_b^2 - 2\mathbf{v}_t \cdot \mathbf{v}_b) \\ &= v_t^2 \frac{m_t(m_t + m_b) - m_t m_b}{m_t + m_b} + v_b^2 \frac{m_b(m_t + m_b) - m_t m_b}{m_t + m_b} + 2\frac{m_t m_b}{m_t + m_b}\mathbf{v}_t \cdot \mathbf{v}_b \\ &= v_t^2 \frac{m_t^2}{m_t + m_b} + v_b^2 \frac{m_b^2}{m_t + m_b} + 2\frac{m_t m_b}{m_t + m_b}\mathbf{v}_t \cdot \mathbf{v}_b \end{aligned} \tag{3.4.36}$$

We can then use

$$\begin{aligned} (m_t + m_b)\mathbf{V} + (m_t - m_b)\mathbf{v}' &= (m_t + m_b)(\mathbf{v}_t + \mathbf{v}_b) + (m_t - m_b)(\mathbf{v}_t - \mathbf{v}_b) \\ &= \mathbf{v}_t\,(m_t + m_b + m_t - m_b) + \mathbf{v}_b\,(m_t + m_b + m_b - m_t) \\ &= 2m_t \mathbf{v}_t + 2m_b \mathbf{v}_b \end{aligned} \tag{3.4.37}$$

$$\frac{1}{4}[(m_t + m_b)\mathbf{V} + (m_t - m_b)\mathbf{v}']^2 = \frac{1}{4}[2m_t \mathbf{v}_t + 2m_b \mathbf{v}_b]^2 = m_t^2 v_t^2 + m_b^2 v_b^2 + 2m_t m_b \mathbf{v}_t \cdot \mathbf{v}_b \tag{3.4.38}$$

[35] The larger than expected yield of the hydrogen (thermonuclear fusion) bomb test Castle Bravo was due to scientists ignoring the lithium-7 reactions. Ivy Mike was the first hydrogen bomb test, but used cryogenic (liquid) deuterium directly, while Castle Bravo used lithium deuteride [i.e., a solid composed of deuterons and lithium].

so that

$$2k_BT\left(\frac{v_t^2}{v_{\mathrm{th}_t}^2}+\frac{v_b^2}{v_{\mathrm{th}_b}^2}-\frac{|\mathbf{v}'|^2}{v_{\mathrm{th}}^2}\right)=\frac{[(m_t+m_b)\mathbf{V}+(m_t-m_b)\mathbf{v}']^2}{4(m_t+m_b)} \tag{3.4.39}$$

We thus have (since $\mathbf{v}_t$ and $\mathbf{v}_b$ vary from $-\infty$ to ∞ in all directions, then $\mathbf{v}'$ and $\mathbf{V}$ will as well since they are just linear combinations of $\mathbf{v}_t$ and $\mathbf{v}_b$)[36]

$$R_r=\frac{n_tn_b}{8\pi^3v_{\mathrm{th}_t}^3v_{\mathrm{th}_b}^3}\iiint_{-\infty}^{\infty}\mathrm{d}^3v'\iiint_{-\infty}^{\infty}\mathrm{d}^3V\;\sigma(v')v'\exp\left(-\frac{[(m_t+m_b)\mathbf{V}+(m_t-m_b)\mathbf{v}']^2}{8k_BT(m_t+m_b)}\right)\exp\left(-\frac{v'^2}{v_{\mathrm{th}}^2}\right) \tag{3.4.40}$$

Now we can do the integral over $\mathbf{V}$ (d^3V). We can rewrite it as

$$\iiint_{-\infty}^{\infty}\mathrm{d}^3V\;\exp\left(-\alpha(\mathbf{V}+\beta\mathbf{C})^2\right) \tag{3.4.41}$$

$$\alpha=\frac{m_t+m_b}{8k_BT} \tag{3.4.42}$$

$$\beta=\frac{m_t-m_b}{m_t+m_b} \tag{3.4.43}$$

Because $\mathbf{V}$ and $\mathbf{v}'$ are now independent we note that within the $\mathbf{V}$ integral that we can use a change of variables $\mathbf{x}=\mathbf{V}+\beta\mathbf{C}$ and thus eliminate the $\mathbf{v}'$ dependence in the exponential for the $\mathbf{V}$ integral.[37] We use the fundamental Gaussian relation

$$\int_{-\infty}^{\infty}\mathrm{d}x\;e^{-\alpha x^2}=\sqrt{\frac{\pi}{\alpha}} \tag{3.4.44}$$

and

$$\iiint_{-\infty}^{\infty}\mathrm{d}^3x\;\exp\left(-\alpha|\mathbf{x}|^2\right)=\left(\frac{\pi}{\alpha}\right)^{3/2}=\left(\frac{8k_BT\pi}{m_t+m_b}\right)^{3/2} \tag{3.4.45}$$

and so

$$R_r=\frac{8^{3/2}(k_BT)^{3/2}n_tn_b}{8(m_t+m_b)^{3/2}\pi^{3/2}v_{\mathrm{th}_t}^3v_{\mathrm{th}_b}^3}\iiint_{-\infty}^{\infty}\mathrm{d}^3v'\;\sigma(v')v'\exp\left(-\frac{v'^2}{v_{\mathrm{th}}^2}\right) \tag{3.4.46}$$

We note the pre-factor ($8^{3/2}/8=2^{3/2}$) can be written as

$$\begin{aligned}\frac{2^{3/2}(k_BT)^{3/2}}{(m_t+m_b)^{3/2}v_{\mathrm{th}_t}^3v_{\mathrm{th}_b}^3}&=\frac{\cancel{2^{3/2}(k_BT)^{3/2}}}{(m_t+m_b)^{3/2}\left(\frac{\cancel{2k_BT}}{m_t}\right)^{3/2}\left(\frac{2k_BT}{m_b}\right)^{3/2}}\\&=\frac{m_t^{3/2}m_b^{3/2}}{(m_t+m_b)^{3/2}(k_BT)^{3/2}}=\left(\frac{\mu}{2k_BT}\right)^{3/2}=\frac{1}{v_{\mathrm{th}}^3}\end{aligned} \tag{3.4.47}$$

[36] By using $\mathbf{V}=\mathbf{v}_t+\mathbf{v}_b$ instead of $(\mathbf{v}_t+\mathbf{v}_b)/2$, our Jacobian for the transformation introduces a factor of 1/8. You should calculate this for yourself.

[37] This is a non-trivial statement. It means that the integral over d^3V gives the same answer so long as $\mathbf{C}$ is constant with respect to $\mathbf{V}$. It doesn't matter that $\mathbf{C}$ will vary in the outer integral because the integration removes any $\mathbf{C}$ dependence.

We can then convert the integral into spherical v' coordinates since the only dependence is on the magnitude of $\mathbf{v}'$.

$$\begin{aligned} R_r &= \frac{n_t n_b}{\pi^{3/2} v_{\text{th}}^3} \iiint_{-\infty}^{\infty} \mathrm{d}^3 v' \; \sigma(v') v' \exp\left(-\frac{v'^2}{v_{\text{th}}^2}\right) \\ &= \frac{n_t n_b}{\pi^{3/2} v_{\text{th}}^3} \int_0^{\infty} \mathrm{d}v' \; \sigma(v') v' \exp\left(-\frac{v'^2}{v_{\text{th}}^2}\right) \int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \; v'^2 \sin\theta \\ &= 4\pi \frac{n_t n_b}{\pi^{3/2} v_{\text{th}}^3} \int_0^{\infty} \mathrm{d}v' \; \sigma(v') v'^3 \exp\left(-\frac{v'^2}{v_{\text{th}}^2}\right) \\ &= \frac{4 n_t n_b}{\pi^{1/2} v_{\text{th}}^3} \int_0^{\infty} \mathrm{d}v' \; \sigma(v') v'^3 \exp\left(-\frac{v'^2}{v_{\text{th}}^2}\right) \end{aligned} \tag{3.4.48}$$

Finally, it can be useful to use impinging particle energy $\epsilon_b = \frac{m_b v'^2}{2}$ instead of using v' because energy is used more often in experiments and so the data is usually given in units of energy. We then use $\mathrm{d}\epsilon_b = m_b v' \, \mathrm{d}v'$ and

$$R_r = \frac{4 n_t n_b}{\pi^{1/2} v_{\text{th}}^3} \int_0^{\infty} \mathrm{d}v' \; \sigma(v') v'^3 \exp\left(-\frac{v'^2}{v_{\text{th}}^2}\right) = n_b n_t \sqrt{\frac{2\mu^3}{\pi (k_B T)^3}} \int_0^{\infty} \mathrm{d}v' \; \sigma(v') v'^3 \exp\left(-\frac{v'^2}{v_{\text{th}}^2}\right) \tag{3.4.49}$$

$$R_r = \frac{8 n_t n_b}{m_b^2 \pi^{1/2} v_{\text{th}}^3} \int_0^{\infty} \mathrm{d}\epsilon_b \; \sigma(\epsilon_b) \epsilon_b \exp\left(-\frac{2\epsilon_b}{m_b v_{\text{th}}^2}\right) = \frac{n_b n_t}{m_b^2} \sqrt{\frac{8\mu^3}{\pi (k_B T)^3}} \int_0^{\infty} \mathrm{d}\epsilon_b \; \sigma(\epsilon_b) \epsilon_b \exp\left(-\frac{\mu \epsilon_b}{m_b k_B T}\right) \tag{3.4.50}$$

Remember the above formula only holds for Maxwellian distributions for both species! If the beam is actually a beam, you will get a different answer. For thermonuclear fusion in magnetic confinement devices, using both the beam and target as Maxwellian is a fairly decent approximation. One last useful form normalizes the energy form with $y = \mu \epsilon_b / (m_b k_B T)$

$$\begin{aligned} R_r &= \frac{n_b n_t}{m_b^2} \sqrt{\frac{8\mu^3}{\pi (k_B T)^3}} \frac{m_b^2 (k_B T)^2}{\mu^2} \int_0^{\infty} \mathrm{d}y \; \sigma\left(\frac{m_b k_B T}{\mu} y\right) y \exp(-y) \\ &= n_b n_t \sqrt{\frac{8 k_B T}{\mu \pi}} \int_0^{\infty} \mathrm{d}y \; \sigma\left(\frac{m_b k_B T}{\mu} y\right) y \exp(-y) \end{aligned} \tag{3.4.51}$$

There is another frame (quite frankly it is more sensible from a theoretical perspective) that is often used for cross section data. This is the center of momentum frame.[38] In this case we normalize

[38] Often abbreviated COM or CM. Some call this a center of mass frame, but center of mass is somewhat imprecise in special relativity (is it the "relativistic mass" or rest mass?), whereas center of momentum is unambiguous. Technically, a center of momentum frame does not have to have the center of mass at the origin, whereas a center of mass frame does. In any case, you can use these terms interchangeably since in practice hardly anyone pays attention to the difference. The important point is that when I say center of momentum, I mean the total momentum in the frame is $\mathbf{0}$.

	D-D (3.4.11)	D-D (3.4.12)	D-T (3.4.10)	D-$^3_2$He (3.4.13)	T-T (3.4.22)	$^3_2$He-T (3.4.24)
A_1	46.097	47.88	45.95	89.27	38.39	123.1
A_2	372	482	50200	25900	448	11250
A_3	4.36×10^{-4}	3.08×10^{-4}	1.368×10^{-2}	3.98×10^{-3}	1.02×10^{-3}	0
A_4	1.220	1.177	1.076	1.297	2.09	0
A_5	0	0	409	647	0	0

Table 3.1: The "Duane" coefficients[5] as given by the NRL Plasma Formulary[8]. Note that the Duane coefficients consider the heavier particle to be the target and use energies ϵ_b

according to the relative energy given by $\epsilon_\mu = \mu v'^2/2$. We then use $\mathrm{d}\epsilon_\mu = \mu v'\,\mathrm{d}v'$ and find

$$R_r = \frac{4n_t n_b}{\pi^{1/2} v_{\mathrm{th}}^3} \int_0^\infty \mathrm{d}v'\ \sigma(v') v'^3 \exp\left(-\frac{v'^2}{v_{\mathrm{th}}^2}\right) = n_b n_t \sqrt{\frac{2\mu^3}{\pi (k_B T)^3}} \int_0^\infty \mathrm{d}v'\ \sigma(v') v'^3 \exp\left(-\frac{v'^2}{v_{\mathrm{th}}^2}\right) \tag{3.4.52}$$

$$R_r = \frac{8n_t n_b}{\pi^{1/2} v_{\mathrm{th}}^3} \int_0^\infty \mathrm{d}\epsilon_\mu\ \sigma(\epsilon_\mu) \epsilon_\mu \exp\left(-\frac{2\epsilon_\mu}{\mu v_{\mathrm{th}}^2}\right) = n_b n_t \sqrt{\frac{8\mu^3}{\pi (k_B T)^3}} \int_0^\infty \mathrm{d}\epsilon_\mu\ \sigma(\epsilon_\mu) \epsilon_\mu \exp\left(-\frac{\epsilon_\mu}{k_B T}\right) \tag{3.4.53}$$

with the normalized $y = \epsilon_\mu/(k_B T)$ giving

$$R_r = n_b n_t \sqrt{\frac{8 k_B T}{\mu \pi}} \int_0^\infty \mathrm{d}y\ \sigma\left(k_B T y\right) y \exp(-y) \tag{3.4.54}$$

If you wish to compare two data sets with one using ϵ_b and the other ϵ_μ, then you must convert the values. To convert ϵ_b to ϵ_μ you use that $\frac{m_b}{\mu}\epsilon_b = \epsilon_\mu$. One can use that $\frac{m_b}{\mu} = \frac{m_t + m_b}{m_t} = 1 + \frac{m_b}{m_t} > 1$ to see that the center of momentum energy will be greater in general.

Because cross sections are not easy to calculate theoretically, empirical relations with fitting functions are used. There are a variety of references to use for this. I will follow the plasma formulary with Duane coefficients and then the more recent and accurate Bosch-Hale fit, but it should always be remembered that these are fits to data. Thus, for all of the reactions we have talked about above, we can use for impinging particle of energy ϵ_b or the center of momentum frame energy ϵ_μ. One must carefully check the references to determine which frame is appropriate and so which formula to use.

We'll consider both models. First, we use a fitting by Duane[5]. This model uses ϵ_b (measured in keV) and writes the cross sections as (I just write ϵ rather than ϵ_b for the formulas)

$$\sigma(\epsilon) = \frac{A_5 + \frac{A_2}{(A_4 - A_3\epsilon)^2 + 1}}{\epsilon[\exp(A_1 \epsilon^{-1/2}) - 1]} \tag{3.4.55}$$

The above formula yields σ in units of barns ($1 \times 10^{-28}\,\mathrm{m}^2$) for the coefficients as given in Table 3.1.

The next model, which is more recent and so more accurate is due to Bosch-Hale (BH)[3]. The parameters can be easily accessed through the MIT Magnetic Fusion Energy Formulary[7]. This

	D-D (3.4.11)	D-D (3.4.12)
B_G	31.3970	31.3970
A_1	5.5576×10^4	5.3701×10^4
A_2	2.1054×10^2	3.3027×10^2
A_3	-3.2638×10^{-2}	-1.2706×10^{-1}
A_4	1.4987×10^{-6}	2.9327×10^{-5}
A_5	1.8181×10^{-10}	-2.5151×10^{-9}
B_1	0.0	0.0
B_2	0.0	0.0
B_3	0.0	0.0
B_4	0.0	0.0
VER	0.5-5000	0.5-4900

Table 3.2: The Bosch-Hale (BH)[3] coefficients for DD reactions as given by the original paper. Here VER means Valid Energy Range in keV. BH uses the center of momentum frame, and so ϵ_μ.

	D-T (3.4.10)	D-T (3.4.10)	D-$^3_2$He (3.4.13)	D-$^3_2$He (3.4.13)
B_G	34.3827	34.3827	68.7508	68.5708
A_1	6.927×10^4	-1.4714×10^6	5.7501×10^{-6}	-8.3993×10^5
A_2	7.454×10^8	0.0	2.5226×10^3	0.0
A_3	2.050×10^6	0.0	4.5566×10^1	0.0
A_4	5.2002×10^4	0.0	0.0	0.0
A_5	0.0	0.0	0.0	0.0
B_1	6.38×10^1	-8.4127×10^{-3}	-3.1995×10^{-3}	-2.6830×10^{-3}
B_2	-9.95×10^{-1}	4.7983×10^{-6}	-8.5530×10^{-6}	1.1633×10^{-6}
B_3	6.981×10^{-5}	-1.0748×10^{-9}	5.9014×10^{-8}	-2.1332×10^{-10}
B_4	1.728×10^{-4}	8.5184×10^{-14}	0.0	1.4250×10^{-14}
VER	0.5-550	550-4700	0.3-900	900-4800

Table 3.3: The Bosch-Hale (BH)[3] coefficients as given by the original paper. It is best to match the D-T formulas at 530 keV as they agree at this value. Similarly, it is best to match the D-$^3_2$He data at 900 keV, though it is not ever a continuous match. Here VER means Valid Energy Range in keV. The Bosch-Hale coefficients use a center of momentum frame and so use ϵ_μ for energies.

uses ϵ_μ (measured in keV), the center of momentum frame, and uses the fit[39]

$$\sigma(\epsilon) = \frac{S(\epsilon)}{\epsilon \exp(B_G \epsilon^{-1/2})} \tag{3.4.56}$$

$$S(\epsilon) = \frac{A_1 + \epsilon(A_2 + \epsilon(A_3 + \epsilon(A_4 + \epsilon A_5)))}{1 + \epsilon(B_1 + \epsilon(B_2 + \epsilon(B_3 + \epsilon B_4)))} \tag{3.4.57}$$

Despite many years of progress, if we convert the Duane values into the center of momentum frame, we see that only at the largest energies (which will probably be outside the largest permissible energy value in the target-beam frame) do we get substantial disagreement. The largest difference can be seen for Deuterium reactions in Figure 3.4.In fact, looking at the relative error between them, one can find that the worst relative error is about 100% for Duane, but that this is the

[39]The formula for $S(\epsilon)$ comes from a Padé approximation.

worst possible error and on a log-log plot this is not easy to see. Over the majority of these energy ranges, the values are very comparable. The most problematic values are at high energy, and because those come from $\epsilon_b < \epsilon_\mu$, we have gone beyond the valid energy range for the fitted data for ϵ_b in the plots.

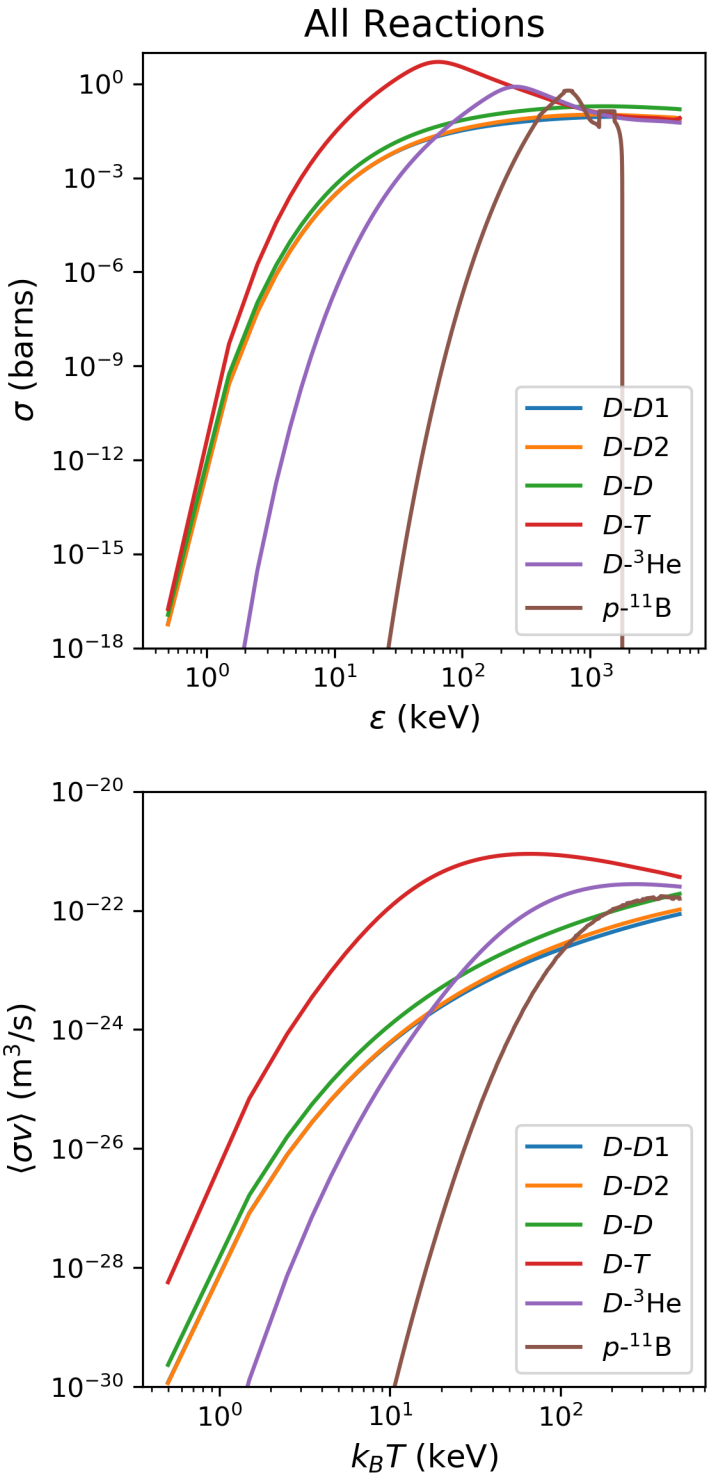

Figure 3.3: This gives the BH values of cross section and reactivity for DD, DT, and D-$^3_2$He reactions ($DD1$ is for (3.4.11) and $DD2$ is for (3.4.12)). It demonstrates why DT is preferred, with this highest reactivity over the temperature range of interest. The proton-boron reactions use[13] for an empirical fit of the cross-section.

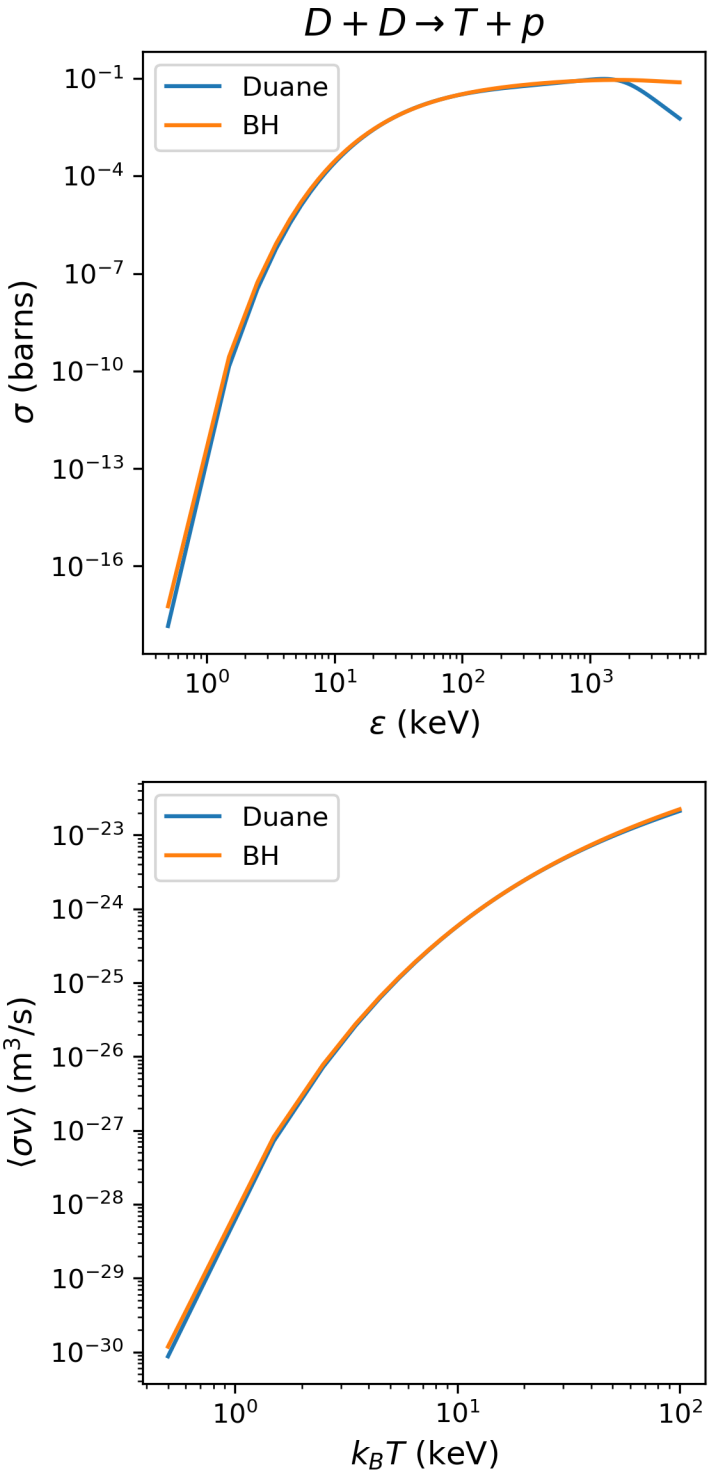

Figure 3.4: The cross sections and reactivities for Duane and BH fits for a DD reaction. The x axis is center of mass energy ϵ_μ and that is why the Duane values are incorrect past 1 MeV. The ϵ_b is past the valid energy range for Duane.

3.4.2.2 Lawson Criteria

Now we can finally get into what is necessary to get fusion going. As we can see in Figures 3.4-3.5, the larger the temperatures that can be achieved, the better off we are in these reactions (although DT has diminishing returns in reactivity as we approach 100 keV). In some sense, we are lucky that the tail of the Maxwellian energy distribution is all that is needed to get fusion reactions. On the other hand, the bulk of the particles will not be participating in reactions since the bulk are at an energy near the given temperature of the plasma. We see this in Figure 3.5 which shows for the DT reaction that the Maxwellian distribution $\epsilon \exp(-\epsilon/(k_B T))$ has its tail contributing to the integrand for the reactivity due to the cross section being higher in the tail.

Let's concentrate on DT for now because we have seen it has one of the best cross sections and reactivities at the temperatures we are looking at, about 1 keV to 100 keV (see Figure 3.3). In that case, we'd like to know the power output in a certain volume and time. We use

$$\mathcal{P}_{DT} = n_D n_T \langle \sigma v \rangle_{DT} E_{DT} \tag{3.4.58}$$

which is simply the number density of the two species times the reactivity and the energy released per reaction E_{DT}. Let's write the total density of tritons and deuterons as $n = n_D + n_T$. Then

$$\mathcal{P}_{DT} = n_D (n - n_D) \langle \sigma v \rangle_{DT} E_{DT} \tag{3.4.59}$$

We'd like to find the value of n_D such that P_{DT} is maximum. Then

$$\frac{\partial \mathcal{P}_{DT}}{\partial n_D} = 0 \Rightarrow n - 2n_D = 0 \Rightarrow n = n_D/2 \tag{3.4.60}$$

which means we have an even mix of n_T and n_D ideally. In that case the optimal power density is given by

$$\mathcal{P}_{DT} = \frac{n^2}{4} \langle \sigma v \rangle_{DT} E_{DT} \tag{3.4.61}$$

We then need to ask how well does the plasma hold energy. Suppose P_L is the natural power loss of the plasma and V is the total plasma volume. If the total energy of the plasma is E_P then we define the energy confinement time as

$$\tau_E = \frac{E_P}{P_L} \tag{3.4.62}$$

From a practical level, if you are keeping a plasma in a steady-state energy balance then the power provided will equal the power loss (so long as there is no fusion power adding to the mix...). There is an empirical relation for the energy confinement time τ_E in tokamaks given by[10]

$$\tau_E^{H98(y,2)} = 0.0562 I_p(\text{MA})^{0.93} B_{0\phi}(\text{T})^{0.15} R(\text{m})^{1.97} \epsilon^{0.58} \kappa^{0.78} n_e (\frac{\text{m}^{-3}}{10^{19}})^{0.41} P(\text{MW})^{-0.69} M^{0.19} \tag{3.4.63}$$

where the parentheses indicate the units to be used[40] with I_p the plasma current, $B_{0\phi}$ the toroidal magnetic field at the major radius, R the major radius, $\epsilon = r/R$ the inverse aspect ratio, κ the plasma elongation, n_e the central line-averaged number density (measured in units of $1 \times 10^{19}\,\text{m}^{-3}$),

[40] The powers are on the variables, not on the units. So $R(\text{m})^{1.97}$ means take $R^{1.97}$ when R is measured in meters.

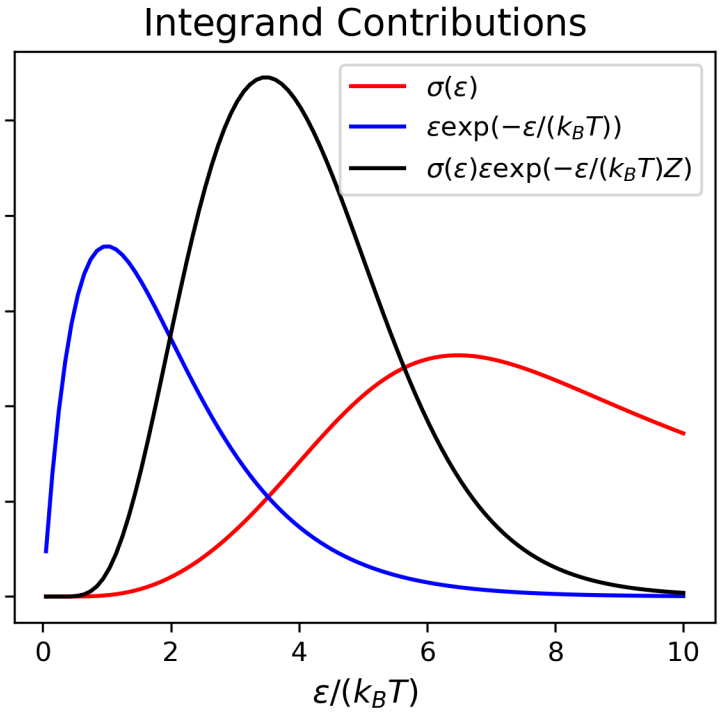

Figure 3.5: The contributions to the reactivity integral are shown. The tail of the Maxwellian distribution ($\epsilon \exp(-\epsilon/(k_BT))$) contributes the most to overall integrand at higher ϵ due to the cross section $\sigma(\epsilon)$ being higher there.

P is the heating power (both fusion and supplied power), and M is the mass number (the number of protons and neutrons for the main ion). We can use ITER[41] values $I_p = 15$, $B_{0\phi} = 5.3$, $R = 6.2$, $\epsilon = 0.32$, $\kappa = 1.8$, $n_e = 10$, $P = 122$, $M = 3$ yield $\tau_E \approx 3\,\mathrm{s}$ which can be considered a typical value.

For stellarators, a different scaling[2][21]

$$\tau_E^{\mathrm{ISS04}} = 0.134 a(\mathrm{m})^{2.28} R(\mathrm{m})^{0.64} n_e(10^{20}\,\mathrm{m}^{-3})^{0.54} B(\mathrm{T})^{0.84} \iota_{2/3}^{0.41} P^{-0.61}(\mathrm{MW}) \tag{3.4.64}$$

with the same meaning except $\iota_{2/3}$ is the rotational transform at radius $r = 2a/3$, and the minor radius is defined by the plasma volume being given by $2\pi R(\pi a^2)$.[42] For a stellarator like W7X, this would yield $a = 0.53$, $R = 5.5$, $n_e = 0.8$, $B = 3$, $\iota_{2/3} \approx 1$, $P = 15$ to 30. This leads to a confinement time of 0.027 s to 0.040 s. W7X has actually attained a confinement time of about 0.1 s[4], so the confinement time scaling should not be taken as too accurate. The scaling gives an idea of why tokamaks are still preferred, though. Tokamaks have tended to be bigger and had fewer problems with particle confinement, and so have had better scaling historically. As can be seen, this may no longer be true in the future.

For DT fusion 14.1 MeV of the energy goes into the neutron which leaves the plasma and so does not add to the energy in the plasma (assuming it does not collide with anything on the way out). Thus we are left with the helium α particles[43] with their $E_\alpha = 3.54\,\mathrm{MeV}$ of kinetic energy. Thus

$$\mathcal{P}_\alpha = \frac{n^2}{4} \langle \sigma v \rangle_{DT} E_\alpha \tag{3.4.65}$$

$$P_\alpha = \iiint \mathrm{d}^3x \, \frac{n^2}{4} \langle \sigma v \rangle_{DT} E_\alpha \tag{3.4.66}$$

We can then consider "ignition" of the plasma, and call the plasma an ignited plasma. This is the point where heating due to α's is enough such that no other external power source is required. This means that $P_\alpha \geq P_L$.

We use this condition to define the ignition point and beyond via

$$P_L \leq P_\alpha \tag{3.4.67}$$

$$\frac{E_P}{\tau_E} \leq \iiint \mathrm{d}^3x \, \frac{n^2}{4} \langle \sigma v \rangle_{DT} E_\alpha \tag{3.4.68}$$

This form is rather unenlightening. We will look at the equality point (it should be obvious which direction for quantities is still ignited). We can define an average term via

$$P_\alpha = \frac{E_\alpha}{4} \overline{n^2 \langle \sigma v \rangle_{DT}} V \tag{3.4.69}$$

$$\overline{q} = \frac{1}{V} \iiint \mathrm{d}^3x \, q \tag{3.4.70}$$

Let hats over a function such as $\hat{T}$ or $\hat{n}$ indicate a peak value for whatever the profile is of T or n and we can write[44]

$$P_\alpha = C_{P1} \frac{E_\alpha}{4} (\hat{n})^2 \widehat{\langle \sigma v \rangle}_{DT} V \tag{3.4.71}$$

[41] This gives us an idea of the best we can do in 2020 and the near future for tokamaks.

[42] That is, given a plasma volume and an a, we determine R by $V = 2\pi R(\pi a^2)$.

[43] They are sometimes called helium ash because they come from "burning", that is fusing, deuterium and tritium. If the helium ash becomes too numerous it starts to cause problems with the fusion process.

[44] Note I wrote $(\hat{n})^2$ rather than $\widehat{n^2}$. I do this so I can divide by $\hat{n}$ rather than dealing with averages of squares. There is no harm in writing it this way so long as we take it into account when determining C_{P1}.

with C_{P1} representing a dimensionless number which depends on all the profiles involved (which are parameterized through their peak values with hats). The energy of the plasma is given by

$$E_P = \iiint \mathrm{d}^3x \left[\sum_s \frac{3}{2} n_s k_B T_s\right] \tag{3.4.72}$$

with s a species index and using equipartition of energy in three dimensions to get $\frac{1}{2}nk_BT$ for each degree of freedom for each species (here we can consider the deuterons and tritons).

$$E_P = 3\overline{nk_BT}V = 3k_BC_{P2}\hat{n}\hat{T}V \tag{3.4.73}$$

where we use the hats and a dimensionless profile dependent C_{P2}. Thus

$$P_L < P_\alpha \Rightarrow \frac{3C_{P2}k_B\hat{n}\hat{T}}{\tau_E}V < \frac{C_{P1}}{4}\hat{n}^2\widehat{\langle\sigma v\rangle}_{DT}E_\alpha V \tag{3.4.74}$$

$$\hat{n}\tau_E > \frac{C_{P2}}{C_{P1}}\frac{12k_B\hat{T}}{\widehat{\langle\sigma v\rangle}_{DT}E_\alpha} \tag{3.4.75}$$

We can set $C_P = C_{P2}/C_{P1}$ and note for flat profiles that $C_P = 1$ for a rough estimate. This rough estimate is often then used to determine what conditions are necessary. It is also usually presented without the hats on top to make it look a little cleaner. This rough estimate has many grains of salt since we are ignoring profile characteristics (unless things are uniform), but this gives a good enough idea of what is required.

We can read this as $n\tau_E$ must exceed the value given by

$$\frac{12k_BT}{\langle\sigma v\rangle_{DT}E_\alpha} \tag{3.4.76}$$

We can find a minimum value for the right hand side to try and find an optimal $n\tau_E$ but τ_E is a function of n and T, so in order to find the actual optimal value we need to take this into account. Since higher energy is expected to lead to worse confinement, we can expect for some real number k that $\tau_E \sim (k_BT)^{-|k|}$, and so an optimal T will be given a little below the optimum from the right-hand side alone. In any case, if we ignore this dependence and plot $12k_BT/\langle\sigma v\rangle_{DT}$ we get Figure 3.6 leading to a minimum at $T \approx 25.9\,\mathrm{keV}$ and so at this minimum temperature ignition would require

$$n\tau_E > 1.50 \times 10^{20}\,\mathrm{s/m^3} \tag{3.4.77}$$

In addition, Wesson[19] uses that for k_BT in keV in the range 10 keV to 25 keV we have

$$\langle\sigma v\rangle_{DT} \approx (k_BT)^2(1.1 \times 10^{-24}\,\mathrm{m^3/s}) \tag{3.4.78}$$

and so we can form a "triple-product" also called the Lawson criterion in some instances (all energies in keV)

$$nk_BT\tau_E > \frac{12}{(1.1 \times 10^{-24}\,\mathrm{m^3/(s\,keV^2)})(3.5 \times 10^3\,\mathrm{keV})} \approx 3.1 \times 10^{21}\,\mathrm{keV\,s} \tag{3.4.79}$$

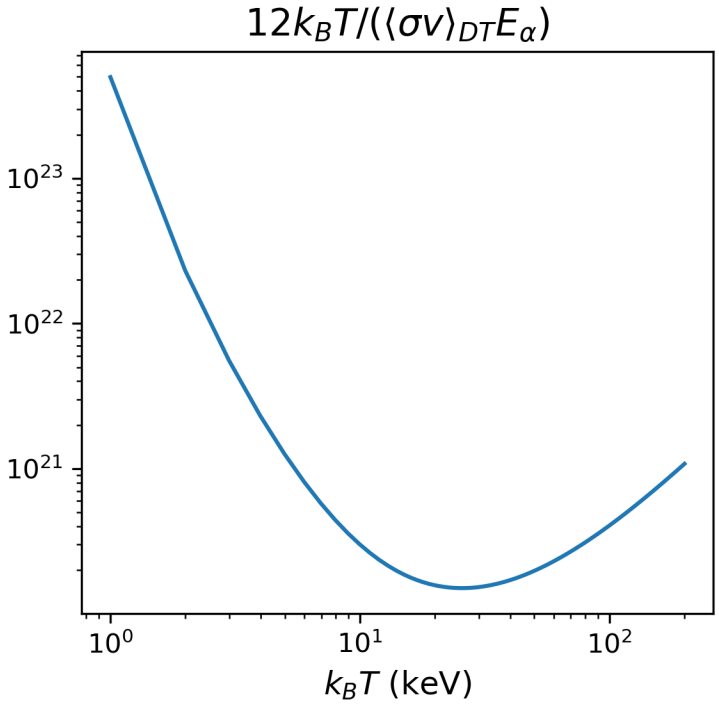

Figure 3.6: This shows the values $nk_BT\tau_E$ must meet in order to meet the requirements for ignition. Note how the minimum is between 20 keV and 30 keV with a value of about $1 \times 10^{20}\,\mathrm{s/m^3}$ to $2 \times 10^{20}\,\mathrm{s/m^3}$.

This leads to a different minimum at a temperature of about $T = 14\,\mathrm{keV}$. Note how this is a good bit smaller than for the $n\tau_E$ criterion. We will see that the left hand side $nk_BT\tau_E$ is a better parameter because it changes less with changes in n and T than $n\tau_E$. To meet this triple product criterion we see that about $n = 1 \times 10^{20}\,\mathrm{m}^{-3}$, $k_BT = 10\,\mathrm{keV}$, and $\tau_E = 3.1\,\mathrm{s}$ would be required for ignition.

We can show for tokamaks that $nk_BT\tau_E$ is a better parameter by using that τ_E scales as

$$\tau_E \propto n^{1/3} P_{\mathrm{in}}^{-2/3} \tag{3.4.80}$$

for tokamaks, where P_{in} is the input power. For an ignited plasma P_{in} is simply $P_\alpha \propto n^2T^2$ [remember (3.4.78)] so in fact

$$nk_BT\tau_E \propto \cancel{n}T(\cancel{n^{1/3}}[\cancel{n^{-4/3}}T^{-4/3}]) = T^{-1/3} \tag{3.4.81}$$

which is a fairly weak dependence on temperature and so the minimum of the right hand side of the triple product is near optimal.

There are other ways of analyzing this with more in common with Lawson's original proposal that does not use ignition. I used Lawson criteria for the name of this section rather than criterion because sometimes the ignition condition is used with $n\tau_E$ or $nk_BT\tau_E$, and sometimes a simple power in and out of the plasma is used instead. This latter approach is what we will now explore in the spirit of Lawson's original expression.

We could instead define a physicists' Q given by the ratio of fusion power to total heating power externally supplied to the plasma P_{ext} [not the power actually drawn to create this heating, see the "engineering" comment below] so

$$Q = \frac{C_{P1}\frac{\hat{n}^2}{4}\langle\sigma v\rangle_{DT} E_{DT} V}{P_{\mathrm{ext}}} \tag{3.4.82}$$

We should note that engineering Q uses the total power required to supply the plasma with the needed heating (that is, what power do you need to draw "from the wall" in order to get the required heating in the plasma). Engineering Q is more important for energy production but we can see that it is simply related to the efficiency of total external energy going into the plasma, and so physicists' Q is usually a factor of 3 to 5 times larger than engineering Q. While this distinction is somewhat important, in fusion research it is assumed we can adjust the value of Q to get the desired result and so physicists' $Q > 1$ is an important measure of progress. If I speak of Q without specifying which type in this chapter, assume it is physicists' Q.[45] One can also adjust things by considering energy loss mechanisms as well, but I won't because while this is important, it complicates the analysis.

Because $E_{DT} \approx 5E_\alpha$, this is often written as

$$Q = \frac{5P_\alpha}{P_{\mathrm{ext}}} \tag{3.4.83}$$

and then we see $Q = 1$ means that the

$$5P_\alpha = P_{\mathrm{ext}} \tag{3.4.84}$$

[45] For the next chapter, I will mostly deal with engineering Q, so do not get confused! Also do not confuse it with the nuclear reaction Q.

Then

$$\frac{P_\alpha}{P_\alpha + P_{\text{ext}}} = \frac{P_\alpha}{6P_\alpha} = \frac{1}{6} \tag{3.4.85}$$

and so P_α provides 1/6 or about 16.7% of the total power and 1/5 or 20% of the externally supplied power. $Q = 1$ is often called scientific breakeven because this is the point at which an appreciable amount of the heating (power) is due to fusion. This is a somewhat arbitrary location, but is important for understanding the physics of magnetic confinement with fusion occurring. To hit engineering breakeven then, we need $Q > 5$ so that $P_\alpha \approx P_{\text{ext}}$. Notice that $P_{\text{ext}} \to 0$ and $Q \to \infty$ at ignition. $Q > 1$ but $Q < \infty$ is often called a burning plasma to indicate that an appreciable amount of the heating comes from fusion, but that external power is still being applied.

It should also be noted that current fusion plasma experiments usually use DD reactions and so extrapolate Q based on assuming similar conditions (n, $k_B T$, etc.) could be achieved with a DT plasma. This is a caveat to most quoted Q values (TFTR and JET being exceptions as they actually had tritons in their plasma).

If we were to do an analysis for a burning but not ignited plasma, then instead of requiring $P_\alpha > P_L$ we would use

$$P_L < P_\alpha + P_{\text{ext}} \tag{3.4.86}$$

$$P_L < P_\alpha + \frac{5P_\alpha}{Q} \tag{3.4.87}$$

$$\frac{E_P}{\tau_E} < \frac{(5+Q)}{Q} P_\alpha \tag{3.4.88}$$

$$\tag{3.4.89}$$

and then follow the same steps as before for the Lawson criteria and find

$$\hat{n}\tau_E > \left(\frac{C_P Q}{Q+5}\right) \frac{12 k_B \hat{T}}{\widehat{\langle \sigma v \rangle}_{DT} E_\alpha} \tag{3.4.90}$$

It is easy to see that as $Q \to \infty$ that $Q/(Q+5) \to 1$. Thus, if we only require $Q = 5$ to 10 then we can simply multiply our Lawson criteria by 0.5 to 0.67 which allows us to get away with energy production without using fusion power to solely power the heating.

In general, if we let $E_{\text{fus.}}$ be the total energy per fusion reaction and the fraction used in heating the plasma κ then we still use

$$Q = \frac{C_{P1} \frac{\hat{n}^2}{4} \langle \sigma v \rangle E_{\text{fus.}} V}{P_{\text{ext}}} = \frac{P_{\text{fus.}}}{P_{\text{ext.}}} \tag{3.4.91}$$

but when we calculate the balance of energy losses and gains we find

$$\frac{E_P}{\tau_E} < \kappa P_{\text{fus.}} + P_{\text{ext.}} = \kappa P_{\text{fus.}} + \frac{P_{\text{fus.}}}{Q} \tag{3.4.92}$$

$$\frac{E_P}{\tau_E} < \frac{\kappa Q + 1}{Q} P_{\text{fus.}} \tag{3.4.93}$$

$$\frac{E_P}{\tau_E} < \frac{\kappa Q + 1}{\kappa Q} P_{\text{fh}} \tag{3.4.94}$$

leading to

$$\hat{n}\tau_E > \frac{C_P \kappa Q}{\kappa Q + 1} \frac{12 k_B \hat{T}}{\widehat{\langle \sigma v \rangle} E_{\text{fh}}} \tag{3.4.95}$$

which also means that we do not need to hit the ignition Lawson criterion. However, we must be careful to consider how much external power is needed. Most likely the external power will be electrical energy produced by the fusion, reducing the amount available to sell.

3.4.2.3 Tritium Breeding and Energy Release per Mass

While deuterium is available in the world's oceans, the supply of tritium is quite scarce. Tritium has a 12.3 yr half-life so that it rapidly diminishes. As previously stated, we can use

$$n + {}^6_3\text{Li} \rightarrow T + {}^4_2\text{He} + 4.78\,\text{MeV} \qquad ((3.4.28))$$
$$n + {}^7_3\text{Li} \rightarrow T + {}^4_2\text{He} + n - 2.76\,\text{MeV} \qquad ((3.4.29))$$

to recover tritium using some of the neutron energy from the *DT* fusion reactions. The ${}^6_3$Li is especially attractive as it would release an additional 4.78 MeV of energy for the power output. The issue of lithium reserves are left for later. The cross section for the above reactions at approximately 14 MeV is[46] about 380 millibarn[18][47] for ${}^7_3$Li and is about 30 millibarn[18] for ${}^6_3$Li at that energy. However, ${}^6_3$Li has a much better cross section at lower energies, with a cross section of about 5 barns at 1 keV neutrons. An interesting report on the difficulties and methods of getting a cross section for these neutron mediated reactions was done by Wyman[20] which gets values similar to modern ones while explaining how these values are arrived at.

3.5 Beam Fusion

In this section we will explain why beam fusion cannot work. We will begin with beam-target fusion which is the same essential analysis as beam-beam fusion. In addition, this analysis will force us to consider losses in the fusion process that we ignored in our previous analysis. This will lead us to new estimates for the Lawson criteria.

3.5.1 Beam-Target Fusion

Rather than heating up particles to 1 keV to 100 keV temperatures, why not just shoot a beam of particles at each other? We will consider deuterium-tritium fusion again, because it is the most optimistic. Then because tritium is the rarer material, we will designate it a target, and deuterium a beam that we will accelerate to the cross section peak. The fusion reactivity will clearly be $\langle \sigma v \rangle_{DT} = \sigma v_D$ where v_D is simply the deuterium velocity. For keV temperatures, we don't need to worry about relativity as the mass of a deuteron is approximately $2\,\text{GeV}/c^2$. The cross-section peak is at 113 keV with a cross section of 4.9 barn. So we need the deuterons to be at a velocity

$$v_D = \sqrt{\frac{2E}{m_D}} = \sqrt{\frac{2(113\,\text{keV})}{1.88\,\text{GeV}/c^2}} = (0.011)c \approx 3.3 \times 10^6\,\text{m/s} \tag{3.5.1}$$

[46] The 14 MeV neutrons correspond to a velocity of 5.18×10^7 m/s.
[47] I used the website https://wwwndc.jaea.go.jp/jendl/j40/j40.html to find the data.

We find the number of reactions (assuming the beam just hits the target tritium nuclei[48]) via the usual means

$$R = n_D n_T \langle \sigma v \rangle_{DT} \tag{3.5.2}$$

with

$$\langle \sigma v \rangle_{DT} = \sigma v_D = (4.9\,\text{barn})(3.3 \times 10^6\,\text{m/s})(1 \times 10^{-28}\,\text{m}^2/\text{barn}) = 1.6 \times 10^{-21}\,\text{m}^3/\text{s} \tag{3.5.3}$$

Given that the density of liquid tritium is given by $\rho = 0.13\,\text{g/cm}^3$ with molar mass $M_m = 3.02\,\text{g/mol}$. Thus the number density of the tritium is given by (with N_A Avogadro's number)

$$n_T = \frac{\rho}{M_m} N_A \approx \frac{0.13\,\text{g/cm}^3}{3.02\,\text{g/mol}}(6.022 \times 10^{23}\,\text{mol}^{-1})(1 \times 10^6\,\text{cm}^3/\text{m}^3) \approx 2.6 \times 10^{28}\,\text{m}^{-3} \tag{3.5.4}$$

Deuterium's number density can also be determined via deuterium gas's density $\rho = 164\,\text{g/m}^3$ with $M_m = 2.01\,\text{g/mol}$ so

$$n_D = \frac{\rho}{M_m} N_A \approx \frac{164\,\text{g/m}^3}{2.01\,\text{g/mol}}(6.022 \times 10^{23}\,\text{mol}^{-1}) \approx 4.9 \times 10^{25}\,\text{m}^{-3} \tag{3.5.5}$$

Thus the reactions per unit volume would be

$$R = n_D n_T \langle \sigma v \rangle_{DT} \approx (4.9 \times 10^{25}\,\text{m}^{-3})(2.6 \times 10^{28}\,\text{m}^{-3})(1.6 \times 10^{-21}\,\text{m}^3/\text{s}) = 2.0 \times 10^{33}\,\text{m}^{-3}\text{s}^{-1} \tag{3.5.6}$$

The total energy density released per second (that is, the power density) will be

$$P_{\text{tot}} = E_{DT} R = (17.6\,\text{MeV})(2.0 \times 10^{33}\,\text{m}^{-3}\text{s}^{-1}) \approx 5.6 \times 10^{21}\,\text{Js}^{-1}\text{m}^{-3} \tag{3.5.7}$$

The power density to accelerate the deuterons from rest to 113 keV every second yields an input power density

$$P_{\text{D,r}} = (2.0 \times 10^{33}\,\text{m}^{-3}\text{s}^{-1})(113\,\text{keV}) \approx 3.6 \times 10^{19}\,\text{Js}^{-1}\text{m}^3 \tag{3.5.8}$$

There will of course be a requirement to cool the tritium so that it is a liquid, but for now let's just look at how much more energy per volume one gets versus the energy required to get the deuterons up to speed. The power density gain G is then

$$G_P = \frac{P_{\text{tot}}}{P_{D,r}} \approx 155 \tag{3.5.9}$$

This is a fairly impressive gain per volume. Note that in this case we could just consider a single fusion event to find this. If we consider per incident deuteron we'd clearly have $G = (17.6\,\text{MeV})/(0.133\,\text{MeV}) \approx 155$ which tells us per deuteron we get a significant gain which leads to large gains per volume.

One other important consideration is the mean free path of the deuterium in the tritium. This tells us what length of tritium we need. We find this via

$$\lambda_{\text{mfp}} = \frac{1}{n_T \sigma} \approx \frac{1}{(2.6 \times 10^{28}\,\text{m}^{-3})(4.9 \times 10^{-28}\,\text{m}^2)} \approx 8 \times 10^{-2}\,\text{m} = 80\,\text{mm} \tag{3.5.10}$$

[48] What could go wrong? Do you think they'll just hit them?

which is quite feasible.

So why doesn't anyone just do this? Well, I hinted before that we were ignoring something. We are ignoring losses due to scattering. Scattering of deuterons on tritons (or vice versa), deuterons on other deuterons, and deuterons/tritons on electrons. To study this, let's just consider the Coulomb interaction. I will simply use the differential cross section for Rutherford scattering as a given, because it is calculated in numerous physics textbooks. The Rutherford integral cross section for scattering at an angle greater than θ (which we will set to a cutoff value θ_c) is given by

$$\begin{aligned}\sigma_s &= \frac{\pi}{4}\left(\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\frac{2}{m_0 v_0^2}\right)^2\left[\csc\left(\frac{\theta}{2}\right)^2 - 1\right] = \frac{\pi}{4}\left(\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\frac{2}{m_0 v_0^2}\right)^2 \cot\left(\frac{\theta}{2}\right)^2 \\ &= \frac{\pi}{4}\left(\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\frac{2}{m_0 v_0^2}\right)^2 \frac{1+\cos\theta}{1-\cos\theta}\end{aligned} \tag{3.5.11}$$

If we do it in a center of momentum frame then replace $\frac{1}{2}m_0 v_0^2$ with $\frac{1}{2}\mu v^2$ where v is the relative velocity and $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass. I will explore the center of momentum approach later, but we will consider a beam-target idea here. You can note that for a small enough angle, the above expression for σ diverges. This is because small θ corresponds to a large impact parameter.[49] We can cut off θ based on an appropriate impact parameter b, where

$$b = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 m_0 v_0^2}\cot\left(\frac{\theta}{2}\right) \tag{3.5.12}$$

Then our cutoff angle θ_c is given by

$$\theta_c = 2\,\mathrm{arccot}\left(\frac{4\pi\epsilon_0 m_0 v_0^2 b}{Z_1 Z_2 e^2}\right) \tag{3.5.13}$$

which is generally speaking a small angle, but not zero. We are concerned with two nuclei hitting each other in the deuteron hitting another deuteron case. We will therefore use an impact parameter b given by the nuclear radius which is typically $1\times10^{-14}\,\mathrm{m}$ or $10\,\mathrm{fm}$. Then

$$\theta_c = 1.1\,\mathrm{rad} \tag{3.5.14}$$

Then plugging in our numbers we find with $K = \frac{1}{2}m_0 v_0^2$ measured in keV that

$$\sigma_s \approx \frac{6.9\times10^{-23}\,\mathrm{m}^2}{K(\mathrm{keV})^2} \approx \frac{6.9\times10^5\,\mathrm{barn}}{K(\mathrm{keV})^2} \tag{3.5.15}$$

For $K = 113\,\mathrm{keV}$ we then find $\sigma_s \approx 54\,\mathrm{barn}$. Two things should be noted. First, the scattering cross section rapidly decreases as energy goes up. This means the faster the particles, the less likely they are to collide. Second is that they have comparable cross sections to fusion cross sections at the temperatures of interest.

We thus have found the scattering cross section, which means that

$$\frac{\sigma_s}{\sigma_{DT}} \approx 11 \tag{3.5.16}$$

[49] Impact parameter b is the perpendicular distance of the projectile from the target particle when the projectile is far away.

How would this degrade the gain factor from before, though? Well, we have that the total cross section is $\sigma_s + \sigma_{DT}$, and so we need to think about how many particles are removed from the beam without fusing. Therefore we have $\sigma_{DT}/(\sigma_s + \sigma_{DT})$ as the proportion now yielding power. That is before we had the reaction rate per volumes perfectly cancelling (E_D is the kinetic energy of the deuteron beam)

$$G = \frac{P_{\text{tot}}}{P_{D,r}} = \frac{RE_{DT}}{RE_D} = \frac{E_{DT}}{E_D} \tag{3.5.17}$$

where R is the reaction rate per volume. That is the input energy on bottom only required enough to produce fusion. Now the input power must also include the total cross section because we have to put more power in because scattered particles do not contribute to the power density gain in the numerator. Because the R before were the same in all respects, we could essentially just cancel them. But now they have different cross sections applied. If we write R_f and R_t for the fusion reactions and total fusion reactions and scatterings, respectively, then we must take into account this difference. Then we must have

$$G_{\text{reduced}} = \frac{R_f E_{DT}}{R_t E_D} = \frac{\sigma_{DT} E_{DT}}{(\sigma_{DT} + \sigma_s) E_D} = \frac{\sigma_{DT}}{\sigma_{DT} + \sigma_s} G \tag{3.5.18}$$

Thus the power density (and so the energy density) released will be this fraction. The new gain is thus

$$G_{\text{reduced}} = \frac{\sigma_{DT}}{\sigma_{DT} + \sigma_s} \frac{E_{\text{tot}}}{E_{D,r}} \approx (0.083)(155) \approx 12 \tag{3.5.19}$$

which is not great, but we see that we still have some room for a gain of energy from this process. This assumes that the first scattering completely removes the beam from a possible fusion event. This is incredibly pessimistic. If this were all, it would be worth it to calculate a more realistic loss.

Unfortunately, this isn't the whole story. What about the electrons in the target tritium? We have to use an impact paramter b of about a_0, an atomic radius (that is, using a Bohr radius) because these electrons are bound to their atoms, and so will interact with a passing particle if it is near the atomic size. Then we find

$$\theta_c \approx 2.5 \times 10^{-4}\,\text{rad} \tag{3.5.20}$$

$$\sigma_s \approx \frac{1.7 \times 10^{-15}\,\text{m}^2}{K(\text{keV})^2} \approx \frac{1.7 \times 10^{13}\,\text{barn}}{K(\text{keV})^2} \tag{3.5.21}$$

and so the new $\sigma_s \approx 1.3 \times 10^9$ barn, a gigantic cross section. Then we have the new power density gain with electron scattering as

$$G_{\text{reduced}} = \frac{\sigma_{DT}}{\sigma_{DT} + \sigma_s} \frac{E_{\text{tot}}}{E_{D,r}} \approx (3.7 \times 10^{-9})(155) \approx 5.7 \times 10^{-7} \tag{3.5.22}$$

which means that we are not even close to energy breakeven. Even if it required multiple collisions. If we look at a collision,[50] we will see that the electron gains most of the kinetic energy from the deuteron and so it really does only require one collision to destroy our chances of beam-target fusion.

[50] Remember the non-relativistic formula we derived in (3.1.15)?

3.5.1.1 Beam-Beam Fusion

Beam-beam fusion is essentially the same as beam-target fusion in a center of momentum frame. Therefore we can consider the exact same analysis as for the beam-target case. Experimentally there would be some problems since we would need to form a tritium beam, but the underlying scattering processes would remain the same.

What is worth exploring is how to calculate a similar cutoff angle for the center of momentum frame. We will introduce a cutoff when the relative velocity kinetic energy is equal to the Coulomb potential energy. This will give us an effective radius which we can use as our cutoff. This just means we ignore interactions that are beyond the effective radius R, because the interaction is so minor. In equations it says

$$\frac{1}{2}\mu v^2 = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 R} \tag{3.5.23}$$

$$R = \frac{Z_1 Z_2 e^2}{2\pi\epsilon_0 \mu v^2} \tag{3.5.24}$$

and we use this as a cutoff.

Also, remember how we talked about the reduced de Broglie wavelength? If we simply assume that we have quantized levels for the angular momentum levels around the center of momentum so $\mu v R = n\hbar$ then the de Broglie wavelength shows up as $\hbar/(\mu v)$. Remember this comes from requiring standing waves for matter waves so that the $2\pi R = n\lambda$ with λ the regular de Broglie wavelength so $R = \lambda\!\!\!^{-}_{dB}$ which shows the relationship even more simply. So if we are using a quantum cutoff, the de Broglie wavelength is what makes the most sense as the cutoff.

For a deuteron on deuteron collision, we find

$$\begin{aligned} R_{DD} &= \frac{e^2}{2\pi\epsilon_0 \frac{m_D}{2} v^2} \\ &= \frac{(1.60\times10^{-19}\,\mathrm{C})^2}{2\pi(8.85\times10^{-12}\,\mathrm{F\,m^{-1}})(3.34\times10^{-27}\,\mathrm{kg})(3.3\times10^{6}\,\mathrm{m\,s^{-1}})^2} \approx 2.5\times10^{-14}\,\mathrm{m} = 25\,\mathrm{fm} \end{aligned} \tag{3.5.25}$$

in line with our estimate of 10 fm above.

Had we used a quantum cutoff, we'd find

$$R^q_{DD} = \lambda\!\!\!^{-}_{dB,DD} = \frac{\hbar}{\mu\nu} \approx \frac{1.055\times10^{-34}\,\mathrm{J\,s}}{(3.34\times10^{-27}\,\mathrm{kg})(3.36\times10^{6}\,\mathrm{m\,s^{-1}})} \approx 9.4\times10^{-15}\,\mathrm{m} = 9.4\,\mathrm{fm} \tag{3.5.26}$$

which is not all that different.

For an electron on deuteron collision, however, we find (same velocity, assuming the electrons are on average at zero velocity on the scale of an atom) we have only a different reduced mass. So we multiply the previous result by

$$\frac{m_D}{2}\frac{1}{\mu} = \frac{m_D}{2}\frac{m_D + m_e}{m_e m_D} = \frac{m_D + m_e}{2m_e} = \frac{m_D}{2m_e} + \frac{1}{2} \approx \frac{3.34\times10^{-27}\,\mathrm{kg}}{9.11\times10^{-31}\,\mathrm{kg}} + \frac{1}{2} \approx 3700 \tag{3.5.27}$$

and so get an estimate of

$$R_{De} \approx 3700 \overbrace{(2.5 \times 10^{-14}\,\mathrm{m})}^{R_{DD}} \approx 9.3 \times 10^{-11}\,\mathrm{m} = 93\,\mathrm{pm} \tag{3.5.28}$$

similar to the Bohr radius (53 pm) estimate we had above. The quantum estimate will be larger by the same factor of about 3700 and so yields

$$R^q_{De} = \lambda_{dB,De} \approx 3700 \overbrace{(9.4 \times 10^{-15}\,\mathrm{m})}^{\lambda_{dB,DD}} \approx 3.5 \times 10^{-11}\,\mathrm{m} = 35\,\mathrm{pm} \tag{3.5.29}$$

which is again in the same ballpark.

3.5.2 What About Thermonuclear Fusion?

You might naturally ask yourself, how do things improve for thermonuclear fusion? The answer is that the electrons are now free of the nuclei, and so it is not as easy to hit the electron. There are still problems with hitting the electrons, since they absorb the kinetic energy, but the electrons are no longer "stationary" with respect to a region of an atom, but can roam freely and so are more difficult to collide with. You can think of this because now the v^2 in $\frac{1}{2}\mu v^2$ is much larger because we have really hot ions and electrons, and so the effective cross section is much smaller. In addition, because the electrons and ions are in a Maxwellian at some temperature, the exchange of collisions will average out and so scattering is not as big of a problem. In other words, the particles are still confined and not immediately lost in thermonuclear fusion. One important loss is still due to radiation, though. Remember that accelerating charges radiate energy, and scattering will cause electrons and ions to accelerate or decelerate causing some Bremsstrahlung. This is a loss mechanism if the radiation leaves the plasma region, which can easily happen, especially at the plasma edges. The original Lawson criterion used an estimate for radiation power losses to derive a Q.

3.6 Muon Catalyzed Fusion

You may have heard that an abundance of muons would help make fusion easy. This has elements of truth to it, but would not solve our problems. If you are unfamiliar, a muon is basically a more massive version of an electron, but it is unstable in nature.

First, why do muons catalyze fusion? If you read some treatments the incorrect argument goes roughly like this: the muon makes the atomic radius smaller and so the nuclei (say a deuteron and triton or deuteron and deuteron) are closer together, which allows fusion to happen more easily. To see this look at Bohr's formula for the orbit of an electron. We have uniform circular motion and the Coulomb law. Thus we must have (use the center of momentum frame with μ the reduced mass, r the relative distance between the particles, Z the charge of the nucleus, and v the relative velocity)

$$\frac{\mu v^2}{r} = \frac{Ze^2}{4\pi\epsilon_0 r^2} \tag{3.6.1}$$

$$\mu v^2 r = \frac{Ze^2}{4\pi\epsilon_0} \tag{3.6.2}$$

The final step is to use $\mu v r = n\hbar$, that is quantized angular momentum. We'd like the radius so $v = n\hbar/(\mu r)$ and

$$\mu \frac{n^2\hbar^2}{\mu^2 r^2} r = \frac{Ze^2}{4\pi\epsilon_0} \tag{3.6.3}$$

$$r = \frac{4n^2\hbar^2\pi\epsilon_0}{Z\mu e^2} \tag{3.6.4}$$

For electrons it is quite accurate to assume it is orbiting the nucleus, so $\mu = m_e$, $Z = 1$ and so for $n = 1$ we find

$$r_{e1} = \frac{4\hbar^2\epsilon_0}{m_e e^2} \tag{3.6.5}$$

the Bohr radius. For muons we actually need to use $\mu = (m_n m_\mu)/(m_n + m_\mu)$ with m_n the nucleus mass and m_μ the muon mass. It turns out that $m_\mu \approx 1.88 \times 10^{-28}\,\text{kg}$ or about 10% of a proton. Thus the new radius of the atom (say a deuteron) with a muon is

$$r_{\mu 1} = \frac{4\hbar^2\epsilon_0}{\mu e^2} = \frac{m_e}{\mu} r_{e1} \approx \frac{1}{196} r_{e1} \approx 0.005 r_{e1} \ll r_{e1} \tag{3.6.6}$$

And so the atomic radius with a muon is smaller than that for an electron.

The problem with this treatment for the given claim is that the previous argument only shows that the atomic radius is smaller, not that the nucleons in the nucleus are any closer together. It is rather difficult to imagine why the nuclei would suddenly get closer together due to this, though, it's not completely implausible. In any case, the smaller atomic radius means that particles can now get closer to the nucleus before they are scattered because the muon screens the nucleus's charge. This is the real key: particles can now get closer to the nucleus before feeling a huge Coulomb barrier. For if we looked at the effective radius R from above, we'd find that $\frac{m_D}{2}\frac{m_D+m_\mu}{m_D m_\mu} \approx 18$, which yields a much more favorable cutoff. It is this aspect that makes it possible for fusion events to occur, even at room temperatures. The even better part is that if a muon penetrates past an electron, the electron in an atom is then ejected because it is no longer bound to the atom (the muon will stay closer to the nucleus and screens the charge from the electron).

Muons have two major limitations, however. First, they last 2.2 µs in their rest frame. This means there are not a whole lot of sources of muons in large numbers, because all the muons decay away. But suppose we found a process that did produce muons in the necessary numbers. There is a second, more fatal, flaw. If we consider DT fusion, then an α particle, ^{4}He is produced. The muon can easily be attracted to the α, and so "stick" to it after a fusion event rather than go free and stick to some other deuteron or triton. Jackson[9] calculated that the probability after a fusion event of the muon sticking to the α to be about 1%. This means that a muon can only help about 100 fusion events to occur before it stops catalyzing fusion. This is not enough reactions to produce net energy when accounting for muon scarcity and lifetimes.

3.7 Further Reading

For a general overview of nuclear, atomic, and particle physics, I highly recommend Eisberg[6]. If you are interested in both the history of fission then Rhodes[15] is a good resource, though it

focuses on the weapons angle. Stellar fusion can be learned about in Phillips[14] among other online resources such as Knapp[12]. Wesson[19] has a good overview of terrestrial fusion. For beam fusion, the lecture notes by Schuster[16] are very clear. Muon catalyzed fusion can of course mostly be looked at through Jackson[9], though online resources tend also to be good.

3.8 Problem Set

3.1. For Section 3.1.

3.1.1. Calculate (3.1.8) and (3.1.9) using the momentum four-vector formalism. That is define (for the standard basis (see Section 1.2.6)

$$\mathbb{P}_W = \frac{E_W}{c}\hat{\mathbf{e}}_0 + \mathbf{p}_W \tag{3.8.1}$$

$$\mathbb{P}_X = \frac{E_X}{c}\hat{\mathbf{e}}_0 + \mathbf{p}_X \tag{3.8.2}$$

$$\mathbb{P}_Y = \frac{E_Y}{c}\hat{\mathbf{e}}_0 + \mathbf{p}_Y \tag{3.8.3}$$

$$\mathbb{P}_Z = \frac{E_Z}{c}\hat{\mathbf{e}}_0 + \mathbf{p}_Z \tag{3.8.4}$$

$$\mathbb{P}_I = \frac{E_I}{c}\hat{\mathbf{e}}_0 + \mathbf{p}_I = \mathbb{P}_W + \mathbb{P}_X \tag{3.8.5}$$

$$\mathbb{P}_F = E_F\hat{\mathbf{e}}_0 + \mathbf{p}_F = \mathbb{P}_Y + \mathbb{P}_Z \tag{3.8.6}$$

using that $\hat{\mathbf{e}}_0 \cdot \hat{\mathbf{e}}_0 = -1$, and for a single particle (any of W, X, Y, or Z) $\mathbb{P} \cdot \mathbb{P} = m_0^2c^2$ for that particle's rest mass.

3.1.2. Derive (3.1.12) and starting just with classical energy conservation and classical momentum conservation. You will still have to assume that the initial kinetic energy is essentially zero in comparison to the final kinetic energy.

3.1.3. Write definitions for nuclide and isotope and see if you understand what the difference between them could be.

3.1.4. Sometimes people refer to radionuclides. These are nuclides that are unstable, and so radioactively decay. Look at a table of nuclides and think about the ratio of radionuclides to stable nuclides. Is it surprising how many radionuclides there are?

3.2. For Section 3.2.

3.2.1. Look up what double decays are possible. Find some example decays.

3.2.2. The SI units for radiation are the becquerel Bq, gray Gy, and sievert Sv. Is there a difference between a becquerel and a hertz? Explain what the differences are between the three units I mentioned. It may help to figure out which refers to a radioactive decay rate (activity referred to a radionuclide), which is an absorbed dose, and which is a dose equivalent.

3.2.3. Radiation units have several other older units used. Determine how a curie Ci, a roentgen R, and a rem rem are related to the SI units in the previous problem. These are often used units despite the availability of SI equivalents.

3.2.4. Find what the lethal doses of radiation are. Is there a single measure that determines this? Do you see why different types of radiation can cause different amounts of biological tissue damage?

3.2.5. The banana equivalent dose (BED) is sometimes used to indicate that radioactivity is all around us. It is usually given as 0.1 µSv, based on the potassium-40 in bananas. What is this in comparison to background doses? Against dangerous exposure levels? Against a CT scan?

3.2.6. Explain the limitations of the BED. A human body keeps the amount of potassium in the body constant. This means that the excess amount of potassium-40 from eating a banana is quickly balanced by expelling an amount equal to the amount gained (assuming that the banana wasn't enriched with excess potassium-40). This means eating an additional banana does not increase the radiation dose a person would receive (except for the couple of hours of "processing" the banana). Explain how this would then adjust what a BED would be if we were to take this expulsion into account. What other limitations are there? [Hint: are all radiation exposures equivalent?]

3.2.7. Learn about the linear threshold model for radioactivity. Do you think it should be used to determine accumulated doses? That is, should a precautionary principle apply? Or do you think that this is being too cautious?

3.2.8. What is the relationship between the $1/e$ time (mean decay time) and the half life? Consider that there are N particles of a radionuclide. Then the differential equation is given by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\mu N$$

for μ the decay constant (one over the mean decay time). What is the solution for N? Then find the relationship between half-lives and decay times.

3.2.9. What if you have a chain of decays? that is given radionuclides A, B, and C with a number of nuclei given by N_A, N_B, and N_C. Assume A decays into B only, and B decays into C only. We assume $N = N_A + N_B + N_C$ is conserved. Thus the differential equations are

$$\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}N_A}{\mathrm{d}t} &= -\mu_A N_A \\
\frac{\mathrm{d}N_B}{\mathrm{d}t} &= -\mu_B N_B + \mu_A N_A \\
\frac{\mathrm{d}N_C}{\mathrm{d}t} &= -\mu_C N_C + \mu_B N_B
\end{aligned}$$

What are the solutions for N_B and N_C?

3.3. For Section 3.3.

3.3.1. Derive the differential cross section for Rutherford scattering. Assume that there is no φ dependence for scattering, as we can rotate to any φ and it will not affect the

differential cross section. To do so, it is simplest to find the relationship between b and θ. Then derive $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\frac{\mathrm{d}b}{\mathrm{d}\theta}$. Remember that we enforce $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} > 0$ so enforce that the previous expression is positive.

3.3.2. Calculate R_r for f_t and f_b two beams at different constant velocities $\mathbf{v}_{0t}$ and $\mathbf{v}_{0b}$. Thus $f_t(\mathbf{v}_t) = \delta(\mathbf{v}_t - \mathbf{v}_{0t})$ and $f_b(\mathbf{v}_b) = \delta(\mathbf{v}_b - \mathbf{v}_{0b})$. Do you get what you would expect naively from the notation $\langle\sigma v\rangle$ for v constant?

3.4. For Section 3.4.

3.4.1. Learn where the Boltzmann distribution makes sense. The Gibbs distribution in statistical mechanics is where you may want to look. The canonical partition function is also worth looking at.

3.4.2. Do you think the Boltzmann distribution makes sense for particles in a plasma such as in the sun? Do you think the number density of particles in the sun should also follow such a distribution?

3.4.3. Find E_C, what I called the Gamow energy, by applying the JWKB approximation to the Coulomb potential for two particles scattering off of each other. Consider spherical symmetry and so only consider $\psi(r,\theta,\varphi) = \xi(r)$ using

$$\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}^2\xi}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\xi}{\mathrm{d}r}\right) = \left(V(r) + \frac{\hbar^2}{2m}\frac{n(n+1)}{r^2} - E\right)\xi$$

with $V(r)$ standing for the Coulomb potential and E the total energy (here an eigenvalue).

3.4.4. I gave the number density distribution as an exponential factor $\exp(-\frac{x\ln 2}{0.1R_\odot})$. Where does this exponential equal 1/2? Do you see why I wrote it this way?

3.4.5. Look up some of the CNO chains and see how carbon, nitrogen, and oxygen act as catalysts. If our sun was dominated by the CNO chain, would it live longer or shorter? Why or why not?

3.4.6. What is the cross section for the proton-proton beginning reaction? I have given you the reaction rate r, n_t the sun's central density, and the thermal velocity as an estimate of v_b previously. Do you think such a cross section can be measured in a laboratory?

3.4.7. Look at Figure 3.3. Which reaction looks most promising to you? Consider the fusion cross sections in this figure against the previous stellar cross section. Which process would you choose to build a nuclear reactor if all you had were fusion cross sections to go off of?

3.4.8. Find the distribution of kinetic energy into the products for the ${}^3_2\mathrm{He} + T \to \alpha + D$ and for all the lithium reactions.

3.4.9. For the Lawson criteria we used flat profiles. What happens if you choose everything to have a simple quadratic profile around a central value. That is,

$$q(\mathbf{x}) = q(r) = \lambda q_0(1 - \frac{r}{a})^2$$

for any quantity q with a the minor radius of the plasma for a toroidal with circular cross section plasma. Here λ is a constant. You will need to find the $\hat{q}$ values and C_{P1} and C_{P2}. Note that $\hat{q} = \lambda q_0$.

3.4.10. What if we have $q(\mathbf{x}) = q(r) = q_0 \exp(-\alpha r)$? What happens to C_{P1} and C_{P2}?

3.5. For Section 3.5.

3.5.1. Suppose someone comes to you with a plan to bombard tritium with deuterium. How do you explain to them that this will not work?

3.5.2. Suppose for the sake of argument, that there were no electrons in the deuterium-tritium beam fusion experiment. We calculated the loss assuming one collision removes a deuteron from the beam. Suppose instead it takes effectively N collisions before it is removed from the beam. As a first approximation, don't alter the cross section after each collision, so that each collision is just barely changing the velocity of the deuterons. We showed that even if only one collision removed a deuteron we would have a gain greater than one. How many collisions are required before our gain factor is within one percent of its original?

3.5.3. Determine how much velocity would on average be lost by a deuteron undergoing a collision. Assume this would be true for each subsequent collision. How many collisions would you then estimate would be necessary to essentially remove the deuteron from a fusion event?

3.6. For Section 3.6.

3.6.1. Explain how muons catalyze fusion reactions.

3.6.2. What would the gain factor be for a muon catalyzed fusion assuming we had all the muons we needed for free?

3.6.3. What is more limiting for muon catalyzed fusion, the lifetime of a muon or the fact that it gets stuck to α particles?

References

[1] United States. Defense Atomic Support Agency et al. *The Effects of Nuclear Weapons*. The Effects of Nuclear Weapons. U.S. Atomic Energy Commission, 1962. URL: https://books.google.com/books?id=0vu108BraNUC.

[2] Allen H Boozer. "Stellarators as a fast path to fusion energy". In: *arXiv preprint arXiv:1912.06289* (2019).

[3] H-S Bosch and GM Hale. "Improved formulas for fusion cross-sections and thermal reactivities". In: *Nuclear fusion* 32.4 (1992), p. 611.

[4] A Dinklage et al. "Magnetic configuration effects on the Wendelstein 7-X stellarator". In: *Nature Physics* 14.8 (2018), pp. 855–860.

[5] BH Duane. "Fusion cross section theory". In: *Annual Report on CTR Technology* (1972).

[6] R.M. Eisberg and R. Resnick. *Quantum physics of atoms, molecules, solids, nuclei, and particles*. Quantum Physics of Atoms, Molecules, Solids, Nuclei and Particles. Wiley, 1985. ISBN: 9780471873730. URL: https://books.google.com/books?id=1v1QAAAAMAAJ.

[7] Z Hartwig and YA Podpaly. "The magnetic fusion energy formulary". In: (2012).

[8] J D Huba. *NRL PLASMA FORMULARY Supported by The Office of Naval Research*. Washington, DC: Naval Research Laboratory, 2013, pp. 1–71. ISBN: 9781234126070. URL: http://wwwppd.nrl.navy.mil/nrlformulary/.

[9] John David Jackson. "Catalysis of nuclear reactions between hydrogen isotopes by μ- mesons". In: *Physical Review* 106.2 (1957), p. 330.

[10] OJWF Kardaun. "On estimating the epistemic probability of realizing Q= Pfus/Paux larger than a specified lower bound in ITER". In: *Nuclear fusion* 42.7 (2002), p. 841.

[11] H.A. Klein. *The Science of Measurement: A Historical Survey*. Dover Books on Mathematics. Dover Publications, 2012. ISBN: 9780486144979. URL: https://books.google.com/books?id=CrmuSiCFyikC.

[12] Jill Knapp. "The physics of fusion in stars". In: (2011). URL: https://www.astro.princeton.edu/~gk/A403/fusion.pdf.

[13] George H Miley, Harry Towner, and Nenad Ivich. *Fusion cross sections and reactivities*. Tech. rep. Illinois Univ., Urbana (USA), 1974.

[14] A.C. Phillips. *The Physics of Stars*. Manchester Physics Series. Wiley, 2013. ISBN: 9781118723272. URL: https://books.google.com/books?id=ue2D%5C_%5C_e06XkC.

[15] R. Rhodes. *The making of the atomic bomb*. Touchstone book. Simon & Schuster, 1986. ISBN: 9780671441333. URL: https://books.google.com/books?id=IsIPAQAAMAAJ.

[16] Eugenio Schuster. "Nuclear fusion and radiation". In: (2019). URL: https://www.lehigh.edu/~eus204/teaching/ME362/lectures/lectures.html.

[17] R. Serber et al. *The Los Alamos Primer: The First Lectures on how to Build an Atomic Bomb*. University of California Press, 1992. ISBN: 9780520075764. URL: https://books.google.com/books?id=7TciOvRNamUC.

[18] Keiichi SHIBATA et al. "JENDL-4.0: A New Library for Nuclear Science and Engineering". In: *Journal of Nuclear Science and Technology* 48.1 (2011), pp. 1–30. DOI: 10.1080/18811248.2011.9711675. eprint: https://doi.org/10.1080/18811248.2011.9711675. URL: https://doi.org/10.1080/18811248.2011.9711675.

[19] John Wesson and David J Campbell. *Tokamaks*. Vol. 149. Oxford university press, 2011.

[20] ME Wyman. *An Integral Experiment to Measure the Tritium Production from 7Li by 14-Mev Neutrons in a Lithium Deuteride Sphere*. Tech. rep. LANL (Los Alamos National Laboratory (LANL), Los Alamos, NM (United States)), 1972.

[21] H Yamada et al. "Characterization of energy confinement in net-current free plasmas using the extended International Stellarator Database". In: *Nuclear Fusion* 45.12 (2005), p. 1684.

Chapter 4

Resource Needs

> *Very few facts are able to tell their own story, without comments to bring out their meaning.*
>
> — John Stuart Mill[15, p. 22]

This chapter is still about nuclear fusion, but focusing on requirements for making nuclear fusion a major electrical power source rather than on the physics requirements for creating such an electrical power source.

I will begin with an analysis for a DT plasma to produce fusion energy at a level that would be useful for the world. I will take fairly optimistic projections, not because I think they are necessarily the most realistic, but because it gives a baseline that we can then subtract inefficiencies from. The other method of using a pessimistic prediction may on occasion be used, but it provides less information. It is easy to assume the worst and show no useful energy will come out. It's a much stronger statement to assume the best and still not find useful energy production.

It is then worth looking in a bit more detail at tritium production for DT plasmas. After that we will explore DD plasmas for producing fusion energy. This analysis can easily be extended to other possible reactions, but DD is of special interest because of the relative abundance of deuterium on Earth and in the universe. Finally, I will examine some other nuclear fusion power plant potential problems.

4.1 Deuterium-Tritium Necessary Resources

> *He who knows only his own side of the case, knows little of that.*
>
> — John Stuart Mill[15, p. 41]

The first question to ask yourself, and, indeed, the first question physicists asked themselves, was do we have enough fuel? We will first calculate how much deuterium, lithium, and tritium there is in the world. It is often claimed that the amount of deuterium and lithium on Earth is enough to satisfy fusion needs for millions of years, and that fusion therefore has an essentially unlimited fuel supply[6]. There are various ways to understand essentially unlimited, but if resources last a million years that seems like a good enough definition for me. That is more than 100 times longer

than recorded human history, at the very least. The question, though, is where are these resources and how easily can they be extracted.

To start with, let's look at tritium. This means we need to look at lithium resources. If we use the most optimistic assumption that all of the lithium will be converted in reactions, and that the mined lithium is refined to be pure lithium-6, we can calculate how much energy and tritium will be produced. Remember the lithium reactions are given by

$$n + {}^{6}_{3}\mathrm{Li} \to T + {}^{4}_{2}\mathrm{He} + 4.78\,\mathrm{MeV} \tag{3.4.28}$$

$$n + {}^{7}_{3}\mathrm{Li} \to T + {}^{4}_{2}\mathrm{He} + n - 2.76\,\mathrm{MeV} \tag{3.4.29}$$

Let's do this per kilogram of lithium-6. The number of atoms of ^{6}Li in a kilogram is

$$\frac{1\,\mathrm{kg}}{6.015\,\mathrm{g\,/mol}} 6.022 \times 10^{23}\,\mathrm{mol}^{-1} \approx 1.00 \times 10^{26} \tag{4.1.1}$$

If all the lithium were to be used up in reactions, then the energy release in 1 kg of lithium-6 would then be

$$(1.00 \times 10^{26})(4.8\,\mathrm{MeV})(1.60 \times 10^{-22}\,\mathrm{GJ\,MeV}^{-1}) \approx 7.7 \times 10^{4}\,\mathrm{GJ} = 77\,\mathrm{TJ} \tag{4.1.2}$$

This will also clearly produce the same number of tritons. It is useful to keep in mind that 1 MJ is approximately the amount of energy in a glazed doughnut. That is a 1 GW power plant is a fairly large power plant, with a US household using about 1 MW h per month[1] or in a nice pure SI unit, 1.37 kW.[2] Thus the above amount of electrical energy could supply a single household for nearly 1800 years. And this is just from the energy release of the lithium-6.

The molar mass of atomic tritium[3] is about 3.02 g /mol and so a kilogram of lithium will produce

$$1\,\mathrm{kg}\frac{3.02\,\mathrm{g\,/mol}}{6.015\,\mathrm{g\,/mol}} \approx 0.5\,\mathrm{kg} \tag{4.1.3}$$

or just about half a kilogram of tritium.

Now we can consider the requirements for the *DT* fusion itself. Suppose we had a kilogram of tritium, then we know for a perfect burn we need the same number of deuterons as tritons. This means the mass of deuterium required is given by

$$1\,\mathrm{kg}\frac{2.01\,\mathrm{g\,/mol}}{3.02\,\mathrm{g\,/mol}} \approx 0.67\,\mathrm{kg} \tag{4.1.4}$$

This way there is the same number of tritons and deuterons. In other words, for this perfect balance, a kilogram of deuterium requires 1.5 kg of tritium which requires 3.0 kg of lithium-6. There are

$$\frac{1\,\mathrm{kg}}{2.01\,\mathrm{g\,/mol}} 6.022 \times 10^{23}\,\mathrm{mol}^{-1} \approx 3.00 \times 10^{26} \tag{4.1.5}$$

[1] The electrical units MW h, kW h, etc., should really be cursed as abominable complications. We have a unit for this already called the joule! But, life is not fair, and the electrical industry has decided to report the energy delivered in this terribly unnecessary unit. You can use 1 W h = 3600 J.

[2] A leisurely 120 doughnuts every day each month (of 30 days). Do not take nutritional advice from a physics textbook.

[3] Note that tritium and deuterium do not usually come as atoms, but as molecules attached to themselves. And in fact, even then they are often attached to other atoms, such as in heavy water.

possible reactions with a kilogram of deuterium. Each *DT* reaction can release 17.6 MeV per reaction. Thus, if we include the energy from the lithium-6 reaction we release 17.6 MeV + 4.78 MeV = 22.4 MeV per reaction or about a 27.3% increase in energy released (or a factor of 1.273) relative to just the *DT* fusion burn. If we ignore the lithium-6 reactions we find the energy release of just *D* and *T* yields

$$(3.00 \times 10^{26})(17.6\,\text{MeV})(1.60 \times 10^{-22}\,\text{GJ MeV}^{-1}) \approx 8.4 \times 10^{5}\,\text{GJ} = 840\,\text{TJ} \tag{4.1.6}$$

Thus, the total energy release if the tritium was produced from lithium-6 would be

$$(3.00 \times 10^{26})(22.4\,\text{MeV})(1.60 \times 10^{-22}\,\text{GJ MeV}^{-1}) \approx 1.08 \times 10^{6}\,\text{GJ} = 1080\,\text{TJ} \tag{4.1.7}$$

This ideal burn requires approximately 3 kg of lithium-6 per kilogram of deuterium, but that does not tell us how much lithium needs to be mined. Since lithium-6 has a natural abundance of 7.6% and lithium-7 a natural abundance of 92.4%, we see that we actually require $(1/0.076) \approx 13$ times the amount of natural lithium to get an appropriate amount of lithium-6. So in reality, for every kilogram of deuterium to fully burn from generated tritium (after the initial tritium is used, of course to start the process), we require (13)(3 kg) or 29 kg of natural lithium.

As an optimistic look, let's consider that the entirety of the energy in these processes can be extracted. Then we can apply factors for efficiency, but it is useful to see what the most energy we could get out is. We will consider 1 TW as a nice benchmark.[4] This is approximately two times the US's electrical production and about half of the world's total electrical production per year in the 2010 decade. We will also consider with and without the extra energy released by lithium-6 since it does not require much extra work. It just entails whether we omit a factor of 1.273. How much deuterium would be required to supply 1 TW for one year. That is

$$(1\,\text{TW})(1\,\text{yr}) = (1\,\text{TW})(3.154 \times 10^{7}\,\text{s}) \approx 3.154 \times 10^{19}\,\text{J} = 3.154 \times 10^{10}\,\text{GJ} \tag{4.1.8}$$

and so the minimum mass of deuterium required is

$$\frac{3.154 \times 10^{10}\,\text{GJ}}{8.4 \times 10^{5}\,\text{GJ/kg}} \approx 3.8 \times 10^{4}\,\text{kg} = 38\,\text{mt} \tag{4.1.9}$$

with 1 mt being one metric ton.[5] This means we require

$$m_{D,\text{min. req.}} \approx 3.8 \times 10^{4}\,\text{kg} \tag{4.1.10}$$
$$m_{T,\text{min. req.}} \approx 5.7 \times 10^{4}\,\text{kg} \tag{4.1.11}$$
$$m_{Li6,\text{min. req.}} \approx 1.1 \times 10^{5}\,\text{kg} \tag{4.1.12}$$
$$m_{Li,\text{min. req.}} \approx 1.5 \times 10^{6}\,\text{kg} \tag{4.1.13}$$

If we include the energy from ^{6}Li reactions we can lower the requirement to (I put the bar over the quantities to represent lithium-6 energy release included)

$$\bar{m}_{D,\text{min. req.}} \approx 3.0 \times 10^{4}\,\text{kg} \tag{4.1.14}$$
$$\bar{m}_{T,\text{min. req.}} \approx 4.5 \times 10^{4}\,\text{kg} \tag{4.1.15}$$
$$\bar{m}_{Li6,\text{min. req.}} \approx 9.0 \times 10^{4}\,\text{kg} \tag{4.1.16}$$
$$\bar{m}_{Li,\text{min. req.}} \approx 1.2 \times 10^{6}\,\text{kg} \tag{4.1.17}$$

[4] The electrical companies would probably say something like about 8760 TW h/yr.

[5] I will try to avoid metric tons from now on since they are just 1000 kg, and could just as easily be called 1 Mg. They also add unneeded confusion about long tons, short tons, and metric tons.

We can compare these results to those that appear in the literature. Hartley[10] says that for a 1 TW fusion power capacity, it would require consumption of about 0.34 mt /GW yr^{-1} of lithium-6 or 3.4×10^5 kg/TW yr^{-1} which is similar to our estimate. The amount of lithium required in each plant is much higher, however. They estimate that about 4×10^5 mt to 1.2×10^6 mt [4×10^8 kg to 1.2×10^9 kg] of lithium for 1 TW power capacity. This is about a thousand times more lithium than is needed as fuel, and a substantial portion of the available lithium on Earth (outside of the oceans). Hartley says it is about 1/3 of known US lithium resources.[6] Hartley also uses that for 1 GW one requires about 50 kg of deuterium per year. This translates to 5×10^4 kg for 1 TW power capacity which is also close to our estimates.

In reality, we need to take into account some inefficiencies for a "realistic" optimistic estimate. Only 80% of the energy is available for electrical energy at best,[7] and conversions of this to electrical energy will lead at best to only about 25% of the 80% available actually being converted.[8] Most likely some of this electrical energy will have to be diverted for heating in current configurations, so let's use that only 15% of the total energy will make it. This means for a constant 1 TW production of electricity, we require per year

$$m_{D,1\,\mathrm{TW\,yr}} = 2.5 \times 10^5\,\mathrm{kg} \tag{4.1.18}$$
$$m_{T,1\,\mathrm{TW\,yr}} = 3.8 \times 10^5\,\mathrm{kg} \tag{4.1.19}$$
$$m_{Li6,1\,\mathrm{TW\,yr}} = 7.6 \times 10^6\,\mathrm{kg} \tag{4.1.20}$$
$$m_{Li,1\,\mathrm{TW\,yr}} = 9.9 \times 10^7\,\mathrm{kg} \tag{4.1.21}$$

If we include lithium-6 energy releases we still get

$$\bar{m}_{D,1\,\mathrm{TW\,yr}} = 2.0 \times 10^5\,\mathrm{kg} \tag{4.1.22}$$
$$\bar{m}_{T,1\,\mathrm{TW\,yr}} = 3.0 \times 10^5\,\mathrm{kg} \tag{4.1.23}$$
$$\bar{m}_{Li6,1\,\mathrm{TW\,yr}} = 6.0 \times 10^5\,\mathrm{kg} \tag{4.1.24}$$
$$\bar{m}_{Li,1\,\mathrm{TW\,yr}} = 7.8 \times 10^6\,\mathrm{kg} \tag{4.1.25}$$

Let us now consider how much of these materials there are in the world, and just as importantly, where they are.

4.1.1 Material and Location

We can first look at deuterium. Virtually all the deuterium that is needed is in the oceans at a concentration of $c_D = 300$ ppm (by mass). The volume of the ocean is estimated at $V_o = 1.37 \times 10^{18}\,\mathrm{m}^3$ with a mean density of $\rho_o = 1030\,\mathrm{kg/m}^3$. This means that there is mass of deuterium M_D in the oceans of

$$M_D = V_o \rho_o c_D = \left(1.37 \times 10^{18}\,\mathrm{m}^3\right)\left(1030\,\mathrm{kg/m}^3\right)\left(3 \times 10^{-4}\right) \approx 4.2 \times 10^{17}\,\mathrm{kg} \tag{4.1.26}$$

[6] Resources that are still not being used because of their poor quality (as of 2020).

[7] Remember we are using some of the energy to heat the plasma and keep the reactions going.

[8] You could argue that we should be using 40% as a best case scenario and then cut it down to about 30% when thinking about the power used by the plant itself. If you believe this, you can cut our resource needs in half, and so resources will last twice as long. These better numbers are not impossible, but seem rather optimistic to me. Since we really can only claim to be getting order of magnitude estimates, the factor of two shouldn't drastically change our results.

For 1 TW of constant power each year (with power losses put in), we estimated that this would require about $m_{D,1\,\mathrm{TW\,yr}} \approx 2.5 \times 10^5\,\mathrm{kg/yr}$ of deuterium and $3.8 \times 10^5\,\mathrm{kg/yr}$ of tritium. This means that the supply of deuterium would last

$$t_{D,L} = \frac{M_D}{m_{D,1\,\mathrm{TW\,yr}}} \approx \frac{4.2 \times 10^{17}\,\mathrm{kg}}{2.5 \times 10^5\,\mathrm{kg/yr}} \approx 1.7 \times 10^{12}\,\mathrm{yr} \tag{4.1.27}$$

or about 2 trillion years. Even if we required 1000 TW of power per year,[9] there would be 2 billion years, much longer than our million year horizon for essentially unlimited.

If instead, we turn our attention to lithium, then the concentration is $c_L = 0.2\,\mathrm{ppm}$ (by mass) and so the mass of lithium in the ocean M_L is

$$M_L = V_o \rho_o c_L = \left(1.37 \times 10^{18}\,\mathrm{m^3}\right)\left(1030\,\mathrm{kg/m^3}\right)\left(2 \times 10^{-7}\right) \approx 2.8 \times 10^{14}\,\mathrm{kg} \tag{4.1.28}$$

The amount of lithium needed for 1 TW each year (this includes that most lithium is lithium-7 and lithium-6 is required to breed tritium, so this mass is the needed to be mined assuming we can separate with 100% efficiency) was calculated at $m_{Li,1\,\mathrm{TW\,yr}} = 9.8 \times 10^7\,\mathrm{kg}$. Then our supply would then be

$$t_{L,L} = \frac{M_L}{m_{L,1\,\mathrm{TW\,yr}}} \approx \frac{2.8 \times 10^{14}\,\mathrm{kg}}{9.8 \times 10^7\,\mathrm{kg/yr}} \approx 2.9 \times 10^6\,\mathrm{yr} \tag{4.1.29}$$

or about 3 million years.[10] This meets our requirements for virtually unlimited, and is what will be presented if you challenge someone about lithium supplies.

However, extracting something with such a low concentration from ocean water is a difficult task. Lithium is already highly prized for lithium-ion batteries, but this lithium comes from brine lake deposits in South America and a few other areas[14][17] and from particularly good ore deposits in Australia. Practically all the world's supply of lithium comes from just these two sources right now. If we could access ocean lithium economically, it would almost certainly already be done. The counterargument is that with cheap fusion energy, the economics of lithium extraction from ocean water will change so that it is economically viable. This could be true, but it is difficult to predict in advance. Before moving on to the relevant resource numbers, let's consider some difficulties with extracting small amounts of material from a vast ocean. We will see that for lithium it just might be doable without losing energy, and so it just might be possible for this to be economically worth it.

4.1.2 Extraction from Ocean Water

As a beginning cautionary tale, consider gold. The concentration is a measly $c_{\mathrm{Au}} = 0.005\,\mathrm{ppb}$ (by mass), but that means there is

$$M_{\mathrm{Au}} = V_o \rho_o c_{\mathrm{Au}} = \left(1.37 \times 10^{18}\,\mathrm{m^3}\right)\left(1030\,\mathrm{kg/m^3}\right)\left(5 \times 10^{-12}\right) \approx 7.0 \times 10^9\,\mathrm{kg} \tag{4.1.30}$$

Compare this with $1.7 \times 10^8\,\mathrm{kg}$ that has been mined by humanity. People have tried to get gold from seawater, but in a cubic kilometer of seawater there is only about 5 kg of gold. Try to imagine

[9] The current global energy production is 2 TW and so 1 TW is a significant production of electrical energy. If we include all power (not just electrical) then the world uses about 20 TW.

[10] Even if we include the extra energy from lithium-6 we only get 3.7 million years.

a cubic kilometer of seawater. Suppose we could process 1000 Olympic size swimming pools of water[11] each day. It would still take over a year (400 days) to process a cubic kilometer of water. That rate translates into about $30\,\mathrm{m^3\,s^{-1}}$, which is about two thousandths (0.002) of the volume discharged by the Mississippi River ($16\,800\,\mathrm{m^3\,s^{-1}}$).[12]

With this lesson in hand, let us reexamine deuterium and lithium from the oceans. Suppose we have a volume of some main substance with density ρ_m with another substance of concentration $c \ll 1$ and density ρ_c and we then take out a small volume $\mathrm{d}V$ and extract some of the substance out of this volume with efficiency α. Let the mixture's concentration be ρ_0. That is we extract a mass of

$$\mathrm{d}m_c = \alpha c \rho_0 \,\mathrm{d}V \tag{4.1.31}$$

Initially the entire mass was

$$M_0 = \rho_0 V_0 = M_0 = M_m + M_c \tag{4.1.32}$$

$$c = \frac{M_c}{M_m + M_c} = \frac{M_c}{M_0} \approx \frac{M_c}{M_m} \tag{4.1.33}$$

where $M_c \ll M_m$ since $c \ll 1$. After removing the small amount of mass the new concentration will be

$$c_n = \frac{M_c - \mathrm{d}m_c}{M_0 - \mathrm{d}m_c} \tag{4.1.34}$$

Because $M_0 \gg M_c$ then this simply says

$$\mathrm{d}c = c_n - c = \frac{M_c - \mathrm{d}m_c}{M_0} - \frac{M_c}{M_0} = -\frac{\mathrm{d}m_c}{M_0} = -\frac{\alpha\rho_0 c\,\mathrm{d}V}{M_0} \tag{4.1.35}$$

Thus, as time changes

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\frac{\alpha\rho_c c}{M_0}\frac{\mathrm{d}V}{\mathrm{d}t} \tag{4.1.36}$$

where $\frac{\mathrm{d}V}{\mathrm{d}t}$ is the volume rate of change, that is, the volume of the main substance that must be processed, alternatively called a discharge or volume flow rate. We can use $M_0 = \rho_0 V_0$ to then write

$$V_0\frac{\mathrm{d}c}{\mathrm{d}t} = -c\frac{\alpha\rho_0}{\rho_0}\frac{\mathrm{d}V}{\mathrm{d}t} \equiv -c\alpha\frac{\mathrm{d}V}{\mathrm{d}t} \tag{4.1.37}$$

Note that if we used ppm by volume instead of mass, we would also arrive at the same equation with $c \ll 1$,[13]

$$V_0\frac{\mathrm{d}c}{\mathrm{d}t} = -c\alpha\frac{\mathrm{d}V}{\mathrm{d}t} \tag{4.1.38}$$

[11] The minimum volume of an Olympic size swimming pool is $2500\,\mathrm{m^3}$ which is what I will use.

[12] You can look at the data yourself as provided by NOAA and NWS at https://waterdata.usgs.gov/usa/nwis/uv?site_no=07374000.

[13] The derivation follows the same form as that for an article examining uranium from ocean water in Abbott[1].

Now if we are extracting and want a certain amount of mass per year, then $\alpha\frac{\mathrm{d}V}{\mathrm{d}t} \propto 1/c$ because the less concentrated the substance is, the more flow we need to get the same mass out. Because of this, we can simply define the flow rate to be $\frac{\mathrm{d}F}{\mathrm{d}t} = \alpha\frac{\mathrm{d}V}{\mathrm{d}t}$. Then $\frac{\mathrm{d}F}{\mathrm{d}t} = k/c$ and we have the equation

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\frac{k}{V_0} \tag{4.1.39}$$

$$c = \frac{-k}{V_0}t + c_0 \tag{4.1.40}$$

with c_0 the initial concentration. We also have that $\frac{\mathrm{d}F_0}{\mathrm{d}t} \equiv \alpha\frac{\mathrm{d}V}{\mathrm{d}t}|_{t=0} = k/c_0 \equiv G_0$ so that if we solve for the flow volume required for a specified mass per year we find

$$\frac{k}{\frac{\mathrm{d}F}{\mathrm{d}t}} = \frac{-k}{V_0}t + c_0 \tag{4.1.41}$$

$$\left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)^{-1} = \frac{-t}{V_0} + \frac{c_0}{k} \tag{4.1.42}$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{1}{\frac{1}{G_0} - \frac{t}{V_0}} = \frac{V_0 G_0}{V_0 - G_0 t} \tag{4.1.43}$$

And so the total volume processed will be

$$V = \frac{F}{\alpha} = \frac{1}{\alpha}\int_0^T \mathrm{d}t\ \frac{V_0 G_0}{V_0 - G_0 t} = \frac{V_0 G_0}{\alpha G_0}\ln\left(\frac{V_0}{V_0 - G_0 T}\right) = -\frac{V_0}{\alpha}\ln\left(1 - \frac{G_0 T}{V_0}\right) \tag{4.1.44}$$

Note a problem with this equation. Remember $G_0 = \alpha\frac{\mathrm{d}V}{\mathrm{d}t}|_{t=0}$. So when $T = V_0/G_0$ we get an infinite amount of water processed. What this is saying is that as we get close to removing all of our desired material out of the water, it requires more and more water to extract the little that is left.

We can then note the initial flow rate required if we are to produce a constant mass per second $r_{m,\text{cons.}}$ is

$$V_0 = \frac{G_0}{\alpha} = \frac{r_{m_s,\text{cons.}}}{\alpha c \rho_0} \tag{4.1.45}$$

This is arrived at by asking what flow rate would initially be required to produce the mass we desire. We want $m_{\text{cons.}}$ and so we initially have $c\rho_0$ of the desired mass per volume. Thus, we take the initial desired mass per time and divide it by the given mass per volume to find the volume per time. Remember that F_0 takes into account that we can only get α of the $c\rho_0$ out, and so have to multiply by $1/\alpha$ to get the correct initial volume flow rate. So we get the formula (4.1.45).

Let's look at this with deuterium. For deuterium, let's assume that we have 100% efficiency ($\alpha = 1$). The initial flow required for our $m_{D,1\,\text{TW yr}} = 2.5 \times 10^5\,\text{kg/yr}$ for 1 TW each year then requires

$$V_0 = \frac{r_{m_D,1\,\text{TW yr}}}{c_D \rho_0} = \frac{(2.5 \times 10^5\,\text{kg/yr})}{(3.15 \times 10^7\,\text{yr/s})(3 \times 10^{-4})(1030\,\text{kg m}^{-3})} \approx 2.5 \times 10^{-2}\,\text{m}^3\,\text{s}^{-1} \tag{4.1.46}$$

This is a small value and would not require enormous expense to produce a volume discharge this large. The total volume discharge of rivers in the world is approximately $5 \times 10^5\,\text{m}^3\,\text{s}^{-1}$.[14] We can

[14] It should be noted that the Amazon river contributes $2 \times 10^5\,\text{m}^3\,\text{s}^{-1}$ of this discharge.

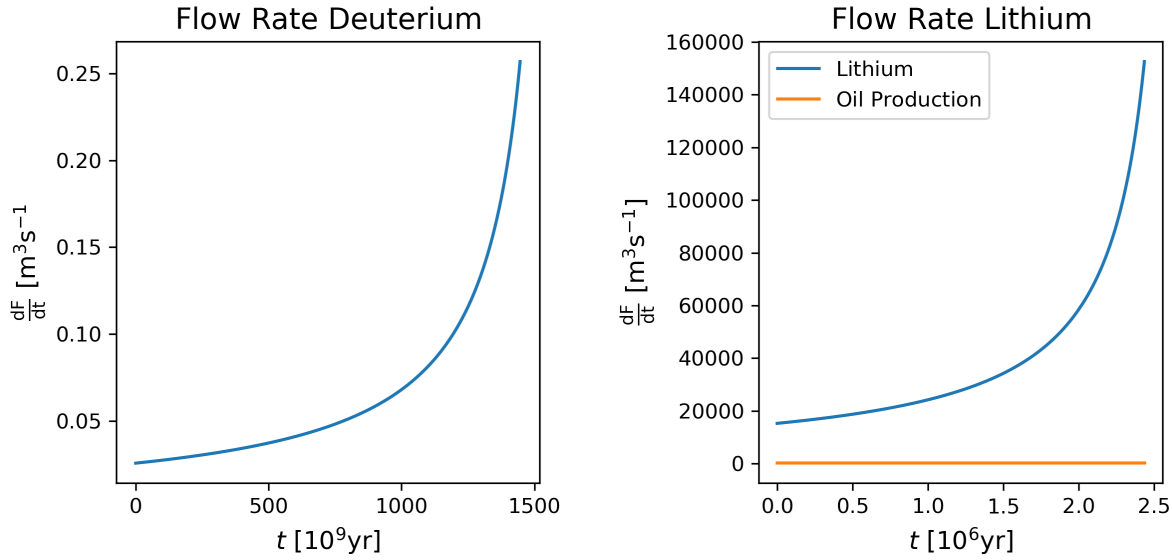

Figure 4.1: The required flow rates for deuterium on the left and lithium on the right are shown over time. This assumes inefficiency in electrical production (15% of total energy release goes out as electrical energy), but perfect removal from sea-water ($\alpha = 1$), so on balance leads to an optimistic estimate. Note that deuterium requires very little volume flow per second and is quite manageable. On the other hand, lithium requires flows of nearly $10^4\,\mathrm{m^3\,s^{-1}}$ or $1\,\mathrm{km^3\,d^{-1}}$, a massive volume of water to process! The oil production of $150\,\mathrm{m^3\,s^{-1}}$ is shown on the lithium figure to show the scale.

instead compare it to oil extraction. In 2018 about 80 or 85 billion barrels of oil were produced.[15] This corresponds to about $150\,\mathrm{m^3\,s^{-1}}$ using US Energy Information Administration data.[16] One other point of information is the water desalinated per year. This is about $10^{13}\,\mathrm{kg\,yr^{-1}}$[3] or $310\,\mathrm{m^3\,s^{-1}}$. We can then graph the required volume discharge rate for deuterium (and lithium) from the oceans in Figure 4.1. The good news is that there will not be substantial increases in the volume discharge rate for millions of years in both cases. The bad news, as we will calculate, is that lithium will require a gigantic volume discharge.

For lithium we require $m_{L,1\,\mathrm{TW\,yr}} = 9.9 \times 10^7\,\mathrm{kg}$, so with 100% efficiency,

$$G_0 = \frac{m_{L,\mathrm{cons.}}}{c_L \rho_0} = \frac{(9.9 \times 10^6\,\mathrm{kg/yr})}{(3.15 \times 10^7\,\mathrm{yr/s})(2 \times 10^{-7})(1030\,\mathrm{kg/m^3})} \approx 1.5 \times 10^4\,\mathrm{m^3/s} = 15\,000\,\mathrm{m^3/s} \tag{4.1.47}$$

This is similar to the discharge of the Mississippi River ($16\,800\,\mathrm{m^3\,s^{-1}}$). Even if we include lithium-6 energy release

$$\bar{G}_0 = \frac{\bar{m}_{L,\mathrm{cons.}}}{c_L \rho_0} = \frac{(7.8 \times 10^6\,\mathrm{kg/yr})}{(3.15 \times 10^7\,\mathrm{yr/s})(2 \times 10^{-7})(1030\,\mathrm{kg/m^3})} \approx 1.2 \times 10^4\,\mathrm{m^3/s} = 12\,000\,\mathrm{m^3/s} \tag{4.1.48}$$

[15] A barrel of oil is $0.1359\,\mathrm{m^3}$.
[16] The data is somewhat confusing, but I used world oil production at about 80 billion barrels per day.

We see that this flow is still less than that of all the rivers of the world, but it is also 100 times larger than the amount of oil extracted currently. If we have less than 100% extraction efficiency this comparison only gets worse.

It seems fairly clear from this that deuterium should not be difficult to extract, even with poor efficiencies, it would be surprising if extracting it from the oceans would be as energy intensive as burning it as fusion fuel.

For ocean lithium, it seems unlikely to be processed efficiently enough to justify its use in fusion, though it just might be doable. That is a lot of water, but fusion plants produce a lot of energy. Note that if we simply assume an efficiency of 50% (this is incredibly optimistic since at best 20% efficiency may have been shown), suddenly even with our most optimistic scenario [perfect energy consumption and $\alpha = 1$] we find

$$\bar{G}_0^{\text{best}} \approx \frac{(0.15)\bar{G}_0}{0.5} \approx (0.3)\bar{G}_0 \approx 4000\,\text{m}^3\,\text{s}^{-1} \tag{4.1.49}$$

Cutting this in half with a more optimistic power plant efficiency might get you down to $2000\,\text{m}^3\,\text{s}^{-1}$. Remember that total oil production is about $150\,\text{m}^3\,\text{s}^{-1}$ as quoted before and that is a massive industry. This is no small amount of water to process, and we have not actually considered the energy efficiency of the extraction process. And remember this is for only 1 TW of power. If fusion were to meet the world's needs in the future, it might require as much as 15 TW, so 15 times the discharge volume rates we've calculated. What this says is that lithium extraction from sea water would need to become an industry bigger than the oil industry is today.

4.1.3 Literature Review of Resources for Fusion

Given these limitations on lithium, we can look at the world's likely resources on land. This is still fairly optimistic, as they are not reserves (that is the subset of resources that are economically viable). There are numerous estimates of resources and reserves. Hartley[10] cites US lithium reserves between $3.7 \times 10^8\,\text{kg}$ to $3.2 \times 10^9\,\text{kg}$. He claims $2.5 \times 10^{16}\,\text{kg}$ of lithium in the ocean, but cites $1.7 \times 10^{-4}\,\text{kg}\,\text{m}^{-3}$ which with our ocean volume estimate of $V_o = 1.3 \times 10^{18}\,\text{m}^3$ yields $2.2 \times 10^{14}\,\text{kg}$ which is essentially our previous estimate. Hartley tends to be quite optimistic that the $3.2 \times 10^9\,\text{kg}$ of US lithium supplies are viable reserves. Hartley essentially assumes so by saying that lithium produces enough energy for fusion such that it will be economically exploitable, and so will become a reserve. I feel as if this is assuming fusion energy is cheap, but you can decide for yourself. Supposing that these resources indeed were reserves, under our optimistic scenario (15% efficiency) we would have $3.2 \times 10^9\,\text{kg}/7.8 \times 10^6\,\text{kg/yr} = 410\,\text{yr}$.

Fasel[7] says lithium resource estimates vary widely from $9.4 \times 10^9\,\text{kg}$ to $2.1 \times 10^{10}\,\text{kg}$. Fasel also explains that Australia has most of the mineral resources, i.e., from rocks, while Chile has most of the brines that supply lithium to the world. Some of the resource estimates include difficult to extract lithium, as well. He also considers ocean resources with the accurate estimation of $2.3 \times 10^{14}\,\text{kg}$ of lithium in the ocean. Fasel then cites a Japanese experiment that claimed to recover usable lithium with 20% efficiency from seawater. This must not have turned out as well as hoped given that it has been nearly 15 years and this method does not appear to have been adopted.[17] Fasel estimates $6.3 \times 10^3\,\text{kg}$ to $8.9 \times 10^3\,\text{kg}$ of lithium consumed for 1.5 GW during

[17] I am dubious of the relevance of this experiment to general extraction of lithium from ocean water. As Fasel states, "Unfortunately, no details exist on the process energy requirements and the estimated production price."

8000 h or 1.37 GW continuously during a year. This translates to 4.6×10^6 kg to 6.5×10^6 kg which is within our most optimistic estimate.

Vikström[18][18] estimates resources as 21.3×10^9 kg to 65.3×10^9 kg with estimated reserves at 6.5×10^9 kg to 29.3×10^9 kg. For our most optimistic scenario assuming every resource becomes a reserve we would find

$$\frac{6.5 \times 10^{10}\,\text{kg}}{7.8 \times 10^6\,\text{kg/yr}} = 8300\,\text{yr} \tag{4.1.50}$$

for a constant 1 TW each year. Vikström also points out the absurdly large volumes of water required to process seawater and gives an estimate from Bardi[3] of 2.5×10^7 kg per year of lithium would require about 1.5×10^3 TW h of energy, or 2.2×10^{11} J kg^{-1} which is 0.22 TJ kg^{-1} of lithium. [19] Bardi estimates two different methods: pumping the water directly through membranes like in desalination plants, and putting membranes in the seawater and using ocean currents (rather than pumping water) to push water through a membrane. The analysis does not include the energy needed to create the membranes or of the process of actually extracting the lithium from the membranes.[20] Pumping water directly would require 9000 J per kilogram of water pumped according to the paper. For 7.8×10^6 kg of lithium corresponding to at best

$$(4000\,\text{m}^3\,\text{s}^{-1})(1\,\text{yr})(1030\,\text{kg}\,\text{m}^{-3}) \approx 1.2 \times 10^{14}\,\text{kg} \tag{4.1.51}$$

of seawater this would require 10^{18} J or 10^6 TJ to process that amount of water into lithium. Thus it requires about 0.13 TJ per kilogram of lithium. This would still compare favorably to our previous burn for *DT* of about 270 TJ with about 30 kg of lithium which would only require about 3.8 TJ of this. Thus the energy requirements are not altogether unfavorable to lithium extraction from seawater if efficient-enough membranes and separation processes can be discovered. You would still need to mass produce them and get 4000 m^3 s^{-1} of water processed. This rate would go through a cubic kilometer of seawater about once every three days.

It is best to just use the words in Vikström, "In summary, extraction from seawater appears not feasible and not something that should be considered viable in practice, at least not in an imminent future. A major portion of sound scepticism should accompany all thoughts about rapid developments of large-scale Li-extraction from seawater."[18]

So we find that in our optimistic (though not extremely optimistic) scenario, we have about 800 years of lithium available. Of course, 800 years is a long time, but it is not even close to our

[18] I consider Vikström to be the best paper on resources among these estimates. The paper is clear, comprehensive, and is fairly careful in its analysis. It also has lithium resources and reserves data for most countries on Earth.

[19] Remember that about 29 kg of natural lithium was needed for a perfect burn of 1 kg of deuterium and 1.5 kg of tritium, which releases 1080 TJ. So the lithium would require about 9 TJ of this. Of the total theoretical energy production, maybe 25% will be available for electrical energy, leaving 270 TJ for electrical energy and so the natural lithium extraction would require at least 3.7% of the fusion reactor's energy. This doesn't include how much land use and the initial set up of the plants, but gives a flavor of why some are optimistic. Even if lithium alone took 10% of the energy, that could theoretically leave room for abundant energy production. I think Bardi's estimate should be closer to 1%, though he is assuming perfect efficiency of lithium extraction and is looking only at moving membranes or water through membranes.

[20] The Bardi paper's primary analysis is for uranium and so its lithium estimate is difficult to follow. It appears to me Bardi overestimates the lithium consumed by a fusion reactor by a factor of 10. Using 3000 kg of membranes required per 1 kg of lithium produced with 5 kW h per kilogram of membrane to move the membranes (this does not include processing energy) yields around 10^2 TW h. However, processing would almost certainly be significant, so that the extra factor of 10 is not perhaps a terrible estimate. In any case, the case is not great, but nor is it so dire as to be impossible to get lithium from seawater for fusion.

virtually unlimited claim of a million years. In addition, that is lithium consumption. The amount of lithium required for the blanket is about 1000 times more than the amount consumed. This would imply that basically the entirety of the world's lithium would be used in fusion power plants if we had a 1 TW fusion electrical energy capacity.[21] It is certainly fair to point out that 800 years is much longer than any technological prediction could be made, but fission power could claim the same advantages on this time frame and it would not theoretically require all of the world's lithium. Remember that I have been using fairly optimistic values.[22] Lithium is a precious resource for *DT* fusion and largely limits what is possible for *DT* fusion energy production.

We can also consider some of the other resource constraints. Lead and beryllium will also probably be needed for the blanket for tritium production. Here, Bradshaw[5] is a great article to consider resource constraints. The paper estimates a lithium-6 consumption of 266 kg per year for 2.385 GW or 1.1×10^5 kg TW^{-1} per year, which actually matches our estimate when not including energy released from the lithium-6 reactions. In addition, the initial lithium-6 inventory is estimated at about 1×10^4 kg to 2.5×10^4 kg for the 2.385 GW plant. So again, we see almost 1000 times more is needed than what is consumed.[23] In addition, Bradshaw considers the lead inventory. It is estimated that lead reserves are 79×10^9 kg and resources are 1.5×10^{12} kg. As an example of lead required, the DEMO reactor considered requires an inventory of 4×10^6 kg to 5×10^6 kg and the lead is burned up at 3.1×10^3 kg per year. Bradshaw then assumes 2760 fusion power stations for 30% of future energy production (about 2.74 TW continuous operation yearly), and with a burn up of about 8×10^5 kg of lithium-6 finds with 9.9×10^9 kg of regular lithium reserves lasting 990 years. But the lithium inventory would be nearly 6.9×10^7 kg of lithium-6, requiring about one tenth of the 7.6×10^8 kg reserves of lithium-6 as estimated in Bradshaw. Beryllium is an even worse case, requiring 3×10^8 kg in inventory and consuming about 5×10^5 kg annually. Note that 3×10^8 kg would most likely be between 90% and 100% of all beryllium, exceeding the Bradshaw beryllium resource estimate. Lead lasts for 175 000 yr, which is better. It would require 1.5×10^{12} kg of lead as inventory, though.

The good news is that lithium-6 is the badly needed part for fusion reactors. Perhaps blankets for tritium production can use something other than lithium with the lithium-6 to get the necessary tritium production.[24] The demand from lithium-ion batteries and fusion may spur conversion of resources to reserves. If they enabled seawater extraction, then there would be far fewer problems, at least from a running out of resources perspective. The seawater extraction would spawn an industry at least as large as oil extraction. To say that this is impossible is wrong, but it would require incredible investment.

This is perhaps a long-winded way of saying that the virtually unlimited claim is not well-justified. On the other hand, it seems reasonable to assume around 200 yr of less than 1 TW production from

[21] Unless we get a different battery technology, this is pretty clearly not ideal since lithium-ion batteries would be needed and require some of the lithium. Of course, the more abundant lithium-7 would be available for batteries, so it is may not be quite so dire.

[22] I am sure people will quibble with this, but even you assume double the efficiency of what I did, you would not be close to essentially unlimited, and you would still be holding most of the world's lithium in fusion power plants.

[23] If beryllium is used in the blanket (to reduce the amount of lithium needed), then 1.20×10^5 kg is required and 190 kg (again consistent with an approximate factor of 1000 between the amount consumed in reactions and the needed inventory).

[24] Although the considered replacements often have problems. Beryllium has a scarcity problem as well, as we saw. Hartley[10] gives the reserves at 4×10^8 kg [compare 8×10^7 kg for Bradshaw[5]] with a consumption of 6×10^5 kg to 1.1×10^6 kg at 1 TW every year. The required inventory of beryllium would again be about 1000 times the consumption, and so we find once again that nearly the entire world's beryllium supply would be in the reactors.

DT fusion is quite possible. With the intervening time, other approaches may present themselves.

One final comment is that this is under the optimistic scenario that the fuels deuterium and tritium are recycled with near 100% efficiency. Much of the fuel will not burn before it escapes confinement because fusion reactions are mostly done by particles in the tail of the Maxwellian. This means most of the fuel just flies around inside the reactor and does not fuse. If the fuels cannot be recycled (and in the case of the lithium blanket, extracted) with perfect efficiency, then the mass required per year increases (although the amount burned per year is fairly minimal), and the number of years of fusion are reduced.[25]

4.2 Tritium Breeding

Precious tritium is what makes this project go.

— Dr. Otto Octavius in *Spider-Man 2*

As mentioned in Section 3.4.2, we need lithium in order to breed tritium for *DT* fusion to work. The details of how much tritium we get requires careful analysis but the basic picture is that we get a tritium breeding ratio (TBR) based on the ratio of the tritium atom production over tritium atoms burned. Then the net TBR (TBR_{net}) is sometimes defined as the ratio of tritium atoms recovered for production over the tritium atoms burned. The net TBR needs to be greater than 1.01 for fusion to really be viable. It certainly must be greater than 1 for physical viability. Let the net TBR be B_T (as I will define it below). Then let f be the fraction of the time the plant is running. If we assume each plant requires at least a mass of tritium M_{T0} to begin running, we can calculate the doubling time for the mass given that the plant is only running a fraction f of the time (if $f = 0.01$, this corresponds to a plant running 1% of the time). The net TBR says that if there are r reactions then there are net TBR times r tritons produced, i.e. $B_T r$ tritons produced.

Before estimating the tritium production for a power plant, let us take a minute to explain what is going on. A fusion reactor needs tritium for *DT* reactions, which needs to come from somewhere. It is imagined lithium-6 is an ideal candidate. Thus, we use the high energy neutrons that come out of *DT* reactions, and have it hit a "blanket" of some material that will produce tritium. This blanket is made up of lithium-6 and other materials so that it can produce tritium. I have mentioned lead and beryllium as possible constituents. As we have seen, there are serious questions about whether these blankets can be produced at a scale needed for widespread fusion energy production. In any case, the blanket also serves to absorb neutrons, produce the heat for electrical energy production, and prevent neutron damage beyond the blanket.

We can estimate the reaction rate per volume near the ignition limit of $k_B T = 10\,\text{keV}$ and $n_T = n_D = n/2 = 5 \times 10^{19}\,\text{m}^{-3}$ as

$$\begin{aligned} \mathcal{R}_{DT} &= n_T n_D \langle \sigma v \rangle_{DT} \approx n_T n_D (10)^2 (1.1 \times 10^{-24}\,\text{m}^3\,\text{s}^{-1}) \\ &\approx (2.5 \times 10^{39}\,\text{m}^{-6})(1.1 \times 10^{-22}\,\text{m}^3\,\text{s}^{-1}) \approx 2.75 \times 10^{17}\,\text{m}^{-3} \end{aligned} \tag{4.2.1}$$

and so the number of reactions (per second) is given by

$$r_{DT} = V \mathcal{R}_{DT} \tag{4.2.2}$$

[25] Assuming that the unburnt and unrecycled fuel is irretrievable.

We then have $N_{rDT} = r_{DT}(1\,\mathrm{s})$ reactions in a second and so for $B_T = \mathrm{TBR}_{\mathrm{net}}$ we must now have bred $N_{btDT} = B_T N_{rDT}$ total triton atoms. Because the molar mass of a triton $w_T = 3 \times 10^{-3}\,\mathrm{kg\,mol^{-1}}$ is fixed, we can multiply both sides by the molar mass to find the mass of bred tritons. We can call this mass of bred tritons M_{btDT}, which were created by the mass of tritons burned in fusion reactions M_{rDT}. Then we find

$$N_{btDT} = B_T N_{rDT} \tag{4.2.3}$$
$$M_{rDT} = w_T N_{rDT} \tag{4.2.4}$$
$$M_{btDT} = w_T N_{btDT} \tag{4.2.5}$$
$$M_{btDT} = B_T M_{rDT} \tag{4.2.6}$$

Clearly after one second of reactions we will then have B_T times the initial burn amount given by M_{TB0} and after N seconds we will have a mass $M_T = (B_T)^N M_{TB0}$. It is also worth noting that if the initial amount of tritium in the reactor is M_{T0}, then $M_{TB0} \ll M_{T0}$ because we will have a lot of initial tritium that will not be burned up in reactions. Thus, if we wish to find the number of seconds N for a doubling period (with respect to the burned tritium) given B_T we use

$$\frac{M_T}{M_{TB0}} = 2 = (B_T)^N = (e^{\ln B_T})^N = e^{N \ln B_T} \tag{4.2.7}$$
$$N = \frac{\ln 2}{\ln B_T} \tag{4.2.8}$$

Thus it will take N seconds to double the initial mass burned assuming that the reaction rate remains constant throughout this process. If we only allow reactions to happen with duty cycle f then clearly it will take more time in proportion to the inverse of the duty cycle, $1/f$. Thus

$$N = \frac{\ln 2}{f \ln B_T} \tag{4.2.9}$$

As a check, for a duty cycle of half, $f = 0.5$ then it will take double the amount of time, which makes complete sense. This leads to Figure 4.2 for the doubling time for a given net TBR, B_T.

Our previous calculations suggested a likely tritium fusion reactivity near 10 keV yields $\langle \sigma v \rangle_{DT} \approx 1.1 \times 10^{-22}\,\mathrm{m^3\,s^{-1}}$. So for a tritium and deuterium density of $5 \times 10^{19}\,\mathrm{m^{-3}}$ this yields a reaction rate (per volume) of

$$\mathcal{R}_{DT} = (25 \times 10^{38}\,\mathrm{m^{-6}})(1.1 \times 10^{-22}\,\mathrm{m^3\,s^{-1}}) \approx 2.75 \times 10^{17}\,\mathrm{m^{-3}\,s^{-1}} \tag{4.2.10}$$

A DEMO-like size[12][26] would lead to a plasma volume of $2500\,\mathrm{m^3}$ and[27] so

$$r_{DT} \approx (2500\,\mathrm{m^3})(2.75 \times 10^{17}\,\mathrm{m^{-3}\,s^{-1}}) \approx (6.9 \times 10^{20}\,\mathrm{s^{-1}}) \tag{4.2.11}$$

The initial mass in such a reactor can be estimated by assuming a rather flat number density throughout the plasma volume yielding (this will be somewhat of an overestimate, but given that you would want to have extra tritium available at all times this is not a terrible overestimate)

$$M_{T0} = n_T V m_T \approx (5 \times 10^{19}\,\mathrm{m^{-3}})(2500\,\mathrm{m^3})(2.14 \times 10^{-27}\,\mathrm{kg}) \approx 2.7 \times 10^{-4}\,\mathrm{kg} = 270\,\mathrm{mg} \tag{4.2.12}$$

[26] The website www.fz-juelich.de/iek/iek-4/EN/Research/07_DEMO/artikel_2014.html gave an estimate of $2500\,\mathrm{m^3}$ for DEMO. Anywhere from about $1800\,\mathrm{m^3}$ to $3500\,\mathrm{m^3}$ seem like reasonable estimates given that DEMO is not a set design[12].

[27] Note that ITER, a rather large device, has a plasma volume of about $830\,\mathrm{m^3}$.

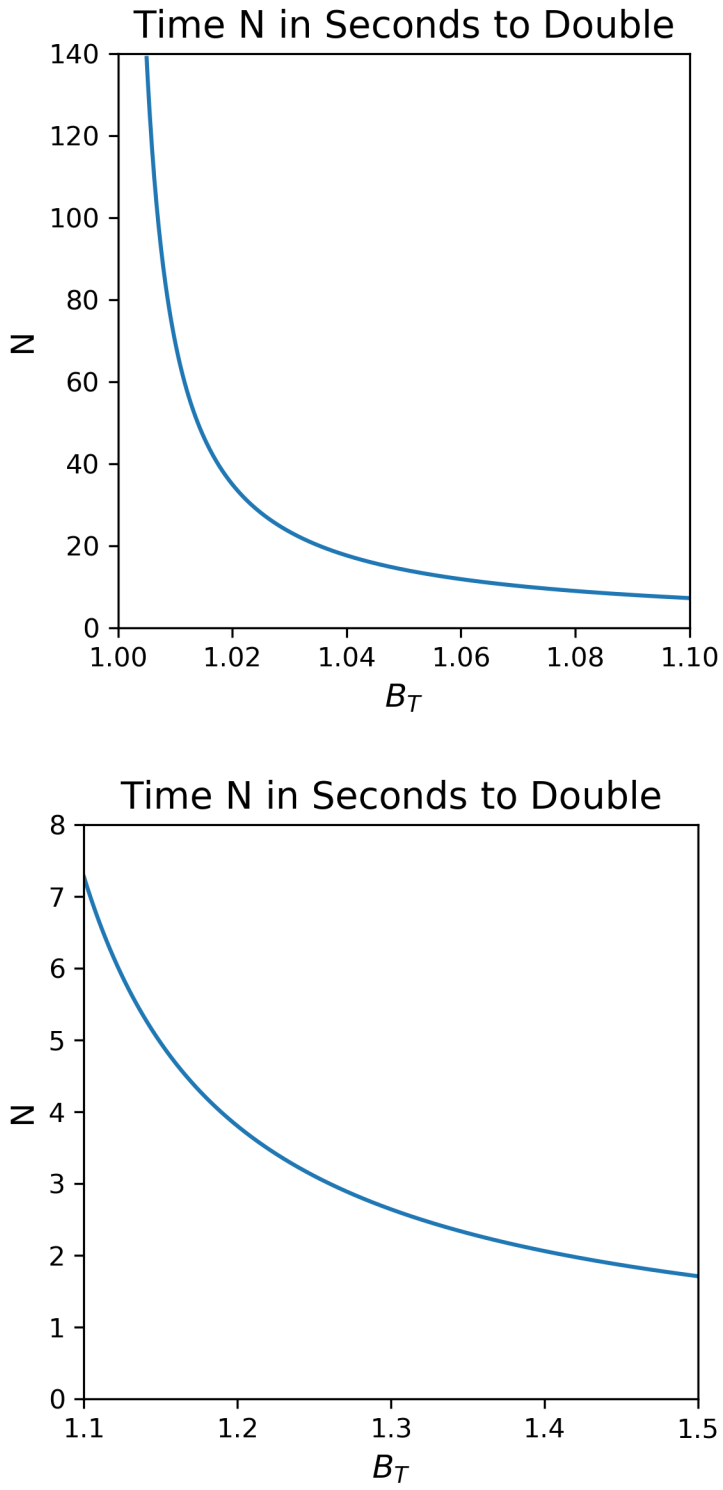

Figure 4.2: Given a net TBR equal to B_T, we see the number of seconds to double the amount of tritium with a full duty cycle $f = 1$ (i.e., no time off). Note that it would be ideal to have shorter times to ensure that the machine does not need to be operating continuously in order to produce enough tritium for future reactors.

Note that the mass burned per second would be given by (N_A is Avogadro's constant)

$$M_{T,\text{burn}} = \frac{r_{DT}}{N_A} w_T \approx \frac{6.9 \times 10^{20}\,\text{s}^{-1}}{6.022 \times 10^{23}\,\text{mol}^{-1}} (3 \times 10^{-3}\,\text{kg mol}^{-1}) \approx 3.4 \times 10^{-6}\,\text{kg} = 3.4\,\text{mg} \quad (4.2.13)$$

Suppose we have a pretty good net TBR, say around $B_T = 1.2$. This means that we have $N = 3.8$ to gain $M_{TB0} = 3.4\,\text{mg}$ of material. If we require $M_{T0} = 270\,\text{mg}$ to start a new tokamak, it will then require $\frac{270}{3.4} N(1\,\text{s}) \approx (79)(3.8\,\text{s}) \approx 300\,\text{s}$ of plasma time to produce enough tritium for a new plant. Even with a poor duty cycle of $f = 0.1$ this would only require 3000 s or about 50 min. On the other hand, suppose that there are problems with $B_T = 1.01$, then $N \approx 70$ and so it will take 5600 s of time to produce enough for a new reactor. Then with a duty cycle of $f = 0.1$ it would take 15 h to produce enough tritium for a new plant.

These are representative of the problem in general, and show that tritium production is not so terrible of a problem even under fairly pessimistic scenarios. It could turn out, of course, that the TBR is just always going to be unfavorable, but that seems unlikely.

As we have seen, the net TBR is fairly important for understanding how quickly new plants would need to be made. For reasonable duty cycles and even fairly pessimistic fusion projections we see that the limiting factor will not be tritium if there are operational fusion power plants. Even if one required ten times the amount of tritium to be placed in the device at any time with $f = 0.1$ and $B_T = 1.01$ it would only require about 6.5 days to produce enough tritium for a second plant.

Of course, this is assuming 1.01 is a pessimistic net TBR. I think it is, but for the truly cynical, assume that in fact the TBR is an abysmal 1.001. Suddenly $N = 690$ so it takes 690 seconds of fusion to double. If we require ten times the mass in the reactor as an inventory then we need 2700 mg. We find that the time the reactor must be operating to produce enough for a new reactor is $2700/3.4 \cdot 690\,\text{s} \approx 5.5 \times 10^5\,\text{s}$. If the duty cycle was then $f = 0.1$ it would require $5.5 \times 10^6\,\text{s}$ or about 64 days. Compared to the time of building a fusion power plant this is nothing, but it would no longer be an insignificant amount of time to replenish lost tritium either.

4.3 Deuterium-Deuterium Fusion

A [numerical] computation is a temptation that should be resisted as long as possible.

— J. P. Boyd[4] [This is a paraphrase of T. S. Elliot.]

Given the resource limitations of DT fusion given in the previous section 4.1, we should consider the difficulties of DD fusion. We have seen that deuterium resources are essentially unlimited. The reactions of interest are then

$$D + T \to {}^4_2\text{He}_2 + n + 17.6\,\text{MeV} \quad (3.4.10)$$
$$D + D \to T + p + 4.03\,\text{MeV} \quad (3.4.11)$$
$$D + D \to {}^3_2\text{He} + n + 3.27\,\text{MeV} \quad (3.4.12)$$
$$D + {}^3_2\text{He} \to {}^4_2\text{He} + p + 18.3\,\text{MeV} \quad (3.4.13)$$

That is, if we have the two $D + D$ reactions we produce ${}^3_2\text{He}$ and T so under ideal conditions[28] we

[28] This is again assuming that we completely consume the fuels or recycle them back into confinement until they are consumed.

can combine these into a "single" reaction

$$6D \rightarrow 2\,{}^4_2\mathrm{He} + 2p + 2n + 43.2\,\mathrm{MeV} \tag{4.3.1}$$

Thus, a kilogram of atomic deuterium in this case which has 3.00×10^{26} deuterons could ideally release

$$\frac{(3.00 \times 10^{26})}{6}(43.2\,\mathrm{MeV})(1.6 \times 10^{-22}\,\mathrm{GJ/MeV}) \approx 3.5 \times 10^5\,\mathrm{GJ} = 350\,\mathrm{TJ} \tag{4.3.2}$$

Compared with the DT results, which could ideally yield 840 TJ we see that DT releases about 2.4 times more energy per kilogram of deuterium than DD releases. This is a bit of an unfair comparison though since DT uses two different reactants. If we instead consider two kilograms of deuterium fully reacting we'd only have around 690 TJ and so DT is only 1.2 times more energy per kilogram (excluding the lithium-6 energy release). Of course, DT reactions occur in this DD cycle, but the tritium is not ever directly supplied.

The other thing to consider is what will eventually contribute to the electrical energy. With tritium, if we only extract neutron energy we get 670 TJ from the neutrons. With deuterium alone, if we only extract electrical energy from neutrons, then 2 kg of deuterium reacting yields

$$2\frac{(3.00 \times 10^{26})}{6}(14.1\,\mathrm{MeV} + 2.45\,\mathrm{MeV})(1.6 \times 10^{-22}\,\mathrm{GJ/MeV}) \approx 2.6 \times 10^5\,\mathrm{GJ} = 260\,\mathrm{TJ} \tag{4.3.3}$$

If we were to only use α's (^{4}He) for heating (so we somehow harnessed proton and neutrons for energy) we'd have

$$\begin{aligned} &2\frac{(3.00 \times 10^{26})}{6}(14.1\,\mathrm{MeV} + 2.45\,\mathrm{MeV} + 3.02\,\mathrm{MeV} + 14.6\,\mathrm{MeV})(1.6 \times 10^{-22}\,\mathrm{GJ/MeV}) \\ &\approx 5.5 \times 10^5\,\mathrm{GJ} = 550\,\mathrm{TJ} \end{aligned} \tag{4.3.4}$$

which is still unfavorable compared to using DT, but still yields an appreciable yield.

The neutron only electrical energy extraction is probably the most likely route at this point.

The next thing to consider is what the Lawson criteria is for these DD reactions. We use the methods from before in Section 3.4.2.2. First, we note that we have α's and protons that will come out of (4.3.1) we need to find the energy in them out of the total. We can write

$$D + T \rightarrow {}^4_2\mathrm{He}_2(3.54\,\mathrm{MeV}) + n(14.1\,\mathrm{MeV}) \tag{4.3.5}$$

$$D + D \rightarrow T(1.01\,\mathrm{MeV}) + p(3.02\,\mathrm{MeV}) \tag{4.3.6}$$

$$D + D \rightarrow {}^3_2\mathrm{He}(0.820\,\mathrm{MeV}) + n(2.45\,\mathrm{MeV}) \tag{4.3.7}$$

$$D + {}^3_2\mathrm{He} \rightarrow {}^4_2\mathrm{He}(3.68\,\mathrm{MeV}) + p(14.6\,\mathrm{MeV}) \tag{4.3.8}$$

And so if we use all the non-neutrons for heating the plasma, we'd get 24.8 MeV of heating. If we instead imagine harnessing the protons for electrical energy production in some way, we still have 7.22 MeV of heating from α's. We will consider both scenarios. If we look at the cross sections and reactivities in Figure 3.3 we see that at 20 keV to 30 keV that DD and $D\,^3$He have similar reactivities and less than DT. Let's consider the two DD reactions (with protons counting as heating so that they can be considered separately) with their heating contribution and then the D^3He.

	Reaction	L_C	$\delta_{ss'}$	E_{heat}
$DD1$	$D + D \rightarrow T + p$	6	1	3.02 MeV
$DD2$	$D + D \rightarrow {}^3_2\text{He} + n$	6	1	0.82 MeV
DD	Both DD reactions	6	1	0.82 MeV and 3.02 MeV
$D\,{}^3\text{He}$	$D + {}^3_2\text{He} \rightarrow {}^4_2\text{He} + p$	12	0	3.68 MeV
DT	$D + T \rightarrow {}^4_2\text{He} + n$	12	0	3.54 MeV

Table 4.1: The parameters used in the Lawson criteria analysis. Generally proton contributions are assumed to be harnessed as energy except in the DD1 reaction and DD reactions.

	$n\tau_E >$	$T^{\text{min}}_{n\tau_E}$	$nk_BT\tau_E >$	$nk_BT\tau_E >$	$T^{\text{min}}_{nk_BT\tau_E}$
$DD1$	$8.8 \times 10^{21}\,\text{s/m}^3$	118 keV	$3.2 \times 10^{23}\,\text{keV s/m}^3$	500 atm s	16 keV
$DD2$	$2.7 \times 10^{22}\,\text{s/m}^3$	136 keV	$1.1 \times 10^{24}\,\text{keV s/m}^3$	1700 atm s	17.0 keV
DD	$6.6 \times 10^{21}\,\text{s/m}^3$	123 keV	$2.5 \times 10^{23}\,\text{keV s/m}^3$	390 atm s	16.0 keV
$D\,{}^3\text{He}$	$1.2 \times 10^{21}\,\text{s/m}^3$	62 keV	$5.4 \times 10^{22}\,\text{keV s/m}^3$	85 atm s	33.0 keV
DT	$1.5 \times 10^{20}\,\text{s/m}^3$	26.0 keV	$2.7 \times 10^{21}\,\text{keV s/m}^3$	4.3 atm s	14.0 keV

Table 4.2: The results of the Lawson criteria fits.

We can restate a generalized Lawson criteria with

$$\hat{n}\tau_E > \frac{C_{P2}}{C_{P1}} \frac{\frac{12}{1+\delta_{ss'}} k_B \hat{T}}{\widehat{\langle \sigma v \rangle}_{ss'} E_{\text{heat}}} \tag{4.3.9}$$

$$\hat{n}\tau_E > \frac{L_C k_B \hat{T}}{\widehat{\langle \sigma v \rangle}_{ss'} E_{\text{heat}}} \tag{4.3.10}$$

$$L_C = \frac{C_{P2}}{C_{P1}} \frac{12}{1+\delta_{ss'}} \tag{4.3.11}$$

where s and s' are the two species, $\delta_{ss'} = 1$ if $s = s'$ and 0 otherwise, and E_{heat} is the energy of the particles that go into heating the plasma and L_C is a dimensionless number dependent on the profiles and species. Table 4.1 shows the relevant parameters for the reactions when using flat profiles. Figure 4.3 shows the Lawson criterion under these conditions. Note that the DD reaction uses

$$\hat{n}\tau_E > \frac{L_C k_B \hat{T}}{\widehat{\langle \sigma v \rangle}_{DD1}(3.02\,\text{MeV}) + \widehat{\langle \sigma v \rangle}_{DD2}(2.45\,\text{MeV})} \tag{4.3.12}$$

for the Lawson criteria.

We can then perform the same analysis but with the triple product for a more optimal calculation

$$\hat{n} k_B \hat{T} \tau_E > \frac{L_C k_B \hat{T}^2}{\widehat{\langle \sigma v \rangle}_{ss'} E_{\text{heat}}} \tag{4.3.13}$$

yielding Figure 4.4. All of this information is summarized in Table 4.2.

There are quite a few things that we can see from this analysis. The first is that the pure DD reactions (both of them) are ignited only when conditions are roughly 100 times more favorable than for pure DT conditions. When this occurs, the DT and D-^{3}He reactions will be in an ignition state but the restrictive conditions on DD means that something in the triple product must be improved far beyond what even DT fusion conditions would need.

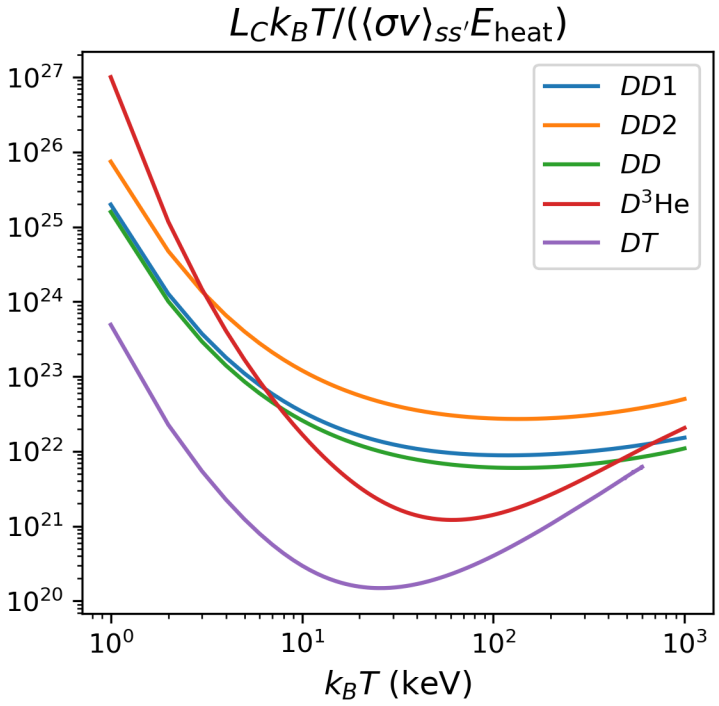

Figure 4.3: This shows the values $n\tau_E$ must meet in order to meet the requirements for ignition. Note how DT reactions are by far the most favorable. The D^3He curve uses the Duane coefficients while the others use BH coefficients.

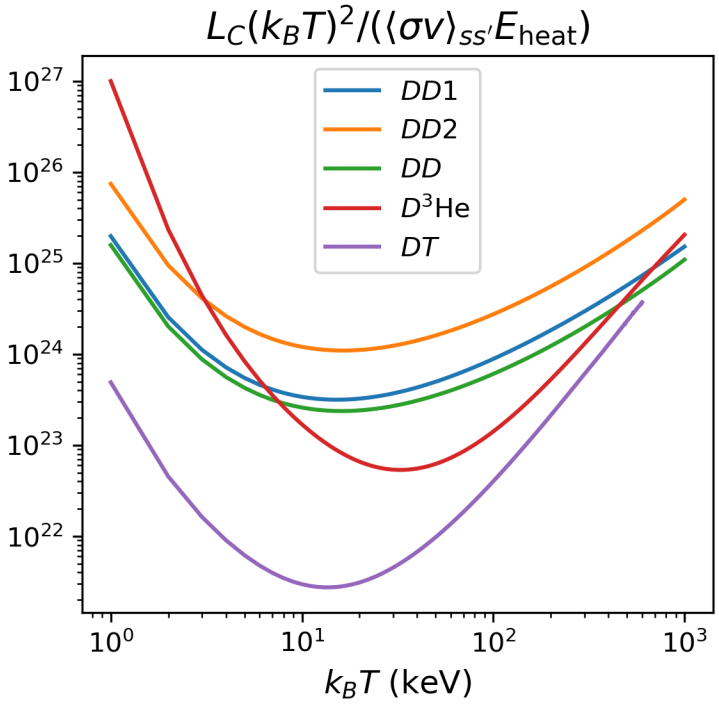

Figure 4.4: This shows the values $nk_BT\tau_E$ must meet in order to meet the requirements for ignition. Note how DT reactions are by far the most favorable. The D-^{3}He curve uses the Duane coefficients while the others use BH coefficients.

4.3.1 Full Deuterium-Deuterium Fusion Cycle Analysis

Finally, while all of these analyzed separately are useful, it is also good to consider the entire fuel cycle for ignition. I will assume that the α's are used for heating, the neutrons and protons somehow being utilized for electrical energy (It may be that protons and α's will be somehow used for energy production or that protons and α's are used for heating). We can write it as (n is n_D)

$$\begin{aligned}\mathcal{P}_{DD,all} &= \overbrace{\frac{n^2}{2}(\langle\sigma v\rangle_{DD1} E_{DD1} + \langle\sigma v\rangle_{DD2} E_{DD2})}^{\mathcal{P}_{DD12}} + nn_T \langle\sigma v\rangle_{DT} E_{DT} + nn_{^3\mathrm{He}} \langle\sigma v\rangle_{D\,^3\mathrm{He}} E_{D\,^3\mathrm{He}} \\ &= \mathcal{P}_{DD12} + n(\frac{nn}{2}\langle\sigma v\rangle_{DD1})\langle\sigma v\rangle_{DT} E_{DT} + n(\frac{nn}{2}\langle\sigma v\rangle_{DD2})\langle\sigma v\rangle_{D\,^3\mathrm{He}} E_{D\,^3\mathrm{He}} \\ &= \frac{n^2}{2}(\langle\sigma v\rangle_{DD1} E_{DD1} + \langle\sigma v\rangle_{DD2} E_{DD2} + \phi_{DT}\langle\sigma v\rangle_{DT} E_{DT} + \phi_{D\,^3\mathrm{He}}\langle\sigma v\rangle_{D\,^3\mathrm{He}} E_{D\,^3\mathrm{He}})\end{aligned} \tag{4.3.14}$$

where I have defined ϕ_{DT} and $\phi_{D\,^3\mathrm{He}}$ by the above equation. We see $E_{DD1} = E_{DD2} = 0$ as there are no α's. For simplicity we can assume a steady state of creation per second of ^{3}He and T. The ϕ are determined by the density via

$$\phi_{DT} = n\langle\sigma v\rangle_{DD1} \tag{4.3.15}$$
$$\phi_{DT} = n\langle\sigma v\rangle_{DD1} \tag{4.3.16}$$
$$\widehat{\phi}_{DT} = \hat{n}\widehat{\langle\sigma v\rangle}_{DD1} \tag{4.3.17}$$
$$\widehat{\phi}_{DT} = \hat{n}\widehat{\langle\sigma v\rangle}_{DD1} \tag{4.3.18}$$

with the hats assuming the same temperature and number density profile as for other quantities (the $\langle\sigma v\rangle$ are in fact multiplied by 1 s above to see how many new particles are generated per second which all burn up within one second to stay in steady state). Note that this is saying

$$\frac{n_T}{n} = \frac{n}{2}\langle\sigma v\rangle_{DD1} t_I \tag{4.3.19}$$
$$\frac{n_{^3\mathrm{He}} n}{n} = \frac{n}{2}\langle\sigma v\rangle_{DD2} t_I \tag{4.3.20}$$

where t_I is the unit of time used in $\langle\sigma v\rangle$. There are two important points here. The first is that the relative fraction of n_T or $n_{^3\mathrm{He}}$ is proportional to n, the deuterium number density. Thus at higher deuterium number density we get more and more tritium and helium-3 as a proportion of the plasma. Second, we are assuming that the density of tritium and helium-3 is completely due to their production via DD reactions.

We can then use

$$\hat{n}\tau_E > \frac{k_B\hat{T}}{\sum_{ss'}\frac{\phi_{ss'}\widehat{\langle\sigma v\rangle}_{ss'}E_{ss',\mathrm{heat}}}{L_{Css'}}} \tag{4.3.21}$$
$$\hat{n}k_B\hat{T}\tau_E > \frac{(k_B\hat{T})^2}{\sum_{ss'}\frac{\phi_{ss'}\widehat{\langle\sigma v\rangle}_{ss'}E_{ss',\mathrm{heat}}}{L_{Css'}}} \tag{4.3.22}$$

which we can write out as only using the α's for heating as

$$\hat{n}^2\tau_E > \frac{k_B\hat{T}}{\frac{\widehat{\langle\sigma v\rangle}_{DD1}\widehat{\langle\sigma v\rangle}_{DT}(3.54\,\text{MeV})}{12} + \frac{\widehat{\langle\sigma v\rangle}_{DD2}\widehat{\langle\sigma v\rangle}_{D\,^3\text{He}}(3.68\,\text{MeV})}{12}} \tag{4.3.23}$$

$$\hat{n}^2 k_B\hat{T}\tau_E > \frac{(k_B\hat{T})^2}{\frac{\widehat{\langle\sigma v\rangle}_{DD1}\widehat{\langle\sigma v\rangle}_{DT}(3.54\,\text{MeV})}{12} + \frac{\widehat{\langle\sigma v\rangle}_{DD2}\widehat{\langle\sigma v\rangle}_{D\,^3\text{He}}(3.68\,\text{MeV})}{12}} \tag{4.3.24}$$

a very generalized form of the Lawson criteria. This produces Figure 4.5. It leads to conditions

$$n^2\tau_E > 1.5\times 10^{43}\,\text{s/m}^6 \tag{4.3.25}$$

$$T^{\min}_{n^2\tau_E} = 77.9\,\text{keV} \tag{4.3.26}$$

$$n^2 k_B T\tau_E > 6.8\times 10^{44}\,\text{keV s/m}^6 = 1.07\times 10^{24}\,\text{atm s/m}^3 \tag{4.3.27}$$

$$T^{\min}_{n^2 k_B T\tau_E} = 29.0\,\text{keV} \tag{4.3.28}$$

Remember that τ_E has a number density dependence, however. Still, these are useful estimates. If we take $n = 10^{20}\,\text{m}^{-3}$ with $k_BT = 29.0\,\text{keV}$ then we require

$$\tau_E > 2300\,\text{s} \tag{4.3.29}$$

which leads again to an estimate that τ_E needs to be about 1000 times better than for DT fusion in addition to the temperature being almost double that required for DT fusion.

4.3.1.1 Neutron Heating Only

We can go through the same process assuming that only neutrons contribute to electrical energy and that all protons and α's are used to heat the plasma as another possibility.

The generalized Lawson criteria then state

$$\hat{n}\tau_E > \frac{k_B\hat{T}}{\frac{\widehat{\langle\sigma v\rangle}_{DD1}(3.02\,\text{MeV})}{6} + \frac{\hat{n}\widehat{\langle\sigma v\rangle}_{DD1}\widehat{\langle\sigma v\rangle}_{DT}(3.54\,\text{MeV})}{12} + \hat{n}\frac{\widehat{\langle\sigma v\rangle}_{DD2}\widehat{\langle\sigma v\rangle}_{D\,^3\text{He}}(3.68\,\text{MeV}+14.6\,\text{MeV})}{12}} \tag{4.3.30}$$

$$\hat{n}\tau_E > \frac{k_B\hat{T}}{\frac{\widehat{\langle\sigma v\rangle}_{DD1}(3.02\,\text{MeV})}{6} + \hat{n}\frac{\widehat{\langle\sigma v\rangle}_{DD1}\widehat{\langle\sigma v\rangle}_{DT}(3.54\,\text{MeV})}{12} + \hat{n}\frac{\widehat{\langle\sigma v\rangle}_{DD2}\widehat{\langle\sigma v\rangle}_{D\,^3\text{He}}(18.3\,\text{MeV})}{12}} \tag{4.3.31}$$

$$\hat{n}k_B\hat{T}\tau_E > \frac{(k_B\hat{T})^2}{\frac{\widehat{\langle\sigma v\rangle}_{DD1}(3.02\,\text{MeV})}{6} + \hat{n}\frac{\widehat{\langle\sigma v\rangle}_{DD1}\widehat{\langle\sigma v\rangle}_{DT}(3.54\,\text{MeV})}{12} + \hat{n}\frac{\widehat{\langle\sigma v\rangle}_{DD2}\widehat{\langle\sigma v\rangle}_{D\,^3\text{He}}(18.3\,\text{MeV})}{12}} \tag{4.3.32}$$

We see that in this case there is no simple way to eliminate $\hat{n}$ dependence on the right-hand side of these equations. Thus, we need to create a contour plot in order to see the full variation. This is shown in Figure 4.6 and for a couple different number densities in Figure 4.7

The higher the achievable number densities the better the results, as we might expect. If we impose $n = 10^{20}\,\text{m}^{-3}$ then the minimum temperature for $n\tau_E$ is 120 keV and $n\tau_E > 7.9\times 10^{21}\,\text{s/m}^3$. This then means $\tau_E > 79\,\text{s}$ which is only an improvement of about 10 from DT required values, however, the temperature is far beyond DT required values. At the same number density $nk_BT\tau_E$ has a minimum temperature of 15.9 keV and $nk_BT\tau_E > 3.1\times 10^{23}\,\text{keV s/m}^3$ which implies $\tau_E > 190\,\text{s}$

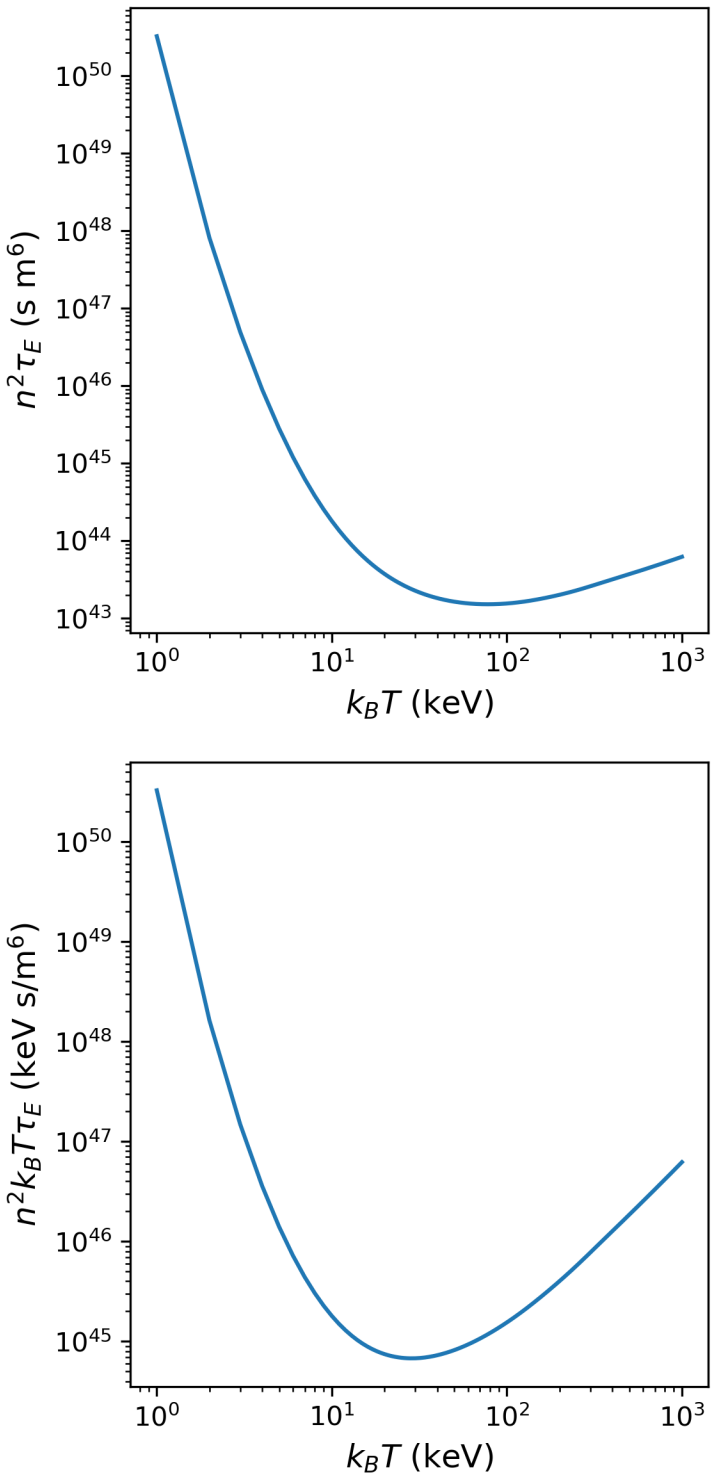

Figure 4.5: This shows the values of $n^2\tau_E$ and $n^2k_BT\tau_E$. These curves use BH coefficients for DD reactions and Duane coefficients for other curves (appropriately translated into the center-of-momentum frame for BH).

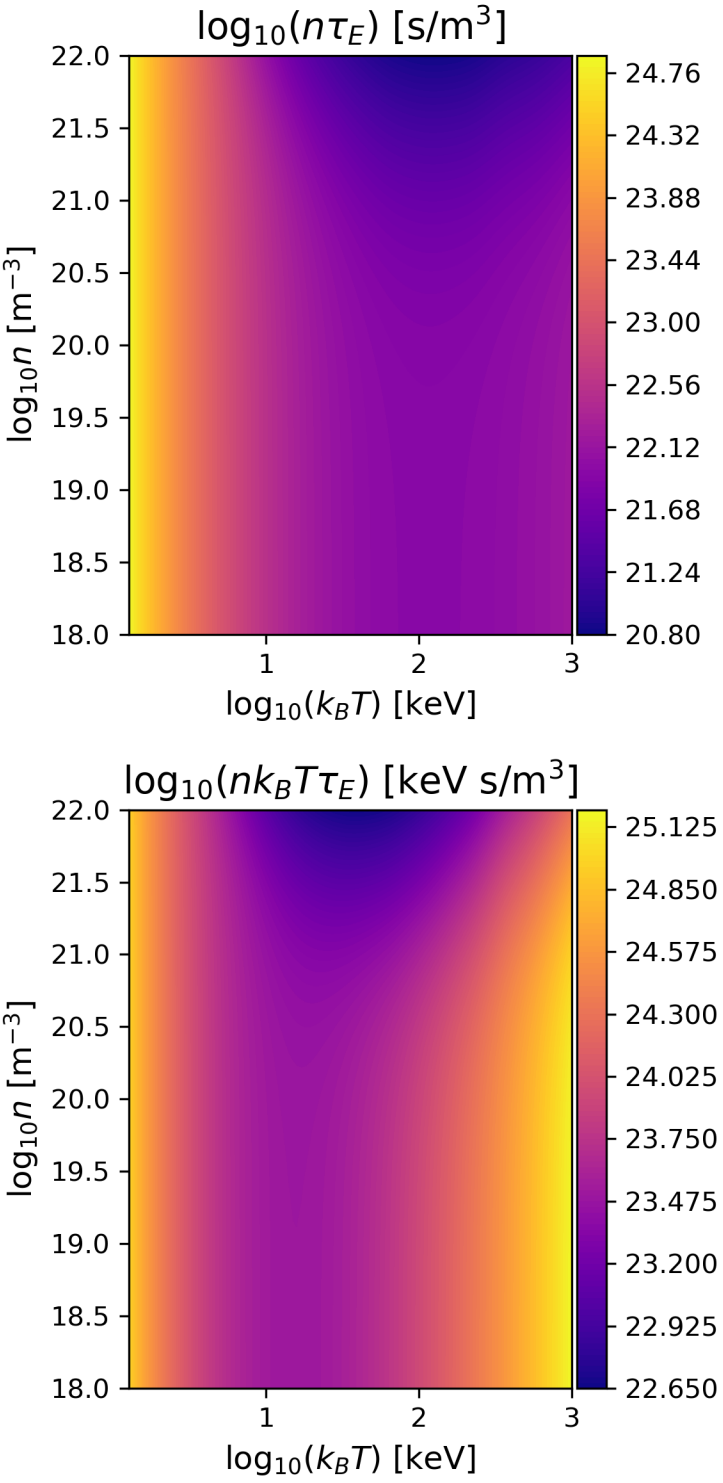

Figure 4.6: This shows the values of the $\log_{10}$ of $n\tau_E$ and $nk_BT\tau_E$ (the units given are for the quantities in the logarithms). The contours show that at low density there is no minimum except at very high temperatures. At higher number densities there is a minimum in a lower temperature range. Otherwise for densities below $10^{21}\mathrm{m}^{-3}$ both $n\tau_E$ and $nk_BT\tau_E$ do not vary much with number density. These curves use BH coefficients for DD reactions and Duane coefficients for the other curves.

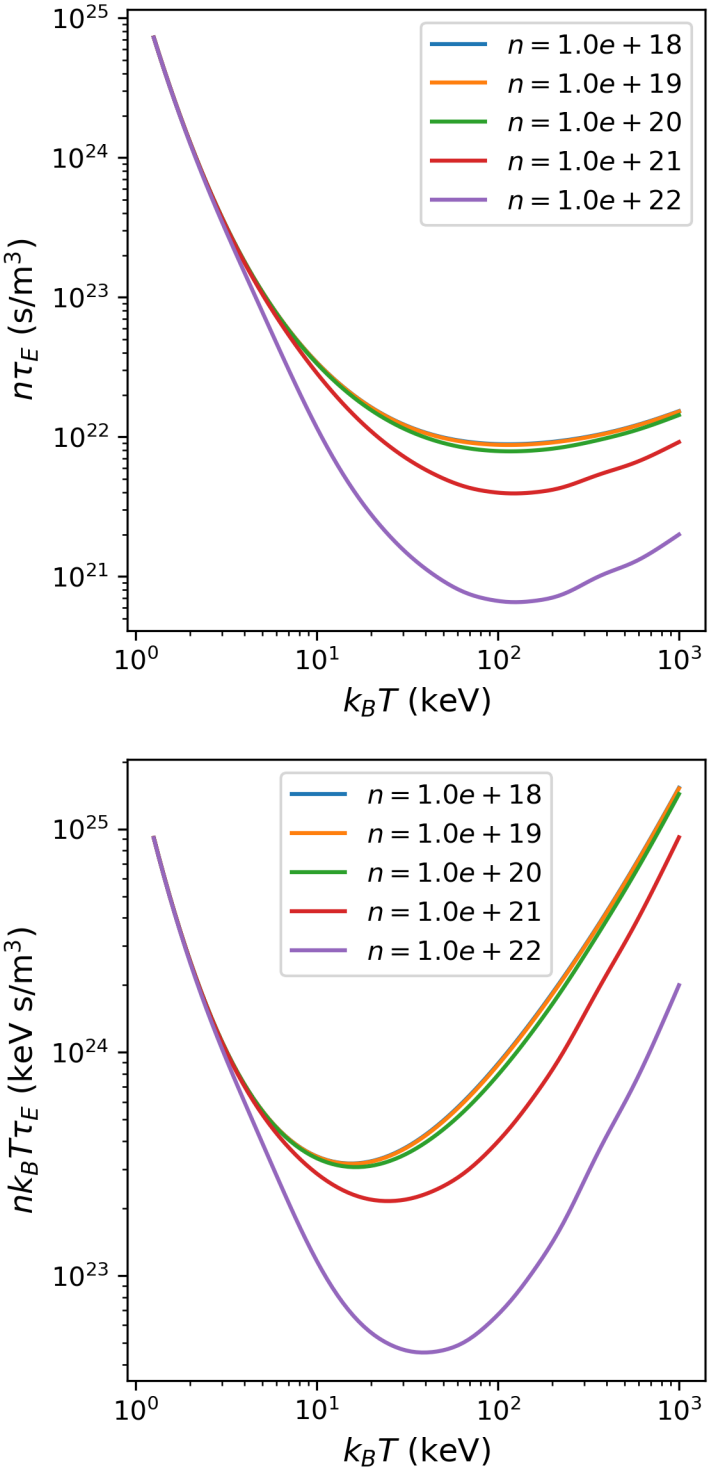

Figure 4.7: This shows the values of $n\tau_E$ and $nk_BT\tau_E$ for various given number densities. These curves use BH coefficients for DD reactions and Duane coefficients for the other curves.

	n	$n\tau_E >$	$T^{\min}_{n\tau_E}$	τ_E
DT	$10^{20}\,\mathrm{m}^{-3}$	$1.50 \times 10^{20}\,\mathrm{s/m}^3$	26 keV	1.5 s
DD_α	$10^{20}\,\mathrm{m}^{-3}$	$1.5 \times 10^{23}\,\mathrm{s/m}^3$	78 keV	1500 s
$DD_{\alpha,p}$	$10^{20}\,\mathrm{m}^{-3}$	$7.9 \times 10^{21}\,\mathrm{s/m}^3$	120 keV	79 s
$DD_{\alpha,p}$	$10^{21}\,\mathrm{m}^{-3}$	$3.9 \times 10^{20}\,\mathrm{s/m}^3$	120 keV	0.39 s
	n	$nk_BT\tau_E >$	$T^{\min}_{nk_BT\tau_E}$	τ_E
DT	$10^{20}\,\mathrm{m}^{-3}$	$3.1 \times 10^{21}\,\mathrm{keV\,s/m}^3$	14 keV	2.2 s
DD_α	$10^{20}\,\mathrm{m}^{-3}$	$6.8 \times 10^{24}\,\mathrm{keV\,s/m}^3$	29 keV	2300 s
$DD_{\alpha,p}$	$10^{20}\,\mathrm{m}^{-3}$	$3.1 \times 10^{23}\,\mathrm{keV\,s/m}^3$	16 keV	190 s
$DD_{\alpha,p}$	$10^{21}\,\mathrm{m}^{-3}$	$2.2 \times 10^{23}\,\mathrm{keV\,s/m}^3$	24 keV	9.2 s

Table 4.3: A summary of representative DD results. DD_α indicates heating only from α particles while $DD_{\alpha,p}$ means both protons and α's are used for fusion heating.

which is not as encouraging, since this requires 100 times better than DT confinement times, though the temperature is now easily in the DT range. The problem really is that confinement times are only expected to be about 1 s to 10 s for a device like ITER. One other factor to consider is how much tritium and ^{3}He are in this steady state. For the $n = 10^{20}\,\mathrm{m}^{-3}$ case, $n_T/n = 0.0013$ and $n_{^3\mathrm{He}}/n = 0.0016$ so that only a very small amount of tritium and helium-3 needs to be seeded in to start this DD fusion cycle process at this level. That or there needs to be a brief time of increased temperature to jumpstart DD reactions until such concentration of tritium and helium-3 are produced.

If we consider $n = 10^{21}\,\mathrm{m}^{-3}$ then for $n\tau_E$ the required temperature is 120 keV again and $n\tau_E > 3.9 \times 10^{20}\,\mathrm{s/m}^3$ or $\tau_E = 0.39$ s. For $nk_BT\tau_E$ the required temperature is 24 keV and $nk_BT\tau_E > 2.2 \times 10^{23}\,\mathrm{keV\,s/m}^3$ or $\tau_E > 9$ s. This is perhaps a more realistic scenario, though a bit more challenging for the temperature. It also requires $n_T/n = 0.013$ and $n_{^3\mathrm{He}}/n = 0.016$ which means that some tritium and helium-3 may need to be generated at higher temperature or supplied when beginning a run. Because tokamak plasmas run into density limits and all current experiments use lower number densities than $10^{21}\,\mathrm{m}^{-3}$, while this looks like a promising scenario, it actually requires substantial improvements in current values.

All of our results so far suggest that improvements are required for DD fusion to be feasible, and that DT fusion is still the most achievable.

4.3.1.2 Deuterium-Deuterium Summary

This analysis shows why DT fusion is considered the best way to start fusion energy. While DD would have a substantial advantage in resource allocation over DT, it demands either better temperatures, densities, or confinement times (or a combination thereof). These usually require improvements in $nk_BT\tau_E$ on the order of 100 to 1000 and so would seem to be impractical for the near future. A comparison of some representative cases is shown in Table 4.3.

4.3.2 Proliferation Concerns

Another consideration for fusion power is that it produces fast neutrons. Fast neutrons can be used to breed plutonium.[29] You will notice specifically that the most easily attainable fusion

[29]Please consult elsewhere for details. This is a book on fusion energy not on plutonium production.

process, DT, creates fast neutrons. Thus, any process that produces fusion energy for other types of reactions should be able to switch to DT and begin producing fast neutrons. Once DD fusion conditions can be met there are really only two barriers to producing these fast neutrons: monitoring of the fusion facilities to make sure there is no fissile material going in or out and the intrinsic cost of the fusion facility.

Most analyses of fusion proliferation concerns point out that there are much simpler ways of producing fast neutrons to produce plutonium. Left implicit in this premise is that fusion facilities are an uneconomical way of producing fast neutrons. If fusion technology were to progress such that it was a way of producing cheap electricity, such an argument no longer would hold. One could use a fusion facility for energy production and for proliferation without losing money if the energy production is economical.[30] As far as I know, there is no actual reason if fusion becomes economical why it would not lead to an easier time of producing plutonium. This is of course true of any technology that could efficiently and cheaply produce fast neutrons, but fusion advocacy tends to claim that proliferation is less of a concern with fusion facilities with studies such as Goldston[9]. It is true that uranium should not be lying around in a fusion facility, and so in that sense would be easy to detect, but this again would require facility monitoring, just as fission plants require.

Another factor is whether fusion facilities intrinsically need to be large.[31] Current thoughts are that fusion facilities need to be large in order to run and so cannot be (easily) hidden. There would be tell-tale signs of a fusion facility being constructed. This again, assumes that fusion power will remain with large devices for the foreseeable future.

It seems to me that fusion energy is in the unenviable position of the more efficient (and smaller) fusion power plants become, the more they become proliferation risks. The major advantage over fission is that with proper monitoring, there is no reason for fissile materials to be on a fusion power plant premises. Should fusion require large, marginally economical facilities, then it won't be a proliferation risk, but that means fusion is probably not a major supplier of electrical energy either.

4.4 Power Plant Considerations

> *There can be no doubt that our descendants will learn to exploit the energy of fusion for peaceful purposes even before its use becomes necessary for the preservation of human civilization.*
>
> — Lev Artsimovich

Fusion power plants are usually large. This is because with a larger volume one can produce more fusion reactions, which are necessary because creating fusion conditions is power intensive, radiation losses can be significant, and generally speaking, the bigger the reactor, the more efficient it gets. This is partially because the confinement time goes up. The improvement is also due to the larger the volume you go to, the less energy you lose on the surface. This is easily seen because

[30] Perhaps one would have to sacrifice energy production for plutonium production, but so long as this is not fatal to energy production, it does not seem like it would be insurmountable for a determined entity.

[31] This is often considered a factor against them from a technological point of view, and so many engineers and scientists have tried to come up with ways of making smaller fusion power plants. Should they succeed, it would actually lead to proliferations risks as it would be easier to hide a small fusion facility.

volume scales as L^3 and surface area as L^2 for a characteristic distance L. So as L increases the volume increases much more quickly than the surface area.

It is instructive to consider a most optimistic scenario and see how long it would take for fusion energy to supply the world's electrical energy needs. As previously mentioned, the current world electrical energy requirements needs a power of 2 TW or 2000 GW. A typical fusion power plant would probably be around 0.5 GW to 1 GW. Suppose it was the larger power. How long would it take to build a single power plant? Suppose we are true optimists and we think that a power plant will be built in about 50 months[32] or around 4 years. We previously showed that this gives plenty of time to produce enough tritium for the future power plants. Suppose that this plant is so enticing that when it is immediately finished that the world decides to double down, and when those are done, they decide to double down. How long will it take to meet the world's needs? We find $2^{11} = 2048$ so approximately 11 doubling periods. This means that it will take about 550 months or about 46 years. This is at a relatively breakneck speed for a complicated and new technology. It is quite unrealistic to assume that fusion reactors would be built in this way, of course. Note that in the last 50 months of those 46 years, 1024 power plants will be built. In reality the process would be more evenly spread after an initial success, but 50 years is still a pretty optimistic estimate given that 50 months for a single plant is an optimistic estimate.

The lesson to be learned from this is that a fusion energy dominated future is far away even under an optimistic construction schedule. In reality, solar and wind power are improving and so fusion will have to beat both of them economically (in addition to fission power) to fuel such widespread construction.

This will probably only happen if those first reactors are enormous successes with few to no problems, so that they can be copied on a wide scale. That is not impossible, but the history of fusion and fission advancements do not support it as most likely. There is certainly no reason yet that it is impossible, but neither is there compelling evidence to suggest that fusion electrical energy will be that much cheaper than all other forms of energy production.

There is one more reason for caution. While the comparison between fusion and fission power plants is not a perfect comparison, it has enough similarities that it is instructive. If anything, fission power plants are simpler in energy production and so this should somewhat compensate for fission's negatives such as radioactive contamination. I'm not saying it's a fair comparison, just that the comparison is not terrible. We calculated earlier that the most optimistic energy released from a kilogram of deuterium and 1.5 kg tritium is 840 TJ. The reaction for ${}^{235}_{92}\mathrm{U}$ is given by

$$n + {}^{235}_{92}\mathrm{U} \rightarrow {}^{141}_{56}\mathrm{Ba}_{+}\, {}^{92}_{36}\mathrm{Kr} + 3n + 202.5\,\mathrm{MeV} \tag{4.4.1}$$

The molar mass[33] of uranium-235 is given by $0.235\,\mathrm{kg\,mol^{-1}}$ and so we'd find the energy output of one kilogram of uranium-235 will be

$$\begin{aligned} &\overbrace{(6.022\times 10^{23}\,\mathrm{mol^{-1}})}^{N_A}(202.5\times 10^{6}\,\mathrm{eV})(1.60\times 10^{-19}\,\mathrm{J\,eV^{-1}})\left(\frac{1\,\mathrm{kg}}{0.235\,\mathrm{kg\,mol^{-1}}}\right) \\ &\approx 8.3\times 10^{13}\,\mathrm{J} = 83\,\mathrm{TJ} \end{aligned} \tag{4.4.2}$$

[32] South Korea has built multiple fission nuclear plants in this time period, so this is not an unprecedented time, though it is far better than the average of the US or most of Europe.

[33] You can usually estimate the molar mass by the mass or atomic number. That is an element with atomic number X, will have a molar mass of $X/1000$ in $\mathrm{kg\,mol^{-1}}$ or just X in $\mathrm{g\,mol^{-1}}$.

Energy Type	Lazard 12.0
SPV Res Roof	160 to 267
SPV Best	36 to 46
Solar Thermal	98 to 181
Wind	29 to 56
Geothermal	71 to 111
Fission	112 to 189
Coal	60 to 143
Natural Gas	41 to 74

Energy Type	IRENA
SPV	58 to 219
Geothermal	60 to 143
Solar Thermal	109 to 272
Offshore Wind	102 to 198
Onshore Wind	44 to 100
Hydro	30 to 136

Energy Type	EIA
SPV	60.0
Geothermal	41.0
Solar Thermal	157.1
Offshore Wind	130.4
Onshore Wind	55.9
Hydro	39.1
Advanced Fission	77.5
Natural Gas	41.2 to 89.3
Coal CCS	98.6 to 104.3

Table 4.4: This shows the levelized cost of energy (LCOE) estimates for different energy types in \$/(MW h) (US dollars per megawatt-hour). Lazard[13], IRENA[11] and EIA[2]are the sources. Here SPV is solar photovoltaics, with SPV Best referring to thin film utility scale solar photovoltaics. SPV Res Roof is for solar paneling on residential roofs. For EIA data, CCS means carbon sequestration (between 30% and 90%). The Natural Gas estimate for EIA is between all sorts of different natural gas plant types. EIA data is for plants planning on coming into service in 2023. Lazard is for 2018 and IRENA for 2019 plants. All data is the unsubsidized LCOE.

Therefore per kilogram there is more energy in deuterium (though per reaction uranium-235 clearly wins). Given that uranium-235 is only about 0.72% of all uranium, one needs about 138.9 kg of natural uranium to actually have a single kilogram of uranium-235. The fact that currently nuclear fission energy is a struggling industry is not reassuring. This is only a factor of 10 difference in energy per mass, which while sizeable, is mostly wiped out by the complexities of a fusion power plant. With fission, you mostly just leave a metal in a box. Much of the fission complexities come from making sure that the metal in the box doesn't melt stuff and release radiation.[34] With deuterium and tritium reactions this is ameliorated a bit by having less radioactive waste (or less dangerous waste, at the very least) and by the reaction being able to shut down very quickly. The extra factor of 10 gives at least some leeway with all the extra machinery required to keep the plasma fusing, but we need to assume that fusion will retain an advantage over fission. Otherwise we would be left with energy prices at fission levels. Fission energy is not super expensive, but neither is it very cheap, which does not leave one full of hope.

One way of comparing energy costs across sectors is to find the levelized cost of energy (LCOE). This is essentially the price of electricity one would need to charge over the entirety of a plant's lifetime in order to ensure that you simply broke even. That is the break even price when accounting for the costs of building the power plant and how long the power plant will last. Now there are problems with this measure. First, it does not actually tell you if your plant will necessarily be able to meet the electrical demands to break even. Second, determining what the costs of the plant are tends to be difficult and so different assumptions (for example, are taxes included? do you discount future money?) can yield different results. That said, multiple firms perform these analyses fairly often and one can look up their reports to find the details of their calculations. See Table 4.4 for unsubsidized cost comparisons.

As I said, the comparison should not be too heavily weighted, since the complexities of nuclear fusion power plants and the complexities of nuclear fission power plants lie in different areas, but

[34]Because the metal is so good at releasing energy, it is actually hard to turn off quickly.

unless you think fusion can use its factor of ten advantage in energy per mass to keep an advantage over fission energy production, the story of nuclear fission energy should give you a sense for why people are not always believers in fusion energy.

4.5 Final Thoughts

> *What goes on in science is not that we try to have theories that accommodate our experiences; it's closer that we try to have experiences that adjudicate among our theories.*
>
> — Paul Rosenbaum[16, p. 126]

> *All learning is cumulative. It is not something that one comes to realize in a morning or an evening.*
>
> — Sakuma Shōzan in *Reflection on My Errors*, quoted from Marius Jansen, *The Making of Modern Japan*

I hope that this book does not turn you off of fusion energy research. That said, I do hope that it gives you a healthy appreciation for just how difficult it is to say that fusion energy will be cheap, and that there are many challenges that could prevent it from being so. Most fusion researchers are aware of the difficulties, but I never felt like they were compiled together in an easily accessible way. In addition, many of the estimates for fusion resources have always seemed absurdly optimistic to me. I wrote this chapter to counter some of the (what I would call) absurd claims and give you a more balanced picture. The numbers do not prove that fusion is impossible. I rather doubt there will ever be a simple calculation that will show fusion energy is impossible, and so fusion researchers are right to be skeptical of those that say fusion energy never will be. The question is more whether fusion is economically practical.

The economics of fusion is a tough topic. All of my previous calculations being just surface scans of a deeply complicated question of whether fusion reactors will be economical. Because we do not know what a fusion reactor will look like in all of its details, it is not possible to do much better than make statements about what a generic reactor is likely to look like. We can only look at a variety of scenarios and see if they seem like they could be profitable. I have mostly avoided talking about cost because it can be a bit of a red herring. What the cost of energy is today will change tomorrow, and how energy is consumed could change greatly on the scale of decades and centuries. Instead, I tried to look at resource allocation. This is much more concrete. If there is not enough lithium for breeder blankets, then it is clear it does not matter what it *would* cost for lithium that does not exist. As you have read, I am not very optimistic about the *DT* process as a long-term energy prospect (though in the short term [of decades to centuries], it seems like it could work). Unfortunately, I do not know if *DD* will ever be technologically and economically feasible. It seems like it could be, and it would require some great advancements. It does not seem infeasible.

Perhaps it is best left there. There have been high hopes for fusion for well over seven decades. Enormous changes and advancements have been made, but the best that we can now state is that it does not seem infeasible. If you prefer, I can remove the negative qualifiers in that sentence and say that fusion energy seems feasible.[35] As long as it is not impossible, the question is whether we (as a society and as scientists) think it is worth trying.

[35] "Fusion energy seems feasible" rather than the statement "fusion energy does not seem infeasible".

4.6 Further Reading

As far as I am aware, there is no book that explains the resource needs and challenges of economic fusion. There are many books and papers on the technical challenges of a fusion reactor, but far fewer on resource needs, understandably. For the resource requirements, Hartley[10], Fasel[7], Vikström[18], and Bradshaw[5] are great for laying out resource requirements. For tritium breeding, there is a large amount of literature looking at the problem. A good overview of material challenges for fusion is provided in Zinkle[19]. The extraction of tritium from the blanket does not appear to be a large problem[8]. It may not be entirely solved, but neither has it shown signs of becoming a large concern. There are a number of DEMO papers, and one can easily search and find them. Zohm[20] is a good reference for explaining the needed size of DEMO.

4.7 Problem Set

4.1. For Section 4.1.

4.1.1. Some lithium-7 will probably be wanted in the blanket so that the TBR is greater than 1. Consider a blanket that is α of pure lithium-6 and $1-\alpha$ of lithium-7. At what α do we get no energy gain and no energy loss from incoming neutrons. For simplicity, assume that the cross section is equal for lithium-6 and lithium-7 reactions.

4.1.2. Use the given cross sections in Section 3.4.2.3 to adjust the probabilities for the previous problem.

4.1.3. Investigate the possible efficiency of converting steam into electrical energy. Does 25% seem like a good estimate to you?

4.2. For Section 4.1.1

4.2.1. Consider other elements in seawater. Lead has a concentration (all by mass) 5×10^{-3} ppm, uranium a concentration 1.6×10^{-3} ppm, and beryllium has a concentration 2.1×10^{-7} ppm. What volume is required to have a single gram of the substance in the water?[36]

4.2.2. What initial volume flow is necessary for each of the elements in the previous problem for 1 TW continuous of fusion energy production?

4.2.3. Derive $V_0\frac{\mathrm{d}c}{\mathrm{d}t}=-c\alpha\frac{\mathrm{d}V}{\mathrm{d}t}$ when c is concentration by volume rather than mass.

4.3. For Section 4.2.

4.3.1. Why is beryllium considered for tritium breeding? Do the following reactions give a hint?

$$\begin{aligned}{}^{9}_{4}\mathrm{Be}+n&\rightarrow\alpha+{}^{6}_{2}\mathrm{He}\\ {}^{6}_{2}\mathrm{He}&\rightarrow{}^{6}_{3}\mathrm{Li}+\beta^{-}\\ {}^{6}_{3}\mathrm{Li}+n&\rightarrow\alpha+T\end{aligned}$$

[36] See https://web.stanford.edu/group/Urchin/mineral.html and https://www3.mbari.org/chemsensor/be/beryllium.html.

4.3.2. Investigate the different types of blankets. Why is lead often used? Are there any other elements you see when talking of blankets for nuclear reactors? What sorts of compositions do you see in terms of lithium-6, lithium-7, beryllium, and lead.

4.4. For Section 4.3.

4.4.1. For a quadratic profile, $q(r) = \lambda q_0(1 - r/a)^2$, do our numbers for the DD and other reactions change much? Are there any differences from when we did the DT calculation?

4.4.2. Do larger number densities make sense as a steady state? That is if $n = 10^{24}\,\mathrm{m}^{-3}$, what is the relative density n_T/n and $n_{^3\mathrm{He}}/n$?

4.4.3. Using the scaling laws for τ_E from (3.4.64) and (3.4.63), what would be necessary to get a τ_E of about 1000 s, making DD fusion a possibility? First try changing just a and R to do this. How large is the resulting tokamak or stellarator?

4.4.4. There are some thoughts of using fusion to provide neutrons for fission reactors. Would this be more or less of a proliferation concern than a pure fission or pure fusion reactor? What are the advantages of this method? What about the disadvantages?

4.5. For Section 4.4.

4.5.1. Look up some typical times for fission reactors to be built in the US, France, and South Korea. Do you think these will be good estimates for fusion reactor build times?

4.5.2. Look at the build time for ITER, TFTR, and JET. Do you think a DEMO would be closer to an ITER or TFTR time?

References

[1] Derek Abbott et al. "Is nuclear power globally scalable". In: *Proceedings of the IEEE* 99.10 (2011), pp. 1611–1617.

[2] US Energy Information Administration. *Levelized Cost and Levelized Avoided Cost of New Generation Resources*. 2019. URL: https://www.eia.gov/outlooks/aeo/pdf/electricity_generation.pdf.

[3] Ugo Bardi. "Extracting minerals from seawater: an energy analysis". In: *Sustainability* 2.4 (2010), pp. 980–992.

[4] J.P. Boyd. *Chebyshev and Fourier Spectral Methods: Second Revised Edition*. Dover Books on Mathematics. Dover Publications, 2013. ISBN: 9780486141923. URL: https://books.google.com/books?id=b4TCAgAAQBAJ.

[5] Alexander Marian Bradshaw, Thomas Hamacher, and Ulrich Fischer. "Is nuclear fusion a sustainable energy form?" In: *Fusion Engineering and Design* 86.9-11 (2011), pp. 2770–2773.

[6] Morris W. Donné A.J.H. et al. "European research roadmap to the realisation of fusion energy". In: (2018).

[7] D Fasel and MQ Tran. "Availability of lithium in the context of future D–T fusion reactors". In: *Fusion engineering and design* 75 (2005), pp. 1163–1168.

[8] Michael A Fütterer et al. "Tritium technology for blankets of fusion power plants". In: *Fusion engineering and design* 49 (2000), pp. 735–743.

[9] Robert J Goldston and Alexander Glaser. "Proliferation risks of magnetic fusion energy: clandestine production, covert production and breakout". In: *Nuclear Fusion* 52.4 (2012), p. 043004.

[10] JN Hartley, BF Gore, and JR Young. "Potential lithium requirements for fusion power plants". In: *Lithium Needs and Resources*. Elsevier, 1978, pp. 337–346.

[11] IREA IRENA. "Renewable power generation costs in 2018". In: *Report, International Renewable Energy Agency, Abu Dhabi* (2019). URL: https://www.irena.org/publications/2019/May/Renewable-power-generation-costs-in-2018.

[12] R Kemp et al. "DEMO design point studies". In: (2014). URL: https://www.euro-fusion.org/fileadmin/user_upload/Archive/wp-content/uploads/2014/12/3.-R.-Kemp-DEMO-Design-Point-Studies.pdf.

[13] Lazard's. *Lazard's Levelized Cost of Energy Analysis (LCOE 12.0)*. 2018. URL: https://www.lazard.com/perspective/levelized-cost-of-energy-and-levelized-cost-of-storage-2018/.

[14] Laura Millan Lombrana. "Lithium Battery Dreams Get a Rude Awakening in South America". In: *Bloomberg* (Oct. 2019). URL: https://www.bloomberg.com/news/articles/2019-10-11/lithium-battery-dreams-get-a-rude-awakening-in-south-america.

[15] J.S. Mill. *On Liberty*. Ticknor and Fields, 1863. URL: https://books.google.com/books?id=7DcoAAAAYAAJ.

[16] P.R. Rosenbaum. *Observation and Experiment*. Harvard University Press, 2017. ISBN: 9780674975576. URL: https://books.google.com/books?id=NnOuDwAAQBAJ.

[17] David Stringer and Laura Millan Lombrana. "These Mining Superpowers Supply the World's Lithium. Now They Want to Make Batteries, Too". In: *Bloomberg* (June 2019). URL: https://www.bloombergquint.com/technology/lithium-miners-australia-chile-see-riches-as-ev-battery-makers.

[18] Hanna Vikström, Simon Davidsson, and Mikael Höök. "Lithium availability and future production outlooks". In: *Applied Energy* 110 (2013), pp. 252–266.

[19] Steven J Zinkle. "Challenges in developing materials for fusion technology-past, present and future". In: *Fusion Science and Technology* 64.2 (2013), pp. 65–75.

[20] Hartmut Zohm. "On the minimum size of DEMO". In: *Fusion science and technology* 58.2 (2010), pp. 613–624.

Appendices

Appendix A

Fourier and Laplace Transform Tables

The discussion in Sections 1.3.3 and 1.3.4 explain some of the theory for Fourier and Laplace transforms. The following tables give common transforms of functions and some of the known identities.

Also note that the convolution for Fourier transforms is given by

$$f(x) * g(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(t-\tau)\, \mathrm{d}\tau = \int_{-\infty}^{\infty} f(t-\tau)\, g(t)\, \mathrm{d}\tau\,, \tag{A.1}$$

while convolution for the Laplace transform is given by

$$f(t) * g(t) = (f * g)(t) = \int_{0}^{t} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau = \int_{0}^{t} f(t-\tau)\, g(t)\, \mathrm{d}\tau\,. \tag{A.2}$$

function	Fourier transform
$f(x)$	$\widetilde{f}(k)$
$af(x) + b\,g(x)$	$a\widetilde{f}(k) + b\,\widetilde{g}(k)$
$f(x-a)$	$e^{-ika}\widetilde{f}(k)$
$e^{iax}f(x)$	$\widetilde{f}(k-a)$
$f(ax)$	$\frac{1}{\lvert a\rvert}\widetilde{f}\left(\frac{k}{a}\right)$
$\widetilde{f}(x)$	$2\pi f(-k)$
$\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$	$(ik)^n\widetilde{f}(k)$
$x^n f(x)$	$i^n\frac{\mathrm{d}\widetilde{f}(k)}{\mathrm{d}k^n}$
$f(x) * g(x)$	$\widetilde{f}(k)\,\widetilde{g}(k)$
$f(x)g(x)$	$\frac{1}{2\pi}\widetilde{f}(k) * \widetilde{g}(k)$
$f(x), \Im\{f(x)\} = 0$	$\widehat{f}(-k) = \widehat{f}(k)^*$

Table A.1: Fourier transforms of general functions (n is an integer). The $*$ is the convolution while * is the complex conjugate.

function	Fourier transform	Comment
$\sin(ax)$	$-i\pi\left[\delta(k-a)-\delta(k+a)\right]$	
1	$2\pi\,\delta(k)$	
$\delta(x)$	1	
e^{iax}	$2\pi\,\delta(k-a)$	
$\cos(ax)$	$\pi\left[\delta(k-a)+\delta(k+a)\right]$	
$e^{-ax}\Theta(x)$	$\frac{1}{a+ik}$	
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}\,e^{-k^2/(4a)}$	
$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+k^2}$	
$\operatorname{sech}(ax)$	$\frac{\pi}{a}\operatorname{sech}\left(\frac{\pi}{2a}k\right)$	
$\cos(ax^2)$	$\sqrt{\frac{\pi}{a}}\cos\left(\frac{k^2}{4a}-\frac{\pi}{4}\right)$	
$\sin(ax^2)$	$-\sqrt{\frac{\pi}{a}}\sin\left(\frac{k^2}{4a}-\frac{\pi}{4}\right)$	
$x^n f(x)$	$(2\pi)\,i^n\delta^{(n)}(k)\,\widetilde{f}(k)$	$n=0,1,2,\ldots$, see caption for $\delta^{(n)}(k)$
$\frac{1}{x}$	$-i\pi\,\operatorname{sgn}\{k\}$	
$\frac{1}{x^n}$	$-i\pi\frac{(-ik)^{n-1}}{(n-1)!}]\,\operatorname{sgn}\{k\}$	$n=0,1,2,\ldots$
$\lvert x\rvert^a$	$-2\frac{\sin(\pi a/2)\Gamma(a+1)}{\lvert k\rvert^{a+1}}$	$0<a<1$
$\operatorname{sgn}(x)$	$\frac{2}{ik}$	
$\Theta(x)$	$\pi\left[\frac{1}{ik\pi}+\delta(k)\right]$	
$\ln\lvert x\rvert$	$-\frac{\pi}{\lvert k\rvert}-2\pi\gamma\,\delta(k)$	$\gamma\approx 0.5722$
$(\mp ix)^{-a}$	$\frac{2\pi}{\Gamma(a)}\,\Theta(\pm k)\,(\pm k)^{a-1}$	$0<a<1$

Table A.2: Fourier transforms of various functions. The $*$ is the convolution, $\Theta(x)$ is the Heaviside step function, sgn is the sign function, and $\delta^{(n)}$ is the nth distributional derivative of δ. Note that $\gamma = 0.57721566$ is the Euler-Mascheroni constant.

function	Laplace transform	
$f(t)$	$F(s)$	$\widehat{f}(\omega)$
$af(t) + b\,g(t)$	$aF(s) + b\,F(s)$	$a\widehat{f}(\omega) + b\,\widehat{g}(\omega)$
$tf(t)$	$-F'(s)$	$-i\widehat{f}(\omega)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$(-i)^n \widehat{f}^{(n)}(\omega)$
$f'(t)$	$sF(s) - f(0)$	$-i\omega\widehat{f}(\omega) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0)$	$(-i\omega)^n \widehat{f}(\omega) - \sum_{k=1}^{n} (-i\omega)^{k-1} f^{(n-k)}(0)$
$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{s}{a}\right)$	$\frac{1}{\lvert a\rvert}\widehat{f}\left(\frac{\omega}{a}\right)$
$e^{at} f(t)$	$F(s-a)$	$\widehat{f}(\omega - a)$
$f(t-a)\,\Theta(t-a)$	$e^{-as}F(s)$	$e^{ia\omega}\widehat{f}(\omega)$
$f(t)^*$	$F(s^*)^*$	$\widehat{f}(-\omega^*)^*$

Table A.3: Laplace transforms of general functions. The $f^{(n)}(t)$ is the nth derivative of $f(t)$.

function	Laplace transform		Convergence
$\delta(t)$	1	1	all s,ω
$\delta(t-\tau)$	$e^{-\tau s}$	$e^{i\tau\omega}$	all s,ω
$\Theta(t)$	$\frac{1}{s}$	$\frac{i}{\omega}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > 0,$
$\Theta(t-\tau)$	$\frac{e^{-\tau s}}{s}$	$\frac{ie^{i\tau\omega}}{\omega}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > 0$
$t\,\Theta(t)$	$\frac{1}{s^2}$	$-\frac{1}{\omega^2}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > 0$
$t^n\,\Theta(t)$	$\frac{n!}{s^{n+1}}$	$\frac{n!}{(-i\omega)^{n+1}}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > 0, n > -1$
$t^q\,\Theta(t)$	$\frac{\Gamma(q+1)}{s^{q+1}}$	$\frac{\Gamma(q+1)}{(-i\omega)^{q+1}}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > 0, \Re\{q) > -1$
$t^n e^{-at}\Theta(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\frac{n!}{(a-i\omega)^{n+1}}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > -a$
$(t-\tau)^n e^{-a(t-\tau)}\Theta(t-\tau)$	$\frac{n!\,e^{-\tau s}}{(s+a)^{n+1}}$	$\frac{n!\,e^{i\tau\omega}}{(a-i\omega)^{n+1}}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > -a$
$e^{-at}\Theta(t)$	$\frac{1}{s+a}$	$\frac{1}{a-i\omega}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > -a$
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2-s^2}$	$\frac{2a}{a^2+\omega^2}$	$-a < \begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} < a$
$\sin(\alpha t)\,\Theta(t)$	$\frac{\alpha}{s^2+\alpha^2}$	$\frac{\alpha}{\alpha^2-\omega^2}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > 0$
$\cos(\alpha t)\,\Theta(t)$	$\frac{\alpha}{s^2+\alpha^2}$	$\frac{\alpha}{\alpha^2-\omega^2}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > 0$
$\sinh(\alpha t)\,\Theta(t)$	$\frac{\alpha}{s^2-\alpha^2}$	$\frac{-\alpha}{\alpha^2+\omega^2}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > \lvert\alpha\rvert$
$\cosh(\alpha t)\,\Theta(t)$	$\frac{\alpha}{s^2-\alpha^2}$	$\frac{-\alpha}{\alpha^2+\omega^2}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > \lvert\alpha\rvert$
$e^{-at}\sin(\alpha t)\,\Theta(t)$	$\frac{\alpha}{(s+a)^2+\alpha^2}$	$\frac{\alpha}{(a-i\omega)^2+\alpha^2}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > -a$
$e^{-at}\cos(\alpha t)\,\Theta(t)$	$\frac{\alpha}{(s+a)^2+\alpha^2}$	$\frac{\alpha}{(a-i\omega)^2+\alpha^2}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > -a$
$\ln(t)\,\Theta(t)$	$-\frac{1}{s}\left[\ln(s)+\gamma\right]$	$\frac{1}{i\omega}\left[\ln(i\omega)+\gamma\right]$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > 0$
$J_n(\alpha t)\,\Theta(t)$	$\frac{\left(\sqrt{s^2+\alpha^2}-s\right)^n}{\alpha^n\sqrt{s^2+\alpha^2}}$	$\frac{\left(\sqrt{\alpha^2-\omega^2}+i\omega\right)^n}{\alpha^n\sqrt{\alpha^2-\omega^2}}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > 0, n > -1$
$\mathrm{erf}(t)\,\Theta(t)$	$\frac{e^{s^2/4}[1-\mathrm{erf}(s/2)]}{s}$	$i\frac{e^{-\omega^2/4}[1-\mathrm{erf}(-i\omega/2)]}{\omega}$	$\begin{matrix}\Re\{s\}\\ \Im\{\omega\}\end{matrix} > 0$

Table A.4: Laplace transforms of various functions. Whenever n appears it is an integer, while q is complex. Again $\gamma = 0.57721566$ is the Euler-Mascheroni constant.

Appendix B

Vector/Tensor Identities

B.1 Vector Identities

In the following $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are vectors, ψ and ϕ are two scalar functions, and the del operator, denoted ∇, is used so $\nabla\psi$ and $\nabla\phi$ are the gradients of the two functions, respectively.

B.1.1 Identities Without Derivatives

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \tag{B.1}$$
$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \tag{B.2}$$
$$(\psi\mathbf{A})\cdot(\phi\mathbf{B}) = \psi\phi(\mathbf{A}\cdot\mathbf{B}) \tag{B.3}$$
$$\mathbf{A}\times\mathbf{B} = -\,\mathbf{B}\times\mathbf{A}, \quad \mathbf{A}\times\mathbf{A} = \mathbf{0} \tag{B.4}$$
$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C} \tag{B.5}$$
$$(\psi\mathbf{A})\times(\phi\mathbf{B}) = \psi\phi(\mathbf{A}\times\mathbf{B}) \tag{B.6}$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{B.7}$$
$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) \tag{B.8}$$
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{D})(\mathbf{A}\cdot\mathbf{C}) \tag{B.9}$$
$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= \mathbf{C}(\mathbf{A}\times\mathbf{B}\cdot\mathbf{D}) - \mathbf{D}(\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}) \\ &= \mathbf{B}(\mathbf{C}\times\mathbf{D}\cdot\mathbf{A}) - \mathbf{A}(\mathbf{C}\times\mathbf{D}\cdot\mathbf{B})\end{aligned} \tag{B.10}$$

Suppose we choose to project a vector $\mathbf{A}$ along the direction $\mathbf{B}$ with $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}| = \mathbf{B}/B$. Then

$$\mathbf{A} = A_{\parallel}\hat{\mathbf{b}} + \mathbf{A}_{\perp} \tag{B.11}$$
$$A_{\parallel} = \mathbf{A}\cdot\hat{\mathbf{b}} \tag{B.12}$$
$$\mathbf{A}_{\perp} = -\,\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times\mathbf{A}) = (\hat{\mathbf{b}}\times\mathbf{A})\times\hat{\mathbf{b}} \tag{B.13}$$

B.1.2 Identities With the Gradient Alone

$$\nabla(\psi+\phi) = \nabla\psi + \nabla\phi \tag{B.14}$$
$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi \tag{B.15}$$
$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A} + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) \tag{B.16}$$
$$\nabla(B^2/2) = \nabla(\mathbf{B}\cdot\mathbf{B}/2) = \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\cdot\nabla\mathbf{B} = (\nabla\mathbf{B})\cdot\mathbf{B} \tag{B.17}$$
$$\mathbf{A}\cdot\nabla\mathbf{A} = \nabla(A^2/2) - \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A}) \tag{B.18}$$
$$\mathbf{B}\cdot\nabla(\mathbf{A}\times\mathbf{C}) = (\mathbf{B}\cdot\nabla\mathbf{A})\times\mathbf{C} + \mathbf{A}\times(\mathbf{B}\cdot\nabla\mathbf{C}) \tag{B.19}$$

B.1.3 Identities With the Divergence Alone

Let $\boldsymbol{\nabla}\cdot$ indicate the divergence. Then,

$$\boldsymbol{\nabla}\cdot(\mathbf{A}+\mathbf{B}) = \boldsymbol{\nabla}\cdot\mathbf{A} + \boldsymbol{\nabla}\cdot\mathbf{B} \tag{B.20}$$
$$\boldsymbol{\nabla}\cdot(\psi\mathbf{A}) = \nabla\psi\cdot\mathbf{A} + \psi\boldsymbol{\nabla}\cdot\mathbf{A} \tag{B.21}$$
$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) \tag{B.22}$$

B.1.4 Identities With the Curl Alone

Let $\boldsymbol{\nabla}\times$ indicate the curl. Then,

$$\boldsymbol{\nabla}\times(\mathbf{A}+\mathbf{B}) = \boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}\times\mathbf{B} \tag{B.23}$$
$$\boldsymbol{\nabla}\times(\psi\mathbf{A}) = \psi\boldsymbol{\nabla}\times\mathbf{A} + \nabla\psi\times\mathbf{A} \tag{B.24}$$
$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{B}\cdot\nabla\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{B} \tag{B.25}$$
$$\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) = (\nabla\mathbf{B})\cdot\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{B.26}$$
$$\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \nabla(A^2/2) - (\mathbf{A}\cdot\nabla)\mathbf{A} \tag{B.27}$$

Note that in Einsteinn summation notation ($\partial_i = \frac{\partial}{\partial x_i}$), in Cartesian coordinates, the relation (B.26) yields (for clarity)

$$\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) = \hat{\mathbf{x}}_i\epsilon_{ijk}A_j\epsilon_{klm}\partial_l B_m = \hat{\mathbf{x}}_i(\partial_i B_j)A_j - \hat{\mathbf{x}}_i A_j\partial_j B_i \tag{B.28}$$

Also remember that if we prove an identity with Cartesian coordinates and can translate into general vector notation, it is true generally.

B.1.5 Other Derivatives

$$\boldsymbol{\nabla}\times\nabla\psi = 0 \tag{B.29}$$
$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0 \tag{B.30}$$
$$\boldsymbol{\nabla}\cdot\nabla\psi = \nabla^2\psi \tag{B.31}$$
$$\nabla^2\mathbf{A} = \boldsymbol{\nabla}\cdot\nabla\mathbf{A} = \nabla(\boldsymbol{\nabla}\cdot\mathbf{A}) + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) \tag{B.32}$$

$$\nabla^2(\boldsymbol{\nabla}\cdot\mathbf{A}) = \boldsymbol{\nabla}\cdot\,\nabla(\boldsymbol{\nabla}\cdot\mathbf{A}) = \boldsymbol{\nabla}\cdot(\,\nabla^2\mathbf{A}) \tag{B.33}$$

$$\boldsymbol{\nabla}\cdot(\phi\,\nabla\psi) = \phi\,\nabla^2\psi + \,\nabla\phi\cdot\,\nabla\psi \tag{B.34}$$

$$\boldsymbol{\nabla}\cdot(\psi\,\nabla\phi - \phi\,\nabla\psi) = \psi\,\nabla^2\phi - \phi\,\nabla^2\psi \tag{B.35}$$

$$\nabla^2(\phi\psi) = \boldsymbol{\nabla}\cdot\,\nabla(\phi\psi) = \phi\,\nabla^2\psi + 2\,\nabla\phi\cdot\,\nabla\psi + \psi\,\nabla^2\phi\,. \tag{B.36}$$

B.2 Useful Derivatives of Position and Velocity Vectors

Let $\mathbf{x} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, $r = |\mathbf{r}|$ (and note that we get the same results exchanging $\mathbf{X} \leftrightarrow \mathbf{v}$ with $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$ the velocity vector). In many textbooks $\mathbf{r} = \mathbf{x}$ is used to reduce possible confusion so that the modulus of $\mathbf{r}$ is written r as $r = |\mathbf{x}|$. If we wrote $x = |\mathbf{x}|$ it would be confusing whether it meant the Cartesian direction x or the modulus of vector $\mathbf{x}$.

$$\boldsymbol{\nabla} \cdot \mathbf{x} = \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{x} = 3 \tag{B.37}$$

$$\boldsymbol{\nabla}_{\mathbf{v}} \cdot \mathbf{v} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{v} = 3 \tag{B.38}$$

$$\boldsymbol{\nabla} \times \mathbf{x} = \frac{\partial}{\partial \mathbf{x}} \times \mathbf{x} = \mathbf{0} \tag{B.39}$$

$$\boldsymbol{\nabla}_{\mathbf{v}} \times \mathbf{v} = \frac{\partial}{\partial \mathbf{v}} \times \mathbf{v} = \mathbf{0} \tag{B.40}$$

$$\nabla|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|} \tag{B.41}$$

$$\nabla v = \frac{\mathbf{v}}{v} \tag{B.42}$$

$$\nabla \frac{1}{|\mathbf{x}|} = \frac{\partial}{\partial \mathbf{x}} \left(\frac{1}{|\mathbf{x}|} \right) = -\frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{B.43}$$

$$\nabla_{\mathbf{v}} \frac{1}{v} = \frac{\partial}{\partial \mathbf{v}} \left(\frac{1}{v} \right) = -\frac{\mathbf{v}}{v^3} \tag{B.44}$$

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|^3} \right) = \frac{\partial}{\partial \mathbf{x}} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|^3} \right) = 4\pi\, \delta(\mathbf{x}) \tag{B.45}$$

$$\boldsymbol{\nabla}_{\mathbf{v}} \cdot \left(\frac{\mathbf{v}}{v^3} \right) = \frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\mathbf{v}}{v^3} \right) = 4\pi\, \delta_v(\mathbf{v}) \tag{B.46}$$

$$\nabla \mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}} = \mathbb{1} \tag{B.47}$$

$$\nabla_{\mathbf{v}} \mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{v}} = \mathbb{1} \tag{B.48}$$

where $\mathbb{1}$, the identity tensor. In addition, using $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ which is a position vector that can be interpreted as pointing from $\mathbf{x}'$ to $\mathbf{x}$ (with $R = |\mathbf{x} - \mathbf{x}'|$) we can find further identities. We can use that $\mathbf{u} = \mathbf{v} - \mathbf{v}'$ analogously. Note how the partial derivative like notation leads to an easy to understand chain rule for some of the derivations.

$$\begin{aligned} \frac{\partial R}{\partial \mathbf{x}} &= \frac{\partial}{\partial \mathbf{x}} |\mathbf{x} - \mathbf{x}'| = \frac{\partial}{\partial \mathbf{x}} \sqrt{(\mathbf{x} - \mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}')} = \frac{1}{\cancel{2}} \frac{\cancel{2}(\mathbf{x} - \mathbf{x}') \cdot \overbrace{\frac{\partial}{\partial \mathbf{x}} (\mathbf{x} - \mathbf{x}')}^{\mathbb{1}}}{\sqrt{(\mathbf{x} - \mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}')}} \\ &= \frac{\overbrace{(\mathbf{x} - \mathbf{x}')}^{\mathbf{x}} \cdot \mathbb{1}}{R} = \frac{\mathbf{R}}{R} \end{aligned} \tag{B.49}$$

$$\frac{\partial u}{\partial \mathbf{v}} = \frac{\mathbf{u}}{u} \tag{B.50}$$

$$\frac{\partial}{\partial \mathbf{x}} \frac{1}{R} = -\frac{1}{R^2} \frac{\partial R}{\partial \mathbf{x}} = -\frac{\mathbf{R}}{R^3} \tag{B.51}$$

$$\frac{\partial}{\partial \mathbf{x}}\frac{1}{u} = -\frac{\mathbf{u}}{u^3} \tag{B.52}$$

$$\frac{\partial^2 R}{\partial \mathbf{x}\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\frac{\partial R}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\frac{\mathbf{R}}{R} = \frac{1}{R}\overbrace{\frac{\partial \mathbf{R}}{\partial \mathbf{x}}}^{\mathbb{1}} + \left(\frac{\partial}{\partial \mathbf{x}}\frac{1}{R}\right)\mathbf{R} = \frac{\mathbb{1}}{R} + \left(-\frac{\mathbf{R}}{R^3}\right)\mathbf{R} = \frac{R^2\mathbb{1} - \mathbf{R}\mathbf{R}}{R^3} \tag{B.53}$$

$$\frac{\partial^2 u}{\partial \mathbf{v}\partial \mathbf{v}} = \frac{u^2\mathbb{1} - \mathbf{u}\mathbf{u}}{u^3} \tag{B.54}$$

$$\nabla^2\frac{1}{R} \equiv \frac{\partial}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{x}}\frac{1}{R} = -\frac{\partial}{\partial \mathbf{x}}\cdot\frac{\mathbf{R}}{R^3} = -4\pi\,\delta(\mathbf{R}) = -4\pi\,\delta(\mathbf{x}-\mathbf{x}') \tag{B.55}$$

$$\nabla_{\mathbf{v}}^2\frac{1}{u} = -4\pi\,\delta_v(\mathbf{u}) = -4\pi\,\delta_v(\mathbf{v}-\mathbf{v}') \tag{B.56}$$

$$\nabla^2 R = \frac{\partial}{\partial \mathbf{x}}\cdot\frac{\partial R}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\cdot\frac{\mathbf{R}}{R} = \frac{1}{R}\underbrace{\frac{\partial}{\partial \mathbf{x}}\cdot\overbrace{\mathbf{R}}^{\mathbf{x}-\mathbf{x}'}}_{3} - \mathbf{R}\cdot\frac{\partial}{\partial \mathbf{x}}\frac{1}{R} = \frac{3}{R} - \frac{R^2}{R^3} = \frac{2}{R} \tag{B.57}$$

$$\nabla_{\mathbf{v}}^2 u = \frac{2}{u} \tag{B.58}$$

$$\nabla^2\,\nabla^2 R = \nabla_{\mathbf{v}}^2\frac{2}{R} = -8\pi\,\delta(\mathbf{x}-\mathbf{x}') \tag{B.59}$$

$$\nabla_{\mathbf{v}}^2\,\nabla_{\mathbf{v}}^2 u = -8\pi\,\delta_v(\mathbf{v}-\mathbf{v}') \tag{B.60}$$

$$\frac{\partial}{\partial \mathbf{x}}\cdot\left(\frac{R^2\mathbb{1} - \mathbf{R}\mathbf{R}}{R^3}\right) = \frac{\partial}{\partial \mathbf{x}}\cdot\frac{\partial^2 R}{\partial \mathbf{x}\partial \mathbf{x}} = \underbrace{\frac{\partial}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{x}}}_{\nabla^2}\frac{\partial R}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\underbrace{\nabla^2 R}_{\frac{2}{R}} = \frac{\partial}{\partial \mathbf{x}}\frac{2}{R} = -\frac{2\mathbf{R}}{R^3} \tag{B.61}$$

$$\frac{\partial}{\partial \mathbf{v}}\cdot\left(\frac{u^2\mathbb{1} - \mathbf{u}\mathbf{u}}{u^3}\right) = -\frac{2\mathbf{u}}{u^3} \tag{B.62}$$

This form also nicely shows that $\frac{\partial}{\partial \mathbf{x}'} = -\frac{\partial}{\partial \mathbf{x}}$ and that $\frac{\partial}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{R}}$ for objects depending on $\mathbf{R}$ (similarly $\frac{\partial}{\partial \mathbf{v}'} = -\frac{\partial}{\partial \mathbf{v}}$ and $\frac{\partial}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{u}}$ for objects depending on $\mathbf{u}$) That is,

$$\frac{\partial}{\partial \mathbf{x}} = \frac{\partial \mathbf{R}}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{R}} = \frac{\partial\,(\mathbf{x}-\mathbf{x}')}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{R}} = \mathbb{1}\cdot\frac{\partial}{\partial \mathbf{R}} = \frac{\partial}{\partial \mathbf{R}} \tag{B.63}$$

$$\frac{\partial}{\partial \mathbf{v}} = \frac{\partial \mathbf{u}}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{u}} = \frac{\partial\,(\mathbf{v}-\mathbf{v}')}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{u}} = \mathbb{1}\cdot\frac{\partial}{\partial \mathbf{u}} = \frac{\partial}{\partial \mathbf{u}} \tag{B.64}$$

$$\frac{\partial}{\partial \mathbf{x}'} = \frac{\partial \mathbf{R}}{\partial \mathbf{x}'}\cdot\frac{\partial}{\partial \mathbf{R}} = \frac{\partial\,(\mathbf{x}-\mathbf{x}')}{\partial \mathbf{x}'}\cdot\frac{\partial}{\partial \mathbf{R}} = -\mathbb{1}\cdot\frac{\partial}{\partial \mathbf{R}} = -\frac{\partial}{\partial \mathbf{R}} \tag{B.65}$$

$$\frac{\partial}{\partial \mathbf{v}'} = \frac{\partial \mathbf{u}}{\partial \mathbf{v}'}\cdot\frac{\partial}{\partial \mathbf{u}} = \frac{\partial\,(\mathbf{v}-\mathbf{v}')}{\partial \mathbf{v}'}\cdot\frac{\partial}{\partial \mathbf{u}} = -\mathbb{1}\cdot\frac{\partial}{\partial \mathbf{u}} = -\frac{\partial}{\partial \mathbf{u}}\,. \tag{B.66}$$

B.3 Tensor Identities

B.3.1 Dyads

A dyad is two vectors adjoined. For example, $\mathbf{AB}$ is a dyad, sometimes written $\mathbf{A}\otimes\mathbf{B}$ to indicate the adjoining. Note that $\mathbf{AB}\neq\mathbf{BA}$ in general. Also let $\overleftrightarrow{\mathbf{T}}=T_{ij}$ be a second order tensor. The dyadic identity tensor (or more generally, the order 2 identity tensor) is here defined as $\mathbb{1}$ which has the property that for any vector that $\mathbf{A}=\mathbf{A}\cdot\mathbb{1}=\mathbb{1}\cdot\mathbf{A}$. For orthogonal coordinates, $\mathbb{1}=\delta^i_j$ in which δ^i_j is the Kroencker delta.

Note that when taking $\boldsymbol{\nabla}\cdot$ on a tensor or dyad, it works on the first index. That is,

$$\boldsymbol{\nabla}\cdot(\mathbf{AB})=(\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{B}+(\mathbf{A}\cdot\nabla)\mathbf{B} \tag{B.67}$$

$$\boldsymbol{\nabla}\cdot(f\overleftrightarrow{\mathbf{T}})=\nabla f\cdot\overleftrightarrow{\mathbf{T}}+f\boldsymbol{\nabla}^l\cdot\overleftrightarrow{\mathbf{T}} \tag{B.68}$$

For a tensor $\overleftrightarrow{\mathbf{T}}=\hat{\mathbf{x}}_iT_{ij}\hat{\mathbf{x}}_j=\mathbf{AB}=\hat{\mathbf{x}}_iA_iB_j\hat{\mathbf{x}}_j$ using a Cartesian coordinate system we find

$$\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}=\hat{\mathbf{x}}_k\cdot\partial_k(\hat{\mathbf{x}}_iT_{ij}\hat{\mathbf{x}}_j)=\hat{\mathbf{x}}_k\cdot\hat{\mathbf{x}}_i\partial_k(T_{ij})\hat{\mathbf{x}}_j=\delta_{ik}\partial_k(T_{ij})\hat{\mathbf{x}}_j=\partial_i(T_{ij})\hat{\mathbf{x}}_j \tag{B.69}$$

$$\nabla f\cdot\overleftrightarrow{\mathbf{T}}=\partial_i(f)T_{ij}\hat{\mathbf{x}}_j \tag{B.70}$$

This is not necessarily a standard definition (there is no standard, but this conforms to the NRL plasma formulary's definition and most plasma literature. The other possible definition is for Cartesian coordinates to have $\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}=\partial_j(T_{ij})\hat{\mathbf{x}}_i$).

Note that we may now define **:**, the double dot operator:

$$\mathbb{1}:\mathbf{AB}\equiv(\mathbb{1}\cdot\mathbf{A})\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{B} \tag{B.71}$$

$$\mathbf{AB}:\mathbf{CD}\equiv\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})\mathbf{D}=(\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D})=\mathbf{D}\cdot(\mathbf{AB})\cdot\mathbf{C}=\mathbf{B}\cdot(\mathbf{CD})\cdot\mathbf{A}\,. \tag{B.72}$$

We also have

$$\overleftrightarrow{\mathbf{T}}:\mathbb{1}=\mathbb{1}:\overleftrightarrow{\mathbf{T}}=T_{ii}=\mathrm{Tr}[\overleftrightarrow{\mathbf{T}}]\,, \tag{B.73}$$

where $\mathrm{Tr}\,\overleftrightarrow{\mathbf{T}}$ is the trace (sum of the diagonal elements) of the tensor $\overleftrightarrow{\mathbf{T}}$.

So we see that

$$\overleftrightarrow{\mathbf{T}}:\overleftrightarrow{\mathbf{S}}\equiv T_{ij}S_{ji} \tag{B.74}$$

Once again, note that while this is the convention followed here, there is no general standard in the literature. Some references may use $\mathbf{AB}:\mathbf{CD}=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})$. However, in plasma literature, the convention used here (B.72) is more common. Other useful identities of the **:** operator are

$$\overleftrightarrow{\mathbf{T}}:\overleftrightarrow{\mathbf{T}}=|\mathbf{T}|^2=T_{ij}T_{ji} \tag{B.75}$$

$$\overleftrightarrow{\mathbf{T}}:\mathbf{AB}=(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A})\cdot\mathbf{B}=\mathbf{B}\cdot\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} \tag{B.76}$$

$$\mathbf{AB}:\overleftrightarrow{\mathbf{T}}=\mathbf{A}\cdot(\mathbf{B}\cdot\overleftrightarrow{\mathbf{T}})=\mathbf{B}\cdot\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} \tag{B.77}$$

$$\overleftrightarrow{\mathbf{T}}:\mathbf{AB}=\mathbf{AB}:\overleftrightarrow{\mathbf{T}} \tag{B.78}$$

$$\mathbf{B}\times\overleftrightarrow{\mathbf{T}}:\overleftrightarrow{\mathbf{W}}=-(\overleftrightarrow{\mathbf{T}}\cdot\overleftrightarrow{\mathbf{W}})^{\intercal}:\mathbf{B}\times\mathbb{1}\,, \tag{B.79}$$

where the $^\intercal$ denotes the transpose (sometimes a capital T or T is used), so that for $\overleftrightarrow{\mathbf{T}} = T_{ij}$, $\overleftrightarrow{\mathbf{T}}^\intercal = T_{ji}$.

In addition, one can define new dot and cross operators as the cross-dot $\overset{\times}{\cdot}$, dot-cross $\overset{\cdot}{\times}$, and cross-cross $\overset{\times}{\times}$ product operators between two dyads or second order tensors. I have decided to keep with what seems to be the small amount of literature using these, and employ definitions that are symmetric or antisymmetric. We then have

$$\mathbf{VW} \overset{\times}{\cdot} \mathbf{YZ} = (\mathbf{V} \times \mathbf{Y})(\mathbf{W} \cdot \mathbf{Z}) \tag{B.80}$$

$$\overleftrightarrow{\mathbf{T}} \overset{\times}{\cdot} \mathbf{VW} = (-\mathbf{V} \times \overleftrightarrow{\mathbf{T}}) \cdot \mathbf{W} \tag{B.81}$$

$$\mathbf{VW} \overset{\cdot}{\times} \mathbf{YZ} = (\mathbf{V} \cdot \mathbf{Y})(\mathbf{W} \times \mathbf{Z}) \tag{B.82}$$

$$\overleftrightarrow{\mathbf{T}} \overset{\cdot}{\times} \mathbf{VW} = \mathbf{V} \cdot \overleftrightarrow{\mathbf{T}} \times \mathbf{W} \tag{B.83}$$

$$\mathbf{VW} \overset{\times}{\times} \mathbf{YZ} = (\mathbf{V} \times \mathbf{Y})(\mathbf{W} \times \mathbf{Z}) \tag{B.84}$$

$$\overleftrightarrow{\mathbf{T}} \overset{\times}{\times} \mathbf{VW} = -\mathbf{V} \times \overleftrightarrow{\mathbf{T}} \times \mathbf{W} \tag{B.85}$$

In general we use that with Cartesian components the definition is

$$\overleftrightarrow{\mathbf{T}} \overset{\times}{\cdot} \overleftrightarrow{\mathbf{S}} = \epsilon_{ijk} T_{jl} \delta_{lm} S_{km} \hat{\mathbf{x}}_i = \epsilon_{ijk} T_{jm} S_{km} \hat{\mathbf{x}}_i \tag{B.86}$$

$$\overleftrightarrow{\mathbf{T}} \overset{\cdot}{\times} \overleftrightarrow{\mathbf{S}} = \delta_{lm} \epsilon_{ijk} T_{lj} S_{mk} \hat{\mathbf{x}}_i = \epsilon_{ijk} T_{mj} S_{mk} \hat{\mathbf{x}}_i \tag{B.87}$$

$$\overleftrightarrow{\mathbf{T}} \overset{\times}{\times} \overleftrightarrow{\mathbf{S}} = \epsilon_{ijk} T_{jm} \epsilon_{lmn} S_{kn} \hat{\mathbf{x}}_i \hat{\mathbf{x}}_l \tag{B.88}$$

Note that with these definitions

$$\mathbf{VW} \overset{\times}{\cdot} \mathbf{YZ} = -\mathbf{YZ} \overset{\times}{\cdot} \mathbf{VW} = -(\mathbf{Y} \times \mathbf{V})(\mathbf{Z} \cdot \mathbf{W}) \tag{B.89}$$

$$\mathbf{VW} \overset{\cdot}{\times} \mathbf{YZ} = -\mathbf{YZ} \overset{\cdot}{\times} \mathbf{VW} = -(\mathbf{Y} \cdot \mathbf{V})(\mathbf{Z} \times \mathbf{W}) \tag{B.90}$$

$$\mathbf{VW} \overset{\times}{\times} \mathbf{YZ} = \mathbf{YZ} \overset{\times}{\times} \mathbf{VW} = (\mathbf{Y} \times \mathbf{V})(\mathbf{Z} \times \mathbf{W}) \tag{B.91}$$

These are sometimes useful for factoring out quantities from integrals or derivatives, otherwise they have limited applications. Because these operators are so rare, sometimes different definitions are used so that (B.89) and (B.90) are in fact equalities so be careful if using these operators in any context and inspect the definition.

Note that any second order tensor $\overleftrightarrow{\mathbf{T}}$ can be decomposed into its symmetric $\overleftrightarrow{\mathbf{T}}_\mathrm{S}$ and antisymmetric (often called skew-symmetric) $\overleftrightarrow{\mathbf{T}}_\mathrm{A}$ components in the following way

$$\overleftrightarrow{\mathbf{T}}_\mathrm{S} = \frac{1}{2}(\overleftrightarrow{\mathbf{T}} + \overleftrightarrow{\mathbf{T}}^\intercal)\,, \quad \overleftrightarrow{\mathbf{T}}_{\mathrm{S},ij} = \overleftrightarrow{\mathbf{T}}_{\mathrm{S},ji}\,, \tag{B.92}$$

$$\overleftrightarrow{\mathbf{T}}_\mathrm{A} = \frac{1}{2}(\overleftrightarrow{\mathbf{T}} - \overleftrightarrow{\mathbf{T}}^\intercal)\,, \quad \overleftrightarrow{\mathbf{T}}_{\mathrm{A},ij} = -\overleftrightarrow{\mathbf{T}}_{\mathrm{S},ji}\,, \tag{B.93}$$

$$\overleftrightarrow{\mathbf{T}} = \overleftrightarrow{\mathbf{T}}_\mathrm{S} + \overleftrightarrow{\mathbf{T}}_\mathrm{A}\,. \tag{B.94}$$

There are two more operations that are useful for finding identities once tensors have been introduced. The first is often called the vector operation, denoted $\mathrm{vec}(\overleftrightarrow{\mathbf{T}})$, or, as denoted here, $\overleftrightarrow{\mathbf{T}}_\times$. This operation is defined by

$$\mathrm{vec}(\overleftrightarrow{\mathbf{T}}) = \overleftrightarrow{\mathbf{T}}_\times = \mathcal{J}\epsilon^{ijk}T_{ij}\mathbf{e}^k = \frac{\epsilon_{ijk}}{\mathcal{J}}T^{ij}\mathbf{e}_k = T^i_{\cdot j}\mathbf{e}_i \times \mathbf{e}^j = T^{\cdot i}_j \mathbf{e}^j \times \mathbf{e}^i \tag{B.95}$$

Note for a dyad this just yields the cross product definition so

$$\mathrm{vec}(\mathbf{AB}) = (\mathbf{AB})_\times = \mathbf{A} \times \mathbf{B} \tag{B.96}$$

suggesting the $\times$ product's use as the appropriate notation.

Now let's note that (it is important that we have index order ik for both tensors below) and using Cartesian coordinates for ease

$$\mathbb{1} \times \mathbf{A} = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_k\delta_{ij}\epsilon_{kjl}A_l = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_k\epsilon_{kil}A_l = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_k\epsilon_{ilk}A_l \tag{B.97}$$

$$\mathbf{A} \times \mathbb{1} = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_k\epsilon_{ijl}A_j\delta_{lk} = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_k\epsilon_{ijk}A_j = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_k\epsilon_{ilk}A_l \tag{B.98}$$

$$\mathbb{1} \times \mathbf{A} = \mathbf{A} \times \mathbb{1} \tag{B.99}$$

contrary to our expectations from the cases for the cross product between two vectors ($\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$). In general, using Cartesian coordinates we find

$$\mathbf{A} \times \overleftrightarrow{\mathbf{T}} = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_l\epsilon_{ijk}A_jT_{kl} = \hat{\mathbf{x}}_k\hat{\mathbf{x}}_l\epsilon_{ijk}A_iT_{jl} \tag{B.100}$$

$$\overleftrightarrow{\mathbf{T}} \times \mathbf{A} = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_l\epsilon_{ljk}T_{ij}A_k = \hat{\mathbf{x}}_k\hat{\mathbf{x}}_l\epsilon_{ijl}T_{ki}A_j \tag{B.101}$$

and so usually $\mathbf{A} \times \overleftrightarrow{\mathbf{T}} \neq \overleftrightarrow{\mathbf{T}} \times \mathbf{A}$ unless $\overleftrightarrow{\mathbf{T}}$ is a symmetric tensor. Also note the tensor index order (either il or kl) for both entries, as it is very important when computing transposes.

Now we can define the dyad operator by (and see the Cartesian coordinate definition)

$$\mathrm{dyad}(\mathbf{A}) = -\frac{1}{2}\mathbb{1} \times \mathbf{A} = -\frac{1}{2}\delta_{ij}\epsilon_{kjl}A_l = -\frac{1}{2}\epsilon_{ilk}A_l \tag{B.102}$$

This is chosen to so that the antisymmetric part of a tensor is recovered under the operation $\mathrm{dyad}(\overleftrightarrow{\mathbf{T}}_\times)$. Thus, omitting the Cartesian coordinate $\hat{\mathbf{x}}_i$ unit vectors for simplicity in our derivation,

$$\mathrm{dyad}(\overleftrightarrow{\mathbf{T}}_\times) = \mathrm{dyad}(\epsilon_{ijk}T_{jk}) = -\frac{1}{2}\epsilon_{lim}\epsilon_{ijk}T_{jk} = \frac{1}{2}\,\epsilon_{ilm}\epsilon_{ijk}T_{jk} = \frac{1}{2}\,(\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{jm})\,T_{jk} \tag{B.103}$$

$$= \frac{1}{2}\,(T_{lm} - T_{ml}) = \frac{1}{2}\,(\overleftrightarrow{\mathbf{T}} - \overleftrightarrow{\mathbf{T}}^\intercal) = \overleftrightarrow{\mathbf{T}}_{\mathrm{A}}\,. \tag{B.104}$$

It is also chosen so that we reasonably get $\mathrm{dyad}(\mathbf{A})_\times = \mathrm{vec}[\mathrm{dyad}(\mathbf{A})] = \mathbf{A}$, again omitting Cartesian coordinate $\hat{\mathbf{x}}_i$ for simplicity,

$$\mathrm{dyad}(\mathbf{A})_\times = -\frac{1}{2}\,(\mathbb{1} \times \mathbf{A})_\times = -\frac{1}{2}\,(\delta_{ij}\epsilon_{kjl}A_l)_\times = -\frac{1}{2}\,\epsilon_{mik}\epsilon_{kjl}\delta_{ij}A_l \tag{B.105}$$

$$= -\frac{1}{2}\,(\epsilon_{mjk}\epsilon_{kjl}A_l) = \frac{1}{2}\,\epsilon_{jkm}\epsilon_{jkl}A_l = \frac{1}{2}\,(\delta_{kk}\delta_{ml} - \delta_{km}\delta_{kl})\,A_l \tag{B.106}$$

$$= \frac{1}{2}\,(3\delta_{ml} - \delta_{lm})\,A_l = A_m = \mathbf{A}\,. \tag{B.107}$$

Now we may use [the index ordering inside the tensor cross product is il (not li) for both], again omitting Cartesian coordinate $\hat{\mathbf{x}}_i$ unit vectors for simplicity.

$$(\mathbf{A}\times\overleftrightarrow{\mathbf{T}})_\times = (\epsilon_{ijk}A_jT_{kl})_\times = \epsilon_{mil}\epsilon_{ijk}A_jT_{kl} = \epsilon_{ilm}\epsilon_{ijk}A_jT_{kl} = (\delta_{lj}\delta_{mk}-\delta_{lk}\delta_{jm})A_jT_{kl} \tag{B.108}$$

$$= A_lT_{ml} - A_mT_{kk} = \overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} - \mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}}) \tag{B.109}$$

$$(\overleftrightarrow{\mathbf{T}}\times\mathbf{A})_\times = (\epsilon_{ljk}T_{ij}A_k)_\times = \epsilon_{mil}\epsilon_{ljk}A_kT_{ij} = \epsilon_{lmi}\epsilon_{ljk}A_kT_{ij} = (\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ji})A_kT_{ij} \tag{B.110}$$

$$= A_kT_{km} - A_mT_{jj} = \mathbf{A}\cdot\overleftrightarrow{\mathbf{T}} - \mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}}) \tag{B.111}$$

and so

$$(\mathbf{A}\times\overleftrightarrow{\mathbf{T}} - \overleftrightarrow{\mathbf{T}}\times\mathbf{A})_\times = \overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} - \mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}}) - [\mathbf{A}\cdot\overleftrightarrow{\mathbf{T}} - \mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}})] = \overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} - \mathbf{A}\cdot\overleftrightarrow{\mathbf{T}} \tag{B.112}$$

$$(\overleftrightarrow{\mathbf{T}}\times\mathbf{A} - \mathbf{A}\times\overleftrightarrow{\mathbf{T}})_\times = \mathbf{A}\cdot\overleftrightarrow{\mathbf{T}} - \mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}}) - [\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} - \mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}})] = \mathbf{A}\cdot\overleftrightarrow{\mathbf{T}} - \overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} \tag{B.113}$$

and thus we see that

$$\mathrm{dyad}[(\mathbf{A}\times\overleftrightarrow{\mathbf{T}})_\times] = \mathrm{dyad}[\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} - \mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}})] = -\frac{1}{2}\mathbb{1}\times\left[\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} - \mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}})\right] \tag{B.114}$$

$$\mathrm{dyad}[(\mathbf{A}\times\overleftrightarrow{\mathbf{T}})_\times] = (\mathbf{A}\times\overleftrightarrow{\mathbf{T}})_\mathrm{A} = \frac{1}{2}[(\mathbf{A}\times\overleftrightarrow{\mathbf{T}}) - (\mathbf{A}\times\overleftrightarrow{\mathbf{T}})^\intercal] \tag{B.115}$$

$$(\mathbf{A}\times\overleftrightarrow{\mathbf{T}}) - (\mathbf{A}\times\overleftrightarrow{\mathbf{T}})^\intercal = \mathbb{1}\times\left[\mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}}) - \overleftrightarrow{\mathbf{T}}\cdot\mathbf{A}\right]. \tag{B.116}$$

or, similarly

$$(\overleftrightarrow{\mathbf{T}}\times\mathbf{A}) - (\overleftrightarrow{\mathbf{T}}\times\mathbf{A})^\intercal = \mathbb{1}\times\left[\mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}}) - \mathbf{A}\cdot\overleftrightarrow{\mathbf{T}}\right]. \tag{B.117}$$

Collected below are some useful tensor identities.

B.3.2 Tensor Identities Without Derivatives

$$\mathbb{1}:\mathbf{AB} = (\mathbb{1}\cdot\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{B} \tag{B.118}$$

$$\mathbf{AB}:\mathbf{CD} \equiv \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})\mathbf{D} = (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D}) = \mathbf{D}\cdot(\mathbf{AB})\cdot\mathbf{C} = \mathbf{B}\cdot(\mathbf{CD})\cdot\mathbf{A} \tag{B.119}$$

$$\mathbb{1}:\overleftrightarrow{\mathbf{T}} = \mathrm{Tr}(\overleftrightarrow{\mathbf{T}}) \tag{B.120}$$

$$\overleftrightarrow{\mathbf{T}}:\overleftrightarrow{\mathbf{T}} = |\overleftrightarrow{\mathbf{T}}|^2 \tag{B.121}$$

$$\overleftrightarrow{\mathbf{T}}:\mathbf{AB} = (\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A})\cdot\mathbf{B} = \mathbf{B}\cdot\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} = \mathbf{A}\cdot(\mathbf{B}\cdot\overleftrightarrow{\mathbf{T}}) = \mathbf{AB}:\overleftrightarrow{\mathbf{T}} \tag{B.122}$$

$$\mathbf{A}\times\overleftrightarrow{\mathbf{T}}:\overleftrightarrow{\mathbf{S}} = -(\overleftrightarrow{\mathbf{T}}\cdot\overleftrightarrow{\mathbf{S}})^\intercal:\mathbf{A}\times\mathbb{1} \tag{B.123}$$

$$\mathbb{1}\cdot\overleftrightarrow{\mathbf{T}} = \overleftrightarrow{\mathbf{T}}\cdot\mathbb{1} = \overleftrightarrow{\mathbf{T}} \tag{B.124}$$

$$\mathbf{A}\cdot\overleftrightarrow{\mathbf{T}} = \overleftrightarrow{\mathbf{T}}^\intercal\cdot\mathbf{A} \tag{B.125}$$

$$\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A} = \mathbf{A}\cdot\overleftrightarrow{\mathbf{T}}^\intercal \tag{B.126}$$

$$\mathbf{B}\cdot(\mathbb{1}\times\mathbf{A})=\mathbf{B}\cdot(\mathbf{A}\times\mathbb{1})=\mathbf{B}\times\mathbf{A} \tag{B.127}$$
$$(\mathbf{A}\times\mathbb{1})\cdot\mathbf{B}=(\mathbb{1}\times\mathbf{A})\cdot\mathbf{B}=\mathbf{A}\times\mathbf{B} \tag{B.128}$$
$$\mathbf{A}\cdot(\mathbf{CB}-\mathbf{BC})=\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \tag{B.129}$$
$$\mathbb{1}\cdot\mathbf{AB}=(\mathbb{1}\cdot\mathbf{A})\mathbf{B}=\mathbf{AB} \tag{B.130}$$
$$\mathbf{AB}\cdot\mathbb{1}=\mathbf{A}(\mathbf{B}\cdot\mathbb{1})=\mathbf{AB} \tag{B.131}$$
$$\mathbb{1}\times\mathbf{A}=\mathbf{A}\times\mathbb{1} \tag{B.132}$$
$$\mathbf{A}\times(\mathbf{BC})=(\mathbf{A}\times\mathbf{B})\mathbf{C} \tag{B.133}$$
$$(\mathbf{A}\times\mathbf{B})\cdot\overleftrightarrow{\mathbf{T}}=\mathbf{A}\cdot(\mathbf{B}\times\overleftrightarrow{\mathbf{T}})=-\mathbf{B}\cdot(\mathbf{A}\times\overleftrightarrow{\mathbf{T}}) \tag{B.134}$$
$$\overleftrightarrow{\mathbf{T}}\cdot(\mathbf{A}\times\mathbf{B})=(\overleftrightarrow{\mathbf{T}}\times\mathbf{A})\cdot\mathbf{B}=-(\overleftrightarrow{\mathbf{T}}\times\mathbf{B})\cdot\mathbf{A} \tag{B.135}$$
$$\mathbf{A}\times\overleftrightarrow{\mathbf{T}}\cdot\mathbf{B}=-(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{B})\times\mathbf{A} \tag{B.136}$$
$$(\mathbf{A}\times\mathbf{B})\times\mathbb{1}=\mathbb{1}\times(\mathbf{A}\times\mathbf{B})=\mathbf{BA}-\mathbf{AB} \tag{B.137}$$
$$(\mathbf{A}\times\overleftrightarrow{\mathbf{T}})^{\intercal}=-\overleftrightarrow{\mathbf{T}}^{\intercal}\times\mathbf{A} \tag{B.138}$$
$$(\overleftrightarrow{\mathbf{T}}\times\mathbf{A})^{\intercal}=-\mathbf{A}\times\overleftrightarrow{\mathbf{T}}^{\intercal} \tag{B.139}$$
$$(\mathbf{A}\times\overleftrightarrow{\mathbf{T}})-(\mathbf{A}\times\overleftrightarrow{\mathbf{T}})^{\intercal}=\mathbb{1}\times[\mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}})-\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A}]=2\,(\mathbf{A}\times\overleftrightarrow{\mathbf{T}})_{\mathrm{A}} \tag{B.140}$$
$$(\overleftrightarrow{\mathbf{T}}\times\mathbf{A})-(\overleftrightarrow{\mathbf{T}}\times\mathbf{A})^{\intercal}=\mathbb{1}\times[\mathbf{A}\,\mathrm{Tr}(\overleftrightarrow{\mathbf{T}})-\mathbf{A}\cdot\overleftrightarrow{\mathbf{T}}]=2\,(\overleftrightarrow{\mathbf{T}}\times\mathbf{A})_{\mathrm{A}} \tag{B.141}$$
$$\overleftrightarrow{\mathbf{T}}_{\mathrm{S}}=\frac{1}{2}(\overleftrightarrow{\mathbf{T}}+\overleftrightarrow{\mathbf{T}}^{\intercal}) \tag{B.142}$$
$$\overleftrightarrow{\mathbf{T}}_{\mathrm{A}}=\frac{1}{2}(\overleftrightarrow{\mathbf{T}}-\overleftrightarrow{\mathbf{T}}^{\intercal}) \tag{B.143}$$
$$\overleftrightarrow{\mathbf{T}}=\overleftrightarrow{\mathbf{T}}_{\mathrm{S}}+\overleftrightarrow{\mathbf{T}}_{\mathrm{A}} \tag{B.144}$$

B.3.3 Tensor Identities With Derivatives

$$\boldsymbol{\nabla}\cdot(\mathbf{AB})=(\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{B}+(\mathbf{A}\cdot\nabla)\mathbf{B} \tag{B.145}$$
$$\boldsymbol{\nabla}\cdot(f\overleftrightarrow{\mathbf{T}})=\nabla f\cdot\overleftrightarrow{\mathbf{T}}+f\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}} \tag{B.146}$$
$$\mathbf{A}\times\mathbb{1}:\nabla\mathbf{B}=\mathbf{A}\cdot\boldsymbol{\nabla}\times\mathbf{B} \tag{B.147}$$
$$\mathbb{1}\cdot\nabla\mathbf{B}=\nabla\mathbf{B}\cdot\mathbb{1}=\nabla\mathbf{B} \tag{B.148}$$
$$\mathbb{1}:\nabla\mathbf{B}=\nabla\mathbf{B}:\mathbb{1}=\boldsymbol{\nabla}\cdot\mathbf{B} \tag{B.149}$$
$$\mathbb{1}\overset{\times}{\cdot}\nabla\mathbf{A}=\boldsymbol{\nabla}\times\mathbf{A} \tag{B.150}$$
$$\mathbb{1}\overset{\cdot}{\times}\nabla\mathbf{A}=-\boldsymbol{\nabla}\times\mathbf{A} \tag{B.151}$$
$$\boldsymbol{\nabla}\cdot(\mathbb{1}\times\mathbf{A})=\boldsymbol{\nabla}\times\mathbf{A} \tag{B.152}$$
$$\boldsymbol{\nabla}\cdot(f\mathbb{1})=\nabla f \tag{B.153}$$
$$\boldsymbol{\nabla}\cdot\mathbb{1}=0 \tag{B.154}$$
$$\nabla\mathbb{1}=\mathbf{0} \tag{B.155}$$
$$\boldsymbol{\nabla}\times(\mathbf{AB})=(\boldsymbol{\nabla}\times\mathbf{A})\mathbf{B}-\mathbf{A}\times\nabla\mathbf{B} \tag{B.156}$$
$$\nabla(\mathbf{A}\times\mathbf{B})=\nabla\mathbf{A}\times\mathbf{B}-\nabla\mathbf{B}\times\mathbf{A} \tag{B.157}$$

$$\boldsymbol{\nabla}\cdot[\mathbf{B}\cdot\nabla\mathbf{A}] = \mathbf{B}\cdot\nabla(\boldsymbol{\nabla}\cdot\mathbf{A}) - (\boldsymbol{\nabla}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\mathbf{B}) \tag{B.158}$$

$$\boldsymbol{\nabla}\times[\mathbf{A}\cdot\nabla\mathbf{A}] = \mathbf{A}\cdot\nabla(\boldsymbol{\nabla}\times\mathbf{A}) + (\boldsymbol{\nabla}\cdot\mathbf{A})(\boldsymbol{\nabla}\times\mathbf{A}) - [\boldsymbol{\nabla}\times\mathbf{A}\cdot\nabla]\mathbf{A} \tag{B.159}$$

$$\begin{aligned}(\mathbf{CA}-\mathbf{AC}):\nabla\mathbf{B} &= \mathbf{CA}:\nabla\mathbf{B} - \mathbf{AC}:\nabla\mathbf{B} = (\mathbf{A}\times\mathbf{C})\cdot(\boldsymbol{\nabla}\times\mathbf{B})\\ \mathbf{A}\cdot\nabla\mathbf{B}\cdot\mathbf{C} - \mathbf{C}\cdot\nabla\mathbf{B}\cdot\mathbf{A} &= (\mathbf{A}\times\mathbf{C})\cdot(\boldsymbol{\nabla}\times\mathbf{B})\end{aligned} \tag{B.160}$$

$$\begin{aligned}\mathbf{CA}:\nabla\mathbf{B} + \mathbf{BA}:\nabla\mathbf{C} &= \nabla[(\mathbf{B}\cdot\mathbf{C})\mathbf{A}] - (\mathbf{B}\cdot\mathbf{C})\boldsymbol{\nabla}\cdot\mathbf{A}\\ \mathbf{A}\cdot\nabla\mathbf{B}\cdot\mathbf{C} + \mathbf{A}\cdot\nabla\mathbf{C}\cdot\mathbf{B} &= \nabla[(\mathbf{B}\cdot\mathbf{C})\mathbf{A}] - (\mathbf{B}\cdot\mathbf{C})\boldsymbol{\nabla}\cdot\mathbf{A}\end{aligned} \tag{B.161}$$

$$\begin{aligned}\mathbf{CA}:\nabla\mathbf{B} &= \nabla\left(\frac{\mathbf{A}\cdot\mathbf{B}}{2}\right)\cdot\mathbf{C} + (\boldsymbol{\nabla}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{C}) + (\boldsymbol{\nabla}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{C})\\ \mathbf{A}\cdot\nabla\mathbf{B}\cdot\mathbf{C} &= \nabla\left(\frac{\mathbf{A}\cdot\mathbf{B}}{2}\right)\cdot\mathbf{C} + (\boldsymbol{\nabla}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{C}) + (\boldsymbol{\nabla}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{C})\end{aligned} \tag{B.162}$$

$$\begin{aligned}2\mathbf{CA}:\nabla\mathbf{B} = 2\mathbf{A}\cdot\nabla\mathbf{B}\cdot\mathbf{C} &= \mathbf{A}\cdot\nabla(\mathbf{B}\cdot\mathbf{C}) - \mathbf{B}\cdot\nabla(\mathbf{C}\cdot\mathbf{A}) + \mathbf{C}\cdot\nabla(\mathbf{A}\cdot\mathbf{B})\\ &\quad - (\mathbf{A}\times\mathbf{B})\cdot(\boldsymbol{\nabla}\times\mathbf{C}) + (\mathbf{B}\times\mathbf{C})\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - (\mathbf{C}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\mathbf{B})\end{aligned} \tag{B.163}$$

Note that (B.159) comes most easily from using (B.18) inside of the curl operator and then employing Einstein notation.

B.3.4 Useful Symbols

One can also use the Kronecker delta

$$\delta_{ij} = \delta^{ij} = \delta^i_j \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{B.164}$$

where only δ^i_j is in fact a tensor. The other two Kronecker deltas are more accurately called symbols because they do not transform as covariant or contravariant tensors. Using the Einstein summation notation then a dot product is easily given as

$$\mathbf{A}\cdot\mathbf{B} = \delta^i_j A_i B^j = \delta^j_i A^i B_j \tag{B.165}$$

where δ^i_j is said to contract the index on $\mathbf{A}$ or $\mathbf{B}$. Note that $\delta^i_j = \delta^j_i$ by its definition. The δ notation is suggestive as one can think that in the limit of the continuum case (i no longer has discrete values 1,2,3,..., but a continuum of values) the Kronecker delta becomes a Dirac delta function.

For a cross product one uses the Levi-Civita symbol ϵ_{ijk}.

$$\epsilon_{ijk} = \epsilon^{ijk} \begin{cases} 1 & \text{for } ijk = 123, 231, 312 \\ -1 & \text{for } ijk = 132, 321, 213 \\ 0 & \text{otherwise.} \end{cases} \tag{B.166}$$

Once again, symbol is the appropriate word because the Levi-Civita symbol is not a third order tensor, as it does not appropriately transform as a third order tensor of any kind. One can show that

$$\mathbf{A}\times\mathbf{B} = \mathcal{J}\epsilon^{ijk}A_jB_k\mathbf{e}^i = \epsilon_{ijk}A_jB^k\mathbf{e}_i/\mathcal{J} \tag{B.167}$$

$$\mathbf{A}\times\mathbf{B} = \mathcal{J}\epsilon^{ijk}A_iB_j\mathbf{e}^k = \epsilon_{ijk}A^iB^j\mathbf{e}_k \tag{B.168}$$

where sums are implied. Note that $\epsilon_{ijk} = -\epsilon_{jik}$ and $\epsilon^{ijk} = -\epsilon^{jik}$. That is, switching any two indices once switches the sign of the term. We see that $\epsilon^{ijk}A_iB_j\mathbf{e}^k = -\epsilon^{jik}A_iB_j\mathbf{e}^k$, which is the well-known result that $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$.

In any case, one can also show that

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk} \tag{B.169}$$
$$\epsilon^{ijk}\epsilon^{imn} = \delta^{jm}\delta^{kn} - \delta^{jn}\delta^{mk} \tag{B.170}$$
$$\epsilon^{ijk}\epsilon_{imn} = \delta^j_m\delta^k_n - \delta^j_n\delta^m_k \tag{B.171}$$
$$\epsilon_{ijk}\epsilon^{imn} = \delta^m_j\delta^n_k - \delta^n_j\delta^k_m \tag{B.172}$$

and also (with significantly more effort)

$$\begin{aligned}\epsilon_{ijk}\epsilon_{lmn} = {}& \delta_{il}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right)\\ &+ \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)\ .\end{aligned} \tag{B.173}$$

$$\begin{aligned}\epsilon^{ijk}\epsilon_{lmn} = {}& \delta^i_l\left(\delta^j_m\delta^k_n - \delta^j_n\delta^k_m\right) - \delta^i_m\left(\delta^j_l\delta^k_n - \delta^j_n\delta^k_l\right)\\ &+ \delta^i_n\left(\delta^j_l\delta_{km} - \delta^j_m\delta^k_l\right)\ .\end{aligned} \tag{B.174}$$

A useful mnemonic (and in fact, the definition for multiple dimensions) is given by

$$\begin{aligned}\epsilon_{ijk}\epsilon_{lmn} &= \begin{vmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{vmatrix}\\ &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)\end{aligned} \tag{B.175}$$

$$\begin{aligned}\epsilon^{ijk}\epsilon_{lmn} &= \begin{vmatrix}\delta^i_l & \delta^i_m & \delta^i_n\\ \delta^j_l & \delta^j_m & \delta^j_n\\ \delta^k_l & \delta^k_m & \delta^k_n\end{vmatrix}\\ &= \delta^i_l(\delta^j_m\delta^k_n - \delta^j_n\delta^k_m) - \delta^i_m(\delta^j_l\delta^k_n - \delta^j_n\delta^k_l) + \delta^i_n\left(\delta^j_l\delta^k_m - \delta^j_m\delta^k_l\right)\end{aligned} \tag{B.176}$$

B.4 Integral Identities

For the following $\mathbf{A}$ is an arbitrary vector, f and g are scalar functions, and $\overleftrightarrow{\mathbf{T}}$ is a second order tensor. $\mathcal{V}$ is a volume enclosed by surface $\mathcal{S}$ with $\hat{\mathbf{n}}$ the unit normal vector out of the volume.

$$\iiint \mathrm{d}V\ \nabla f = \oiint \mathrm{d}S\ \hat{\mathbf{n}} f \tag{B.177}$$
$$\iiint \mathrm{d}V\ \boldsymbol{\nabla}\cdot\mathbf{A} = \oiint \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{A} \tag{B.178}$$
$$\iiint \mathrm{d}V\ \boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}} = \oiint \mathrm{d}S\ \hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{T}} \tag{B.179}$$
$$\iiint \mathrm{d}V\ \boldsymbol{\nabla}\times\mathbf{A} = \oiint \mathrm{d}S\ \hat{\mathbf{n}}\times A \tag{B.180}$$

In the following $\mathcal{S}$ is a surface (not necessarily closed) with bounding curve C with line element $\mathrm{d}\boldsymbol{\ell}$ (sometimes people use $\mathrm{d}\mathbf{x}$ as the line element).

$$\iint_{\mathcal{S}} \mathrm{d}S\ \hat{\mathbf{n}} \times \nabla f = \oint_{C} \mathrm{d}\boldsymbol{\ell}\ f \tag{B.181}$$

$$\iint_{\mathcal{S}} \mathrm{d}S\ \boldsymbol{\nabla} \times \mathbf{A} = \oint_{C} \mathrm{d}\boldsymbol{\ell}\ \cdot \mathbf{A} \tag{B.182}$$

$$\iint_{\mathcal{S}} \mathrm{d}S\ \hat{\mathbf{n}} \cdot \nabla f \times \nabla g = \oint_{C} \mathrm{d}g\ f = -\oint_{C} \mathrm{d}f\ g \tag{B.183}$$

If you are curious as to how to derive the scalar values, define a constant (in space) vector $\mathbf{c}$. Then we can write $\boldsymbol{\nabla} \cdot (f\mathbf{c}) = \nabla f \cdot \mathbf{c} + \cancel{f\boldsymbol{\nabla} \cdot (\mathbf{c})}$. So we apply the identity for $\boldsymbol{\nabla} \cdot (f\mathbf{c})$ and find

$$\mathbf{c} \cdot \iiint_{\mathcal{V}} \mathrm{d}V\ \nabla f = \iiint_{\mathcal{V}} \mathrm{d}V\ \boldsymbol{\nabla} \cdot (f\mathbf{c}) = \oiint_{\mathcal{S}} \mathrm{d}S\ \hat{\mathbf{n}} \cdot f\mathbf{c} = \mathbf{c} \cdot \oiint_{\mathcal{S}} \mathrm{d}S\ \hat{\mathbf{n}} f \tag{B.184}$$

and because $\mathbf{c}$ is a constant, then the quantities it is dotted into must be equal.

B.5 Leibniz Integral Rules

This summarizes the most commonly used Leibniz integral rules. The original Leibniz integral rule is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} \mathrm{d}t\ f(x,t) = f(x,b(x))\frac{\mathrm{d}b}{\mathrm{d}x} - f(x,a(x))\frac{\mathrm{d}a}{\mathrm{d}x} + \int_{a(x)}^{b(x)} \mathrm{d}t\ \frac{\partial f(x,t)}{\partial x} \tag{B.185}$$

Which can easily be generalized to

$$\frac{\partial}{\partial \mathbf{x}}\int_{a(\mathbf{x})}^{b(\mathbf{x})} \mathrm{d}t\ f(\mathbf{x},t) = f(\mathbf{x},b(\mathbf{x}))\frac{\partial b}{\partial \mathbf{x}} - f(\mathbf{x},a(\mathbf{x}))\frac{\partial a}{\partial \mathbf{x}} + \int_{a(\mathbf{x})}^{b(\mathbf{x})} \mathrm{d}t\ \frac{\partial f(\mathbf{x},t)}{\partial \mathbf{x}} \tag{B.186}$$

$$\nabla\int_{a(\mathbf{x})}^{b(\mathbf{x})} \mathrm{d}t\ f(\mathbf{x},t) = f(\mathbf{x},b(\mathbf{x}))\,\nabla b - f(\mathbf{x},a(\mathbf{x}))\,\nabla a + \int_{a(\mathbf{x})}^{b(\mathbf{x})} \mathrm{d}t\ \nabla f(\mathbf{x},t) \tag{B.187}$$

More typically, we are interested in a time derivative, and so the Leibniz integral rule is written

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} \mathrm{d}x\ f(x,t) = f(b(t),t)\frac{\mathrm{d}b}{\mathrm{d}t} - f(a(t),t)\frac{\mathrm{d}a}{\mathrm{d}t} + \int_{a(t)}^{b(t)} \mathrm{d}x\ \frac{\partial f(x,t)}{\partial t} \tag{B.188}$$

In three dimensional space the formula for integration under the integral is often more complicated with some surprising terms from what we are used to in one dimension. Let's list the new rules.

First, let's look at the time derivative of a line integral with a changing path (in time). I am unaware of any specific name given to this rule. So, for vector field $\mathbf{W}(x,t)$ and time dependent path C_t we find (let $\mathbf{V}(\mathbf{x}(t),t)$ be the time dependent velocity of the path)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}_0(t)}^{\mathbf{x}_1(t)} \mathrm{d}\mathbf{x}\ \cdot\mathbf{W} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_{C_t} \mathrm{d}\mathbf{x}\ \cdot\mathbf{W} = -\int_{C_t} \mathrm{d}\mathbf{x}\ \cdot[\mathbf{V}\times(\boldsymbol{\nabla}\times\mathbf{W})] + \int_{C_t} \mathrm{d}\mathbf{x}\ \cdot\frac{\partial \mathbf{W}}{\partial t} \\ + \mathbf{W}(\mathbf{x}_1(t),t)\cdot\mathbf{V}(\mathbf{x}_1(t),t) - \mathbf{W}(\mathbf{x}_0(t),t)\cdot\mathbf{V}(\mathbf{x}_0(t),t)\end{aligned} \tag{B.189}$$

We can prove this fairly easily by using a parameterization s independent of time

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}_0(t)}^{\mathbf{x}_1(t)} \mathrm{d}\mathbf{x}\ \cdot\mathbf{W} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{s_0}^{s_1} \mathrm{d}s\ \frac{\partial \mathbf{x}}{\partial s}\cdot\mathbf{W}(s,t) = \int_{s_0}^{s_1} \mathrm{d}s\ \left[\frac{\partial \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}}{\partial s}\cdot\mathbf{W}(s,t) + \frac{\partial \mathbf{x}}{\partial s}\cdot\frac{\mathrm{d}\mathbf{W}(s,t)}{\mathrm{d}t}\right] \tag{B.190}$$

The second integral is easy to manipulate. In the first we have by definition that $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{V}$ as $\mathbf{V}$ was defined above. We then find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}_0(t)}^{\mathbf{x}_1(t)} \mathrm{d}\mathbf{x}\ \cdot\mathbf{W} = \int_{s_0}^{s_1} \mathrm{d}s\ \frac{\partial \mathbf{V}}{\partial s}\cdot\mathbf{W}(s,t) + \int_{C_t} \mathrm{d}\mathbf{x}\ \cdot\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} \tag{B.191}$$

$$= \int_{C_t} \mathrm{d}\mathbf{x}\ \cdot\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} + \int_{s_0}^{s_1} \mathrm{d}s\ \frac{\partial(\mathbf{V}\cdot\mathbf{W})}{\partial s} - \int_{s_0}^{s_1} \mathrm{d}s\ \mathbf{V}\cdot\frac{\partial \mathbf{W}}{\partial s} \tag{B.192}$$

$$= \int_{C_t} \mathrm{d}\mathbf{x}\ \cdot\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} + [\mathbf{V}\cdot\mathbf{W}]_{\mathbf{x}_0}^{\mathbf{x}_1} - \int_{C_t} \mathrm{d}\mathbf{x}\ \cdot[\nabla\mathbf{W}\cdot\mathbf{V}] \tag{B.193}$$

where the final integral came from $\mathrm{d}s\,\frac{\partial \mathbf{W}}{\partial s}\cdot\mathbf{V} = \mathrm{d}s\,\frac{\partial \mathbf{x}}{\partial s}\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\cdot\mathbf{V}$. We can then use $\nabla\mathbf{W}\cdot\mathbf{V} = \mathbf{V}\times(\boldsymbol{\nabla}\times\mathbf{W}) + \mathbf{V}\cdot\nabla\mathbf{W}$. And so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}_0(t)}^{\mathbf{x}_1(t)}\mathrm{d}\mathbf{x}\ \cdot\mathbf{W} = \int_{C_t}\mathrm{d}\mathbf{x}\ \cdot\left[\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} - \mathbf{V}\times(\boldsymbol{\nabla}\times\mathbf{W}) - \mathbf{V}\cdot\nabla\mathbf{W}\right] + \mathbf{V}(\mathbf{x}_1(t),t)\cdot\mathbf{W}(\mathbf{x}_1(t),t) - \mathbf{V}(\mathbf{x}_0(t),t)\cdot\mathbf{W}(\mathbf{x}_0(t),t) \tag{B.194}$$

We can then use that $\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} = \frac{\partial \mathbf{W}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{W}$ and so the advective part cancels out and we indeed get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}_0(t)}^{\mathbf{x}_1(t)}\mathrm{d}\mathbf{x}\ \cdot\mathbf{W} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_{C_t}\mathrm{d}\mathbf{x}\ \cdot\mathbf{W} = -\int_{C_t}\mathrm{d}\mathbf{x}\ \cdot[\mathbf{V}\times(\boldsymbol{\nabla}\times\mathbf{W})] + \int_{C_t}\mathrm{d}\mathbf{x}\ \cdot\frac{\partial\mathbf{W}}{\partial t} + \mathbf{W}(\mathbf{x}_1(t),t)\cdot\mathbf{V}(\mathbf{x}_1(t),t) - \mathbf{W}(\mathbf{x}_0(t),t)\cdot\mathbf{V}(\mathbf{x}_0(t),t) \tag{B.195}$$

For the time derivative of a flux integral, we have the Helmholtz transport theorem, which we proved in Chapter 2. See the discussion following (2.7.11). This states for vector field $\mathbf{W}$ with velocity vector describing the change of the boundary $\mathbf{V}$ (as above) over the time changing surface S with normal $\hat{\mathbf{n}}$ and bounding curve C as

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{W} = \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{V}\boldsymbol{\nabla}\cdot\mathbf{W} - \oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot(\mathbf{V}\times\mathbf{W}) + \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\frac{\partial\mathbf{W}}{\partial t} \tag{B.196}$$

Finally, we can consider the time derivative of a volume integral. This is usually called the Reynolds transport theorem. This is discussed in Section 2.9.1.1. For vector field $\mathbf{W}$ with $\mathbf{V}$ describing the change of the surface of the given volume $V(t)$ with surface $S = \partial V(t)$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)}\mathrm{d}^3x\ \mathbf{W} = \iiint_{V(t)}\mathrm{d}^3x\ \left[\boldsymbol{\nabla}\cdot(\mathbf{V}\mathbf{W}) + \frac{\partial\mathbf{W}}{\partial t}\right] = \iiint_{V(t)}\mathrm{d}^3x\ \frac{\partial\mathbf{W}}{\partial t} + \oiint_{\partial V(t)}\mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{V}\mathbf{W} \tag{B.197}$$

We can then summarize our identities as

$$\nabla\int_{a(\mathbf{x})}^{b(\mathbf{x})}\mathrm{d}t\ f(\mathbf{x},t) = f(\mathbf{x},b(\mathbf{x}))\,\nabla b - f(\mathbf{x},a(\mathbf{x}))\,\nabla a + \int_{a(\mathbf{x})}^{b(\mathbf{x})}\mathrm{d}t\ \nabla f(\mathbf{x},t) \tag{B.198}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)}\mathrm{d}x\ f(x,t) = f(b(t),t)\frac{\mathrm{d}b}{\mathrm{d}t} - f(a(t),t)\frac{\mathrm{d}a}{\mathrm{d}t} + \int_{a(t)}^{b(t)}\mathrm{d}x\ \frac{\partial f(x,t)}{\partial t} \tag{B.199}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{C_t}\mathrm{d}\mathbf{x}\ \cdot\mathbf{W} = -\int_{C_t}\mathrm{d}\mathbf{x}\ \cdot[\mathbf{V}\times(\boldsymbol{\nabla}\times\mathbf{W})] + \int_{C_t}\mathrm{d}\mathbf{x}\ \cdot\frac{\partial\mathbf{W}}{\partial t} + \mathbf{W}(\mathbf{x}_1(t),t)\cdot\mathbf{V}(\mathbf{x}_1(t),t) - \mathbf{W}(\mathbf{x}_0(t),t)\cdot\mathbf{V}(\mathbf{x}_0(t),t) \tag{B.200}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{W} = \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{V}\boldsymbol{\nabla}\cdot\mathbf{W} - \oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot(\mathbf{V}\times\mathbf{W}) + \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\frac{\partial\mathbf{W}}{\partial t} \tag{B.201}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)}\mathrm{d}^3x\ \mathbf{W} = \iiint_{V(t)}\mathrm{d}^3x\ \left[\boldsymbol{\nabla}\cdot(\mathbf{V}\mathbf{W}) + \frac{\partial\mathbf{W}}{\partial t}\right] = \iiint_{V(t)}\mathrm{d}^3x\ \frac{\partial\mathbf{W}}{\partial t} + \oiint_{\partial V(t)}\mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{V}\mathbf{W} \tag{B.202}$$

Appendix C

Common Coordinate Conversions

This appendix lists the most useful curvilinear coordinate system properties and transformations. It covers (common) cylindrical coordinates, (plasma) cylindrical coordinates, physicists' spherical coordinates, primitive toroidal coordinates, plasma toroidal coordinates, and general toroidal coordinates.

There are in fact quite a few variations in chosen variables, but I have tried to define a consistent set that are minimally confusing. My common cylindrical coordinates use (r, φ, Z) with r axial distance, φ the azimuthal angle, and Z the axial height. Mathematicians typically use (ρ, θ, z) with ρ axial distance, θ the azimuthal angle, and z the axial height. This notation is fine, but can cause confusion later with spherical coordinates. The plasma toroidal coordinates use (R, Z, ζ) where R is an axial distance, Z is an axial height, and ζ is an azimuthal angle. Note that ζ and φ point in opposite directions so that (R, Z, ζ) and (r, φ, Z) are both right-handed coordinates and the reason for the difference is that the (R, Z, ζ) system can be easily translated into primitive toroidal coordinates ($R \to r$, $Z \to \zeta$, $\zeta \to \theta$). The ISO standard for cylindrical coordinates is (ρ, φ, z).

Physicists' spherical coordinates (r, θ, φ) have r the radius, θ the polar angle, and φ the azimuthal angle. The mathematician's spherical coordinates are also often given by (r, θ, φ) but with θ meaning azimuthal angle and φ the polar angle. This should be avoided as then (r, θ, φ) is not a right-handed system. The logic is that mathematicians' cylindrical uses θ for the azimuth and they want to keep it there. The problems are many because of this lack of uniformity. I will always only use the ISO standard, which is the physicists' notation. Physicists' notation is also the only one consistent with how spherical harmonics are compiled. That is spherical harmonics always use θ as the polar angle, and φ as the azimuthal angle. If you are used to the mathematicians' notation, I would strongly recommend unlearning it and becoming comfortable with the physicists' notation because of the right-handedness and spherical harmonics advantages.[1]

The various toroidal coordinate systems are mostly peculiar to plasma situations, though primitive toroidal coordinates are fairly well known even in mathematics. They use (r, θ, ζ) with r the minor radius, θ is the poloidal angle, and ζ is the toroidal angle. The other types of toroidal coordinates are rarely used, even in plasma physics, and so are listed mostly for completeness.

C.1 Generic Coordinate Conversion

Here let's take a coordinate system, (ξ^1, ξ^2, ξ^3) which can be written out in Cartesian coordinates (x, y, z) and assume we know

$$\xi^1 = \xi^1(x, y, z) \tag{C.1}$$
$$\xi^2 = \xi^2(x, y, z) \tag{C.2}$$
$$\xi^3 = \xi^3(x, y, z) \tag{C.3}$$

and assume it is invertible (In other words the Jacobian determinant $|\boldsymbol{\mathcal{J}}| \neq 0$ for this coordinate system transformation)

$$x = x(\xi^1, \xi^2, \xi^3) \tag{C.4}$$
$$y = y(\xi^1, \xi^2, \xi^3) \tag{C.5}$$
$$z = z(\xi^1, \xi^2, \xi^3) \tag{C.6}$$

So we can then find

$$J = \nabla\xi^1 \cdot \nabla\xi^2 \times \nabla\xi^3 \tag{C.7}$$
$$\mathcal{J} = \frac{1}{\nabla\xi^1 \cdot \nabla\xi^2 \times \nabla\xi^3} \tag{C.8}$$

We can then form the covariant components of the metric tensor $g_{ij} = \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^j}$ with $\mathbf{x} = x\mathbf{x} + y\mathbf{y} + z\mathbf{z}$ a position vector. Note we could write

$$\mathbf{x} = x(\xi^1, \xi^2, \xi^3)\hat{\mathbf{x}} + y(\xi^1, \xi^2, \xi^3)\hat{\mathbf{y}} + z(\xi^1, \xi^2, \xi^3)\hat{\mathbf{z}} \tag{C.9}$$

and then we would have as components

$$g_{11} = \left(\left(\frac{\partial x(\xi^1, \xi^2, \xi^3)}{\partial \xi^1}\right)_{\xi^2, \xi^3}\right)^2 + \left(\left(\frac{\partial y(\xi^1, \xi^2, \xi^3)}{\partial \xi^1}\right)_{\xi^2, \xi^3}\right)^2 + \left(\left(\frac{\partial z(\xi^1, \xi^2, \xi^3)}{\partial \xi^1}\right)_{\xi^2, \xi^3}\right)^2 \tag{C.10}$$

$$g_{11} = \left(\frac{\partial x}{\partial \xi^1}\right)^2 + \left(\frac{\partial y}{\partial \xi^1}\right)^2 + \left(\frac{\partial z}{\partial \xi^1}\right)^2 \tag{C.11}$$

$$g_{i'j'} = \left(\frac{\partial x(\xi^1, \xi^2, \xi^3)}{\partial \xi^{i'}}\right)_{\xi^{j'}, \xi^{k'}} \left(\frac{\partial x(\xi^1, \xi^2, \xi^3)}{\partial \xi^{j'}}\right)_{\xi^{i'}, \xi^{k'}} + \left(\frac{\partial y(\xi^1, \xi^2, \xi^3)}{\partial \xi^{i'}}\right)_{\xi^{j'}, \xi^{k'}} \left(\frac{\partial y(\xi^1, \xi^2, \xi^3)}{\partial \xi^{j'}}\right)_{\xi^{i'}, \xi^{k'}} + \left(\frac{\partial z(\xi^1, \xi^2, \xi^3)}{\partial \xi^{i'}}\right)_{\xi^{j'}, \xi^{k'}} \left(\frac{\partial z(\xi^1, \xi^2, \xi^3)}{\partial \xi^{j'}}\right)_{\xi^{i'}, \xi^{k'}} \tag{C.12}$$

with the i', j', k' not a sum but an even permutation of $1, 2, 3$. Note that g_{11} is the same, but (C.10) explicitly shows the objects held constant.

Note that we would find the tangent vector basis (sometimes called the "covariant" vector basis, but remember this is not a great name) as

$$\mathbf{e}_1 = \mathbf{e}_{\xi^1} = \frac{\partial \mathbf{x}}{\partial \xi^1} \tag{C.13}$$
$$\mathbf{e}_2 = \mathbf{e}_{\xi^2} = \frac{\partial \mathbf{x}}{\partial \xi^2} \tag{C.14}$$
$$\mathbf{e}_3 = \mathbf{e}_{\xi^3} = \frac{\partial \mathbf{x}}{\partial \xi^3} \tag{C.15}$$

with $\mathcal{J} = \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3$. Then the tangent-reciprocal vector basis (again, sometimes called the "contravariant" vector basis, but this is a poor name) as

$$\mathbf{e}^1 = \mathbf{e}^{\xi^1} = \nabla\xi^1 \tag{C.16}$$

$$\mathbf{e}^2 = \mathbf{e}^{\xi^2} = \nabla\xi^2 \tag{C.17}$$

$$\mathbf{e}^3 = \mathbf{e}^{\xi^3} = \nabla\xi^3 \tag{C.18}$$

Remember we can use reciprocal relations so that [with (i', j', k') an even cyclic permutation of $(1, 2, 3)$]

$$\mathbf{e}^{i'} = \nabla\xi^{i'} = \frac{\mathbf{e}_{j'} \times \mathbf{e}_{k'}}{\mathbf{e}_{i'} \cdot \mathbf{e}_{j'} \times \mathbf{e}_{k'}} = \frac{\mathbf{e}_{j'} \times \mathbf{e}_{k'}}{\mathcal{J}} = \frac{\frac{\partial \mathbf{x}}{\partial \xi^{j'}} \times \frac{\partial \mathbf{x}}{\partial \xi^{k'}}}{\frac{\partial \mathbf{x}}{\partial \xi^{i'}} \cdot \left(\frac{\partial \mathbf{x}}{\partial \xi^{j'}} \times \frac{\partial \mathbf{x}}{\partial \xi^{k'}} \right)} \tag{1.2.64}$$

$$\mathbf{e}_{i'} = \frac{\partial \mathbf{x}}{\partial \xi^{i'}} = \frac{\mathbf{e}^{j'} \times \mathbf{e}^{k'}}{\mathbf{e}^{i'} \cdot \mathbf{e}^{j'} \times \mathbf{e}^{k'}} = \mathcal{J}\mathbf{e}^{j'} \times \mathbf{e}^{k'} = \frac{\nabla\xi^{j'} \times \nabla\xi^{k'}}{\nabla\xi^{i'} \cdot \nabla\xi^{j'} \times \nabla\xi^{k'}} \tag{1.2.65}$$

We can then form the contravariant components of the metric tensor $g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j = \nabla\xi^i \cdot \nabla\xi^j$. We can define $x^1 = x$, $x^2 = y$, and $x^3 = z$ for convenience, as well. Note we could write

$$g^{11} = \left(\left(\frac{\partial \xi^1(x, y, z)}{\partial x} \right)_{y,z} \right)^2 + \left(\left(\frac{\partial \xi^1(x, y, z)}{\partial y} \right)_{z,x} \right)^2 + \left(\left(\frac{\partial \xi^1(x, y, z)}{\partial z} \right)_{x,y} \right)^2 \tag{C.19}$$

$$g^{11} = \left(\frac{\partial \xi^1}{\partial x} \right)^2 + \left(\frac{\partial \xi^1}{\partial y} \right)^2 + \left(\frac{\partial \xi^1}{\partial z} \right)^2 \tag{C.20}$$

$$\begin{aligned} g^{i'j'} = &\left(\frac{\partial \xi^1(x^1, x^2, x^3)}{\partial x^{i'}} \right)_{x^{j'}, x^{k'}} \left(\frac{\partial \xi^1(x^1, x^2, x^3)}{\partial x^{j'}} \right)_{x^{i'}, x^{k'}} + \left(\frac{\partial \xi^2(x^1, x^2, x^3)}{\partial x^{i'}} \right)_{x^{j'}, x^{k'}} \left(\frac{\partial \xi^2(x^1, x^2, x^3)}{\partial x^{j'}} \right)_{x^{i'}, x^{k'}} \\ &+ \left(\frac{\partial \xi^3(x^1, x^2, x^3)}{\partial x^{i'}} \right)_{x^{j'}, x^{k'}} \left(\frac{\partial \xi^3(x^1, x^2, x^3)}{\partial x^{j'}} \right)_{x^{i'}, x^{k'}} \end{aligned} \tag{C.21}$$

with the i', j', k' not a sum but an even permutation of $1, 2, 3$.

Finally, I will list the Christoffel symbols via

$$\Gamma_{k,ij} = \frac{1}{2} \left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k} \right] \tag{C.22}$$

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{C.23}$$

and list the Christoffel symbols one at a time as a matrix. Thus $\Gamma^{k'}_{ij}$ is listed for each k' as a matrix $\mathbf{M}$ with entries M_{ij} given by $\Gamma^{k'}_{ij}$.

C.2 (Common) Cylindrical Coordinates

We have Cartesian (x, y, z) and cylindrical (r, φ, Z) as our two coordinate systems. ($0 \leq r < \infty$, $0 \leq \varphi \leq 2\pi$, and $-\infty < Z < \infty$)

We use the equations

$$r^2 = x^2 + y^2 \tag{C.1}$$
$$\tan\varphi = \frac{y}{x} \tag{C.2}$$
$$Z = z \tag{C.3}$$

Thus, we find

$$\begin{aligned} r\,\mathrm{d}r &= x\,\mathrm{d}x + y\,\mathrm{d}y \\ \mathrm{d}r &= \frac{x}{r}\,\mathrm{d}x + \frac{y}{r}\,\mathrm{d}y = \cos\varphi\,\mathrm{d}x + \sin\varphi\,\mathrm{d}y \end{aligned} \tag{C.4}$$

$$\begin{aligned} \sec^2\varphi\,\mathrm{d}\varphi &= \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} \\ \mathrm{d}\varphi &= \cos^2\varphi\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} = \frac{x^2}{x^2+y^2}\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = \frac{\cos\varphi}{r}\,\mathrm{d}y - \frac{\sin\varphi}{r}\,\mathrm{d}x \end{aligned} \tag{C.5}$$

$$\mathrm{d}Z = \mathrm{d}z \tag{C.6}$$

and so

$$\mathbf{J} = \frac{\partial(r,\varphi,Z)}{\partial(x,y,z)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\frac{\sin\varphi}{r} & \frac{\cos\varphi}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{C.7}$$

$$J = \frac{\cos\varphi\cos\varphi}{r} - \frac{\sin\varphi\sin\varphi}{r} = \frac{1}{r} \tag{C.8}$$

Note that we then have

$$\mathbf{e}^1 = \mathbf{e}^r = \nabla r = \cos\varphi\,\nabla x + \sin\varphi\,\nabla y \tag{C.9}$$
$$|\nabla r| = 1 \tag{C.10}$$
$$\mathbf{e}^2 = \mathbf{e}^\varphi = \nabla\varphi = -\frac{\sin\varphi}{r}\nabla x + \frac{\cos\varphi}{r}\nabla y \tag{C.11}$$
$$|\nabla\varphi| = \sqrt{\frac{\sin^2\varphi + \cos^2\varphi}{r^2}} = \frac{1}{r} \tag{C.12}$$
$$\mathbf{e}^3 = \mathbf{e}^Z = \nabla Z = \nabla z \tag{C.13}$$
$$|\nabla Z| = 1 \tag{C.14}$$

So that

$$\hat{\mathbf{e}}^1 = \hat{\mathbf{e}}^r = \hat{\mathbf{r}} = \cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} \tag{C.15}$$
$$\hat{\mathbf{e}}^2 = \hat{\mathbf{e}}^\varphi = \hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}} = -\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} \tag{C.16}$$
$$\hat{\mathbf{e}}^3 = \hat{\mathbf{e}}^Z = \hat{\mathbf{Z}} = \hat{\mathbf{z}} \tag{C.17}$$

The metric tensor is given by $g^{ij} = \sum_{k=1}^{3} \frac{\partial \xi^i}{\partial x^k} \frac{\partial \xi^j}{\partial x^k}$. Thus

$$\begin{aligned} g^{rr} &= \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2 \\ &= \cos^2\varphi + \sin^2\varphi + 0^2 = 1 \end{aligned} \tag{C.18}$$

$$\begin{aligned} g^{\varphi\varphi} &= \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \\ &= \frac{\sin^2\varphi}{r^2} + \frac{\cos^2\varphi}{r^2} + 0 = \frac{1}{r^2} \end{aligned} \tag{C.19}$$

$$\begin{aligned} g^{ZZ} &= \left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2 \\ &= 0 + 0 + 1 = 1 \end{aligned} \tag{C.20}$$

$$\begin{aligned} g^{r\varphi} &= \frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \varphi}{\partial z} \\ &= \cos\varphi\frac{-\sin\varphi}{r} + \sin\varphi\frac{\cos\varphi}{r} + 0 = 0 \end{aligned} \tag{C.21}$$

$$\begin{aligned} g^{rZ} &= \frac{\partial r}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial Z}{\partial z} \\ &= \cos\varphi(0) + \sin\varphi(0) + 0(1) = 0 \end{aligned} \tag{C.22}$$

$$\begin{aligned} g^{\varphi Z} &= \frac{\partial \varphi}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial \varphi}{\partial z}\frac{\partial Z}{\partial z} \\ &= \frac{-\sin\varphi}{r}(0) + \frac{\cos\varphi}{r}(0) + 0(1) = 0 \end{aligned} \tag{C.23}$$

Thus

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{C.24}$$

In the other direction we would use

$$x = r\cos\varphi \tag{C.25}$$

$$y = r\sin\varphi \tag{C.26}$$

$$z = Z \tag{C.27}$$

and so

$$\mathrm{d}x = \cos\varphi\,\mathrm{d}r - r\sin\varphi\,\mathrm{d}\varphi \tag{C.28}$$

$$\mathrm{d}y = \sin\varphi\,\mathrm{d}r + r\cos\varphi\,\mathrm{d}\varphi \tag{C.29}$$

$$\mathrm{d}z = \mathrm{d}Z \tag{C.30}$$

$$\mathbf{e}_1 = \mathbf{e}_r = \left(\frac{\partial \mathbf{x}}{\partial r}\right)_{\theta,\varphi} = \cos\varphi\sin\theta\,\nabla x + \sin\varphi\sin\theta\,\nabla y + \cos\theta \tag{C.31}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = r\cos\varphi\cos\theta\,\nabla x + r\sin\varphi\cos\theta\,\nabla y - r\sin\theta\,\nabla z \tag{C.32}$$

$$\mathbf{e}_3 = \mathbf{e}_\varphi = \frac{\partial \mathbf{x}}{\partial \varphi} = -r\sin\varphi\sin\theta\,\nabla x + r\cos\varphi\sin\theta\,\nabla y \tag{C.33}$$

and so we then have

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(r,\varphi,Z)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial Z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{C.34}$$

$$\mathcal{J} = r\cos\varphi\cos\varphi + r\sin\varphi\sin\varphi = r \tag{C.35}$$

Note that we then have

$$\hat{\mathbf{x}} = \cos\varphi\,\nabla r - r\sin\varphi\,\nabla\varphi = \cos\varphi\hat{\mathbf{r}} - \sin\varphi\hat{\boldsymbol{\varphi}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{r}} - \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\varphi}} \tag{C.36}$$

$$\hat{\mathbf{y}} = \sin\varphi\,\nabla r + r\cos\varphi\,\nabla\varphi = \sin\varphi\hat{\mathbf{r}} + \cos\varphi\hat{\boldsymbol{\varphi}} = \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{r}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\varphi}} \tag{C.37}$$

$$\hat{\mathbf{z}} = \nabla Z = \hat{\mathbf{Z}} \tag{C.38}$$

The other metric tensor is given by $g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}$. Thus

$$\begin{aligned} g_{rr} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \\ &= \cos^2\varphi + \sin^2\varphi + 0^2 = 1 \end{aligned} \tag{C.39}$$

$$\begin{aligned} g_{\varphi\varphi} &= \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 \\ &= r^2\sin^2\varphi + r^2\cos^2\varphi + 0 = r^2 \end{aligned} \tag{C.40}$$

$$\begin{aligned} g_{ZZ} &= \left(\frac{\partial x}{\partial Z}\right)^2 + \left(\frac{\partial y}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 \\ &= 0 + 0 + 1 = 1 \end{aligned} \tag{C.41}$$

$$\begin{aligned} g_{r\varphi} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \varphi} \\ &= \cos\varphi(-r\sin\varphi) + \sin\varphi(r\cos\varphi) + 0 = 0 \end{aligned} \tag{C.42}$$

$$\begin{aligned} g_{rZ} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial Z} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial Z} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial Z} \\ &= \cos\varphi(0) + \sin\varphi(0) + 0(1) = 0 \end{aligned} \tag{C.43}$$

$$\begin{aligned} g_{\varphi Z} &= \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial Z} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial Z} + \frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial Z} \\ &= -r\sin\varphi(0) + r\cos\varphi(0) + 0(1) = 0 \end{aligned} \tag{C.44}$$

Thus

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{C.45}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right] \tag{C.46}$$

$$\Gamma_{r,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.47}$$

$$\Gamma_{\varphi,ij} = \begin{bmatrix} 0 & r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.48}$$

$$\Gamma_{Z,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.49}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{C.50}$$

$$\Gamma^r_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.51}$$

$$\Gamma^\varphi_{ij} = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.52}$$

$$\Gamma^Z_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.53}$$

C.3 (Plasma/Toroidal System) Cylindrical Coordinates

We have Cartesian (x, y, z) and cylindrical (R, Z, ζ) as our two coordinate systems. ($0 \le R < \infty$, $-\infty < Z < \infty$, and $0 \le \zeta \le 2\pi$)

We use the equations

$$R^2 = x^2 + y^2 \tag{C.1}$$

$$\tan(-\zeta) = \frac{y}{x} \tag{C.2}$$

$$Z = z \tag{C.3}$$

Thus, we find

$$\begin{aligned} R\,\mathrm{d}R &= x\,\mathrm{d}x + y\,\mathrm{d}y \\ \mathrm{d}R &= \frac{x}{R}\,\mathrm{d}x + \frac{y}{R}\,\mathrm{d}y = \cos(-\zeta)\,\mathrm{d}x + \sin(-\zeta)\,\mathrm{d}y = \cos\zeta\,\mathrm{d}x - \sin\zeta\,\mathrm{d}y \end{aligned} \tag{C.4}$$

$$\begin{aligned} -\sec^2\zeta\,\mathrm{d}\zeta &= \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} \\ \mathrm{d}\zeta &= \cos^2\zeta\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2} = \frac{x^2}{x^2+y^2}\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2} = \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2+y^2} \\ &= \frac{\sin(-\zeta)}{R}\,\mathrm{d}x - \frac{\cos(-\zeta)}{R}\,\mathrm{d}y = -\frac{\sin\zeta}{R}\,\mathrm{d}x - \frac{\cos\zeta}{R}\,\mathrm{d}y \end{aligned} \tag{C.5}$$

$$\mathrm{d}Z = \mathrm{d}z \tag{C.6}$$

and so

$$\mathbf{J} = \frac{\partial(R,Z,\zeta)}{\partial(x,y,z)} = \begin{bmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\zeta & -\sin\zeta & 0 \\ 0 & 0 & 1 \\ -\frac{\sin\zeta}{R} & -\frac{\cos\zeta}{R} & 0 \end{bmatrix} \tag{C.7}$$

$$J = -(\frac{-\cos\zeta\cos\zeta}{R} - \frac{\sin\zeta\sin\zeta}{R}) = \frac{1}{R} \tag{C.8}$$

Note that we then have

$$\mathbf{e}^1 = \mathbf{e}^R = \nabla R = \cos\zeta\,\nabla x - \sin\zeta\,\nabla y \tag{C.9}$$

$$|\nabla R| = 1 \tag{C.10}$$

$$\mathbf{e}^2 = \mathbf{e}^\zeta = \nabla\zeta = -\frac{\sin\zeta}{R}\nabla x - \frac{\cos\zeta}{R}\nabla y \tag{C.11}$$

$$|\nabla\zeta| = \sqrt{\frac{\sin^2\zeta + \cos^2\zeta}{R^2}} = \frac{1}{R} \tag{C.12}$$

$$\mathbf{e}^3 = \mathbf{e}^Z = \nabla Z = \nabla z \tag{C.13}$$

$$|\nabla Z| = 1 \tag{C.14}$$

So that

$$\hat{\mathbf{e}}^1 = \hat{\mathbf{e}}^R = \hat{\mathbf{R}} = \cos\zeta\hat{\mathbf{x}} - \sin\zeta\hat{\mathbf{y}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} \tag{C.15}$$

$$\hat{\mathbf{e}}^2 = \hat{\mathbf{e}}^\zeta = \hat{\boldsymbol{\zeta}} = -\sin\zeta\hat{\mathbf{x}} - \cos\zeta\hat{\mathbf{y}} = \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} - \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} \tag{C.16}$$

$$\hat{\mathbf{e}}^3 = \hat{\mathbf{e}}^Z = \hat{\mathbf{Z}} = \hat{\mathbf{z}} \tag{C.17}$$

The metric tensor is given by $g^{ij} = \sum_{k=1}^{3} \frac{\partial\xi^i}{\partial x^k}\frac{\partial\xi^j}{\partial x^k}$. Thus

$$\begin{aligned} g^{RR} &= \left(\frac{\partial R}{\partial x}\right)^2 + \left(\frac{\partial R}{\partial y}\right)^2 + \left(\frac{\partial R}{\partial z}\right)^2 \\ &= \cos^2\zeta + \sin^2\zeta + 0^2 = 1 \end{aligned} \tag{C.18}$$

$$\begin{aligned} g^{ZZ} &= \left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2 \\ &= 0 + 0 + 1 = 1 \end{aligned} \tag{C.19}$$

$$\begin{aligned} g^{\zeta\zeta} &= \left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2 \\ &= \frac{\sin^2\theta}{R^2} + \frac{\cos^2\zeta}{R^2} + 0 = \frac{1}{R^2} \end{aligned} \tag{C.20}$$

$$\begin{aligned} g^{RZ} &= \frac{\partial R}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial R}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial Z}{\partial z} \\ &= \cos\zeta(0) + \sin\zeta(0) + 0(1) = 0 \end{aligned} \tag{C.21}$$

$$\begin{aligned} g^{R\zeta} &= \frac{\partial R}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial R}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial \zeta}{\partial z} \\ &= \cos\zeta\frac{-\sin\zeta}{R} - \sin\zeta\frac{-\cos\zeta}{R} + 0 = 0 \end{aligned} \tag{C.22}$$

$$\begin{aligned} g^{Z\zeta} &= \frac{\partial Z}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial Z}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial Z}{\partial z}\frac{\partial \zeta}{\partial z} \\ &= (0)\frac{-\sin\zeta}{R} + (0)\frac{-\cos\varphi}{R} + (1)0 = 0 \end{aligned} \tag{C.23}$$

Thus

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{R^2} \end{bmatrix} \tag{C.24}$$

In the other direction we would use

$$x = R\cos\zeta \tag{C.25}$$

$$y = -R\sin\zeta \tag{C.26}$$

$$z = Z \tag{C.27}$$

$$\mathbf{e}_1 = \mathbf{e}_r = \left(\frac{\partial \mathbf{x}}{\partial R}\right)_{Z,\zeta} = \cos\zeta\,\nabla x - \sin\zeta\,\nabla y \tag{C.28}$$

$$\mathbf{e}_2 = \mathbf{e}_Z = \frac{\partial \mathbf{x}}{\partial Z} = \nabla z \tag{C.29}$$

$$\mathbf{e}_3 = \mathbf{e}_\zeta = \frac{\partial \mathbf{x}}{\partial \zeta} = -R\sin\zeta\,\nabla x - R\cos\zeta\,\nabla y \tag{C.30}$$

and so

$$\mathrm{d}x = \cos\zeta\,\mathrm{d}R - R\sin\zeta\,\mathrm{d}\zeta \tag{C.31}$$

$$\mathrm{d}y = -\sin\zeta\,\mathrm{d}R - R\cos\zeta\,\mathrm{d}\zeta \tag{C.32}$$

$$\mathrm{d}z = \mathrm{d}Z \tag{C.33}$$

and so we then have

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(R,Z,\zeta)} = \begin{bmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial Z} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial Z} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial Z} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \cos\zeta & 0 & -R\sin\zeta \\ -\sin\zeta & 0 & -R\cos\zeta \\ 0 & 1 & 0 \end{bmatrix} \tag{C.34}$$

$$\mathcal{J} = -(-R\cos\zeta\cos\zeta + R\sin\zeta\sin\zeta) = R \tag{C.35}$$

Note that we then have

$$\hat{\mathbf{x}} = \cos\zeta\,\nabla R - R\sin\zeta\,\nabla\zeta = \cos\zeta\hat{\mathbf{R}} - \sin\zeta\hat{\boldsymbol{\zeta}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{R}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\zeta}} \tag{C.36}$$

$$\hat{\mathbf{y}} = -\sin\zeta\,\nabla R - R\cos\zeta\,\nabla\zeta = -\sin\zeta\hat{\mathbf{R}} - \cos\zeta\hat{\boldsymbol{\zeta}} = \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{R}} - \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\zeta}} \tag{C.37}$$

$$\hat{\mathbf{z}} = \nabla Z = \hat{\mathbf{Z}} \tag{C.38}$$

The other metric tensor is given by $g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^k}{\partial \xi^j}$. Thus

$$g_{RR} = \left(\frac{\partial x}{\partial R}\right)^2 + \left(\frac{\partial y}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2 \tag{C.39}$$
$$= \cos^2\zeta + \sin^2\zeta + 0^2 = 1$$
$$g_{ZZ} = \left(\frac{\partial x}{\partial Z}\right)^2 + \left(\frac{\partial y}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 \tag{C.40}$$
$$= 0 + 0 + 1 = 1$$
$$g_{\zeta\zeta} = \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2 \tag{C.41}$$
$$= R^2\sin^2\zeta + R^2\cos^2\zeta + 0 = R^2$$
$$g_{RZ} = \frac{\partial x}{\partial R}\frac{\partial x}{\partial Z} + \frac{\partial y}{\partial R}\frac{\partial y}{\partial Z} + \frac{\partial z}{\partial R}\frac{\partial z}{\partial Z} \tag{C.42}$$
$$= \cos\zeta(0) + (-\sin\zeta)(0) + 0(1) = 0$$
$$g_{R\zeta} = \frac{\partial x}{\partial R}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial R}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial R}\frac{\partial z}{\partial \zeta} \tag{C.43}$$
$$= \cos\varphi(-R\sin\zeta) - \sin\varphi(-R\cos\zeta) + 0 = 0$$
$$g_{Z\zeta} = \frac{\partial x}{\partial Z}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial Z}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial Z}\frac{\partial z}{\partial \zeta} \tag{C.44}$$
$$= (0)(-R\sin\zeta) + (0)(-R\cos\zeta) + (1)0 = 0$$

Thus

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^2 \end{bmatrix} \tag{C.45}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right] \tag{C.46}$$
$$\Gamma_{R,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -R \end{bmatrix} \tag{C.47}$$
$$\Gamma_{Z,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.48}$$
$$\Gamma_{\zeta,ij} = \begin{bmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ R & 0 & 0 \end{bmatrix} \tag{C.49}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{C.50}$$

$$\Gamma^R_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -R \end{bmatrix} \tag{C.51}$$

$$\Gamma^Z_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.52}$$

$$\Gamma^\zeta_{ij} = \begin{bmatrix} 0 & 0 & \frac{1}{R} \\ 0 & 0 & 0 \\ \frac{1}{R} & 0 & 0 \end{bmatrix} \tag{C.53}$$

C.4 (Physicists') Spherical Coordinates

We have Cartesian (x, y, z) and spherical (r, θ, φ) as our two coordinate systems. ($0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$)

We use the equations

$$r^2 = x^2 + y^2 + z^2 \tag{C.1}$$

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z} \Leftrightarrow \cos\theta = \frac{z}{r} \tag{C.2}$$

$$\tan\varphi = \frac{y}{x} \tag{C.3}$$

Thus, we find (using $\frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}\frac{\sqrt{x^2+y^2}}{r} = \cos\varphi\sin\theta$ and similarly for y and that $z = r\cos\theta$ so that $\sqrt{x^2+y^2} = r\sin\theta$)

$$\begin{aligned} r\,\mathrm{d}r &= x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z \\ \mathrm{d}r &= \frac{x}{r}\,\mathrm{d}x + \frac{y}{r}\,\mathrm{d}y = \frac{z}{r}\,\mathrm{d}z = \cos\varphi\sin\theta\,\mathrm{d}x + \sin\varphi\sin\theta\,\mathrm{d}y + \cos\theta\,\mathrm{d}z \end{aligned} \tag{C.4}$$

$$\begin{aligned} \sec^2\theta\,\mathrm{d}\theta &= \frac{\frac{zx\,\mathrm{d}x + zy\,\mathrm{d}y}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}\,\mathrm{d}z}{z^2} = \frac{x}{z\sqrt{x^2+y^2}}\,\mathrm{d}x + \frac{y}{z\sqrt{x^2+y^2}}\,\mathrm{d}y - \frac{\sqrt{x^2+y^2}}{z^2}\,\mathrm{d}z \\ \mathrm{d}\theta &= \frac{zx}{r^2\sqrt{x^2+y^2}}\,\mathrm{d}x + \frac{zy}{r^2\sqrt{x^2+y^2}}\,\mathrm{d}y - \frac{\sqrt{x^2+y^2}}{r^2}\,\mathrm{d}z \\ \mathrm{d}\theta &= \frac{(r\cos\theta)(r\sin\theta\cos\varphi)}{r^3\sin\theta}\,\mathrm{d}x + \frac{(r\cos\theta)(r\sin\theta\sin\varphi)}{r^3\sin\theta}\,\mathrm{d}y - \frac{r\sin\theta}{r^2}\,\mathrm{d}z \\ &= \frac{\cos\varphi\cos\theta}{r}\,\mathrm{d}x + \frac{\sin\varphi\cos\theta}{r}\,\mathrm{d}y - \frac{\sin\theta}{r}\,\mathrm{d}z \end{aligned} \tag{C.5}$$

$$\begin{aligned} \sec^2\varphi\,\mathrm{d}\varphi &= \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} \\ \mathrm{d}\varphi &= \cos^2\varphi\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} = \frac{x^2}{x^2+y^2}\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = -\frac{\sin\varphi}{r\sin\theta}\,\mathrm{d}x + \frac{\cos\varphi}{r\sin\theta}\,\mathrm{d}y \end{aligned} \tag{C.6}$$

and so

$$\mathbf{J} = \frac{\partial(r,\theta,\varphi)}{\partial(x,y,z)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \frac{\cos\varphi\cos\theta}{r} & \frac{\sin\varphi\cos\theta}{r} & -\frac{\sin\theta}{r} \\ \frac{-\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{bmatrix} \tag{C.7}$$

$$J = \frac{\sin\theta}{r}\left(\cos\varphi\sin\theta\frac{\cos\varphi}{r\sin\theta} - \sin\varphi\sin\theta\frac{-\sin\varphi}{r\sin\theta}\right) + \cos\theta\left(\frac{\cos\varphi\cos\theta}{r}\frac{\cos\varphi}{r\sin\theta} - \frac{\sin\varphi\cos\theta}{r}\frac{-\sin\varphi}{r\sin\theta}\right) \tag{C.8}$$

$$= \frac{\sin\theta}{r}\frac{\cos^2\varphi+\sin^2\varphi}{r} + \frac{\cos^2\theta}{r^2\sin\theta}\left(\cos^2\varphi+\sin^2\varphi\right) = \frac{\sin^2\theta+\cos^2\theta}{r^2\sin\theta} = \frac{1}{r^2\sin\theta} \tag{C.9}$$

Note that we then have

$$\mathbf{e}^1 = \mathbf{e}^r = \nabla r = \cos\varphi\sin\theta\,\nabla x + \sin\varphi\sin\theta\,\nabla y + \cos\theta\,\nabla z \tag{C.10}$$

$$|\nabla r| = 1 \tag{C.11}$$

$$\mathbf{e}^2 = \mathbf{e}^\theta = \nabla\theta = \frac{\cos\varphi\cos\theta}{r}\nabla x + \frac{\sin\varphi\cos\theta}{r}\nabla y - \frac{\sin\theta}{r}\nabla z \tag{C.12}$$

$$|\nabla\theta| = \sqrt{\frac{(\cos^2\varphi+\sin^2\varphi)\cos^2\theta+\sin^2\theta}{r^2}} = \sqrt{\frac{1}{r^2}} = \frac{1}{r} \tag{C.13}$$

$$\mathbf{e}^3 = \mathbf{e}^\varphi = \nabla\varphi = -\frac{\sin\varphi}{r\sin\theta}\nabla x + \frac{\cos\varphi}{r\sin\theta}\nabla y \tag{C.14}$$

$$|\nabla\varphi| = \sqrt{\frac{\sin^2\varphi+\cos^2\varphi}{r^2\sin^2\theta}} = \sqrt{\frac{1}{r^2\sin^2\theta}} = \frac{1}{r\sin\theta} \tag{C.15}$$

So that

$$\hat{\mathbf{e}}^1 = \hat{\mathbf{e}}^r = \hat{\mathbf{r}} = \cos\varphi\sin\theta\hat{\mathbf{x}} + \sin\varphi\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \tag{C.16}$$

$$\hat{\mathbf{e}}^2 = \hat{\mathbf{e}}^\theta = \hat{\boldsymbol{\theta}} = \cos\varphi\cos\theta\hat{\mathbf{x}} + \sin\varphi\cos\theta\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \tag{C.17}$$

$$\hat{\mathbf{e}}^3 = \hat{\mathbf{e}}^\varphi = \hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}} \tag{C.18}$$

$$\tag{C.19}$$

The metric tensor is given by $g^{ij} = \sum_{k=1}^{3} \frac{\partial\xi^i}{\partial x^k}\frac{\partial\xi^j}{\partial x^k}$. Thus

$$\begin{aligned} g^{rr} &= \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2 \\ &= \cos^2\varphi\sin^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\theta = 1 \end{aligned} \tag{C.20}$$

$$\begin{aligned} g^{\theta\theta} &= \left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 \\ &= \frac{\cos^2\varphi\cos^2\theta}{r^2} + \frac{\sin^2\varphi\cos^2\theta}{r^2} + \frac{\sin^2\theta}{r^2} = \frac{1}{r^2} \end{aligned} \tag{C.21}$$

$$\begin{aligned} g^{\varphi\varphi} &= \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \\ &= \frac{\sin^2\varphi}{r^2\sin^2\theta} + \frac{\cos^2\varphi}{r^2\sin^2\theta} = \frac{1}{r^2\sin^2\theta} \end{aligned} \tag{C.22}$$

$$\begin{aligned} g^{r\theta} &= \frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \theta}{\partial z} \\ &= \cos\varphi\sin\theta\frac{\cos\varphi\cos\theta}{r} + \sin\varphi\sin\theta\frac{\sin\varphi\cos\theta}{r} + \cos\theta\frac{-\sin\theta}{r} = 0 \end{aligned} \tag{C.23}$$

$$\begin{aligned} g^{r\varphi} &= \frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \varphi}{\partial z} \\ &= \cos\varphi\sin\theta\frac{-\sin\varphi}{r\sin\theta} + \sin\varphi\sin\theta\frac{\cos\varphi}{r\sin\theta} + \cos\theta(0) = 0 \end{aligned} \tag{C.24}$$

$$\begin{aligned} g^{\theta\varphi} &= \frac{\partial \theta}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \theta}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial \theta}{\partial z}\frac{\partial \varphi}{\partial z} \\ &= \frac{\cos\varphi\cos\theta}{r}\frac{-\sin\varphi}{r\sin\theta} + \frac{\sin\varphi\cos\theta}{r}\frac{\cos\varphi}{r\sin\theta} + \frac{-\sin\theta}{r}(0) = 0 \end{aligned} \tag{C.25}$$

Thus

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix} \tag{C.26}$$

In the other direction we would use

$$x = r\cos\varphi\sin\theta \tag{C.27}$$

$$y = r\sin\varphi\sin\theta \tag{C.28}$$

$$z = r\cos\theta \tag{C.29}$$

$$\mathbf{e}_1 = \mathbf{e}_r = \left(\frac{\partial \mathbf{x}}{\partial r}\right)_{\theta,\varphi} = \cos\varphi\sin\theta\,\nabla x - \sin\varphi\sin\theta\,\nabla y \tag{C.30}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = r\cos\varphi\cos\theta\,\nabla x + r\sin\varphi\cos\theta\,\nabla y - r\sin\theta\,\nabla z \tag{C.31}$$

$$\mathbf{e}_3 = \mathbf{e}_\varphi = \frac{\partial \mathbf{x}}{\partial \varphi} = -r\sin\varphi\sin\theta\,\nabla x + r\cos\varphi\sin\theta\,\nabla y \tag{C.32}$$

and so

$$\mathrm{d}x = \cos\varphi\sin\theta\,\mathrm{d}r + r\cos\varphi\cos\theta\,\mathrm{d}\theta - r\sin\varphi\sin\theta\,\mathrm{d}\varphi \tag{C.33}$$

$$\mathrm{d}y = \sin\varphi\sin\theta\,\mathrm{d}r + r\sin\varphi\cos\theta\,\mathrm{d}\theta + r\cos\varphi\sin\theta\,\mathrm{d}\varphi \tag{C.34}$$

$$\mathrm{d}z = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta \tag{C.35}$$

and so we then have

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \cos\varphi\sin\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} \tag{C.36}$$

$$\begin{aligned} \mathcal{J} &= \cos\theta\left((r\cos\varphi\cos\theta)(r\cos\varphi\sin\theta) - (-r\sin\varphi\sin\theta)(r\sin\varphi\cos\theta)\right) \\ &\quad - -r\sin\theta\left((\cos\varphi\sin\theta)(r\cos\varphi\sin\theta) - (-r\sin\varphi\sin\theta)(\sin\varphi\sin\theta)\right) \\ &= r^2\cos^2\theta\sin\theta + r^2\sin^3\theta = r^2\sin\theta \end{aligned} \tag{C.37}$$

Note that we then have

$$\begin{aligned}\hat{\mathbf{x}} &= \cos\varphi\sin\theta\,\nabla r + r\cos\varphi\cos\theta\,\nabla\theta - r\sin\varphi\sin\theta\,\nabla\varphi \\ &= \cos\varphi\sin\theta\hat{\mathbf{r}} + \cos\varphi\cos\theta\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}} \\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} + \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}} - \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\varphi}}\end{aligned} \tag{C.38}$$

$$\begin{aligned}\hat{\mathbf{y}} &= \sin\varphi\sin\theta\,\nabla r + r\sin\varphi\cos\theta\,\nabla\theta + r\cos\varphi\sin\theta\,\nabla\varphi \\ &= \sin\varphi\sin\theta\hat{\mathbf{r}} + \sin\varphi\cos\theta\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}} \\ &= \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} + \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\varphi}}\end{aligned} \tag{C.39}$$

$$\begin{aligned}\hat{\mathbf{z}} &= \cos\theta\,\nabla r - r\sin\theta\,\nabla\theta = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \\ &= \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} - \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}}\end{aligned} \tag{C.40}$$

The other metric tensor is given by $g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}$. Thus

$$\begin{aligned}g_{rr} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \\ &= \cos^2\varphi\sin^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\theta = 1\end{aligned} \tag{C.41}$$

$$\begin{aligned}g_{\theta\theta} &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \\ &= r^2\cos^2\varphi\cos^2\theta + r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\theta = r^2\end{aligned} \tag{C.42}$$

$$\begin{aligned}g_{\varphi\varphi} &= \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 \\ &= r^2\sin^2\varphi\sin^2\theta + r^2\cos^2\varphi\sin^2\theta = r^2\sin^2\theta\end{aligned} \tag{C.43}$$

$$\begin{aligned}g_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\ &= \cos\varphi\sin\theta(r\cos\varphi\cos\theta) + \sin\varphi\sin\theta(r\sin\varphi\cos\theta) + \cos\theta(-r\sin\theta) = 0\end{aligned} \tag{C.44}$$

$$\begin{aligned}g_{r\varphi} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \varphi} \\ &= \cos\varphi\sin\theta(-r\sin\varphi\sin\theta) + \sin\varphi\sin\theta(r\cos\varphi\sin\theta) + \cos\theta(0) = 0\end{aligned} \tag{C.45}$$

$$\begin{aligned}g_{\theta\varphi} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \varphi} \\ &= r\cos\varphi\cos\theta(-r\sin\varphi\sin\theta) + r\sin\varphi\cos\theta(r\cos\varphi\sin\theta) + -r\sin\theta(0) = 0\end{aligned} \tag{C.46}$$

Thus

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix} \tag{C.47}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right] \tag{C.48}$$

$$\Gamma_{r,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{bmatrix} \tag{C.49}$$

$$\Gamma_{\theta,ij} = \begin{bmatrix} 0 & r & 0 \\ r & 0 & 0 \\ 0 & 0 & r^2\sin\theta\cos\theta \end{bmatrix} \tag{C.50}$$

$$\Gamma_{\varphi,ij} = \begin{bmatrix} 0 & 0 & r\sin^2\theta \\ 0 & 0 & r^2\sin\theta\cos\theta \\ r\sin^2\theta & r^2\sin\theta\cos\theta & 0 \end{bmatrix} \tag{C.51}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{C.52}$$

$$\Gamma^r_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{bmatrix} \tag{C.53}$$

$$\Gamma^\theta_{ij} = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{C.54}$$

$$\Gamma^\varphi_{ij} = \begin{bmatrix} 0 & 0 & \frac{1}{r} \\ 0 & 0 & \cot\theta \\ \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \tag{C.55}$$

C.5 Primitive Toroidal Coordinates

We have Cartesian (x, y, z) and primitive toroidal coordinates (r, θ, ζ) as our two coordinate systems. ($0 \le r < \infty$, $0 \le \theta \le 2\pi$, and $0 \le \zeta \le 2\pi$)

We use

$$r^2 = (R - R_0)^2 + z^2 = (\sqrt{x^2 + y^2} - R_0)^2 + z^2 \tag{C.1}$$

$$\tan\theta = \frac{z}{R - R_0} = \frac{z}{\sqrt{x^2 + y^2} - R_0} \tag{C.2}$$

$$\tan(-\zeta) = \frac{y}{x} \tag{C.3}$$

$$R = \sqrt{x^2 + y^2} \tag{C.4}$$

$$\tag{C.5}$$

where $\sqrt{x_0^2 + y_0^2} = R_0 > 0$ is a given constant.

Thus, we find

$$\begin{aligned}
\mathrm{d}r &= \frac{2\left(\sqrt{x^2+y^2}-R_0\right)\left(\frac{2x\,\mathrm{d}x+2y\,\mathrm{d}y}{2\sqrt{x^2+y^2}}\right)+2z\,\mathrm{d}z}{2\sqrt{\left(\sqrt{x^2+y^2}-R_0\right)^2+z^2}} = \frac{\left(\sqrt{(x^2+y^2)}-R_0\right)\left(\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{\sqrt{x^2+y^2}}\right)+z\,\mathrm{d}z}{\sqrt{(\sqrt{x^2+y^2}-R_0)^2+z^2}} \\
&= \frac{(1-\frac{R_0}{\sqrt{x^2+y^2}})(x\,\mathrm{d}x+y\,\mathrm{d}y)+z\,\mathrm{d}z}{\sqrt{(\sqrt{x^2+y^2}-R_0)^2+z^2}} \\
&= \frac{(R-R_0)\cos\zeta}{r}\,\mathrm{d}x - \frac{(R-R_0)\sin\zeta}{r}\,\mathrm{d}y + \sin\theta\,\mathrm{d}z \\
&= \cos\theta\cos\zeta\,\mathrm{d}x - \cos\zeta\sin\zeta\,\mathrm{d}y + \sin\theta\,\mathrm{d}z
\end{aligned} \tag{C.6}$$

$$\sec^2\theta\,\mathrm{d}\theta = \frac{(\sqrt{x^2+y^2}-R_0)\,\mathrm{d}z - z\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{\sqrt{x^2+y^2}}}{(\sqrt{x^2+y^2}-R_0)^2} \tag{C.7}$$

$$\mathrm{d}\theta = \frac{(\sqrt{x^2+y^2}-R_0)\,\mathrm{d}z - z\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{\sqrt{x^2+y^2}}}{(\sqrt{x^2+y^2}-R_0)^2+z^2} \tag{C.8}$$

$$= -\frac{\cos\zeta\sin\theta}{r}\,\mathrm{d}x + \frac{\sin\zeta\sin\theta}{r}\,\mathrm{d}y + \frac{\cos\theta}{r}\,\mathrm{d}z \tag{C.9}$$

$$\sec^2(-\zeta)(-\,\mathrm{d}\zeta) = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} \tag{C.10}$$

$$\begin{aligned}
\mathrm{d}\zeta &= \cos^2(\zeta)\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2} = \frac{y}{x^2+y^2}\,\mathrm{d}x - \frac{x}{x^2+y^2}\,\mathrm{d}y \\
&= -\frac{\sin\zeta}{R}\,\mathrm{d}x - \frac{\cos\zeta}{R}\,\mathrm{d}y
\end{aligned} \tag{C.11}$$

and so

$$\mathbf{J} = \frac{\partial(r,\theta,\zeta)}{\partial(x,y,z)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} & \frac{\partial\theta}{\partial z} \\ \frac{\partial\zeta}{\partial x} & \frac{\partial\zeta}{\partial y} & \frac{\partial\zeta}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\zeta & -\cos\theta\sin\zeta & \sin\theta \\ \frac{-\cos\zeta\sin\theta}{r} & \frac{\sin\zeta\sin\theta}{r} & \frac{\cos\theta}{r} \\ -\frac{\sin\zeta}{R} & -\frac{\cos\zeta}{R} & 0 \end{bmatrix} \tag{C.12}$$

$$\begin{aligned}
J = \sin\theta &\left(\frac{-\cos\zeta\sin\theta}{r}(\frac{-\cos\zeta}{R}) - \frac{\sin\zeta\sin\theta}{r}(\frac{-\sin\zeta}{R})\right) \\
&- \frac{\cos\theta}{r}\left(\cos\theta\cos\zeta(\frac{-\cos\zeta}{R}) - (-\cos\theta\sin\zeta)(\frac{-\sin\zeta}{R})\right) \\
&= \frac{\sin^2\theta}{rR} + \frac{\cos^2\theta}{rR} = \frac{1}{rR}
\end{aligned} \tag{C.13}$$

Note that we then have

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_r = \nabla r = \cos\theta\cos\zeta\,\nabla x - \cos\theta\sin\zeta\,\nabla y + \sin\theta\,\nabla z \tag{C.14}$$

$$|\nabla r| = 1 \tag{C.15}$$

$$\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_\theta = \nabla\theta = -\frac{\cos\zeta\sin\theta}{r}\,\nabla x + \frac{\sin\zeta\sin\theta}{r}\,\nabla y + \frac{\cos\theta}{r}\,\nabla z \tag{C.16}$$

$$|\nabla\theta| = \sqrt{\frac{\cos^2\zeta\sin^2\theta + \sin^2\zeta\sin^2\theta + \cos^2\theta}{r^2}} = \sqrt{\frac{1}{r^2}} = \frac{1}{r} \tag{C.17}$$

$$\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_\zeta = \nabla\zeta = -\frac{\sin\zeta}{R}\nabla x - \frac{\cos\zeta}{R}\nabla y \tag{C.18}$$

$$|\nabla\zeta| = \sqrt{\frac{\sin^2\zeta + \cos^2\zeta}{R^2}} = \sqrt{\frac{1}{R^2}} = \frac{1}{R} \tag{C.19}$$

So that

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_r = \hat{\mathbf{r}} = \cos\theta\cos\zeta\hat{\mathbf{x}} - \cos\theta\sin\zeta\hat{\mathbf{y}} + \sin\theta\hat{\mathbf{z}} \tag{C.20}$$

$$\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_\theta = \hat{\boldsymbol{\theta}} = -\cos\zeta\sin\theta\hat{\mathbf{x}} + \sin\zeta\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \tag{C.21}$$

$$\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_\zeta = \hat{\boldsymbol{\zeta}} = -\sin\zeta\hat{\mathbf{x}} - \cos\zeta\hat{\mathbf{y}} \tag{C.22}$$

$$\tag{C.23}$$

The metric tensor is given by $g^{ij} = \sum_{k=1}^{3} \frac{\partial\xi^i}{\partial x^k}\frac{\partial\xi^j}{\partial x^k}$. Thus

$$\begin{aligned} g^{rr} &= \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2 \\ &= \cos^2\theta\cos^2\zeta + \cos^2\theta\sin^2\zeta + \sin^2\theta = 1 \end{aligned} \tag{C.24}$$

$$\begin{aligned} g^{\theta\theta} &= \left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 \\ &= \frac{\cos^2\zeta\sin^2\theta}{r^2} + \frac{\sin^2\zeta\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2} = \frac{1}{r^2} \end{aligned} \tag{C.25}$$

$$\begin{aligned} g^{\zeta\zeta} &= \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2 + \left(\frac{\partial\zeta}{\partial z}\right)^2 \\ &= \frac{\sin^2\zeta}{R^2} + \frac{\cos^2\zeta}{R^2} = \frac{1}{R^2} \end{aligned} \tag{C.26}$$

$$\begin{aligned} g^{r\theta} &= \frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial\theta}{\partial z} \\ &= \cos\zeta\cos\theta\frac{-\cos\zeta\sin\theta}{r} - \sin\zeta\cos\theta\frac{\sin\zeta\sin\theta}{r} + \sin\theta\frac{\cos\theta}{r} = 0 \end{aligned} \tag{C.27}$$

$$\begin{aligned} g^{r\zeta} &= \frac{\partial r}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial\zeta}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial\zeta}{\partial z} \\ &= \cos\zeta\cos\theta\frac{-\sin\zeta}{R} - \sin\zeta\cos\theta\frac{-\cos\zeta}{R} + \sin\theta(0) = 0 \end{aligned} \tag{C.28}$$

$$\begin{aligned} g^{\theta\zeta} &= \frac{\partial\theta}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\theta}{\partial y}\frac{\partial\zeta}{\partial y} + \frac{\partial\theta}{\partial z}\frac{\partial\zeta}{\partial z} \\ &= \frac{-\cos\zeta\sin\theta}{r}\frac{-\sin\zeta}{R} + \frac{\sin\zeta\sin\theta}{r}\frac{-\cos\zeta}{R} + \frac{\cos\theta}{r}(0) = 0 \end{aligned} \tag{C.29}$$

Thus

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{R^2} \end{bmatrix} \tag{C.30}$$

In the other direction we would use

$$x = R\cos\zeta \tag{C.31}$$
$$y = -R\sin\zeta \tag{C.32}$$
$$z = r\sin\theta \tag{C.33}$$
$$R - R_0 = r\cos\theta \tag{C.34}$$

or combining, if we so wish

$$x = (R_0 + r\cos\theta)\cos\zeta \tag{C.35}$$
$$y = -(R_0 + r\cos\theta)\sin\zeta \tag{C.36}$$
$$z = r\sin\theta \tag{C.37}$$
$$\mathbf{e}_1 = \mathbf{e}_R = \left(\frac{\partial\mathbf{x}}{\partial r}\right)_{\theta,\zeta} = \cos\theta\cos\zeta\,\nabla x - \cos\theta\sin\zeta\,\nabla y + \sin\theta\,\nabla z \tag{C.38}$$
$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial\mathbf{x}}{\partial\theta} = -r\sin\theta\cos\zeta\,\nabla x + r\sin\theta\sin\zeta\,\nabla y + r\cos\theta\,\nabla z \tag{C.39}$$
$$\mathbf{e}_3 = \mathbf{e}_\zeta = \frac{\partial\mathbf{x}}{\partial\zeta} = -(R_0 + r\cos\theta)\sin\zeta\,\nabla x - (R_0 + r\cos\theta)\cos\zeta\,\nabla y \tag{C.40}$$

and so

$$\mathrm{d}x = \cos\zeta\,\mathrm{d}R - R\sin\zeta\,\mathrm{d}\zeta \tag{C.41}$$
$$\mathrm{d}y = -\sin\zeta\,\mathrm{d}R - R\cos\zeta\,\mathrm{d}\zeta \tag{C.42}$$
$$\mathrm{d}z = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta \tag{C.43}$$
$$\mathrm{d}R = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta \tag{C.44}$$
$$\mathrm{d}x = \cos\theta\cos\zeta\,\mathrm{d}r - r\sin\theta\cos\zeta\,\mathrm{d}\theta - R\sin\zeta\,\mathrm{d}\zeta \tag{C.45}$$
$$\mathrm{d}y = -\cos\theta\sin\zeta\,\mathrm{d}r + r\sin\theta\sin\zeta\,\mathrm{d}\theta - R\cos\zeta\,\mathrm{d}\zeta \tag{C.46}$$

and so we then have

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(r,\theta,\zeta)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\zeta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\zeta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\zeta} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\zeta & -r\sin\theta\cos\zeta & -R\sin\zeta \\ -\cos\theta\sin\zeta & r\sin\theta\sin\zeta & -R\cos\zeta \\ \sin\theta & r\cos\theta & 0 \end{bmatrix} \tag{C.47}$$

$$\begin{aligned} \mathcal{J} &= \sin\theta\,((-r\sin\theta\cos\zeta)(-R\cos\zeta) - (-R\sin\zeta)(r\sin\theta\sin\zeta)) \\ &\quad - r\cos\theta\,(\cos\theta\cos\zeta(-R\cos\zeta) - (-R\sin\zeta)(-\cos\theta\sin\zeta)) \\ &= rR\sin^2\theta + rR\cos^2\theta = rR \end{aligned} \tag{C.48}$$

Note that we then have

$$\begin{aligned} \hat{\mathbf{x}} &= \cos\theta\cos\zeta\,\nabla r - r\sin\theta\cos\zeta\,\nabla\theta - R\sin\zeta\,\nabla\zeta \\ &= \cos\theta\cos\zeta\hat{\mathbf{r}} - \sin\theta\cos\zeta\hat{\boldsymbol{\theta}} - \sin\zeta\hat{\boldsymbol{\zeta}} \\ &= \frac{(R-R_0)x}{rR}\hat{\mathbf{r}} - \frac{zx}{rR}\hat{\boldsymbol{\theta}} + \frac{y}{R}\hat{\boldsymbol{\zeta}} \\ &= \frac{(\sqrt{x^2+y^2}-R_0)x}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} - \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\zeta}} \end{aligned} \tag{C.49}$$

$$\begin{aligned}
\hat{\mathbf{y}} &= -\cos\theta\sin\zeta\,\nabla r + r\sin\theta\sin\zeta\,\nabla\theta - R\cos\zeta\,\nabla\zeta \\
&= -\cos\theta\sin\zeta\hat{\mathbf{r}} + \sin\theta\sin\zeta\hat{\boldsymbol{\theta}} - \cos\zeta\hat{\boldsymbol{\zeta}} \\
&= \frac{(R-R_0)y}{rR}\hat{\mathbf{r}} - \frac{yz}{rR}\hat{\boldsymbol{\theta}} - \frac{x}{R}\hat{\boldsymbol{\zeta}} \qquad \text{(C.50)} \\
&= \frac{(\sqrt{x^2+y^2}-R_0)y}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} - \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}} - \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\zeta}} \\
\hat{\mathbf{z}} &= \sin\theta\,\nabla r + r\cos\theta\,\nabla\theta \\
&= \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}} \\
&= \frac{z}{r}\hat{\mathbf{r}} + \frac{R-R_0}{r}\hat{\boldsymbol{\theta}} \qquad \text{(C.51)} \\
&= \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{r}} + \frac{\sqrt{x^2+y^2}-R_0}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{\theta}}
\end{aligned}$$

The other metric tensor is given by $g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}$. Thus

$$\begin{aligned}
g_{rr} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \qquad \text{(C.52)} \\
&= \cos^2\theta\cos^2\zeta + \cos^2\theta\sin^2\zeta + \sin^2\theta = 1 \\
g_{\theta\theta} &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \qquad \text{(C.53)} \\
&= r^2\sin^2\theta\cos^2\zeta + r^2\sin^2\theta\sin^2\zeta + r^2\cos^2\theta = r^2 \\
g_{\zeta\zeta} &= \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2 \qquad \text{(C.54)} \\
&= R^2\sin^2\zeta + R^2\cos^2\zeta + 0 = R^2 \\
g_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \qquad \text{(C.55)} \\
&= \cos\theta\cos\zeta(-r\sin\theta\cos\zeta) - \cos\theta\sin\zeta(r\sin\theta\sin\zeta) + r\sin\theta\cos\theta = 0 \\
g_{r\zeta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \zeta} \qquad \text{(C.56)} \\
&= \cos\theta\cos\zeta(-R\sin\zeta) + \cos\theta\sin\zeta R\cos\zeta + \sin\theta(0) = 0 \\
g_{\theta\zeta} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \zeta} \qquad \text{(C.57)} \\
&= -r\sin\theta\cos\zeta(-R\sin\zeta) + r\sin\theta\sin\zeta(-R\cos\zeta) + r\cos\theta(0) = 0
\end{aligned}$$

Thus

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & R^2 \end{bmatrix} \qquad \text{(C.58)}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right] \qquad \text{(C.59)}$$

$$\Gamma_{r,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -R\cos\theta \end{bmatrix} \tag{C.60}$$

$$\Gamma_{\theta,ij} = \begin{bmatrix} 0 & r & 0 \\ r & 0 & 0 \\ 0 & 0 & rR\sin\theta \end{bmatrix} \tag{C.61}$$

$$\Gamma_{\zeta,ij} = \begin{bmatrix} 0 & 0 & R\cos\theta \\ 0 & 0 & -rR\sin\theta \\ R\cos\theta & -rR\sin\theta & 0 \end{bmatrix} \tag{C.62}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{C.63}$$

$$\Gamma^r_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -R\cos\theta \end{bmatrix} \tag{C.64}$$

$$\Gamma^\theta_{ij} = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & \frac{R}{r}\sin\theta \end{bmatrix} \tag{C.65}$$

$$\Gamma^\zeta_{ij} = \begin{bmatrix} 0 & 0 & \frac{\cos\theta}{R} \\ 0 & 0 & -\frac{r\sin\theta}{R} \\ \frac{\cos\theta}{R} & -\frac{r\sin\theta}{R} & 0 \end{bmatrix} \tag{C.66}$$

C.6 Plasma Toroidal Coordinates

We have Cartesian (x, y, z) and plasma toroidal coordinates (ψ, θ, ζ) as our two coordinate systems. $(1 < \psi < \infty$, $0 < \theta < 2\pi$, and $0 < \zeta < 2\pi)$

We use

$$x = a\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta \tag{C.1}$$

$$y = a\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta \tag{C.2}$$

$$z = a\frac{\sin\theta}{\psi-\cos\theta} \tag{C.3}$$

which means

$$\mathbf{e}_1 = \mathbf{e}_\psi = \left(\frac{\partial\mathbf{x}}{\partial\psi}\right)_{\theta,\zeta} = \frac{a\cos\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\nabla x + \frac{a\sin\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\nabla y - \frac{a\sin\theta}{(\psi-\cos\theta)^2}\nabla z \tag{C.4}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial\mathbf{x}}{\partial\theta} = -\frac{a\sqrt{\psi^2-1}\cos\zeta\sin\theta}{(\psi-\cos\theta)^2}\nabla x - \frac{a\sqrt{\psi^2-1}\sin\zeta\sin\theta}{(\psi-\cos\theta)^2}\nabla y + \frac{a(\psi\cos\theta-1)}{(\psi-\cos\theta)^2}\nabla z \tag{C.5}$$

$$\mathbf{e}_3 = \mathbf{e}_\zeta = \frac{\partial\mathbf{x}}{\partial\zeta} = -\frac{a\sqrt{\psi^2-1}}{\psi-\cos\theta}\nabla x + \frac{a\sqrt{\psi^2-1}}{\psi-\cos\theta}\nabla y \tag{C.6}$$

Taking (define $\beta = \frac{z^2}{x^2+y^2}$ and $\gamma = \frac{(1+r^2/a^2)^2}{(1-r^2/a^2)^2}$ where $r^2 = x^2+y^2+z^2$)

$$\frac{x^2+y^2+z^2}{a^2} = \frac{r^2}{a^2} = \frac{\psi^2 - 1 + \sin^2\theta}{(\psi-\cos\theta)^2} = \frac{\psi^2-\cos^2\theta}{(\psi-\cos\theta)^2} = \frac{(\psi+\cos\theta)\cancel{(\psi-\cos\theta)}}{(\psi-\cos\theta)\cancel{^2}} = \frac{\psi+\cos\theta}{\psi-\cos\theta} \tag{C.7}$$

$$\psi = \frac{-\cos\theta(1+\frac{r^2}{a^2})}{1-\frac{r^2}{a^2}} \Rightarrow \psi^2 = \gamma\cos^2\theta \tag{C.8}$$

$$\frac{y}{z} = \frac{\sqrt{\psi^2-1}}{\sin\theta}\sin\zeta = \frac{\sqrt{\psi^2-1}}{\sin\theta}\frac{y}{\sqrt{x^2+y^2}} \Rightarrow \frac{\sqrt{x^2+y^2}}{z} = \frac{\sqrt{\psi^2-1}}{\sin\theta} \tag{C.9}$$

$$\sin^2\theta = \frac{z^2}{x^2+y^2}(\psi^2-1) = \beta[(1-\sin^2\theta)\gamma - 1] \tag{C.10}$$

$$\sin^2\theta = \frac{\beta(\gamma-1)}{1+\gamma\beta} = \frac{4a^2z^2}{(-a^2+x^2+y^2)^2+2(a^2+x^2+y^2)z^2+z^4} \tag{C.11}$$

$$\sin\theta = \frac{2az}{\sqrt{(-a^2+x^2+y^2)^2+2(a^2+x^2+y^2)z^2+z^4}} \tag{C.12}$$

$$\begin{aligned}\psi^2 &= \left(\frac{1+r^2/a^2}{1-r^2/a^2}\right)^2\left(1-\frac{4a^2z^2}{(-a^2+x^2+y^2)^2+2(a^2+x^2+y^2)z^2+z^4}\right)\\ &= \frac{\left(a^2+x^2+y^2+z^2\right)^2}{2z^2\left(a^2+x^2+y^2\right)+\left(-a^2+x^2+y^2\right)^2+z^4}\end{aligned} \tag{C.13}$$

Thus we can rewrite our expressions as the ugly

$$\psi^2 = \frac{\left(a^2+x^2+y^2+z^2\right)^2}{2z^2\left(a^2+x^2+y^2\right)+\left(-a^2+x^2+y^2\right)^2+z^4} \tag{C.14}$$

$$\sin^2\theta = \frac{\beta(\gamma-1)}{1+\gamma\beta} = \frac{4a^2z^2}{(-a^2+x^2+y^2)^2+2(a^2+x^2+y^2)z^2+z^4} \tag{C.15}$$

$$\tan\zeta = \frac{y}{x} \tag{C.16}$$

So we find

$$\mathrm{d}x = \frac{a\cos\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\,\mathrm{d}\psi - \frac{a\sqrt{\psi^2-1}\cos\zeta\sin\theta}{(\psi-\cos\theta)^2}\,\mathrm{d}\theta - \frac{a\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta\,\mathrm{d}\zeta \tag{C.17}$$

$$\mathrm{d}y = \frac{a\sin\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\,\mathrm{d}\psi - \frac{a\sqrt{\psi^2-1}\sin\zeta\sin\theta}{(\psi-\cos\theta)^2}\,\mathrm{d}\theta + \frac{a\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta\,\mathrm{d}\zeta \tag{C.18}$$

$$\begin{aligned}\mathrm{d}z &= a\frac{\cos\theta(\psi-\cos\theta)\,\mathrm{d}\theta - \sin\theta(\,\mathrm{d}\psi + \sin\theta\,\mathrm{d}\theta)}{(\psi-\cos\theta)^2}\\ &= -a\frac{\sin\theta}{(\psi-\cos\theta)^2}\,\mathrm{d}\psi + a\frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2}\,\mathrm{d}\theta\end{aligned} \tag{C.19}$$

We of course then have

$$\mathbf{J} = \boldsymbol{\mathcal{J}}^{-1} = \frac{\partial(\psi,\theta,\zeta)}{\partial(x,y,z)} = \begin{bmatrix} \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} & \frac{\partial\theta}{\partial z} \\ \frac{\partial\zeta}{\partial x} & \frac{\partial\zeta}{\partial y} & \frac{\partial\zeta}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{\psi^2-1}\cos\zeta(1-\psi\cos\theta)}{a} & \frac{\sqrt{\psi^2-1}(1-\psi\cos\theta)\sin\zeta}{a} & -\frac{\left(\psi^2-1\right)\sin\theta}{a} \\ -\frac{\sqrt{\psi^2-1}\cos\zeta\sin\theta}{a} & -\frac{\sqrt{\psi^2-1}\sin\zeta\sin\theta}{a} & \frac{\psi\cos\theta-1}{a} \\ \frac{(\cos\theta-\psi)\sin\zeta}{a\sqrt{\psi^2-1}} & \frac{\cos\zeta(\psi-\cos\theta)}{a\sqrt{\psi^2-1}} & 0 \end{bmatrix} \tag{C.20}$$

$$J = \frac{1}{\mathcal{J}} = \frac{(\psi-\cos\theta)^3}{a^3} \tag{C.21}$$

Because of the ugliness of calculating g^{ij} directly, I use the results of g_{ij} below (C.56) and invert it to find.

$$g^{ij} = \begin{bmatrix} [(\psi^2-1)(\psi-\cos\theta)^2] & 0 & 0 \\ 0 & (\psi-\cos\theta)^2 & 0 \\ 0 & 0 & (\psi^2-1)(\psi-\cos\theta)^2 \end{bmatrix} \tag{C.22}$$

We can now note that

$$\mathbf{e}^1 = \mathbf{e}^\psi = \nabla\psi = \frac{\sqrt{\psi^2-1}(1-\psi\cos\theta)\cos\zeta}{a}\nabla x + \frac{\sqrt{\psi^2-1}(1-\psi\cos\theta)\sin\zeta}{a}\nabla y - \frac{\sin\theta(\psi^2-1)}{a}\nabla z \tag{C.23}$$

$$\begin{aligned} |\nabla\psi|^2 &= \frac{(\psi^2-1)(1-\psi\cos\theta)^2\cos^2\zeta + (\psi^2-1)(1-\psi\cos\theta)^2\sin^2\zeta + (\psi^2-1)^2\sin^2\theta}{a^2} \\ &= \frac{(\psi^2-1)(1-\psi\cos\theta)^2 + (\psi^2-1)^2\sin^2\theta}{a^2} \\ &= \frac{(\psi^2-1)\left[1 - 2\psi\cos\theta + \cancel{\psi^2\cos^2\theta} + \psi^2 - \cancel{\psi^2\cos^2\theta} - 1 + \cos^2\theta\right]}{a^2} \\ &= \frac{(\psi^2-1)(\psi-\cos\theta)^2}{a^2} \end{aligned} \tag{C.24}$$

$$|\nabla\psi| = \frac{\sqrt{\psi^2-1}(\psi-\cos\theta)}{a} \tag{C.25}$$

$$\mathbf{e}^2 = \mathbf{e}^\theta = \nabla\theta = -\frac{\sqrt{\psi^2-1}\cos\zeta\sin\theta}{a}\nabla x + -\frac{\sqrt{\psi^2-1}\sin\zeta\sin\theta}{a}\nabla y + \frac{\psi\cos\theta-1}{a}\nabla z \tag{C.26}$$

$$\begin{aligned} |\nabla\theta|^2 &= \frac{(\psi^2-1)\cos^2\zeta\sin^2\theta + (\psi^2-1)\sin^2\zeta\sin^2\theta + (1-\psi\cos\theta)^2}{a^2} \\ &= \frac{(\psi^2-1)\sin^2\theta + (1-\psi\cos\theta)^2}{a^2} = \frac{(\psi-\cos\theta)^2}{a^2} \end{aligned} \tag{C.27}$$

$$|\nabla\theta| = \frac{\psi-\cos\theta}{a} \tag{C.28}$$

$$\mathbf{e}^3 = \mathbf{e}^\zeta = \nabla\zeta = \frac{(\cos\theta - \psi)\sin\zeta}{a\sqrt{\psi^2-1}}\nabla x + \frac{\cos\zeta(\psi - \cos\theta)}{a\sqrt{\psi^2-1}}\nabla y \tag{C.29}$$

$$|\nabla\zeta|^2 = \frac{(\psi-\cos\theta)^2\sin^2\zeta + (\psi-\cos\theta)^2\cos^2\zeta}{a^2(\psi^2-1)} = \frac{(\psi-\cos\theta)^2}{a^2(\psi^2-1)} \tag{C.30}$$

$$|\nabla\zeta| = \frac{\psi-\cos\theta}{a\sqrt{\psi^2-1}} \tag{C.31}$$

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(\psi,\theta,\zeta)} = \begin{bmatrix} \frac{\partial x}{\partial\psi} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\zeta} \\ \frac{\partial y}{\partial\psi} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\zeta} \\ \frac{\partial z}{\partial\psi} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\zeta} \end{bmatrix} = \begin{bmatrix} \frac{a\cos\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2} & -\frac{a\sqrt{\psi^2-1}\cos\zeta\sin\theta}{(\psi-\cos\theta)^2} & -\frac{a\sqrt{\psi^2-1}\sin\zeta}{\psi-\cos\theta} \\ \frac{a\sin\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2} & -\frac{a\sqrt{\psi^2-1}\sin\zeta\sin\theta}{(\psi-\cos\theta)^2} & \frac{a\sqrt{\psi^2-1}\cos\zeta}{\psi-\cos\theta} \\ \frac{-a\sin\theta}{(\psi-\cos\theta)^2} & \frac{a(\psi\cos\theta-1)}{(\psi-\cos\theta)^2} & 0 \end{bmatrix} \tag{C.32}$$

$$\begin{aligned}
\mathcal{J} &= \frac{-a\sin\theta}{(\psi-\cos\theta)^2}\left(-\frac{a\sqrt{\psi^2-1}\cos\zeta\sin\theta}{(\psi-\cos\theta)^2}\frac{a\sqrt{\psi^2-1}\cos\zeta}{\psi-\cos\theta} - \frac{a\sqrt{\psi^2-1}\sin\zeta}{\psi-\cos\theta}\frac{a\sqrt{\psi^2-1}\sin\zeta\sin\theta}{(\psi-\cos\theta)^2}\right) \\
&\quad - \frac{a(\psi\cos\theta-1)}{(\psi-\cos\theta)^2}\left(\frac{a\cos\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\frac{a\sqrt{\psi^2-1}\cos\zeta}{\psi-\cos\theta} - \frac{-a\sqrt{\psi^2-1}\sin\zeta}{\psi-\cos\theta}\frac{a\sin\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\right) \\
&= \frac{a^3\sin^2\theta(\psi^2-1)}{(\psi-\cos\theta)^5}\left(\cos^2\zeta+\sin^2\zeta\right) \\
&\quad + \frac{a^3(1-\psi\cos\theta)^2}{(\psi-\cos\theta)^5}\left(\cos\zeta^2+\sin^2\zeta\right) \\
&= \frac{a^3}{(\psi-\cos\theta)^5}\left((\psi^2-1)\sin^2\theta + (1-\psi\cos\theta)^2\right) \\
&= \frac{a^3}{(\psi-\cos\theta)^5}\left((\psi^2-1)(1-\cos^2\theta)+1+2\psi\cos\theta+\psi^2\cos^2\theta\right) = \frac{a^3}{(\psi-\cos\theta)^5}\left(\psi^2+2\psi\cos\theta+\cos^2\theta\right) \\
&= \frac{a^3\left(\psi-\cos\theta\right)^2}{(\psi-\cos\theta)^5} = \frac{a^3}{(\psi-\cos\theta)^3}
\end{aligned} \tag{C.33}$$

Note that we then have (using (C.23) and the following)

$$\begin{aligned}
\hat{\mathbf{x}} &= \frac{a\cos\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\nabla\psi - \frac{a\sqrt{\psi^2-1}\cos\zeta\sin\theta}{(\psi-\cos\theta)^2}\nabla\theta - \frac{a\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta\,\nabla\zeta \\
&= \frac{\cos\zeta(1-\psi\cos\theta)}{\psi-\cos\theta}\hat{\boldsymbol{\psi}} - \frac{\sqrt{\psi^2-1}\cos\zeta\sin\theta}{\psi-\cos\theta}\hat{\boldsymbol{\theta}} - \sin\zeta\hat{\boldsymbol{\zeta}}
\end{aligned} \tag{C.34}$$

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{a\sin\zeta(1-\psi\cos\theta)}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\nabla\psi - \frac{a\sqrt{\psi^2-1}\sin\zeta\sin\theta}{(\psi-\cos\theta)^2}\nabla\theta + \frac{a\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta\,\nabla\zeta \\
&= \frac{\sin\zeta(1-\psi\cos\theta)}{\psi-\cos\theta}\hat{\boldsymbol{\psi}} - \frac{\sqrt{\psi^2-1}\sin\zeta\sin\theta}{\psi-\cos\theta}\hat{\boldsymbol{\theta}} + \cos\zeta\hat{\boldsymbol{\zeta}}
\end{aligned} \tag{C.35}$$

$$\begin{aligned}\hat{\mathbf{z}} &= -a\frac{\sin\theta}{(\psi-\cos\theta)^2}\nabla\psi + a\frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2}\nabla\theta \\ &= -\frac{\sqrt{\psi^2-1}\sin\theta}{\psi-\cos\theta}\hat{\boldsymbol{\psi}} + \frac{\psi\cos\theta-1}{\psi-\cos\theta}\hat{\boldsymbol{\theta}}\end{aligned} \tag{C.36}$$

The metric tensor is given by $g_{ij} = \sum_{k=1}^{3}\frac{\partial x^k}{\partial\xi^i}\frac{\partial x^k}{\partial\xi^j}$. Thus

$$g_{\psi\psi} = \frac{\partial x}{\partial\psi}\frac{\partial x}{\partial\psi} + \frac{\partial y}{\partial\psi}\frac{\partial y}{\partial\psi} + \frac{\partial z}{\partial\psi}\frac{\partial z}{\partial\psi} \tag{C.37}$$

$$= \frac{(1-\psi\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4}\cos^2\zeta + \frac{(1-\psi\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4}\sin^2\zeta + \frac{\sin^2\theta}{(\psi-\cos\theta)^4} \tag{C.38}$$

$$= \frac{(1-\psi\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4} + \frac{\sin^2\theta(\psi^2-1)}{(\psi^2-1)(\psi-\cos\theta)^4} = \frac{1+\psi^2\cos^2\theta-2\psi\cos\theta+\psi^2\sin^2\theta-\sin^2\theta}{(\psi^2-1)(\psi-\cos\theta)^4} \tag{C.39}$$

$$\boxed{g_{\psi\psi} = \frac{\cos^2\theta-2\psi\cos\theta+\psi^2}{(\psi^2-1)(\psi-\cos\theta)^4} = \frac{(\psi-\cos\theta)^2}{(\psi^2-1)(\psi-\cos\theta)^4} = \frac{1}{(\psi^2-1)(\psi-\cos\theta)^2}} \tag{C.40}$$

$$g_{\theta\theta} = \frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\theta} + \frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\theta} + \frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\theta} \tag{C.41}$$

$$= \frac{(\psi^2-1)\sin^2\theta}{(\psi-\cos\theta)^4}\cos^2\zeta + \frac{(\psi^2-1)\sin^2\theta}{(\psi-\cos\theta)^4}\sin^2\zeta + \frac{(1-\psi\cos\theta)^2}{(\psi-\cos\theta)^4} \tag{C.42}$$

$$= \frac{\psi^2\sin^2\theta-\sin^2\theta+1-2\psi\cos\theta+\psi^2\cos^2\theta}{(\psi-\cos\theta)^4} = \frac{\cos^2\theta-2\psi\cos\theta+\psi^2}{(\psi-\cos\theta)^4} \tag{C.43}$$

$$\boxed{g_{\theta\theta} = \frac{(\psi-\cos\theta)^2}{(\psi-\cos\theta)^4} = \frac{1}{(\psi-\cos\theta)^2}} \tag{C.44}$$

$$g_{\zeta\zeta} = \frac{\partial x}{\partial\zeta}\frac{\partial x}{\partial\zeta} + \frac{\partial y}{\partial\zeta}\frac{\partial y}{\partial\zeta} + \frac{\partial z}{\partial\zeta}\frac{\partial z}{\partial\zeta} \tag{C.45}$$

$$\boxed{g_{\zeta\zeta} = \frac{\psi^2-1}{(\psi-\cos\theta)^2}\sin^2\zeta + \frac{\psi^2-1}{(\psi-\cos\theta)^2}\cos^2\zeta = \frac{\psi^2-1}{(\psi-\cos\theta)^2}} \tag{C.46}$$

$$g_{\psi\theta} = \frac{\partial x}{\partial\psi}\frac{\partial x}{\partial\theta} + \frac{\partial y}{\partial\psi}\frac{\partial y}{\partial\theta} + \frac{\partial z}{\partial\psi}\frac{\partial z}{\partial\theta} \tag{C.47}$$

$$\begin{aligned}= &\left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\cos\zeta\right)\left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\cos\zeta\right) \\ &+\left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\sin\zeta\right)\left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\sin\zeta\right) \\ &+\left(\frac{-\sin\theta}{(\psi-\cos\theta)^2}\right)\left(\frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2}\right)\end{aligned} \tag{C.48}$$

$$\boxed{g_{\psi\theta} = \frac{(\psi\cos\theta-1)\sqrt{\psi^2-1}\sin\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^4} - \frac{\sin\theta(\psi\sin\theta\cos\theta-1}{(\psi-\cos\theta)^4} = 0} \tag{C.49}$$

$$g_{\psi\zeta} = \frac{\partial x}{\partial\psi}\frac{\partial x}{\partial\zeta} + \frac{\partial y}{\partial\psi}\frac{\partial y}{\partial\zeta} + \frac{\partial z}{\partial\psi}\frac{\partial z}{\partial\zeta} \tag{C.50}$$

$$= \left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\cos\zeta\right)\left(-\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta\right) + \left(\frac{1-\psi\cos\theta}{\sqrt{\psi^2-1}(\psi-\cos\theta)^2}\sin\zeta\right)\left(\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta\right) + \left(\frac{-\sin\theta}{(\psi-\cos\theta)^2}\right)(0) \tag{C.51}$$

$$\boxed{g_{\psi\zeta} = -\frac{1-\psi\cos\theta}{(\psi-\cos\theta)^3}\sin\zeta\cos\zeta + \frac{1-\psi\cos\theta}{(\psi-\cos\theta)^3}\sin\zeta\cos\zeta = 0} \tag{C.52}$$

$$g_{\theta\zeta} = \frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\zeta} + \frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\zeta} + \frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\zeta} \tag{C.53}$$

$$= \left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\cos\zeta\right)\left(-\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\sin\zeta\right) + \left(-\frac{\sqrt{\psi^2-1}\sin\theta}{(\psi-\cos\theta)^2}\sin\zeta\right)\left(\frac{\sqrt{\psi^2-1}}{\psi-\cos\theta}\cos\zeta\right) + \left(\frac{\psi\cos\theta-1}{(\psi-\cos\theta)^2}\right)(0) \tag{C.54}$$

$$\boxed{g_{\theta\zeta} = \frac{(\psi^2-1)\sin\theta}{(\psi-\cos\theta)^3}(\sin\zeta\cos\zeta - \sin\zeta\cos\zeta) = 0.} \tag{C.55}$$

Hence we have altogether

$$g_{ij} = \begin{bmatrix} [(\psi^2-1)(\psi-\cos\theta)^2]^{-1} & 0 & 0 \\ 0 & (\psi-\cos\theta)^{-2} & 0 \\ 0 & 0 & (\psi^2-1)(\psi-\cos\theta)^{-2} \end{bmatrix} \tag{C.56}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial\xi^j} + \frac{\partial g_{jk}}{\partial\xi^i} - \frac{\partial g_{ij}}{\partial\xi^k}\right] \tag{C.57}$$

$$\Gamma_{\psi,ij} = \begin{bmatrix} 0 & \frac{-\sin\theta}{(\psi-\cos\theta)^3(\psi^2-1)} & 0 \\ \frac{-\sin\theta}{(\psi-\cos\theta)^3(\psi^2-1)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.58}$$

$$\Gamma_{\theta,ij} = \begin{bmatrix} \frac{\sin\theta}{(\psi-\cos\theta)^3(\psi^2-1)} & 0 & 0 \\ 0 & \frac{-\sin\theta}{(\psi-\cos\theta)^3} & 0 \\ 0 & 0 & \frac{(\psi^2-1)\sin\theta}{(\psi-\cos\theta)^3} \end{bmatrix} \tag{C.59}$$

$$\Gamma_{\zeta,ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-(\psi^2-1)\sin\theta}{(\psi-\cos\theta)^3} \\ 0 & \frac{-(\psi^2-1)\sin\theta}{(\psi-\cos\theta)^3} & 0 \end{bmatrix} \tag{C.60}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{C.61}$$

$$\Gamma^\psi_{ij} = \begin{bmatrix} 0 & \frac{-\sin\theta}{\psi-\cos\theta} & 0 \\ \frac{-\sin\theta}{\psi-\cos\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.62}$$

$$\Gamma^{\theta}_{ij} = \begin{bmatrix} \frac{\sin\theta}{(\psi-\cos\theta)(\psi^2-1)} & 0 & 0 \\ 0 & \frac{-\sin\theta}{\psi-\cos\theta} & 0 \\ 0 & 0 & \frac{(\psi^2-1)\sin\theta}{\psi-\cos\theta} \end{bmatrix} \tag{C.63}$$

$$\Gamma^{\zeta}_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-\sin\theta}{\psi-\cos\theta} \\ 0 & \frac{-\sin\theta}{\psi-\cos\theta} & 0 \end{bmatrix} \tag{C.64}$$

C.7 General Toroidal Coordinates

We have Cartesian (x, y, z) and plasma toroidal coordinates (τ, θ, ζ) as our two coordinate systems. $(-\infty < \tau < \infty$, $0 \leq \theta \leq 2\pi$, and $0 \leq \zeta \leq 2\pi)$

We use

$$x = a\frac{\sinh\tau}{\cosh\tau - \cos\theta}\cos\zeta \tag{C.1}$$

$$y = a\frac{\sinh\tau}{\cosh\tau - \cos\theta}\sin\zeta \tag{C.2}$$

$$z = a\frac{\sin\theta}{\cosh\tau - \cos\theta} \tag{C.3}$$

Note that we then have $\sinh\tau = \sqrt{\psi^2 - 1}$ and $\cosh\tau = \sqrt{1 + \sinh^2\tau} = \sqrt{\psi^2} = \psi$ as a connection to our previous coordinates (this would then restrict $0 < \tau < \infty$, which is actually nicer as it removes the $\text{sgn}(\tau)$ functions in some relations).

Thus we can rewrite our expressions as the ugly

$$\cosh^2\tau = \frac{\left(a^2 + x^2 + y^2 + z^2\right)^2}{2z^2\left(a^2 + x^2 + y^2\right) + \left(-a^2 + x^2 + y^2\right)^2 + z^4} \tag{C.4}$$

$$\sin^2\theta = \frac{\beta(\gamma-1)}{1+\gamma\beta} = \frac{4a^2z^2}{(-a^2 + x^2 + y^2)^2 + 2(a^2 + x^2 + y^2)z^2 + z^4} \tag{C.5}$$

$$\tan\zeta = \frac{y}{x} \tag{C.6}$$

These are so painfully ugly that we will calculate the Jacobian matrix via determining the results the "other way" first and inverting the matrix.

Note that one can write

$$\rho^2 = x^2 + y^2 \tag{C.7}$$

$$d_1^2 = (\rho + a)^2 + z^2 \tag{C.8}$$

$$d_2^2 = (\rho - a)^2 + z^2 \tag{C.9}$$

$$e^{\tau} = \frac{d_1}{d_2} \tag{C.10}$$

$$\cos\theta = \frac{d_1^2 + d_2^2 - 4a^2}{d_1 d_2} \tag{C.11}$$

So we find

$$\mathrm{d}x = a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\,\mathrm{d}\tau - a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\,\mathrm{d}\theta - a\frac{\sinh\tau}{\cosh\tau-\cos\theta}\sin\zeta\,\mathrm{d}\zeta \quad \text{(C.12)}$$

$$\mathrm{d}y = a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\,\mathrm{d}\tau - a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\,\mathrm{d}\theta + a\frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta\,\mathrm{d}\zeta \quad \text{(C.13)}$$

$$\mathrm{d}z = -a\frac{\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\,\mathrm{d}\tau + a\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2}\,\mathrm{d}\theta \quad \text{(C.14)}$$

which means

$$\mathbf{e}_1 = \mathbf{e}_\tau = \left(\frac{\partial\mathbf{x}}{\partial\psi}\right)_{\theta,\zeta} = a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\theta\,\nabla x + a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\,\nabla y - \frac{a\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\nabla z \quad \text{(C.15)}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta = \frac{\partial\mathbf{x}}{\partial\theta} = -\frac{a\sinh\tau\sin\theta\cos\zeta}{(\cosh\tau-\cos\theta)^2}\nabla x - \frac{a\sinh\tau\sin\theta\sin\zeta}{(\cosh\tau-\cos\theta)^2}\nabla y + a\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2} \quad \text{(C.16)}$$

$$\mathbf{e}_3 = \mathbf{e}_\zeta = \frac{\partial\mathbf{x}}{\partial\zeta} = -\frac{a\sinh\tau\sin\zeta}{\cosh\tau-\cos\theta}\nabla x + \frac{a\sinh\tau\cos\zeta}{\cosh\tau-\cos\theta} \quad \text{(C.17)}$$

$$\boldsymbol{\mathcal{J}} = \mathbf{J}^{-1} = \frac{\partial(x,y,z)}{\partial(\tau,\theta,\zeta)} = \begin{bmatrix} \frac{\partial x}{\partial\tau} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\zeta} \\ \frac{\partial y}{\partial\tau} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\zeta} \\ \frac{\partial z}{\partial\tau} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\zeta} \end{bmatrix} = \begin{bmatrix} a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta & -a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta & -a\frac{\sinh\tau}{\cosh\tau-\cos\theta}\sin\zeta \\ a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta & -a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta & a\frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta \\ -a\frac{\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2} & a\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2} & 0 \end{bmatrix} \quad \text{(C.18)}$$

$$\begin{aligned}
\mathcal{J} = -a\frac{\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\Bigg(& -a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\frac{a\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta \\
& -\frac{a\sinh\tau\sin\zeta}{\cosh\tau-\cos\theta}\frac{a\sinh\tau\sin\theta\sin\zeta}{(\cosh\tau-\cos\theta)^2}\Bigg) \\
& -a\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2}\Bigg(\frac{a(1-\cosh\tau\cos\theta)\cos\zeta}{(\cosh\tau-\cos\theta)^2}\frac{a\sinh\tau\cos\zeta}{\cosh\tau-\cos\theta} \\
& +\frac{a\sinh\tau\sin\zeta}{\cosh\tau-\cos\theta}\frac{a(1-\cosh\tau\cos\theta)\sin\zeta}{(\cosh\tau-\cos\theta)^2}\Bigg) \qquad \text{(C.19)} \\
= \frac{a^3\sin^2\theta\sinh^3\tau}{(\cosh\tau-\cos\theta)^5}&\left(\cos^2\zeta+\sin^2\zeta\right) + \frac{a^3(1-\cosh\tau\cos\theta)^2\sinh\tau}{(\cosh\tau-\cos\theta)^5}\left(\cos\zeta^2+\sin^2\zeta\right) \\
= \frac{a^3\sinh\tau}{(\cosh\tau-\cos\theta)^5}&\left((\cosh^2\tau-1)(1-\cos^2\theta)+(1-\cos\theta\cosh\tau)^2\right) \\
= \frac{a^3\sinh\tau}{(\cosh\tau-\cos\theta)^5}&(\cosh\tau-\cos\theta)^2 = \frac{a^3\sinh\tau}{(\cosh\tau-\cos\theta)^3}
\end{aligned}$$

Note that we then have (using (C.32) and the following equations)

$$\begin{aligned}\hat{\mathbf{x}} &= a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\,\nabla\tau - a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\,\nabla\theta - a\frac{\sinh\tau}{\cosh\tau-\cos\theta}\sin\zeta\,\nabla\zeta \\ &= \frac{1-\cosh\tau\cos\theta}{\cosh\tau-\cos\theta}\cos\zeta\hat{\boldsymbol{\tau}} - \frac{\sinh\tau\sin\theta}{\cosh\tau-\cos\theta}\cos\zeta\hat{\boldsymbol{\theta}} - \operatorname{sgn}(\tau)\sin\zeta\hat{\boldsymbol{\zeta}}\end{aligned} \tag{C.20}$$

$$\begin{aligned}\hat{\mathbf{y}} &= a\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\,\nabla\tau - a\frac{\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\,\nabla\theta + a\frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta\,\nabla\zeta \\ &= \frac{1-\cosh\tau\cos\theta}{\cosh\tau-\cos\theta}\sin\zeta\hat{\boldsymbol{\tau}} - \frac{\sinh\tau\sin\theta}{\cosh\tau-\cos\theta}\sin\zeta\hat{\boldsymbol{\theta}} + \operatorname{sgn}(\tau)\cos\zeta\hat{\boldsymbol{\zeta}}\end{aligned} \tag{C.21}$$

$$\begin{aligned}\hat{\mathbf{z}} &= -a\frac{\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\nabla\tau + a\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2}\nabla\theta \\ &= -\frac{\sin\theta\sinh\tau}{\cosh\tau-\cos\theta}\hat{\boldsymbol{\tau}} + \frac{\cos\theta\cosh\tau-1}{\cosh\tau-\cos\theta}\hat{\boldsymbol{\theta}}\end{aligned} \tag{C.22}$$

The metric tensor is given by $g_{ij} = \sum_{k=1}^{3}\frac{\partial x^k}{\partial\xi^i}\frac{\partial x^k}{\partial\xi^j}$. Thus

$$\begin{aligned}\frac{g_{\tau\tau}}{a^2} &= \frac{\partial x}{\partial\tau}\frac{\partial x}{\partial\tau} + \frac{\partial y}{\partial\tau}\frac{\partial y}{\partial\tau} + \frac{\partial z}{\partial\tau}\frac{\partial z}{\partial\tau} \\ &= \left(\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\right)^2 + \left(\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\right)^2 + \left(\frac{-\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\right)^2 \\ &= \frac{(1-\cosh\tau\cos\theta)^2+\sin^2\theta\sinh^2\tau}{(\cosh\tau-\cos\theta)^4} = \frac{1-2\cosh\tau\cos\theta+\cosh^2\tau\cos^2\theta+\sin^2\theta\sinh^2\tau}{(\cosh\tau-\cos\theta)^4} \\ &= \frac{1-2\cosh\tau\cos\theta+\cosh^2\tau+\sin^2\theta(\sinh^2\tau-\cosh^2\tau)}{(\cosh\tau-\cos\theta)^4} = \frac{1-2\cosh\tau\cos\theta+\cosh^2\tau-\sin^2\theta}{(\cosh\tau-\cos\theta)^4} \\ &= \frac{\cos^2\theta-2\cosh\tau\cos\theta+\cosh^2\tau}{(\cosh\tau-\cos\theta)^4} = \frac{(\cosh\tau-\cos\theta)^2}{(\cosh\tau-\cos\theta)^4} = \frac{1}{(\cosh\tau-\cos\theta)^2}\end{aligned} \tag{C.23}$$

$$\begin{aligned}\frac{g_{\tau\theta}}{a^2} &= \frac{\partial x}{\partial\tau}\frac{\partial x}{\partial\theta} + \frac{\partial y}{\partial\tau}\frac{\partial y}{\partial\theta} + \frac{\partial z}{\partial\tau}\frac{\partial z}{\partial\theta} \\ &= \frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta \\ &\quad + \frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta \\ &\quad + \frac{-\sin\theta\sinh\tau}{(\cosh\tau-\cos\theta)^2}\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2} \\ &= \frac{(1-\cos\theta\cosh\tau)(\sin\theta\sinh\tau)}{(\cosh\tau-\cos\theta)^4}\left(\cos^2\zeta+\sin^2\zeta-1\right) = 0\end{aligned} \tag{C.24}$$

$$\begin{aligned}\frac{g_{\tau\zeta}}{a^2} &= \frac{\partial x}{\partial\tau}\frac{\partial x}{\partial\zeta} + \frac{\partial y}{\partial\tau}\frac{\partial y}{\partial\zeta} + \frac{\partial z}{\partial\tau}\frac{\partial z}{\partial\zeta} \\ &= -\frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\frac{\sinh\tau}{\cosh\tau-\cos\theta}\sin\zeta + \frac{1-\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta + 0 \\ &= \frac{(1-\cosh\tau\cos\theta)}{(\cosh\tau-\cos\theta)^2}\sin\zeta\cos\zeta\,(-1+1) = 0\end{aligned} \tag{C.25}$$

$$\begin{aligned}
\frac{g_{\theta\theta}}{a^2} &= \frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\theta} + \frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\theta} + \frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\theta} \\
&= \left(\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\cos\zeta\right)^2 + \left(\frac{-\sinh\tau\sin\theta}{(\cosh\tau-\cos\theta)^2}\sin\zeta\right)^2 + \left(\frac{\cos\theta\cosh\tau-1}{(\cosh\tau-\cos\theta)^2}\right)^2 \\
&= \frac{\sinh^2\tau\sin^2\theta + (1-\cos\theta\cosh\tau)^2}{(\cosh\tau-\cos\theta)^4} = \frac{\sinh^2\tau\sin^2\theta + 1 - 2\cosh\tau\cos\theta + \cos^2\theta\cosh^2\tau}{(\cosh\tau-\cos\theta)^4} \\
&= \frac{\sinh^2\tau\sin^2\theta + (1-\sin^2\theta)\cosh^2\tau + 1 - 2\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^4} = \frac{-\sin^2\theta + \cosh^2\tau + 1 - 2\cosh\tau\cos\theta}{(\cosh\tau-\cos\theta)^4} \\
&= \frac{\cosh^2\tau - 2\cosh\tau\cos\theta + \cos^2\theta}{(\cosh\tau-\cos\theta)^4} = \frac{(\cosh\tau-\cos\theta)^2}{(\cosh\tau-\cos\theta)^4} = \frac{1}{(\cosh\tau-\cos\theta)^2}
\end{aligned} \tag{C.26}$$

$$\frac{g_{\theta\zeta}}{a^2} = \frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\zeta} + \frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\zeta} + \frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\zeta} = \frac{g_{32}^2}{a^2} = 0 \tag{C.27}$$

$$\begin{aligned}
\frac{g_{\zeta\zeta}}{a^2} &= \frac{\partial x}{\partial\zeta}\frac{\partial x}{\partial\zeta} + \frac{\partial y}{\partial\zeta}\frac{\partial y}{\partial\zeta} + \frac{\partial z}{\partial\zeta}\frac{\partial z}{\partial\zeta} \\
&= \left(-\frac{\sinh\tau}{\cosh\tau-\cos\theta}\sin\zeta\right)^2 + \left(\frac{\sinh\tau}{\cosh\tau-\cos\theta}\cos\zeta\right)^2 + 0 = \frac{\sinh^2\tau}{(\cosh\tau-\cos\theta)^2}
\end{aligned} \tag{C.28}$$

Thus, we find

$$g_{ij} = \begin{bmatrix} \frac{a^2}{(\cosh\tau-\cos\theta)^2} & 0 & 0 \\ 0 & \frac{a^2}{(\cosh\tau-\cos\theta)^2} & 0 \\ 0 & 0 & \frac{a^2\sinh^2\tau}{(\cosh\tau-\cos\theta)^2} \end{bmatrix}. \tag{C.29}$$

We of course then have

$$\begin{aligned}
\mathbf{J} = \boldsymbol{\mathcal{J}}^{-1} = \frac{\partial(\tau,\theta,\zeta)}{\partial(x,y,z)} &= \begin{bmatrix} \frac{\partial\tau}{\partial x} & \frac{\partial\tau}{\partial y} & \frac{\partial\tau}{\partial z} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} & \frac{\partial\theta}{\partial z} \\ \frac{\partial\zeta}{\partial x} & \frac{\partial\zeta}{\partial y} & \frac{\partial\zeta}{\partial z} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\cos\zeta(1-\cos\theta\cosh\tau)}{a} & \frac{\sin\zeta(1-\cos\theta\cosh\tau)}{a} & -\frac{\sin\theta\sinh\tau}{a} \\ -\frac{\cos\zeta\sin\theta\sinh\tau}{a} & -\frac{\sin\zeta\sin\theta\sinh\tau}{a} & \frac{\cos\theta\cosh\tau-1}{a} \\ \frac{(\cos\theta-\cosh\tau)\operatorname{csch}\tau\sin\zeta}{a} & \frac{(\cosh\tau-\cos\theta)\operatorname{csch}\tau\cos\zeta}{a} & 0 \end{bmatrix}
\end{aligned} \tag{C.30}$$

$$J = \frac{1}{\mathcal{J}} = \frac{(\cosh\tau-\cos\theta)^3}{a^3\sinh\tau} \tag{C.31}$$

This then gives us (utilizing $(1-xy)^2 + (1-x^2)(y^2-1) = (x-y)^2$)

$$\mathbf{e}^1 = \mathbf{e}^\tau = \nabla\tau = \frac{\cos\zeta(1-\cos\theta\cosh\tau)}{a}\nabla x + \frac{\sin\zeta(1-\cos\theta\cosh\tau)}{a}\nabla y + -\frac{\sin\theta\sinh\tau}{a}\nabla z \tag{C.32}$$

$$\begin{aligned}
|\nabla\tau|^2 &= \frac{(1-\cos\theta\cosh\tau)^2\cos^2\zeta + (1-\cos\theta\cosh\tau)^2\sin^2\zeta + \sin^2\theta\sinh^2\tau}{a^2} \\
&= \frac{(1-\cos\theta\cosh\tau)^2 + \sin^2\theta\sinh^2\tau}{a^2} = \frac{(1-\cos\theta\cosh\tau)^2 + (1-\cos^2\theta)(\cosh^2\tau-1)}{a^2} \\
&= \frac{(\cosh\tau-\cos\theta)^2}{a^2}
\end{aligned} \tag{C.33}$$

$$|\nabla\tau| = \frac{\cosh\tau - \cos\theta}{a} \tag{C.34}$$

$$\mathbf{e}^2 = \mathbf{e}^\theta = \nabla\theta = -\frac{\cos\zeta\sin\theta\sinh\tau}{a}\nabla x + -\frac{\sin\zeta\sin\theta\sinh\tau}{a}\nabla y + \frac{\cos\theta\cosh\tau - 1}{a}\nabla z \tag{C.35}$$

$$\begin{aligned} |\nabla\theta|^2 &= \frac{\sinh^2\tau\sin^2\theta\cos^2\zeta + \sinh^2\tau\sin^2\theta\sin^2\zeta + (1-\cosh\tau\cos\theta)^2}{a^2} \\ &= \frac{\sinh^2\tau\sin^2\theta + (1-\cosh\tau\cos\theta)^2}{a^2} = \frac{(\cosh\tau-\cos\theta)^2}{a^2} \end{aligned} \tag{C.36}$$

$$|\nabla\theta| = \frac{\cosh\tau - \cos\theta}{a} \tag{C.37}$$

$$\mathbf{e}^3 = \mathbf{e}^\zeta = \nabla\zeta = \frac{(\cos\theta - \cosh\tau)\operatorname{csch}\tau\sin\zeta}{a}\nabla x + \frac{(\cosh\tau - \cos\theta)\operatorname{csch}\tau\cos\zeta}{a}\nabla y \tag{C.38}$$

$$|\nabla\zeta|^2 = \frac{(\cosh\tau-\cos\theta)^2\operatorname{csch}^2\tau\sin^2\zeta + (\cosh\tau-\cos\theta)^2\operatorname{csch}^2\tau\cos^2\zeta}{a^2} = \frac{(\cosh\tau-\cos\theta)^2}{a^2\sinh^2\tau} \tag{C.39}$$

$$|\nabla\zeta| = \frac{\cosh\tau - \cos\theta}{a|\sinh\tau|} \tag{C.40}$$

Note that

$$g^{ij} = \begin{bmatrix} \frac{(\cosh\tau-\cos\theta)^2}{a^2} & 0 & 0 \\ 0 & \frac{(\cosh\tau-\cos\theta)^2}{a^2} & 0 \\ 0 & 0 & \frac{(\cosh\tau-\cos\theta)^2}{a^2\sinh^2\tau} \end{bmatrix} \tag{C.41}$$

Thus we find for the Christoffel symbols that

$$\Gamma_{k,ij} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial\xi^j} + \frac{\partial g_{jk}}{\partial\xi^i} - \frac{\partial g_{ij}}{\partial\xi^k}\right] \tag{C.42}$$

$$\Gamma_{\tau,ij} = \begin{bmatrix} \frac{-a^2\sinh^2\tau}{(\cosh\tau-\cos\theta)^3} & \frac{-a^2\sin^2\theta}{(\cosh\tau-\cos\theta)^3} & 0 \\ \frac{-a^2\sin^2\theta}{(\cosh\tau-\cos\theta)^3} & \frac{a^2\sinh^2\tau}{(\cosh\tau-\cos\theta)^3} & 0 \\ 0 & 0 & \frac{a^2\sinh\tau(\cosh\tau\cos\theta-1)}{(\cosh\tau-\cos\theta)^3} \end{bmatrix} \tag{C.43}$$

$$\Gamma_{\theta,ij} = \begin{bmatrix} \frac{a^2\sin^2\theta}{(\cosh\tau-\cos\theta)^3} & \frac{-a^2\sinh\tau}{(\cosh\tau-\cos\theta)^3} & 0 \\ \frac{-a^2\sinh\tau}{(\cosh\tau-\cos\theta)^3} & \frac{a^2\sin^2\theta}{(\cosh\tau-\cos\theta)^3} & 0 \\ 0 & 0 & \frac{a^2\sin\theta\sinh^2\tau}{(\cosh\tau-\cos\theta)^3} \end{bmatrix} \tag{C.44}$$

$$\Gamma_{\zeta,ij} = \begin{bmatrix} 0 & 0 & \frac{a^2\sinh\tau(1-\cosh\tau\cos\theta)}{(\cosh\tau-\cos\theta)^3} \\ 0 & 0 & \frac{-a^2\sin\theta\sinh^2\tau}{(\cosh\tau-\cos\theta)^3} \\ \frac{a^2\sinh\tau(1-\cosh\tau\cos\theta)}{(\cosh\tau-\cos\theta)^3} & \frac{-a^2\sin\theta\sinh^2\tau}{(\cosh\tau-\cos\theta)^3} & 0 \end{bmatrix} \tag{C.45}$$

and

$$\Gamma^k_{ij} = g^{kl}\Gamma_{l,ij} \tag{C.46}$$

$$\Gamma^\tau_{ij} = \begin{bmatrix} \frac{-\sinh\tau}{\cosh\tau-\cos\theta} & \frac{-\sin\theta}{\cosh\tau-\cos\theta} & 0 \\ \frac{-\sin\theta}{\cosh\tau-\cos\theta} & \frac{\sinh\tau}{\cosh\tau-\cos\theta} & 0 \\ 0 & 0 & \frac{\sinh\tau(\cos\theta\cosh\tau-1)}{\cosh\tau-\cos\theta} \end{bmatrix} \tag{C.47}$$

$$\Gamma^{\theta}_{ij} = \begin{bmatrix} \frac{\sin\theta}{\cosh\tau-\cos\theta} & \frac{-\sinh\tau}{\cosh\tau-\cos\theta} & 0 \\ \frac{-\sinh\tau}{\cosh\tau-\cos\theta} & \frac{-\sin\theta}{\cosh\tau-\cos\theta} & 0 \\ 0 & 0 & \frac{\sinh\tau\sin\theta}{\cosh\tau-\cos\theta} \end{bmatrix} \tag{C.48}$$

$$\Gamma^{\zeta}_{ij} = \begin{bmatrix} 0 & 0 & \frac{1-\cosh\tau\cos\theta}{\sinh\tau(\cosh\tau-\cos\theta)} \\ 0 & 0 & \frac{-\sin\theta}{\cosh\tau-\cos\theta} \\ \frac{1-\cosh\tau\cos\theta}{\sinh\tau(\cosh\tau-\cos\theta)} & \frac{-\sin\theta}{\cosh\tau-\cos\theta} & 0 \end{bmatrix} \tag{C.49}$$

C.8 Differential Operators in Coordinate Systems

The following will show the gradient, curl, and divergence of quantities in various coordinate systems. To summarize, for scalar f, vector $\mathbf{A}$, and second order tensor $\overset{\leftrightarrow}{\mathbf{T}}$ we find

$$\nabla f = \mathbf{e}^i \frac{\partial f}{\partial \xi^i} \tag{C.1}$$

$$\nabla \mathbf{A} = \left(\frac{\partial A_k}{\partial \xi^j} - A_i \Gamma^i_{kj} \right) \mathbf{e}^j \mathbf{e}^k \tag{C.2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{\mathcal{J}} \frac{\partial}{\partial \xi^i} \left(\mathcal{J} A^i \right) \tag{C.3}$$

$$(\boldsymbol{\nabla} \times \mathbf{A})^k = \frac{\epsilon^{ijk}}{\mathcal{J}} \frac{\partial A_j}{\partial \xi^i} \tag{C.4}$$

$$\boldsymbol{\nabla} \cdot \overset{\leftrightarrow}{\mathbf{T}} = \left(\frac{1}{\mathcal{J}} \frac{\partial \mathcal{J} T^{ij}}{\partial \xi^i} + T^{il} \Gamma^j_{il} \right) \mathbf{e}_j \tag{C.5}$$

$$\boldsymbol{\nabla} \times \overset{\leftrightarrow}{\mathbf{T}} = \frac{\epsilon^{ijk}}{\mathcal{J}} \mathbf{e}_k \mathbf{e}^l \left(\frac{\partial T_{jl}}{\partial \xi^i} + T_{ip} \Gamma^p_{jl} \right) \tag{C.6}$$

I will use that

$$\mathbf{A} = A(1)\hat{\mathbf{e}}^1 + A(2)\hat{\mathbf{e}}^2 + A(3)\hat{\mathbf{e}}^3 \tag{C.7}$$

$$\overset{\leftrightarrow}{\mathbf{T}} = \sum_{i,j=1}^{3} T(i,j)\hat{\mathbf{e}}^i \hat{\mathbf{e}}^j \tag{C.8}$$

to put vectors and tensors in their standard form (the basis vectors are the normalized tangent-reciprocal basis vectors).

C.8.1 (Common) Cylindrical Coordinates

We use the right handed coordinates (r, φ, Z). Here $\mathcal{J} = r$.

C.8.1.1 Gradient

First the gradient of a scalar is found via

$$\begin{aligned} \nabla f &= \mathbf{e}^r \frac{\partial f}{\partial r} + \mathbf{e}^{\varphi} \frac{\partial f}{\partial \varphi} + \mathbf{e}^Z \frac{\partial f}{\partial Z} \\ &= \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \varphi} \hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial Z} \hat{\mathbf{Z}} \end{aligned} \tag{C.9}$$

The gradient of a vector is given by

$$(\nabla\mathbf{A})(r,r) = (\nabla\mathbf{A})_{rr} = \frac{\partial A_r}{\partial r} = \frac{\partial A(r)}{\partial r} \tag{C.10}$$

$$(\nabla\mathbf{A})(r,\varphi) = \frac{1}{r}(\nabla\mathbf{A})_{r\varphi} = \frac{1}{r}\frac{\partial A_\varphi}{\partial r} - \frac{A_\varphi}{r^2} = \frac{1}{r}\frac{\partial[rA(\varphi)]}{\partial r} - \frac{A(\varphi)}{r} = \frac{\partial A(\varphi)}{\partial r} \tag{C.11}$$

$$(\nabla\mathbf{A})(r,Z) = (\nabla\mathbf{A})_{rZ} = \frac{\partial A_Z}{\partial r} = \frac{\partial A(Z)}{\partial r} \tag{C.12}$$

$$(\nabla\mathbf{A})(\varphi,r) = \frac{1}{r}(\nabla\mathbf{A})_{\varphi r} = \frac{1}{r}\left(\frac{\partial A_r}{\partial\varphi} - \frac{A_\varphi}{r}\right) = \frac{\partial A(r)}{\partial\varphi} - \frac{A(\varphi)}{r} \tag{C.13}$$

$$(\nabla\mathbf{A})(\varphi,\varphi) = \frac{1}{r^2}(\nabla\mathbf{A})_{\varphi\varphi} = \frac{1}{r^2}\left(rA_r + \frac{\partial A_\varphi}{\partial\varphi}\right) = \frac{1}{r}\frac{\partial A(\varphi)}{\partial\varphi} + \frac{A(r)}{r} \tag{C.14}$$

$$(\nabla\mathbf{A})(\varphi,Z) = \frac{1}{r}(\nabla\mathbf{A})_{\varphi Z} = \frac{1}{r}\frac{\partial A_Z}{\partial\varphi} = \frac{1}{r}\frac{\partial A(Z)}{\partial\varphi} \tag{C.15}$$

$$(\nabla\mathbf{A})(Z,r) = (\nabla\mathbf{A})_{Zr} = \frac{\partial A_r}{\partial Z} = \frac{\partial A(r)}{\partial Z} \tag{C.16}$$

$$(\nabla\mathbf{A})(Z,\varphi) = \frac{1}{r}(\nabla\mathbf{A})_{Z\varphi} = \frac{1}{r}\left(\frac{\partial A_\varphi}{\partial Z}\right) = \frac{\partial A(\varphi)}{\partial Z} \tag{C.17}$$

$$(\nabla\mathbf{A})(Z,Z) = (\nabla\mathbf{A})_{ZZ} = \frac{\partial A_Z}{\partial Z} = \frac{\partial A(Z)}{\partial Z} \tag{C.18}$$

As a matrix where rows represent the first index and columns the second index

$$\begin{bmatrix} \frac{\partial A(r)}{\partial r} & \frac{\partial A(\varphi)}{\partial r} & \frac{\partial A(Z)}{\partial r} \\ \frac{1}{r}\frac{\partial A(r)}{\partial\varphi} - \frac{A(\varphi)}{r} & \frac{1}{r}\frac{\partial A(\varphi)}{\partial\varphi} + \frac{A(r)}{r} & \frac{1}{r}\frac{\partial A(Z)}{\partial\phi} \\ \frac{\partial A(r)}{\partial z} & \frac{\partial A(\varphi)}{\partial Z} & \frac{\partial A(Z)}{\partial Z} \end{bmatrix} \tag{C.19}$$

C.8.1.2 Divergence

The divergence of a vector is found by

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{1}{r}\frac{\partial(rA^r)}{\partial r} + \frac{1}{r}\frac{\partial(rA^\varphi)}{\partial\varphi} + \frac{1}{r}\frac{\partial(rA^Z)}{\partial Z} \\ &= \frac{1}{r}\frac{\partial(rA(r))}{\partial r} + \frac{1}{r}\frac{\partial A(\varphi)}{\partial\varphi} + \frac{\partial A(Z)}{\partial Z} \end{aligned} \tag{C.20}$$

The divergence of a second order tensor is found by

$$\begin{aligned} (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(r) = (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^r &= \frac{1}{r}\left(\frac{\partial(rT^{rr})}{\partial r} + \frac{\partial(rT^{\varphi r})}{\partial\varphi} + \frac{\partial(rT^{Zr})}{\partial Z}\right) - rT^{\varphi\varphi} \\ &= \frac{1}{r}\frac{\partial[rT(r,r)]}{\partial r} + \frac{1}{r}\frac{\partial T(\varphi,r)}{\partial\varphi} + \frac{\partial T(Z,r)}{\partial Z} - \frac{T(\varphi,\varphi)}{r} \end{aligned} \tag{C.21}$$

$$\begin{aligned} (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(\varphi) = r(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^\varphi &= r\frac{1}{r}\left(\frac{\partial(rT^{r\varphi})}{\partial r} + \frac{\partial(rT^{\varphi\varphi})}{\partial\varphi} + \frac{\partial(rT^{Z\varphi})}{\partial Z}\right) + r\frac{T^{r\varphi} + T^{\varphi r}}{r} \\ &= \frac{\partial T(r,\varphi)}{\partial r} + \frac{1}{r}\frac{\partial T(\varphi,\varphi)}{\partial\varphi} + \frac{\partial T(Z,\varphi)}{\partial Z} + \frac{T(r,\varphi) + T(\varphi,r)}{r} \\ &= \frac{1}{r}\frac{\partial[rT(r,\varphi)]}{\partial r} + \frac{1}{r}\frac{\partial T(\varphi,\varphi)}{\partial\varphi} + \frac{\partial T(Z,\varphi)}{\partial Z} + \frac{T(\varphi,r)}{r} \end{aligned} \tag{C.22}$$

$$(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(Z)=(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{Z}=\frac{1}{r}\left(\frac{\partial(rT^{rZ})}{\partial r}+\frac{\partial(rT^{\varphi Z})}{\partial\varphi}+\frac{\partial(rT^{ZZ})}{\partial Z}\right)$$
$$=\frac{1}{r}\frac{\partial[rT(r,\varphi)]}{\partial r}+\frac{1}{r}\frac{\partial T(\varphi,Z)}{\partial\varphi}+\frac{\partial T(Z,Z)}{\partial Z} \tag{C.23}$$

(C.24)

C.8.1.3 Curl

The curl of a vector is given by

$$(\boldsymbol{\nabla}\times\mathbf{A})(r)=(\boldsymbol{\nabla}\times\mathbf{A})^{r}=\frac{1}{r}\left(\frac{\partial A_Z}{\partial\varphi}-\frac{\partial A_\varphi}{\partial Z}\right)=\frac{1}{r}\left(\frac{\partial A(Z)}{\partial\varphi}-\frac{\partial[rA(\varphi)]}{\partial Z}\right)$$
$$=\frac{1}{r}\frac{\partial A(Z)}{\partial\varphi}-\frac{\partial A(\varphi)}{\partial Z} \tag{C.25}$$

$$(\boldsymbol{\nabla}\times\mathbf{A})(\varphi)=r(\boldsymbol{\nabla}\times\mathbf{A})^{\varphi}=\frac{r}{r}\left(\frac{\partial A_r}{\partial Z}-\frac{\partial A_Z}{\partial r}\right)=\left(\frac{\partial A(r)}{\partial Z}-\frac{\partial A(Z)}{\partial r}\right)$$
$$=\frac{\partial A(r)}{\partial Z}-\frac{\partial A(Z)}{\partial r} \tag{C.26}$$

$$(\boldsymbol{\nabla}\times\mathbf{A})(Z)=(\boldsymbol{\nabla}\times\mathbf{A})^{Z}=\frac{1}{r}\left(\frac{\partial A_\varphi}{\partial r}-\frac{\partial A_r}{\partial\varphi}\right)=\frac{1}{r}\left(\frac{\partial[rA(\varphi)]}{\partial r}-\frac{\partial A(r)}{\partial\varphi}\right)$$
$$=\frac{1}{r}\frac{\partial[rA(\varphi)]}{\partial r}-\frac{1}{r}\frac{\partial A(r)}{\partial\varphi} \tag{C.27}$$

The curl of a second order tensor is given by

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,r)=(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{r}_{\cdot r}=\frac{1}{r}\left(\frac{\partial T_{Zr}}{\partial\varphi}-\frac{\partial T_{\varphi r}}{\partial Z}\right)-\frac{T_{Z\varphi}}{r^2}$$
$$=\frac{1}{r}\frac{\partial T(Z,r)}{\partial\varphi}-\frac{\partial T(\varphi,r)}{\partial Z}-\frac{T(Z,\varphi)}{r} \tag{C.28}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,\varphi)=\frac{1}{r}(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{r}_{\cdot\varphi}=\frac{1}{r^2}\left(\frac{\partial T_{Z\varphi}}{\partial\varphi}-\frac{\partial T_{\varphi\varphi}}{\partial Z}\right)+\frac{T_{Zr}}{r}$$
$$=\frac{1}{r}\frac{\partial T(Z,\varphi)}{\partial\varphi}-\frac{\partial T(\varphi,\varphi)}{\partial Z}+\frac{T(Z,r)}{r} \tag{C.29}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,Z)=(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{r}_{\cdot Z}=\frac{1}{r}\left(\frac{\partial T_{ZZ}}{\partial\varphi}-\frac{\partial T_{\varphi Z}}{\partial Z}\right)$$
$$=\frac{1}{r}\frac{\partial T(Z,Z)}{\partial\varphi}-\frac{\partial T(\varphi,Z)}{\partial Z} \tag{C.30}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\varphi,r)=r(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\varphi}_{\cdot r}=r\frac{1}{r}\left(\frac{\partial T_{rr}}{\partial Z}-\frac{\partial T_{Zr}}{\partial r}\right)$$
$$=\frac{\partial T(r,r)}{\partial Z}-\frac{\partial T(Z,r)}{\partial r} \tag{C.31}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\varphi,\varphi)=(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\varphi}_{\cdot\varphi}=\frac{1}{r}\left(\frac{\partial T_{r\varphi}}{\partial Z}-\frac{\partial T_{Z\varphi}}{\partial r}\right)+\frac{T_{Z\varphi}}{r^2}$$
$$=\frac{\partial T(r,\varphi)}{\partial Z}-\frac{\partial T(Z,\varphi)}{\partial r}+\frac{T(Z,\varphi)}{r} \tag{C.32}$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\varphi,Z) &= r(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\varphi}_{\cdot Z} = r\frac{1}{r}\left(\frac{\partial T_{rZ}}{\partial Z}-\frac{\partial T_{ZZ}}{\partial r}\right)\\
&= \frac{\partial T(r,Z)}{\partial Z}-\frac{\partial T(Z,Z)}{\partial r}
\end{aligned}
\tag{C.33}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(Z,r) &= (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{Z}_{\cdot r} = \frac{1}{r}\left(\frac{\partial T_{\varphi r}}{\partial r}-\frac{\partial T_{rr}}{\partial\varphi}\right)+\frac{T_{r\varphi}}{r^2}\\
&= \frac{\partial T(\varphi,r)}{\partial r}-\frac{1}{r}\frac{\partial T(r,r)}{\partial\varphi}+\frac{T(r,\varphi)}{r}
\end{aligned}
\tag{C.34}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(Z,\varphi) &= \frac{1}{r}(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{Z}_{\cdot\varphi} = \frac{1}{r^2}\left(\frac{\partial T_{\varphi\varphi}}{\partial r}-\frac{\partial T_{r\varphi}}{\partial\varphi}\right)-\frac{T_{rr}}{r}-\frac{T_{\varphi\varphi}}{r^3}\\
&= \frac{\partial T(\varphi,\varphi)}{\partial r}-\frac{1}{r}\frac{\partial T(r,\varphi)}{\partial\varphi}-\frac{T(r,r)}{r}-\frac{T(\varphi,\varphi)}{r}
\end{aligned}
\tag{C.35}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(Z,Z) &= (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{Z}_{\cdot Z} = \frac{1}{r}\left(\frac{\partial T_{\varphi Z}}{\partial r}-\frac{\partial T_{rZ}}{\partial\varphi}\right)\\
&= \frac{\partial T(\varphi,Z)}{\partial r}-\frac{1}{r}\frac{\partial T(r,Z)}{\partial\varphi}
\end{aligned}
\tag{C.36}
$$

C.8.2 (Plasma/Toroidal System) Cylindrical Coordinates

We use the right handed coordinates (R, Z, ζ). Here $\mathcal{J} = R$.

C.8.2.1 Gradient

First the gradient of a scalar is found via

$$\begin{aligned}\nabla f &= \mathbf{e}^R \frac{\partial f}{\partial R} + \mathbf{e}^Z \frac{\partial f}{\partial Z} + \mathbf{e}^\zeta \frac{\partial f}{\partial \zeta} \\ &= \frac{\partial f}{\partial R}\hat{\mathbf{R}} + \frac{\partial f}{\partial Z}\hat{\mathbf{Z}} + \frac{1}{R}\frac{\partial f}{\partial \zeta}\hat{\boldsymbol{\zeta}}\end{aligned} \tag{C.37}$$

The gradient of a vector is given by

$$(\nabla \mathbf{A})(R,R) = (\nabla \mathbf{A})_{RR} = \frac{\partial A_R}{\partial R} = \frac{\partial A(R)}{\partial R} \tag{C.38}$$

$$(\nabla \mathbf{A})(R,Z) = \frac{1}{R}(\nabla \mathbf{A})_{RZ} = \frac{\partial A_Z}{\partial R} = \frac{\partial A(Z)}{\partial R} \tag{C.39}$$

$$(\nabla \mathbf{A})(R,\zeta) = \frac{1}{R}(\nabla \mathbf{A})_{R\zeta} = \frac{1}{R}\left(\frac{\partial A_\zeta}{\partial R} - \frac{A_\zeta}{R}\right) = \frac{1}{R}\frac{\partial [RA(\zeta)]}{\partial R} - \frac{A(\zeta)}{R} = \frac{\partial A(\zeta)}{\partial R} \tag{C.40}$$

$$(\nabla \mathbf{A})(Z,R) = (\nabla \mathbf{A})_{ZR} = \frac{\partial A_R}{\partial Z} = \frac{\partial A(R)}{\partial Z} \tag{C.41}$$

$$(\nabla \mathbf{A})(Z,Z) = (\nabla \mathbf{A})_{ZZ} = \frac{\partial A_Z}{\partial Z} = \frac{\partial A(Z)}{\partial Z} \tag{C.42}$$

$$(\nabla \mathbf{A})(Z,\zeta) = \frac{1}{R}(\nabla \mathbf{A})_{Z\zeta} = \frac{1}{R}\left(\frac{\partial A_\zeta}{\partial Z}\right) = \frac{\partial A(\zeta)}{\partial Z} \tag{C.43}$$

$$(\nabla \mathbf{A})(\zeta,R) = \frac{1}{R}(\nabla \mathbf{A})_{\zeta R} = \frac{1}{R}\left(\frac{\partial A_R}{\partial \zeta} - \frac{A_\zeta}{R}\right) = \frac{1}{R}\frac{\partial A(R)}{\partial \zeta} - \frac{A(\zeta)}{R} \tag{C.44}$$

$$(\nabla \mathbf{A})(\zeta,Z) = \frac{1}{R}(\nabla \mathbf{A})_{\zeta Z} = \frac{1}{R}\left(\frac{\partial A_Z}{\partial \zeta}\right) = \frac{1}{R}\frac{\partial A(Z)}{\partial \zeta} \tag{C.45}$$

$$(\nabla \mathbf{A})(\zeta,\zeta) = \frac{1}{R^2}(\nabla \mathbf{A})_{\zeta\zeta} = \frac{1}{R^2}\left(A_R R + \frac{\partial A_\zeta}{\partial \zeta}\right) = \frac{1}{R}\frac{\partial A(\zeta)}{\partial \zeta} + \frac{A(R)}{R} \tag{C.46}$$

As a matrix where rows represent the first index and columns the second index

$$\begin{bmatrix} \frac{\partial A(R)}{\partial R} & \frac{\partial A(Z)}{\partial R} & \frac{\partial A(\zeta)}{\partial R} \\ \frac{\partial A(R)}{\partial Z} & \frac{\partial A(Z)}{\partial Z} & \frac{\partial A(\zeta)}{\partial Z} \\ \frac{1}{R}\frac{\partial A(R)}{\partial \zeta} - \frac{A(\varphi)}{R} & \frac{1}{R}\frac{\partial A(Z)}{\partial \zeta} & \frac{1}{R}\frac{\partial A(\zeta)}{\partial \zeta} + \frac{A(R)}{R} \end{bmatrix} \tag{C.47}$$

C.8.2.2 Divergence

The divergence of a vector is given by

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{1}{R}\frac{\partial (RA^R)}{\partial R} + \frac{1}{R}\frac{\partial (RA^Z)}{\partial Z} + \frac{1}{R}\frac{\partial (RA^\zeta)}{\partial \zeta} \\ &= \frac{1}{R}\frac{\partial (RA(R))}{\partial R} + \frac{1}{R}\frac{\partial A(Z)}{\partial Z} + \frac{\partial A(\zeta)}{\partial \zeta}\end{aligned} \tag{C.48}$$

The divergence of a second order tensor is given by

$$\begin{aligned}(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})(R) = (\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^R &= \frac{1}{R}\left(\frac{\partial(RT^{RR})}{\partial R}+\frac{\partial(RT^{ZR})}{\partial Z}+\frac{\partial(RT^{\zeta R})}{\partial\zeta}\right)-RT^{\zeta\zeta}\\ &= \frac{1}{R}\frac{\partial(RT(R,R))}{\partial R}+\frac{\partial T(Z,R)}{\partial Z}+\frac{1}{R}\frac{\partial T(\zeta,R)}{\partial\zeta}-\frac{T(\zeta,\zeta)}{R}\end{aligned} \tag{C.49}$$

$$\begin{aligned}(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})(Z) = (\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^Z &= \frac{1}{R}\left(\frac{\partial(RT^{RZ})}{\partial R}+\frac{\partial(RT^{ZZ})}{\partial Z}+\frac{\partial(RT^{\zeta Z})}{\partial\zeta}\right)\\ &= \frac{1}{R}\frac{\partial[RT(R,Z)]}{\partial R}+\frac{\partial T(Z,Z)}{\partial Z}+\frac{1}{R}\frac{\partial T(\zeta,Z)}{\partial\zeta}\end{aligned} \tag{C.50}$$

$$\begin{aligned}(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})(\zeta) = R(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^\zeta &= R\frac{1}{R}\left(\frac{\partial(RT^{R\zeta})}{\partial R}+\frac{\partial(RT^{Z\zeta})}{\partial Z}+\frac{\partial(RT^{\zeta\zeta})}{\partial\zeta}\right)+R\frac{T^{R\zeta}+T^{\zeta R}}{R}\\ &= \frac{\partial T(R,\zeta)}{\partial R}+\frac{\partial T(Z,\zeta)}{\partial Z}+\frac{1}{R}\frac{\partial T(\zeta,\zeta)}{\partial\zeta}+\frac{T(R,\zeta)+T(\zeta,R)}{R}\\ &= \frac{1}{R}\frac{\partial[RT(R,\zeta)]}{\partial R}+\frac{\partial T(Z,\zeta)}{\partial Z}+\frac{1}{R}\frac{\partial T(\zeta,\zeta)}{\partial\zeta}+\frac{T(\zeta,R)}{R}\end{aligned} \tag{C.51}$$

C.8.2.3 Curl

The curl of a vector is given by

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})(R) = (\boldsymbol{\nabla}\times\mathbf{A})^R &= \frac{1}{R}\left(\frac{\partial A_\zeta}{\partial Z}-\frac{\partial A_Z}{\partial\zeta}\right)=\frac{1}{R}\left(\frac{\partial[RA(\zeta)]}{\partial Z}-\frac{\partial A(Z)}{\partial\zeta}\right)\\ &= \frac{\partial A(\zeta)}{\partial Z}-\frac{1}{R}\frac{\partial A(Z)}{\partial\zeta}\end{aligned} \tag{C.52}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})(Z) = (\boldsymbol{\nabla}\times\mathbf{A})^Z &= \frac{1}{R}\left(\frac{\partial A_R}{\partial\zeta}-\frac{\partial A_\zeta}{\partial R}\right)\\ &= \frac{1}{R}\frac{\partial A(R)}{\partial\zeta}-\frac{\partial A(\zeta)}{\partial R}\end{aligned} \tag{C.53}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})(\zeta) = R(\boldsymbol{\nabla}\times\mathbf{A})^\zeta &= \frac{1}{R}\left(\frac{\partial A_R}{\partial Z}-\frac{\partial A_Z}{\partial R}\right)\\ &= \frac{1}{R}\left(\frac{\partial A(R)}{\partial Z}-\frac{\partial A(Z)}{\partial R}\right)\end{aligned} \tag{C.54}$$

The curl of a second order tensor is given by

$$\begin{aligned}(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(R,R) = (\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})^R_{\cdot R} &= \frac{1}{R}\left(\frac{\partial T_{\zeta R}}{\partial Z}-\frac{\partial T_{ZR}}{\partial\zeta}\right)+\frac{T_{Z\zeta}}{R^2}\\ &= \frac{\partial T(\zeta,R)}{\partial Z}-\frac{1}{R}\frac{\partial T(Z,R)}{\partial\zeta}+\frac{T(Z,\zeta)}{R}\end{aligned} \tag{C.55}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(R,Z) = (\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})^R_{\cdot Z} &= \frac{1}{R}\left(\frac{\partial T_{\zeta Z}}{\partial Z}-\frac{\partial T_{ZZ}}{\partial\zeta}\right)\\ &= \frac{\partial T(\zeta,Z)}{\partial R}-\frac{1}{R}\frac{\partial T(Z,Z)}{\partial\zeta}\end{aligned} \tag{C.56}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(R,\zeta)=\frac{1}{R}(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})^{R}_{\cdot\zeta}=\frac{1}{R^2}\left(\frac{\partial T_{\zeta\zeta}}{\partial Z}-\frac{\partial T_{Z\zeta}}{\partial\zeta}\right)-\frac{RT_{ZR}}{R^2}$$
$$=\frac{\partial T(\zeta,\zeta)}{\partial Z}-\frac{1}{R}\frac{\partial T(Z,\zeta)}{\partial\zeta}-\frac{T(Z,R)}{R} \tag{C.57}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(Z,R)=(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{Z}_{\cdot R}=\frac{1}{R}\left(\frac{\partial T_{RR}}{\partial\zeta}-\frac{\partial T_{\zeta R}}{\partial R}\right)-\frac{T_{R\zeta}}{R^2}$$
$$=\frac{1}{R}\frac{\partial T(R,R)}{\partial\zeta}-\frac{\partial T(\zeta,R)}{\partial R}-\frac{T(R,\zeta)}{R} \tag{C.58}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(Z,Z)=(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{Z}\cdot Z=\frac{1}{R}\left(\frac{\partial T_{RZ}}{\partial\zeta}-\frac{\partial T_{\zeta Z}}{\partial R}\right)$$
$$=\frac{1}{R}\frac{\partial T(R,Z)}{\partial\zeta}-\frac{\partial T(\zeta,Z)}{\partial R} \tag{C.59}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(Z,\zeta)=\frac{1}{R}(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{Z}_{\cdot\zeta}=\frac{1}{R^2}\left(\frac{\partial T_{R\zeta}}{\partial\zeta}-\frac{\partial T_{\zeta\zeta}}{\partial R}\right)+\frac{RT_{RR}+\frac{T_{\zeta\zeta}}{R}}{R^2}$$
$$=\frac{1}{R}\frac{\partial T(R,\zeta)}{\partial\zeta}-\frac{\partial T(\zeta,\zeta)}{\partial R}+\frac{T(R,R)+T(\zeta,\zeta)}{R} \tag{C.60}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(\zeta,R)=R(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{\zeta}_{\cdot R}=R\frac{1}{R}\left(\frac{\partial T_{ZR}}{\partial R}-\frac{\partial T_{RR}}{\partial Z}\right)$$
$$=\frac{\partial T(Z,R)}{\partial R}-\frac{\partial T(R,R)}{\partial Z} \tag{C.61}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(\zeta,Z)=R(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{\zeta}_{\cdot Z}=R\frac{1}{R}\left(\frac{\partial T_{ZZ}}{\partial R}-\frac{\partial T_{RZ}}{\partial Z}\right)$$
$$=\frac{\partial T(Z,Z)}{\partial R}-\frac{\partial T(R,Z)}{\partial Z} \tag{C.62}$$

$$(\boldsymbol{\nabla}\times\overset{\leftrightarrow}{\mathbf{T}})(\zeta,\zeta)=(\boldsymbol{\nabla}\cdot\overset{\leftrightarrow}{\mathbf{T}})^{\zeta}_{\cdot\zeta}=\frac{1}{R}\left(\frac{\partial T_{Z\zeta}}{\partial R}-\frac{\partial T_{R\zeta}}{\partial Z}\right)-\frac{T_{Z\zeta}}{R^2}$$
$$=\frac{\partial T(Z,\zeta)}{\partial R}-\frac{\partial T(R,\zeta)}{\partial Z}-\frac{T(Z,\zeta)}{R} \tag{C.63}$$

C.8.3 (Physicists') Spherical Coordinates

We use the right handed coordinates (r, θ, φ). Here $\mathcal{J} = r^2 \sin\theta$.

C.8.3.1 Gradient

First the gradient of a scalar is found via

$$\begin{aligned}\nabla f &= \mathbf{e}^r \frac{\partial f}{\partial r} + \mathbf{e}^\theta \frac{\partial f}{\partial \theta} + \mathbf{e}^\varphi \frac{\partial f}{\partial \varphi} \\ &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}\end{aligned} \tag{C.64}$$

The gradient of a vector is given by

$$(\nabla\mathbf{A})(r,r) = (\nabla\mathbf{A})_{rr} = \frac{\partial A_r}{\partial r} = \frac{\partial A(r)}{\partial r} \tag{C.65}$$

$$(\nabla\mathbf{A})(r,\theta) = \frac{1}{r}(\nabla\mathbf{A})_{r\theta} = \frac{1}{r}\left(\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r}\right) = \frac{1}{r}\frac{\partial[rA(\theta)]}{\partial r} - \frac{A(\theta)}{r} = \frac{\partial A(\theta)}{\partial r} \tag{C.66}$$

$$(\nabla\mathbf{A})(r,\varphi) = \frac{1}{r\sin\theta}(\nabla\mathbf{A})_{r\varphi} = \frac{1}{r\sin\theta}\left(\frac{\partial A_\varphi}{\partial r} - \frac{A_\varphi}{r}\right) = \frac{1}{r}\frac{\partial[rA(\varphi)]}{\partial r} - \frac{A(\varphi)}{r} = \frac{\partial A(\varphi)}{\partial r} \tag{C.67}$$

$$(\nabla\mathbf{A})(\theta,r) = \frac{1}{r}(\nabla\mathbf{A})_{\theta r} = \frac{1}{r}\left(\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r}\right) = \frac{1}{r}\frac{\partial A(r)}{\partial \theta} - \frac{A(\theta)}{r} \tag{C.68}$$

$$(\nabla\mathbf{A})(\theta,\theta) = \frac{1}{r^2}(\nabla\mathbf{A})_{\theta\theta} = \frac{1}{r^2}\left(\frac{\partial A_\theta}{\partial \theta} + A_r r\right) = \frac{1}{r}\frac{\partial A(\theta)}{\partial \theta} + \frac{A(r)}{r} \tag{C.69}$$

$$\begin{aligned}(\nabla\mathbf{A})(\theta,\varphi) &= \frac{1}{r^2\sin\theta}(\nabla\mathbf{A})_{\theta\varphi} = \frac{1}{r^2\sin\theta}\left(\frac{\partial A_\varphi}{\partial \theta} - \frac{A_\varphi\cos\theta}{\sin\theta}\right) \\ &= \frac{1}{r\sin\theta}\frac{\partial[\sin\theta\, A(\varphi)]}{\partial\theta} - \frac{A(\varphi)\cot\theta}{r}\end{aligned} \tag{C.70}$$

$$(\nabla\mathbf{A})(\varphi,r) = \frac{1}{r\sin\theta}(\nabla\mathbf{A})_{\varphi r} = \frac{1}{r\sin\theta}\left(\frac{\partial A_r}{\partial \varphi} - \frac{A_\varphi}{r}\right) = \frac{1}{r\sin\theta}\frac{\partial A(r)}{\partial\varphi} - \frac{A(\varphi)}{r} \tag{C.71}$$

$$(\nabla\mathbf{A})(\varphi,\theta) = \frac{1}{r^2\sin\theta}(\nabla\mathbf{A})_{\varphi\theta} = \frac{1}{r^2\sin\theta}\left(\frac{\partial A_\theta}{\partial\varphi} - A_\varphi\cot\theta\right) = \frac{1}{r\sin\theta}\frac{\partial A(\theta)}{\partial\varphi} - \frac{A(\varphi)\cot\theta}{r} \tag{C.72}$$

$$\begin{aligned}(\nabla\mathbf{A})(\varphi,\varphi) &= \frac{1}{r^2\sin^2\theta}(\nabla\mathbf{A})_{\varphi\varphi} = \frac{1}{r^2\sin^2\theta}\left(\frac{\partial A_\varphi}{\partial\varphi} + A_r r\sin\theta + A_\theta\sin\theta\cos\theta\right) \\ &= \frac{1}{r\sin\theta}\frac{\partial A(\varphi)}{\partial\varphi} + \frac{A(r)}{r\sin\theta} + \frac{A(\theta)\cot\theta}{r}\end{aligned} \tag{C.73}$$

As a matrix where rows represent the first index and columns the second index

$$\begin{bmatrix} \dfrac{\partial A(r)}{\partial r} & \dfrac{\partial A(\theta)}{\partial r} & \dfrac{\partial A(\varphi)}{\partial r} \\ \dfrac{1}{r}\left(\dfrac{\partial A(r)}{\partial\theta} - A(\theta)\right) & \dfrac{1}{r}\left(\dfrac{\partial A(\theta)}{\partial\theta} + A(r)\right) & \dfrac{1}{r\sin\theta}\left(\dfrac{\partial[\sin\theta\, A(\varphi)]}{\partial\theta} - A(\varphi)\cos\theta\right) \\ \dfrac{1}{r\sin\theta}\left(\dfrac{\partial A(r)}{\partial\varphi} - A(\varphi)\sin\theta\right) & \dfrac{1}{r\sin\theta}\left(\dfrac{\partial A(\theta)}{\partial\varphi} - A(\varphi)\cos\theta\right) & \dfrac{1}{r\sin\theta}\left(\dfrac{\partial A(\varphi)}{\partial\varphi} + A(r) + A(\theta)\cos\theta\right) \end{bmatrix} \tag{C.74}$$

C.8.3.2 Divergence

The divergence of a vector is given by

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{1}{r^2\sin\theta}\frac{\partial(r^2\sin\theta A^r)}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial(r^2\sin\theta A^\theta)}{\partial\theta} + \frac{1}{r^2\sin\theta}\frac{\partial(r^2\sin\theta A^\varphi)}{\partial\varphi}\\ &= \frac{1}{r^2}\frac{\partial(r^2A(r))}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial[\sin\theta A(\theta)]}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A(\varphi)}{\partial\varphi}\end{aligned} \tag{C.75}$$

The divergence of a second order tensor is given by

$$\begin{aligned}(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(r) = (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^r &= \frac{1}{r^2\sin\theta}\left(\frac{\partial(\mathcal{J}T^{rr})}{\partial r} + \frac{\partial(\mathcal{J}T^{\theta r})}{\partial\theta} + \frac{\partial(\mathcal{J}T^{\varphi r})}{\partial\varphi} - rT^{\theta\theta}\right) - r\sin^2\theta T^{\varphi\varphi}\\ &= \frac{1}{r^2}\frac{\partial[r^2T(r,r)]}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial[\sin\theta T(\theta,r)]}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T(\varphi,r)}{\partial\varphi} - \frac{T(\theta,\theta)+T(\varphi,\varphi)}{r}\end{aligned} \tag{C.76}$$

$$\begin{aligned}&(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(\theta) = r(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^\theta\\ &= \frac{r}{r^2\sin\theta}\left(\frac{\partial(\mathcal{J}T^{r\theta})}{\partial r} + \frac{\partial(\mathcal{J}T^{\theta\theta})}{\partial\theta} + \frac{\partial(\mathcal{J}T^{\varphi\theta})}{\partial\varphi}\right) + r\frac{T^{r\theta}+T^{\theta r}}{r} - r\sin\theta\cos\theta T^{\varphi\varphi}\\ &= \frac{1}{r}\frac{\partial[rT(r,\theta)]}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial[\sin\theta T(\theta,\theta)]}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T(\varphi,\theta)}{\partial\varphi} + \frac{T(r,\theta)+T(\theta,r)}{r} + \frac{\cot\theta\, T(\varphi,\varphi)}{r}\\ &= \frac{1}{r^2}\frac{\partial[r^2T(r,\theta)]}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial[\sin\theta T(\theta,\theta)]}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T(\varphi,\theta)}{\partial\varphi} + \frac{T(\theta,r)}{r} + \frac{\cot\theta\, T(\varphi,\varphi)}{r}\end{aligned} \tag{C.77}$$

$$\begin{aligned}&(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(\varphi) = r\sin\theta(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^\varphi\\ &= \frac{r\sin\theta}{r^2\sin\theta}\left(\frac{\partial(\mathcal{J}T^{r\varphi})}{\partial r} + \frac{\partial(\mathcal{J}T^{\theta\varphi})}{\partial\theta} + \frac{\partial(\mathcal{J}T^{\varphi\varphi})}{\partial\varphi}\right) + r\sin\theta\left(\frac{T^{r\varphi}+T^{\varphi r}}{r} + \cot\theta[T^{\theta\varphi}+T^{\varphi\theta}]\right)\\ &= \frac{1}{r}\frac{\partial[rT(r,\varphi)]}{\partial r} + \frac{1}{r}\frac{\partial T(\theta,\varphi)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T(\varphi,\varphi)}{\partial\varphi} + \frac{T(r,\varphi)+T(\varphi,r)}{r} + \cot\theta\frac{T(\theta,\varphi)+T(\varphi,\theta)}{r}\\ &= \frac{1}{r^2}\frac{\partial[r^2T(r,\varphi)]}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial[\sin\theta\, T(\theta,\varphi)]}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T(\varphi,\varphi)}{\partial\varphi} + \frac{T(\varphi,r)}{r} + \cot\theta\frac{T(\varphi,\theta)}{r}\end{aligned} \tag{C.78}$$

C.8.3.3 Curl

The curl of a vector is given by

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})(r) = (\boldsymbol{\nabla}\times\mathbf{A})^r &= \frac{1}{r^2\sin\theta}\left(\frac{\partial A_\varphi}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\\ &= \frac{1}{r\sin\theta}\frac{\partial[\sin\theta A(\varphi)]}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A(\theta)}{\partial\varphi}\end{aligned} \tag{C.79}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})(\theta) = r(\boldsymbol{\nabla}\times\mathbf{A})^\theta &= r\frac{1}{r^2\sin\theta}\left(\frac{\partial A_r}{\partial\varphi} - \frac{\partial A_\varphi}{\partial r}\right)\\ &= \frac{1}{r\sin\theta}\frac{\partial A(r)}{\partial\varphi} - \frac{1}{r}\frac{\partial[rA(\varphi)]}{\partial r}\end{aligned} \tag{C.80}$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\mathbf{A})(\varphi) &= r\sin\theta(\boldsymbol{\nabla}\times\mathbf{A})^{\varphi} = \frac{r\sin\theta}{r^2\sin\theta}\left(\frac{\partial A_\theta}{\partial r}-\frac{\partial A_r}{\partial\theta}\right)\\
&= \frac{1}{r}\frac{\partial[rA(\theta)]}{\partial r}-\frac{1}{r}\frac{\partial A(r)}{\partial\theta}
\end{aligned}
\tag{C.81}
$$

The curl of a second order tensor is given by

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,r) &= (\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})^{r}_{\cdot r} = \frac{1}{r^2\sin\theta}\left(\frac{\partial T_{\varphi r}}{\partial\theta}-\frac{\partial T_{\theta r}}{\partial\varphi}\right)+\frac{T_{\theta\varphi}-T_{\varphi\theta}}{r^3\sin\theta}\\
&= \frac{1}{r\sin\theta}\frac{\partial[\sin\theta\,T(\varphi,r)]}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial T(\theta,r)}{\partial\varphi}+\frac{T(\theta,\varphi)-T(\varphi,\theta)}{r}
\end{aligned}
\tag{C.82}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,\theta) &= \frac{1}{r}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})^{r}_{\cdot\theta} = \frac{1}{r^3\sin\theta}\left(\frac{\partial T_{\varphi\theta}}{\partial\theta}-\frac{\partial T_{\theta\theta}}{\partial\varphi}\right)+\frac{\cot\theta\,T_{\theta\varphi}+rT_{\varphi r}}{r^3\sin\theta}\\
&= \frac{1}{r\sin\theta}\frac{\partial[\sin\theta\,T(\varphi,\theta)]}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial T(\theta,\theta)}{\partial\varphi}+\frac{\cot\theta\,T(\theta,\varphi)+T(\varphi,r)}{r}
\end{aligned}
\tag{C.83}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,\varphi) &= \frac{1}{r\sin\theta}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})^{r}_{\cdot\varphi}\\
&= \frac{1}{r^3\sin^2\theta}\left(\frac{\partial T_{\varphi\varphi}}{\partial\theta}-\frac{\partial T_{\theta\varphi}}{\partial\varphi}\right)-\frac{\cot\theta\,T_{\varphi\varphi}+r\sin^2\theta\,T_{\theta r}-\sin\theta\cos\theta\,T_{\theta\theta}}{r^3\sin^2\theta}\\
&= \frac{1}{r\sin^2\theta}\frac{\partial[\sin^2\theta\,T(\varphi,\varphi)]}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial T(\theta,\varphi)}{\partial\varphi}-\frac{\cot\theta[T(\varphi,\varphi)+T(\theta,\theta)]+T(\theta,r)}{r}
\end{aligned}
\tag{C.84}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\theta,r) &= r(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot r} = \frac{r}{r^2\sin\theta}\left(\frac{\partial T_{rr}}{\partial\varphi}-\frac{\partial T_{\varphi r}}{\partial r}\right)-r\frac{T_{r\varphi}}{r^3\sin\theta}\\
&= \frac{1}{r\sin\theta}\frac{\partial T(r,r)}{\partial\varphi}-\frac{1}{r}\frac{\partial[rT(\varphi,r)]}{\partial r}-\frac{T(r,\varphi)}{r}
\end{aligned}
\tag{C.85}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\theta,\theta) &= (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot\theta} = \frac{1}{r^2\sin\theta}\left(\frac{\partial T_{r\theta}}{\partial\varphi}-\frac{\partial T_{\varphi\theta}}{\partial r}\right)+\frac{\frac{1}{r}T_{\varphi\theta}-\cot\theta\,T_{r\varphi}}{r^2\sin\theta}\\
&= \frac{1}{r\sin\theta}\frac{\partial T(r,\theta)}{\partial\varphi}-\frac{1}{r^2}\frac{\partial[r^2T(\varphi,\theta)]}{\partial r}+\frac{T(\varphi,\theta)-\cot\theta\,T(r,\varphi)}{r}
\end{aligned}
\tag{C.86}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\theta,\varphi) &= \frac{1}{\sin\theta}(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot\varphi} = \frac{1}{r^2\sin^2\theta}\left(\frac{\partial T_{r\varphi}}{\partial\varphi}-\frac{\partial T_{\varphi\varphi}}{\partial r}\right)+\frac{\frac{T_{\varphi\varphi}}{r}+r\sin^2\theta\,T_{rr}+\sin\theta\cos\theta\,T_{r\theta}}{r^2\sin^2\theta}\\
&= \frac{1}{r\sin\theta}\frac{\partial T(r,\varphi)}{\partial\varphi}-\frac{1}{r^2}\frac{\partial[r^2T(\varphi,\varphi)]}{\partial r}+\frac{T(\varphi,\varphi)+T(r,r)+T(r,\theta)}{r}
\end{aligned}
\tag{C.87}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\varphi,r) &= r\sin\theta(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\varphi}_{\cdot r} = \frac{r\sin\theta}{r^2\sin\theta}\left(\frac{\partial T_{\theta r}}{\partial r}-\frac{\partial T_{rr}}{\partial\theta}\right)+\frac{r\sin\theta\,T_{r\theta}}{r(r^2\sin\theta)}\\
&= \frac{1}{r}\frac{\partial[rT(\theta,r)]}{\partial r}-\frac{1}{r}\frac{\partial T(r,r)}{\partial\theta}+\frac{T(r,\theta)}{r}
\end{aligned}
\tag{C.88}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\varphi,\theta) &= \sin\theta(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\varphi}_{\cdot\theta} = \frac{\sin\theta}{r^2\sin\theta}\left(\frac{\partial T_{\theta\theta}}{\partial r}-\frac{\partial T_{r\theta}}{\partial\theta}\right)+\sin\theta\frac{\frac{T_{\theta\theta}}{r}-rT_{rr}}{r^2\sin\theta}\\
&= \frac{1}{r^2}\frac{\partial[r^2T(\theta,\theta)]}{\partial r}-\frac{1}{r}\frac{\partial T(r,\theta)}{\partial\theta}+\frac{T(\theta,\theta)-T(r,r)}{r}
\end{aligned}
\tag{C.89}
$$

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\varphi,\varphi) = (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\varphi}_{\varphi} &= \frac{1}{r^2\sin\theta}\left(\frac{\partial T_{\theta\varphi}}{\partial r} - \frac{\partial T_{r\varphi}}{\partial\theta}\right) + \frac{\cot\theta\, T_{r\varphi} - \frac{T_{\theta\varphi}}{r}}{r^2\sin\theta} \\
&= \frac{1}{r^2}\frac{\partial[r^2 T(\theta,\varphi)]}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial[\sin\theta\, T(r,\varphi)]}{\partial\theta} + \frac{\cot\theta\, T(r,\varphi) - T(\theta,\varphi)}{r}
\end{aligned}
\tag{C.90}
$$

C.8.4 Primitive Toroidal Coordinates

We use the right handed coordinates (r, θ, ζ). Here $\mathcal{J} = rR = r(R_0 + r\cos\theta)$.

C.8.4.1 Gradient

First the gradient of a scalar is found via

$$\begin{aligned}\nabla f &= \mathbf{e}^r \frac{\partial f}{\partial r} + \mathbf{e}^\theta \frac{\partial f}{\partial \theta} + \mathbf{e}^\zeta \frac{\partial f}{\partial \zeta} \\ &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{R}\frac{\partial f}{\partial \zeta}\hat{\boldsymbol{\zeta}}\end{aligned} \tag{C.91}$$

The gradient of a vector is given by

$$(\nabla\mathbf{A})(r,r) = (\nabla\mathbf{A})_{rr} = \frac{\partial A_r}{\partial r} = \frac{\partial A(r)}{\partial r} \tag{C.92}$$

$$(\nabla\mathbf{A})(r,\theta) = \frac{1}{r}(\nabla\mathbf{A})_{r\theta} = \frac{1}{r}\left(\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r}\right) = \frac{\partial[rA(\theta)]}{\partial r} - \frac{A(\theta)}{r} = \frac{\partial A(\theta)}{\partial r} \tag{C.93}$$

$$(\nabla\mathbf{A})(r,\zeta) = \frac{1}{R}(\nabla\mathbf{A})_{r\zeta} = \frac{1}{R}\left(\frac{\partial A_\zeta}{\partial r} - \frac{A_\zeta\cos\theta}{R}\right) = \frac{1}{R}\frac{\partial[RA(\zeta)]}{\partial r} - \frac{A(\zeta)\cos\theta}{R} \tag{C.94}$$

$$(\nabla\mathbf{A})(\theta,r) = \frac{1}{r}(\nabla\mathbf{A})_{\theta r} = \frac{1}{r}\left(\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r}\right) = \frac{1}{r}\frac{\partial A(r)}{\partial \theta} - \frac{A(\theta)}{r} \tag{C.95}$$

$$(\nabla\mathbf{A})(\theta,\theta) = \frac{1}{r^2}(\nabla\mathbf{A})_{\theta\theta} = \frac{1}{r^2}\left(\frac{\partial A_\theta}{\partial \theta} + A_r r\right) = \frac{1}{r}\frac{\partial A(\theta)}{\partial \theta} + \frac{A(r)}{r} \tag{C.96}$$

$$\begin{aligned}(\nabla\mathbf{A})(\theta,\zeta) &= \frac{1}{rR}(\nabla\mathbf{A})_{\theta\zeta} = \frac{1}{rR}\left(\frac{\partial A_\zeta}{\partial \theta} - \frac{A_\zeta r\sin\theta}{R}\right) \\ &= \frac{1}{rR}\frac{\partial[RA(\zeta)]}{\partial\theta} - \frac{A(\zeta)\sin\theta}{R}\end{aligned} \tag{C.97}$$

$$(\nabla\mathbf{A})(\zeta,r) = \frac{1}{R}(\nabla\mathbf{A})_{\zeta r} = \frac{1}{R}\left(\frac{\partial A_r}{\partial \zeta} - \frac{A_\zeta\cos\theta}{R}\right) = \frac{1}{R}\frac{\partial A(r)}{\partial \zeta} - \frac{A(\zeta)\cos\theta}{R} \tag{C.98}$$

$$(\nabla\mathbf{A})(\zeta,\theta) = \frac{1}{rR}(\nabla\mathbf{A})_{\zeta\theta} = \frac{1}{rR}\left(\frac{\partial A_\theta}{\partial \zeta} - \frac{A_\zeta r\sin\theta}{R}\right) = \frac{1}{R}\frac{\partial A(\theta)}{\partial \zeta} - \frac{A(\zeta)\sin\theta}{R} \tag{C.99}$$

$$\begin{aligned}(\nabla\mathbf{A})(\zeta,\zeta) &= \frac{1}{R^2}(\nabla\mathbf{A})_{\zeta\zeta} = \frac{1}{R^2}\left(\frac{\partial A_\zeta}{\partial \zeta} + A_r R\cos\theta - \frac{A_\theta R\sin\theta}{r}\right) \\ &= \frac{1}{R}\frac{\partial A(\zeta)}{\partial\zeta} + \frac{A(r)\cos\theta}{R} + \frac{A(\theta)\sin\theta}{R}\end{aligned} \tag{C.100}$$

As a matrix where rows represent the first index and columns the second index

$$\begin{bmatrix} \dfrac{\partial A(r)}{\partial r} & \dfrac{\partial A(\theta)}{\partial r} & \dfrac{1}{R}\left(\dfrac{\partial[RA(\zeta)]}{\partial r} - A(\zeta)\cos\theta\right) \\ \dfrac{1}{r}\left(\dfrac{\partial A(r)}{\partial\theta} - A(\theta)\right) & \dfrac{1}{r}\left(\dfrac{\partial A(\theta)}{\partial\theta} + A(r)\right) & \dfrac{1}{R}\left(\dfrac{1}{r}\dfrac{\partial[RA(\zeta)]}{\partial\theta} - A(\zeta)\sin\theta\right) \\ \dfrac{1}{R}\left(\dfrac{\partial A(r)}{\partial\zeta} - A(\zeta)\cos\theta\right) & \dfrac{1}{R}\left(\dfrac{\partial A(\theta)}{\partial\zeta} - A(\zeta)\sin\theta\right) & \dfrac{1}{R}\left(\dfrac{\partial A(\zeta)}{\partial\zeta} + A(r)\cos\theta + A(\theta)\sin\theta\right) \end{bmatrix} \tag{C.101}$$

C.8.4.2 Divergence

The divergence of a vector is given by

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{1}{rR}\frac{\partial(rRA^r)}{\partial r}+\frac{1}{rR}\frac{\partial(rRA^\theta)}{\partial\theta}+\frac{1}{rR}\frac{\partial(rRA^\varphi)}{\partial\varphi}\\ &= \frac{1}{rR}\frac{\partial[rRA(r)]}{\partial r}+\frac{1}{rR}\frac{\partial[RA(\theta)]}{\partial\theta}+\frac{1}{R}\frac{\partial A(\zeta)}{\partial\zeta}\end{aligned} \tag{C.102}$$

The divergence of a second order tensor is given by

$$\begin{aligned}(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(r) &= (\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^r = \frac{1}{rR}\left(\frac{\partial(\mathcal{J}T^{rr})}{\partial r}+\frac{\partial(\mathcal{J}T^{\theta r})}{\partial\theta}+\frac{\partial(\mathcal{J}T^{\zeta r})}{\partial\zeta}\right)-rT^{\theta\theta}-R\cos\theta\,T^{\zeta\zeta}\\ &= \frac{1}{rR}\frac{\partial[rRT(r,r)]}{\partial r}+\frac{1}{rR}\frac{\partial[RT(\theta,r)]}{\partial\theta}+\frac{1}{R}\frac{\partial T(\zeta,r)}{\partial\zeta}-\frac{T(\theta,\theta)}{r}-\frac{\cos\theta\,T(\zeta\zeta)}{R}\end{aligned} \tag{C.103}$$

$$\begin{aligned}&(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(\theta) = r(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^\theta\\ &= \frac{r}{rR}\left(\frac{\partial(\mathcal{J}T^{r\theta})}{\partial r}+\frac{\partial(\mathcal{J}T^{\theta\theta})}{\partial\theta}+\frac{\partial(\mathcal{J}T^{\zeta\theta})}{\partial\zeta}\right)+r\frac{T^{\theta r}+T^{r\theta}}{r}+r\frac{R}{r}\sin\theta T^{\zeta\zeta}\\ &= \frac{1}{R}\frac{\partial[RT(r,\theta)]}{\partial r}+\frac{1}{rR}\frac{\partial[RT(\theta,\theta)]}{\partial\theta}+\frac{1}{R}\frac{\partial T(\zeta,\theta)}{\partial\zeta}+\frac{T(\theta,r)+T(r,\theta)}{r}+\frac{\sin\theta}{R}T(\zeta,\zeta)\end{aligned} \tag{C.104}$$

$$\begin{aligned}&(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})(\zeta) = R(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^\zeta\\ &= \frac{R}{rR}\left(\frac{\partial(\mathcal{J}T^{r\zeta})}{\partial r}+\frac{\partial(\mathcal{J}T^{\theta\zeta})}{\partial\theta}+\frac{\partial(\mathcal{J}T^{\zeta\zeta})}{\partial\zeta}\right)+R\cos\theta\frac{T^{r\zeta}+T^{\zeta r}}{R}-rR\sin\theta\frac{T^{\theta\zeta}+T^{\zeta\theta}}{R}\\ &= \frac{1}{r}\frac{\partial[rT(r,\zeta)]}{\partial r}+\frac{1}{r}\frac{\partial T(\theta,\zeta)}{\partial\theta}+\frac{1}{R}\frac{\partial T(\zeta,\zeta)}{\partial\zeta}+\cos\theta\frac{T(r,\zeta)+T(\zeta,r)}{R}-\sin\theta\frac{T(\theta,\zeta)+T(\zeta,\theta)}{R}\end{aligned} \tag{C.105}$$

C.8.4.3 Curl

The curl of a vector is given by

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})(r) &= (\boldsymbol{\nabla}\times\mathbf{A})^r = \frac{1}{rR}\left(\frac{\partial A_\zeta}{\partial\theta}-\frac{\partial A_\theta}{\partial\zeta}\right)\\ &= \frac{1}{rR}\frac{\partial[RA(\zeta)]}{\partial\theta}-\frac{1}{R}\frac{\partial A(\theta)}{\partial\zeta}\end{aligned} \tag{C.106}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})(\theta) &= r(\boldsymbol{\nabla}\times\mathbf{A})^\theta = \frac{r}{rR}\left(\frac{\partial A_r}{\partial\zeta}-\frac{\partial A_\zeta}{\partial r}\right)\\ &= \frac{1}{R}\frac{\partial A(r)}{\partial\zeta}-\frac{1}{R}\frac{\partial[RA(\zeta)]}{\partial r}\end{aligned} \tag{C.107}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})(\zeta) &= R(\boldsymbol{\nabla}\times\mathbf{A})^\zeta = \frac{R}{rR}\left(\frac{\partial A_\theta}{\partial r}-\frac{\partial A_r}{\partial\theta}\right)\\ &= \frac{1}{r}\frac{\partial[rA(\theta)]}{\partial r}-\frac{1}{r}\frac{\partial A(r)}{\partial\theta}\end{aligned} \tag{C.108}$$

The curl of a second order tensor is given by

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,r)=(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})^{r}_{\cdot r}=\frac{1}{rR}\left(\frac{\partial T_{\zeta r}}{\partial\theta}-\frac{\partial T_{\theta r}}{\partial\zeta}\right)+\frac{\cos\theta\,T_{\theta\zeta}}{rRR}-\frac{T_{\zeta\theta}}{rrR}$$
$$=\frac{1}{rR}\frac{\partial[RT(\zeta,r)]}{\partial\theta}-\frac{1}{R}\frac{\partial T(\theta,r)}{\partial\zeta}+\frac{\cos\theta\,T(\theta,\zeta)}{R}-\frac{T(\zeta,\theta)}{r} \tag{C.109}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,\theta)=\frac{1}{r}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})^{r}_{\cdot\theta}=\frac{1}{r^2R}\left(\frac{\partial T_{\zeta\theta}}{\partial\theta}-\frac{\partial T_{\theta\theta}}{\partial\zeta}\right)+\frac{T_{\zeta r}r}{r^2R}-\frac{T_{\theta\zeta}r\sin\theta}{Rr^2R}$$
$$=\frac{1}{rR}\frac{\partial[RT(\zeta,\theta)]}{\partial\theta}-\frac{1}{R}\frac{\partial T(\theta,\theta)}{\partial\zeta}+\frac{T(\zeta,r)}{R}-\frac{T(\theta,\zeta)\sin\theta}{R} \tag{C.110}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(r,\zeta)=\frac{1}{R}(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})^{r}_{\cdot\zeta}$$
$$=\frac{1}{rR^2}\left(\frac{\partial T_{\zeta\zeta}}{\partial\theta}-\frac{\partial T_{\theta\zeta}}{\partial\zeta}\right)+\frac{T_{\zeta\zeta}r\sin\theta}{RrR^2}+\frac{T_{\theta\theta}R\sin\theta}{rrR^2}-\frac{T_{\theta r}R\cos\theta}{rR^2}$$
$$=\frac{1}{rR^2}\frac{\partial[R^2T(\zeta,\zeta)]}{\partial\theta}-\frac{1}{R}\frac{\partial T(\theta,\zeta)}{\partial\zeta}+\frac{\sin\theta[T(\zeta,\zeta)\sin\theta+T(\theta,\theta)]-T(\theta,r)\cos\theta}{R} \tag{C.111}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\theta,r)=r(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot r}=\frac{r}{rR}\left(\frac{\partial T_{rr}}{\partial\zeta}-\frac{\partial T_{\zeta r}}{\partial r}\right)-\frac{T_{r\zeta}\cos\theta}{RR}$$
$$=\frac{1}{R}\frac{\partial T(r,r)}{\partial\zeta}-\frac{1}{R}\frac{\partial[RT(\zeta,r)]}{\partial r}-\frac{T(r,\zeta)\cos\theta}{R} \tag{C.112}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\theta,\theta)=(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot\theta}=\frac{1}{rR}\left(\frac{\partial T_{r\theta}}{\partial\zeta}-\frac{\partial T_{\zeta\theta}}{\partial r}\right)+\frac{T_{\zeta\theta}}{rrR}+\frac{T_{r\zeta}r\sin\theta}{RrR}$$
$$=\frac{1}{R}\frac{\partial T(r,\theta)}{\partial\zeta}-\frac{1}{rR}\frac{\partial[rRT(\zeta,\theta)]}{\partial r}+\frac{T(\zeta,\theta)}{r}+\frac{T(r,\zeta)\sin\theta}{R} \tag{C.113}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\theta,\zeta)=\frac{r}{R}(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\theta}_{\cdot\zeta}=\frac{r}{RrR}\left(\frac{\partial T_{r\zeta}}{\partial\zeta}-\frac{\partial T_{\zeta\zeta}}{\partial r}\right)+\frac{T_{\zeta\zeta}\cos\theta}{R^2R}+\frac{T_{rr}R\cos\theta}{R^2}-\frac{T_{r\theta}R\sin\theta}{R^2r}$$
$$=\frac{1}{R}\frac{\partial T(r,\zeta)}{\partial\zeta}-\frac{1}{R^2}\frac{\partial[R^2T(\zeta,\zeta)]}{\partial r}+\frac{[T(\zeta,\zeta)+T(r,r)]\cos\theta-T(r,\theta)\sin\theta}{R} \tag{C.114}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\zeta,r)=R(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\zeta}_{\cdot r}=\frac{R}{rR}\left(\frac{\partial T_{\theta r}}{\partial r}-\frac{\partial T_{rr}}{\partial\theta}\right)+\frac{T_{r\theta}}{rr}$$
$$=\frac{1}{r}\frac{\partial[rT(\theta,r)]}{\partial r}-\frac{1}{r}\frac{\partial T(r,r)}{\partial\theta}+\frac{T(r,\theta)}{r} \tag{C.115}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\zeta,\theta)=\frac{R}{r}(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\zeta}_{\cdot\theta}=\frac{R}{rrR}\left(\frac{\partial T_{\theta\theta}}{\partial r}-\frac{\partial T_{r\theta}}{\partial\theta}\right)-\frac{T_{rr}r}{r^2}-\frac{T_{\theta\theta}}{rr^2}$$
$$=\frac{1}{r^2}\frac{\partial[r^2T(\theta,\theta)]}{\partial r}-\frac{1}{r}\frac{\partial T(r,\theta)}{\partial\theta}-\frac{T(r,r)+T(\theta,\theta)}{r} \tag{C.116}$$

$$(\boldsymbol{\nabla}\times\overleftrightarrow{\mathbf{T}})(\zeta,\zeta)=(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}})^{\zeta}_{\cdot\zeta}=\frac{1}{rR}\left(\frac{\partial T_{\theta\zeta}}{\partial r}-\frac{\partial T_{r\zeta}}{\partial\theta}\right)-\frac{T_{\theta\zeta}\cos\theta+T_{r\zeta}r\sin\theta}{rRR}$$
$$=\frac{1}{rR}\frac{\partial[rRT(\theta,\zeta)]}{\partial r}-\frac{1}{rR}\frac{\partial[RT(r,\zeta)]}{\partial\theta}-\frac{T(\theta,\zeta)\cos\theta+T(r,\zeta)\sin\theta}{R} \tag{C.117}$$

References

[1] Tevian Dray and Corinne A Manogue. "Conventions for spherical coordinates". In: (2002). URL: http://sites.science.oregonstate.edu/math/bridge/papers/spherical.pdf.

Appendix D

Reactivity Calculations Program

The following python program was what I used for calculating reactivity values and cross section values.

Reactivity Calculator

```
#/usr/bin/env python3

import numpy as np
import matplotlib.pyplot as plt
import matplotlib.cm as cm


# Cross section matches using Duane coefficients
# from paper DUANE, B.H., "Fusion cross section theory",
# in Annual Report on CTR Technology 1972
# (WOLKENHAUER, W.C.,Ed.), Rep. BNWL-1685,
#Battelle Pacific Northwest Laboratory,
#Richland, WA (1972).
#####################################################################
# function sigmaDuane
#   input:
#     en (np array): incoming energies in keV
#     reac (string): type of reaction
#                    'dd1' D+D->T+p
#                    'dd2' D+D->3He+n
#                    'dt' D+T->4He+n
#                    'dhe' D+3He->4He+p
#                    'het' 3He+T->4He+p+n and
#                          3He+T->4He+D and
#                          3He+T->5He+p
#                    'dd' D+D->T+p and
#                         D+D->3He+n
#   output: cross section for reaction in barns
#####################################################################
def sigmaDuane(en,reac):

  a1={}
  a2={}
  a3={}
  a4={}
  a5={}
  #D-D reaction 1 D+D->T+p
  a1['dd1']=46.097
  a2['dd1']=372.
  a3['dd1']=4.36e-4
  a4['dd1']=1.220
  a5['dd1']=0.
  #D-D reaction 2 D+D->3He+n
  a1['dd2']=47.88
```

```
45    a2['dd2']=482.
46    a3['dd2']=3.08e-4
47    a4['dd2']=1.177
48    a5['dd2']=0.
49    #D-T reaction
50    a1['dt']=45.95
51    a2['dt']=50200.
52    a3['dt']=1.368e-2
53    a4['dt']=1.076
54    a5['dt']=409.
55    #D-3He reaction
56    a1['dhe']=89.27
57    a2['dhe']=25900.
58    a3['dhe']=3.98e-3
59    a4['dhe']=1.297
60    a5['dhe']=647.
61    #T-T reaction
62    a1['tt']=38.39
63    a2['tt']=448.
64    a3['tt']=1.02e-3
65    a4['tt']=2.09
66    a5['tt']=0.
67    #3He-T reaction all
68    a1['het']=123.1
69    a2['het']=11250.
70    a3['het']=0
71    a4['het']=0
72    a5['het']=0
73    # for 'dd' add the reactivities for both reactions
74    if reac=='dd':
75      dd1=(a5['dd1']+a2['dd1']*((a4['dd1']-a3['dd1']*en)**2+1.)**-1.)/(en*(np.exp(a1['dd1']*en
        **-0.5)-1.))
76      dd2=(a5['dd2']+a2['dd2']*((a4['dd2']-a3['dd2']*en)**2+1.)**-1.)/(en*(np.exp(a1['dd2']*en
        **-0.5)-1.))
77      return dd1+dd2
78    # otherwise just use the formula fit
79    return (a5[reac]+a2[reac]*((a4[reac]-a3[reac]*en)**2+1.)**-1.)/(en*(np.exp(a1[reac]*en**-0.5)
        -1.))
80
81  # Cross sections from Bosch-Hale
82  # Bosch, H.S. and Hale, G.M., 1992.
83  # Improved formulas for fusion cross-sections
84  # and thermal reactivities. Nuclear fusion, 32(4), p.611.
85  ######################################################################
86  # function sigmaBH
87  #   input:
88  #     en (np array): incoming energies in keV
89  #     reac (string): type of reaction
90  #                    'dd1' D+D->T+p
91  #                    'dd2' D+D->3He+n
92  #                    'dt' D+T->4He+n
93  #                    'dhe' D+3He->4He+p
94  #                    'dd' D+D->T+p and
95  #                         D+D->3He+n
96  #   output: cross section for reaction in millibarns
97  ######################################################################
98  def sigmaBH(en,reac):
99    A1={}
100   A2={}
101   A3={}
102   A4={}
103   A5={}
104   B1={}
105   B2={}
106   B3={}
107   B4={}
108   BG={}
109   # errflags lets you know if you are extrapolating
110   errflags=[]
111   # high is for when we switch formulas
112   high=np.shape(en)[0]
```

```
113     #D-D reaction 1 D+D->T+p
114     BG['dd1']=31.3970
115     A1['dd1']=5.5576e4
116     A2['dd1']=2.1054e2
117     A3['dd1']=-3.2638e-2
118     A4['dd1']=1.4987e-6
119     A5['dd1']=1.8181e-10
120     B1['dd1']=0.
121     B2['dd1']=0.
122     B3['dd1']=0.
123     B4['dd1']=0.
124     #D-D reaction 2 D+D->3He + p
125     BG['dd2']=31.3970
126     A1['dd2']=5.3701e4
127     A2['dd2']=3.3027e2
128     A3['dd2']=-1.2706e-1
129     A4['dd2']=2.9327e-5
130     A5['dd2']=-2.5151e-9
131     B1['dd2']=0.
132     B2['dd2']=0.
133     B3['dd2']=0.
134     B4['dd2']=0.
135     #D-T reaction
136     BG['dt']=34.3827
137     A1['dt']=6.927e4
138     A2['dt']=7.454e8
139     A3['dt']=2.050e6
140     A4['dt']=5.2002e4
141     A5['dt']=0.0
142     B1['dt']=6.380e1
143     B2['dt']=-9.950e-1
144     B3['dt']=6.981e-5
145     B4['dt']=1.728e-4
146     #D-3He reaction
147     BG['dhe']=68.7508
148     A1['dhe']=5.7501e6
149     A2['dhe']=2.5226e3
150     A3['dhe']=4.5566e1
151     A4['dhe']=0.0
152     A5['dhe']=0.0
153     B1['dhe']=-3.1995e-3
154     B2['dhe']=-8.5530e-6
155     B3['dhe']=5.9014e-8
156     B4['dhe']=0.0
157     # add error flags for questionable data
158     if reac=='dd1':
159       if (np.max(en)>4900.):
160         errflags.append("Greater than 4.9 MeV, maybe bad data.")
161     if reac=='dd2':
162       if np.max(en)>5000.:
163         errflags.append("Greater than 4.9 MeV, maybe bad data.")
164     if reac=='dt':
165       if np.max(en)>530.:
166         # high tells where to switch to new coefficients
167         high=np.where(en>530)[0][0]
168         BG['dth']=34.3827
169         A1['dth']=-1.4714e6
170         A2['dth']=0.0
171         A3['dth']=0.0
172         A4['dth']=0.0
173         A5['dth']=0.0
174         B1['dth']=-8.4127e-3
175         B2['dth']=4.7983e-6
176         B3['dth']=-1.0748e-9
177         B4['dth']=8.5184e-14
178     if reac=='dhe':
179       if np.max(en)>930.:
180         # high tells where to switch to new coefficients
181         high=np.where(en>930)[0][0]
182         BG['dheh']=68.7508
183         A1['dheh']=-8.3993e5
```

```
184         A2['dheh']=0.0
185         A3['dheh']=0.0
186         A4['dheh']=0.0
187         A5['dheh']=0.0
188         B1['dheh']=-2.6830e-3
189         B2['dheh']=1.1633e-6
190         B3['dheh']=-2.1332e-10
191         B4['dheh']=1.4250e-14
192       # add both reactivities for 'dd'
193       if reac=='dd':
194         S1=(A1['dd1']+en*(A2['dd1']+en*(A3['dd1']+en*(A4['dd1']+en*A5['dd1']))))/(1+en*(B1['dd1']+en*(
          B2['dd1']+en*(B3['dd1']+en*B4['dd1']))))
195         S2=(A1['dd2']+en*(A2['dd2']+en*(A3['dd2']+en*(A4['dd2']+en*A5['dd2']))))/(1+en*(B1['dd2']+en*(
          B2['dd2']+en*(B3['dd2']+en*B4['dd2']))))
196         sig1=S1/(en*np.exp(BG['dd1']*en**-0.5))
197         sig2=S2/(en*np.exp(BG['dd2']*en**-0.5))
198         return sig1+sig2
199       S=(A1[reac]+en*(A2[reac]+en*(A3[reac]+en*(A4[reac]+en*A5[reac]))))/(1.+en*(B1[reac]+en*(B2[reac
          ]+en*(B3[reac]+en*B4[reac]))))
200       sig=S/(en*np.exp(BG[reac]*en**-0.5))
201       # if high is needed change some of the values to the correct higher energy formula
202       if high<(np.shape(en)[0]):
203         S[high:]=(A1[reac+'h']+en[high:]*(A2[reac+'h']+en[high:]*(A3[reac+'h']+en[high:]*(A4[reac+'h'
          ]+en[high:]*A5[reac+'h']))))/(1.+en[high:]*(B1[reac+'h']+en[high:]*(B2[reac+'h']+en[high:]*(B3
          [reac+'h']+en[high:]*B4[reac+'h']))))
204         sig[high:]=S[high:]/(en[high:]*np.exp(BG[reac+'h']*en[high:]**-0.5))
205       if len(errflags)>1:
206         print(errflags)
207       return sig
208
209   ######################################################################
210   # function sigv
211   #   calculates the reactivity <sigma v> in m^3/s
212   #   input:
213   #     params (dictionary): dictionary of parameters
214   #                  'coeff': which cross section fit to use
215   #                           'Duane' for sigmaDuane or
216   #                           'BH' for sigmaBH
217   #               'reaction': which reaction to use, see
218   #                           'reac' for Duane or BH
219   #                     'mb': mass of "beam" particles
220   #                     'mt': mass of "target" particles
221   #                   NOTE: For BH mb and mt can be
222   #                         swapped with no change in
223   #                         output.
224   #                      'T': np array of energy values
225   #                           for cross sections
226   #     lagpoly (integer): which degree of Laguerre-Gauss
227   #                        interpolation to begin with
228   #   output: reactivity for reaction in m^3/s
229   ######################################################################
230   def sigv(params,lagpoly):
231     T=params['T']
232     mb=params['mb']
233     mt=params['mt']
234     reac=params['reaction']
235     # mo and mu are convenience parameters
236     mo=mt/(mb+mt)
237     mu=mo*mb
238     # how many energy values do we require
239     Ts=np.shape(T)[0]
240     # caclulate the weights and roots for laguerre-gauss
241     xi,wi=np.polynomial.laguerre.laggauss(lagpoly)
242     tot=np.zeros(Ts)
243     #kb for converting keV to joules
244     kb=1.6e-16
245     #for each temperature given, calculate the reactivity
246     for i in range(Ts):
247       if params['coeff']=="Duane":
248         # 1e-28 converts barns to m^2
249         sig=(sigmaDuane(xi*T[i]/mo,reac))*1e-28
```

```
250       elif params['coeff']=="BH":
251         # 1e-28*1e-3 converts millibarns to m^2
252         sig=(sigmaBH(xi*T[i],reac))*1e-28*1e-3
253       # this is the formula for doing the integral over the
254       # cross sections
255       tot[i]=(8.*kb*T[i]/(mu*np.pi))**0.5*np.sum(wi*sig*xi)
256     return tot
257
258 ###################################################################
259 # function findsigv
260 #   calculates the reactivity <sigma v> in m^3/s
261 #   and iterates until answer does not vary
262 #   input:
263 #     params (dictionary): dictionary of parameters
264 #                 'coeff': which cross section fit to use
265 #                          'Duane' for sigmaDuane or
266 #                          'BH' for sigmaBH
267 #              'reaction': which reaction to use, see
268 #                          'reac' for Duane or BH
269 #                    'mb': mass of "beam" particles
270 #                    'mt': mass of "target" particles
271 #                    NOTE: For BH mb and mt can be
272 #                          swapped with no change in
273 #                          output.
274 #                     'T': np array of energy values
275 #                          for cross sections
276 #     tol (real): tolerance, how little the relative
277 #                 difference between two iterations
278 #                 must be before the calculation ends
279 #   output (list): input temperatures,
280 #                  reactivity for reaction in m^3/s
281 #                  accurate to tolerance,
282 #                  degree of laguerre-gauss polynomial
283 #                  used in final approximation
284 ###################################################################
285 def findsigv(params,tol=1e-2):
286     err=tol+1
287     lagpoly=5
288     lagpoly1=lagpoly+1
289     maxlagpoly=params.get('maxlagpoly',100)
290     while err>tol:
291       sigv0=sigv(params,lagpoly)
292       sigv1=sigv(params,lagpoly1)
293       err=np.max(np.abs((sigv0-sigv1)/sigv1))
294       lagpoly=lagpoly1
295       lagpoly1=lagpoly+1
296       if lagpoly>maxlagpoly:
297         print('Lagpoly='+str(maxlagpoly))
298         break
299     return params['T'],sigv0,lagpoly
```

Appendix E

SI Units

The International System of Units [abbreviated SI from the French *Système international (d'unités)*] is what is meant when people say the metric system today. It is a wonderful coherent and consistent system that is too often ignored. Coherent here means that the factor between base and derived units is simply one or unity. A non-coherent system will have conversion factors, so, for example, a gallon is not simply inches cubed.

E.1 SI System

The SI system uses seven base units, the second s, the meter m, the kilogram kg, the ampere A, the kelvin K, mole mol, and the candela cd. There are many more derived units such as the newton N, the pascal Pa, and the joule J. In addition there are 20 prefixes that supply a power of 10. Such as kilo meaning 10^3 or giga meaning 10^9. The approved prefixes are listed in Table E.2. When using prefixes, one must of course remember that the new unit is no longer part of a coherent system, but since the prefix tells us the conversion factor, it is still easy to do any conversion. From the base units we can create new units called derived units. A list of some derived units are given in Table E.3. There is an endless number of derived units, but these are the most common ones. Any combination of base units without prefixes leads to a coherent derived unit. As SI states it[2], coherent "means that equations between numerical values of quantities take exactly the same form as the equations between the quantities themselves" (i.e., there are no needed conversion factors). The other important part of the SI system is that every quantity is now based off of fundamental relationships in nature. This means there is no "standard" bar that defines the relationship. For example, time is measured off the frequency of a hyperfine transition in caesium and the meter is defined by the distance that light travels in vacuum in a $1/299\,792\,458$ of a second. Many of the numbers become definitions under this. A list of defined constants for SI is given in Table E.1.

Using the base and derived units consistently means that the conversion factors are one. For mechanical units, it is fairly easy to convert between different systems of units, even English customary or imperial ones, but once electromagnetic units are used, the use of SI should be obligatory to reduce confusion. While a CGS system is consistent, it is only sometimes used in theoretical physics and in astronomy and is rapidly losing popularity. My own opinion is that the SI system should always be used in any application with as few possible deviations as possible allowed. SI is clear and it is easy to look up formulas for SI and what each unit means. In addition,

	Constant definition	SI Relation
$\Delta\nu_{\mathrm{Cs}}$	$9\,192\,631\,770\,\mathrm{Hz}$	$1\,\mathrm{s}=\dfrac{9\,192\,631\,770}{\Delta\nu_{\mathrm{Cs}}}$
c	$299\,792\,458\,\mathrm{m\,s^{-1}}$	$1\,\mathrm{m}=\dfrac{9\,192\,631\,770}{299\,792\,458}\dfrac{c}{\Delta\nu_{\mathrm{Cs}}}$
h	$6.626\,070\,15\times10^{-34}\,\mathrm{J\,s}$	$1\,\mathrm{kg}=\dfrac{(299\,792\,458)^2}{(6.626\,070\,15\times10^{-34})(9\,192\,631\,770)}\dfrac{h\Delta\nu_{\mathrm{Cs}}}{c^2}$
e	$1.602\,176\,634\times10^{-19}\,\mathrm{A\,s}$	$1\,\mathrm{A}=\dfrac{1}{(1.602\,176\,634\times10^{-19})(9\,192\,631\,770)}e\Delta\nu_{\mathrm{Cs}}$
k_B	$1.380\,649\times10^{-23}\,\mathrm{J\,K^{-1}}$	$1\,\mathrm{K}=\dfrac{(1.380\,649\times10^{-23})}{(6.626\,070\,15\times10^{-34})(9\,192\,631\,770)}\dfrac{h\Delta\nu_{\mathrm{Cs}}}{k_B}$
N_A	$6.022\,140\,76\times10^{23}\,\mathrm{mol^{-1}}$	$1\,\mathrm{mol}=\dfrac{6.022\,140\,76\times10^{23}}{N_A}$
K_{cd}	$683\,\mathrm{lm\,W^{-1}}$	$1\,\mathrm{cd}=\dfrac{(\Delta\nu_{\mathrm{Cs}})^2hK_{\mathrm{cd}}}{(9\,192\,631\,770)^2(6.626\,070\,15\times10^{-34})(683)}$

	Explanation
$\Delta\nu_{\mathrm{Cs}}$	unperturbed ground-state hyperfine transition frequency of cesium 133 atom
c	distance light travels in vacuum in $1/299\,792\,458$ of a second
h	Relation between Planck constant and mass
e	Relationship between elementary charge and time
k_B	Relationship between energy and temperature
N_A	Relationship between number of atoms in a mole of a substance
K_{cd}	The candela is given by defining the luminous efficacy K_{cd} of monochromatic radiation of $540\times10^{14}\,\mathrm{Hz}$ to be 683 in $\mathrm{lmW^{-1}}$ equivalent to $\mathrm{cdsrkg^{-1}m^{-2}s^3}$

Table E.1: These are the defined constants used to define the base units of the SI system.

.

Name	Symbol	Base 10
yotta	Y	10^{24}
zetta	Z	10^{21}
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
milli	m	10^{-3}
micro	µ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}
zepto	z	10^{-21}
yocto	y	10^{-24}

Table E.2: The metric prefixes as powers of $10^3 = 1000$. I ignore hecto, deca, deci, and centi as prefixes because they are not used often and are better avoided.

SI keeps some non-SI units as "acceptable for use with SI units". These are given in Table E.4.

In reality, any coherent system with prefixes is essentially as good as another one. CGS and SI both have the problem of one base unit having a prefix on it (centimeter in CGS and kilogram in SI). The real improvement is the widespread reach of SI and that it has governing documents that clearly indicate how the units are measured relative to natural phenomena.

Name	Symbol	Derived Quantity	base units	alternate base
radian	rad	plane angle	$\mathrm{m/m}$	
steradian	sr	solid angle	$\mathrm{m^2\,m^{-2}}$	
hertz	Hz	frequency	$\mathrm{s^{-1}}$	
newton	N	force	$\mathrm{kg\,m\,s^{-2}}$	
pascal	Pa	pressure, stress	$\mathrm{kg\,m^{-1}\,s^{-2}}$	
joule	Pa	energy, work, heat	$\mathrm{kg\,m^2\,s^{-2}}$	$\mathrm{N\,m}$
watt	W	power, radiant flux	$\mathrm{kg\,m^2\,s^{-3}}$	$\mathrm{J\,s^{-1}}$
coulomb	C	electric charge	$\mathrm{A\,s}$	
volt	V	electric voltage/potential difference	$\mathrm{kg\,m^2\,s^{-3}\,A^{-1}}$	$\mathrm{W\,A^{-1}}$
farad	F	electric capacitance	$\mathrm{kg^{-1}\,m^{-2}\,s^{-4}\,A^2}$	$\mathrm{C\,V^{-1}}$
ohm	Ω	electric resistance	$\mathrm{kg\,m^2\,s^{-3}\,A^{-2}}$	$\mathrm{V\,A^{-1}}$
siemens	S	electric conductance	$\mathrm{kg^{-1}\,m^{-2}\,s^3\,A^2}$	$\mathrm{A\,V^{-1}}$
weber	Wb	magnetic flux	$\mathrm{kg\,m^2\,s^{-2}\,A^{-1}}$	$\mathrm{V\,s}$
tesla	T	magnetic flux density, magnetic field	$\mathrm{kg\,s^{-2}\,A^{-1}}$	$\mathrm{Wb\,m^{-2}}$
henry	H	electric inductance	$\mathrm{kg\,m^2\,s^{-2}\,A^{-2}}$	$\mathrm{Wb\,A^{-1}}$
degree Celsius	°C	temperature (offset from Kelvin)	K	
lumen	lm	luminous flux	$\mathrm{cd\,sr}$	
lux	lx	illuminance	$\mathrm{cd\,sr\,m^{-2}}$	$\mathrm{lm\,m^{-2}}$
becquerel	Bq	activity referred to a radionuclide	$\mathrm{s^{-1}}$	
gray	Gy	absorbed dose, kerma	$\mathrm{m^2\,s^{-2}}$	$\mathrm{J\,kg^{-1}}$
sievert	Sv	dose equivalent	$\mathrm{m^2\,s^{-2}}$	$\mathrm{J\,kg^{-1}}$
katal	kat	catalytic activity	$\mathrm{mol\,s^{-1}}$	

Table E.3: These are all of the derived units in the SI system. The Bq is only used for radioactive decay and Hz is used only for periodic phenomena. Sometimes voltage is called electric tension.

Name	Symbol	Quantity	in SI units
minute	min	time	60 s
hour	h	time	60 min=3600 s
day	d	time	24 h=86 400 s
astronomical unit	au	distance	$1.495\,978\,707\,00 \times 10^{11}$ m
degree	°	plane angle	$(\pi/180)$rad
(arc) minute	′	plane angle	$(1/60)°=(\pi/10\,800)$rad
(arc) second	″	plane angle	$(1/60)′=(\pi/648\,000)$rad
hectare	ha	area	$1 \times 10^4\,\mathrm{m^2}$
liter	L	volume	$1 \times 10^{-3}\,\mathrm{m^3}$
metric ton	t	mass	1×10^3 kg
dalton	Da	mass	$1.660\,539\,040 \times 10^{-27}$ kg
atomic mass unit	u	mass	$1.660\,539\,040 \times 10^{-27}$ kg
electronvolt	eV	energy	$1.602\,176\,634 \times 10^{-19}$ J
neper	Np	logarithmic unit	$m = \ln(X/X_0)$ means mNp
bel	B	logarithmic unit	$m = \log_{10}(X/X_0)$ means mB
decibel	dB	logarithmic unit	$m = 10\log_{10}(X/X_0)$ means mdB

Table E.4: These are units that are accepted for use with other SI units. Note that for the logarithmic values X_0 is a base of quantity X that is measured against.

E.2 Conversion Between SI and Gaussian

While I mentioned CGS, there is, in fact, no such system electromagnetically. There are CGS systems with Gaussian units being the most common CGS system (and often used synonymously with CGS). Converting from Gaussian units to SI is always a headache for multiple reasons. One is that something that is dimensionless in Gaussian units may have units in SI. Another reason is that the SI system is "rationalized" because it omits factors of 4π in its definition of laws. This leads to factors of 4π showing up all over the place in conversions. One easy way to start seeing the problem comes from dealing with the Coulomb law in a vacuum. In complete generality we write

$$F = k\frac{q_1 q_2}{r^2} \tag{E.1}$$

with factor k being a proportionality constant relating the charge, distance between the charges, and the force between the charges. In Gaussian units $k = 1$. That is it is dimensionless unity. This means that the units of charge are $\mathrm{g}^{1/2}\mathrm{cm}^{3/2}\mathrm{s}^{-1}$. In SI, current is a base unit and so charge is defined by the coulomb given by 1 A s. For SI $k = 1/(4\pi\epsilon_0)$ with ϵ_0 a constant often called the vacuum permittivity or electric constant.

It is sometimes claimed that because charge is built from mechanical units that it is easier to see the connection between mechanical units and electromagnetic ones as factors of c show up in Gaussian units. In SI one would have to deal with these implicitly through $\epsilon_0\mu_0 = 1/c^2$. My own experience does not support this assertion. Trying to understand what a $\mathrm{g}^{1/2}$ or $\mathrm{cm}^{3/2}$ physically means will not in any way help you see the connection any better. Almost all of the formula in Gaussian units have never helped me see how it relates to mechanical units in any straightforward way. The only exception I can think of is that capacitance and resistance in Gaussian units are in centimeters and seconds. Then a capacitance of 1 cm can be related to the capacitance between infinity and a conducting sphere of radius 1 cm in vacuum. Note how this is still a fairly belabored relation and in practice not all that useful. Gaussian units can also measure $\mathbf{E}$ and $\mathbf{B}$ in the same units whereas $c\mathbf{B}$ has the same units as $\mathbf{E}$ in SI. The statement that $\mathbf{B}$ is easier to compare with $\mathbf{E}$ with these units is then usually applied. Since $\mathbf{E}$ and $\mathbf{B}$ are physically different fields that cause very different physical phenomena I never understood why this would be considered much of an advantage. In practice, Gaussian units use different derived units for electric field (statV/cm) and magnetic field (G), negating this supposed advantage. As previously stated, any coherent system with useable prefixes is just as good as any other in practice and so it is better to use the most widespread such system. Despite this, some excellent textbooks include strange claims like "The SI system has the virtue of overall convenience in practical, large-scale phenomena, especially in engineering applications. The Gaussian system is more suitable for microscopic problems involving the electrodynamics of charged particles, etc."[1, pp. 783, 784]. Note that "suitability" isn't really ever defined, and in fact either system can be used in any application with few problems. SI is convenient because it is more widespread.

A summary of electromagnetic laws and formulas in the Gaussian and SI units are given in Table E.5. When considering the different definitions for electric polarization and magnetization the headaches in conversion multiply.

To convert between systems one must either consult tables of conversion values or look at the equations and start converting. Because quantities are not necessarily using the same units between the two systems, conversion is in some cases a bit of a stretch and it is more like a translation

Law Name	SI	Gaussian
Gauss's Law	$\boldsymbol{\nabla}\cdot\mathbf{E}=\rho_q/\epsilon_0$	$\boldsymbol{\nabla}\cdot\mathbf{E}=4\pi\rho_q$
Divergence Constraint	$\boldsymbol{\nabla}\cdot\mathbf{B}=0$	$\boldsymbol{\nabla}\cdot\mathbf{B}=0$
Faraday's Law	$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$	$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$
Ampère-Maxwell Law	$\boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$	$\boldsymbol{\nabla}\times\mathbf{B}=\frac{4\pi}{c}\mathbf{J}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$
Lorentz Force	$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$	$\mathbf{F}=q(\mathbf{E}+[\mathbf{v}/c]\times\mathbf{B})$
Coulomb's Law	$\mathbf{F}=\frac{q_1q_2}{4\pi\epsilon_0 r^2}$	$\mathbf{F}=\frac{q_1q_2}{r^2}$
Poynting Vector	$\mathbf{E}\times\mathbf{B}/\mu_0$	$c\mathbf{E}\times\mathbf{B}/(4\pi)$

Table E.5: The electromagnetic laws in SI and Gaussian units with conventional definitions of quantities. ρ_q means charge density.

than a direct conversion. You will also need to be careful of electromagnetic vs electrostatic units. Electromagnetic units usually have "ab" at the front of their units while electrostatic have "stat". Electrostatic units are almost the same as Gaussian units but have different names for some units. The Gaussian named units are analogues of SI for the most part. Gaussian units use the Franklin Fr as a derived charge unit. If we use subscript G for quantities measured in Gaussian units and subscript SI for quantities measured in SI, then we have the conversion factors below. The arrow to the right indicates the number of Gaussian units on top to an equivalent SI unit with $c_G = 2.997\,924\,58$ so defined.

$$\frac{q_G}{q_{SI}} = \frac{1}{\sqrt{4\pi\epsilon_0}} \to \frac{c_G\cdot 10^9\mathrm{Fr}}{1\,\mathrm{C}} \tag{E.2}$$

$$\frac{I_G}{I_{SI}} = \frac{1}{\sqrt{4\pi\epsilon_0}} \to \frac{c_G\cdot 10^9\mathrm{Fr/s}}{1\,\mathrm{A}} \tag{E.3}$$

$$\frac{\mathbf{J}_G}{\mathbf{J}_{SI}} = \frac{1}{\sqrt{4\pi\epsilon_0}} \to \frac{c_G\cdot 10^{13}\mathrm{Fr/s\,cm^{-2}}}{1\,\mathrm{A/s^2}} \tag{E.4}$$

$$\frac{\phi_G}{\phi_{SI}} = \sqrt{4\pi\epsilon_0} \to \frac{1\mathrm{statV}}{c_G\cdot 10^2\mathrm{V}} \tag{E.5}$$

$$\frac{\mathbf{E}_G}{\mathbf{E}_{SI}} = \sqrt{4\pi\epsilon_0} \to \frac{1\mathrm{statV/cm}}{c_G\cdot 10^4\mathrm{V/m}} \tag{E.6}$$

$$\frac{\mathbf{D}_G}{\mathbf{D}_{SI}} = \sqrt{\frac{4\pi}{\epsilon_0}} \to \frac{4\pi\cdot c_G\cdot 10^5\mathrm{Fr/cm^2}}{1\mathrm{C/m^2}} \tag{E.7}$$

$$\frac{\mathbf{B}_G}{\mathbf{B}_{SI}} = \sqrt{\frac{4\pi}{4\pi\epsilon_0}} \to \frac{10^4\mathrm{G}}{1\mathrm{T}} \tag{E.8}$$

$$\frac{\mathbf{H}_G}{\mathbf{H}_{SI}} = \sqrt{4\pi\mu_0} \to \frac{4\pi\cdot 10^{-3}\mathrm{Oe}}{1\mathrm{A/m}} \tag{E.9}$$

$$\frac{R_G}{R_{SI}} = 4\pi\epsilon_0 \to \frac{1\mathrm{s}}{(c_G)^2\cdot 10^{11}\Omega} \tag{E.10}$$

$$\frac{C_G}{C_{SI}} = \frac{1}{4\pi\epsilon_0} \to \frac{(c_G)^2\cdot 10^{11}\mathrm{cm}}{\mathrm{F}} \tag{E.11}$$

$$\frac{L_G}{L_{SI}} = 4\pi\epsilon_0 \to \frac{1\mathrm{s}^2/\mathrm{cm}}{(c_G)^2\cdot 10^{11}\mathrm{H}} \tag{E.12}$$

Given a formula one then replaces each quantity with its conversion factor. For example given Ampère's Law without the Maxwell correction

$$\boldsymbol{\nabla} \times \mathbf{B}_{SI} = \mu_0 \mathbf{J}_{SI} \tag{E.13}$$

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B}_G \sqrt{\frac{\mu_0}{4\pi}} &= \mu_0 \sqrt{4\pi\epsilon_0} \mathbf{J}_G \\ \boldsymbol{\nabla} \times \mathbf{B}_G &= 4\pi \sqrt{\mu_0 \epsilon_0} \mathbf{J}_G = \frac{4\pi}{c} \mathbf{J}_G \end{aligned} \tag{E.14}$$

We can then perform the same manipulations in reverse for Gaussian to SI.

E.2.1 Electromagnetic Unit Systems

I will now explain how one can see various relations between different electromagnetic unit systems. One should note that in fact there is a coherent foot-pound-second (FPS) analogue of CGS systems that, as far as I know, has never been used in any useful physical calculation. In this case there are electromagnetic and electrostatic units with fpsm for the electromagnetic and fpse for the electrostatic units, where the difference will be easier to see below.

If we write

$$F = k_C \frac{q_1 q_2}{r^2} \tag{E.15}$$

$$\frac{\mathrm{d}F}{\mathrm{d}x} = 2k_A \frac{I_1 I_2}{r} \tag{E.16}$$

for Coulomb's law and the Ampère force per unit length laws. Note that r is the distance between the two point charges (Coulomb) or very thin wires (Ampère). We will see that electrostatic units choose $k_C = 1$ and electromagnetic units choose $k_A = 1$. From this we can then use that an electric field for a point charge is then defined by

$$E = k_C \frac{q_1}{r^2} \tag{E.17}$$

to define electric field units. To find the relationship between k_C and k_A we can take the ratio of the two laws above to find dimensionally (using the reasonable assumption that the current is charge per time) that (with L length, T time, and $[q]$ indicating dimensions of q)

$$L = \frac{[k_C]}{[k_A]} \frac{T^2}{L} \tag{E.18}$$

$$\frac{[k_C]}{[k_A]} = \frac{L^2}{T^2} \tag{E.19}$$

This means that k_C/k_A must be a velocity squared. One can then compare the known values for free space and find that in fact the velocity to be squared is the speed of light c [note how the factor of 2 was put in to ensure this].

Now we need to define magnetic field units. There is some extra freedom here. We can define the magnetic field via a single nearly infinitely thin wire as

$$B = 2k_A \alpha_B \frac{I}{r} \tag{E.20}$$

System	k_C	k_A	α_B	α_F	$k_A\alpha_B$
Electrostatic (esu)	1	c^{-2}	1	1	c^{-2}
Electromagnetic (emu)	c^{-2}	1	1	1	1
Gaussian	1	c^{-2}	c	c^{-1}	c^{-1}
Heaviside-Lorentz	$\frac{1}{4\pi}$	$\frac{1}{4\pi c^2}$	c	c^{-1}	$\frac{1}{4\pi c}$
SI	$\frac{1}{4\pi\epsilon_0}$	$\frac{\mu_0}{4\pi}$	1	1	$\frac{\mu_0}{4\pi}$

Table E.6: The electromagnetic systems with various proportionality factors chosen.

Then we have for the ratio $[E]/[B]$ that

$$\frac{[E]}{[B]} = \frac{[k_C]}{[k_A][\alpha_B]}\frac{T}{L} = \frac{L^2}{T^2[\alpha_B]}\frac{T}{L} = \frac{L}{T[\alpha_B]} \tag{E.21}$$

Then Faraday's law can be written as

$$\boldsymbol{\nabla}\times\mathbf{E} + \alpha_F\frac{\partial\mathbf{B}}{\partial t} = 0 \tag{E.22}$$

It can readily be seen that $[\alpha_F] = 1/[\alpha_B]$ to be dimensionally consistent. To show that in fact $\alpha_F = 1/\alpha_B$ we write out all of Maxwell's laws with our proportionality coefficients as (Note that the 4π comes from consistently applying Gauss's law with $\boldsymbol{\nabla}\cdot\mathbf{E}$ and (E.15). The other 4π with $\boldsymbol{\nabla}\times\mathbf{B}$ comes from consistently applying Stokes law where a 2π occurs but we have defined our extra factor of 2 for the force in (E.20).)

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi k_C\rho \tag{E.23}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = 4\pi k_A\alpha_B\mathbf{J} + \frac{k_A\alpha_B}{k_C}\frac{\partial\mathbf{E}}{\partial t} \tag{E.24}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\alpha_F\frac{\partial\mathbf{B}}{\partial t} \tag{E.25}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{E.26}$$

Now we can use that in free space we have ($\mathbf{J} = \mathbf{0}$)

$$\begin{aligned}\nabla^2\mathbf{B} =& \nabla(\boldsymbol{\nabla}\cdot\mathbf{B}) - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{B} = \mathbf{0} - \boldsymbol{\nabla}\times\left(\frac{k_A\alpha_B}{k_C}\frac{\partial\mathbf{E}}{\partial t}\right) = -\frac{k_A\alpha_B}{k_C}\frac{\partial}{\partial t}\left(-\alpha_F\frac{\partial\mathbf{B}}{\partial t}\right)\\ =& \frac{k_A\alpha_B\alpha_F}{k_C}\frac{\partial^2\mathbf{B}}{\partial t^2}\end{aligned} \tag{E.27}$$

Because we know that this wave propagates at c then we must have

$$\frac{k_A\alpha_B\alpha_F}{k_C} = \frac{1}{c^2} \tag{E.28}$$

and we have already set $k_C/k_A = c^2$ so $\alpha_B = 1/\alpha_F$.

Then we can summarize different choices of systems as in Table E.6. As one can can see, it is much better to just use one system rather than have to deal with the various unit systems.

With this we can see where the different unit systems start with choosing different proportionality constants as being unity and dimensionless.

Quantity	Sym.	SI	ESU	EMU	Gaussian
electric charge	q	$1\,\mathrm{C}$	$c_G 10^9\,\mathrm{statC}$	$10^{-1}\,\mathrm{abC}$	$c_G 10^9\,\mathrm{Fr}$
electric current	I	$1\,\mathrm{A}$	$c_G 10^9\,\mathrm{statA}$	$10^{-1}\,\mathrm{abC}$	$c_G 10^9\,\mathrm{Fr/s}$
electric potential	ϕ	$1\,\mathrm{V}$	$c_G^{-1}10^{-2}\,\mathrm{statV}$	$10^8\,\mathrm{abV}$	$c_G^{-1}10^{-2}\,\mathrm{statV}$
electric field	$\mathbf{E}$	$1\,\mathrm{V/m}$	$c_G^{-1}10^{-4}\,\mathrm{statV/cm}$	$10^6\,\mathrm{abV/cm}$	$c_G^{-1}10^{-4}\,\mathrm{statV/cm}$
—	$\mathbf{D}$	$1\,\mathrm{C/m^2}$	$4\pi c_G \cdot 10^5\,\mathrm{statC/cm^2}$	$4\pi \cdot 10^{-5}\,\mathrm{abC/cm^2}$	$4\pi c_G \cdot 10^5\,\mathrm{Fr/cm^2}$
magnetic field	$\mathbf{B}$	$1\,\mathrm{T}$	$c_G^{-1}10^{-6}\,\mathrm{statT}$	$10^4\,\mathrm{G}$	$10^4\,\mathrm{G}$
H field	$\mathbf{H}$	$1\,\mathrm{A\,m^{-1}}$	$4\pi c_G \cdot 10^7\,\mathrm{statA/cm}$	$4\pi \cdot 10^{-3}\,\mathrm{Oe}$	$4\pi \cdot 10^{-3}\,\mathrm{Oe}$
magnetic flux	Φ_m	$1\,\mathrm{Wb}$	$c_G^{-1}10^{-2}\,\mathrm{statWb}$	$10^8\,\mathrm{Mx}$	$10^8\,\mathrm{Mx}$
resistance	R	$1\,\Omega$	$c_G^{-2}10^{-11}\,\mathrm{s\,cm^{-1}}$	$10^9\,\mathrm{ab}\Omega$	$c_G^{-2}10^{-11}\,\mathrm{ab}\Omega$
resistivity	ρ	$1\,\Omega\,\mathrm{m}$	$c_G^{-2}10^{-9}\,\mathrm{s}$	$10^{11}\,\mathrm{ab}\Omega\mathrm{cm}$	$c_G^{-2}10^{-9}\,\mathrm{s}$
capacitance	C	$1\,\mathrm{F}$	$c_G^{2}10^{11}\,\mathrm{cm}$	$10^{-9}\,\mathrm{abF}$	$c_G^{2}10^{11}\,\mathrm{cm}$
inductance	L	$1\,\mathrm{H}$	$c_G^{-2}10^{-11}\,\mathrm{cm^{-1}\,s^2}$	$10^9\,\mathrm{abH}$	$c_G^{-2}10^{-11}\,\mathrm{cm^{-1}\,s^2}$

Table E.7: Conversion factors between various electromagnetic unit systems. Here $c_G = 2.997\,984\,52$ and Sym. means commonly used symbol.

In electromagnetic units, one starts with Ampère's Law with a unit of current being defined. A Biot Bi is then given by the constant current in two straight parallel conductors of infinite length, with almost no circular cross-section placed one centimeter apart in a vacuum, produces a force equal to two dynes per centimeter of length of the conductors. Thus a Biot is the square root of a dyne so $\mathrm{Bi}=\mathrm{g^{1/2}cm^{1/2}s^{-1}}$ and sometimes called an abampere or emu current. The charge is $\mathrm{Bi\,s}=\mathrm{g^{1/2}cm^{1/2}}$ sometimes called an abcoulomb or emu charge.

In electrostatic units, one instead starts with Coulomb's Law and so that the unit of charge is called either a Franklin, statcoulomb, or an esu charge. Then a Franklin is defined as the charge of each of two equal point charges set a centimeter apart such that the force between them is one dyne. It has units of $\mathrm{g^{1/2}cm^{3/2}s^{-1}}$. The current is then given by $1\,\mathrm{Fr/s}$ which is called either a statampere or an esu current and has units of $\mathrm{g^{1/2}cm^{3/2}s^{-2}}$.

To convert from an esu unit to an emu unit requires factors of c. This is summarized in Table E.7.

E.3 The Name Current Density

I have often been bothered by the name current density since in almost every other circumstance density means $\mathrm{m^{-3}}$ or an equivalent per volume measurement. Yet current density means current per surface area. The reason for this is rather mundane. It is an area density, so that it really should be called a current area density. Most likely it was considered too long of a name.

The important thing to remember is that density is actually ambiguous as it depends on the space you are considering. Linear density, area density, and volume density could all be options for the meaning of density. Because current densities are a flow through a surface the density refers to an area density.

References

[1] John David Jackson. *Classical electrodynamics*. 1999.

[2] David B Newell and Eite Tiesinga. *The International System of Units (SI)*. Tech. rep. Technical report, National Institute of Standards and Technology, 2019, 2019.

List of Terms

DD Refers to processes with deuterium and deuterium. Often means deuterium and deuterium fusion. 375

DT Refers to processes with deuterium and tritium. Usually means deuterium and tritium fusion. 375

Airy function An Airy function is the solution $u(x)$ to the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - xu(x) = 0$$

with the two solutions denoted $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$. See section 1.4.2 for the details on these functions. 109

alpha particle An alpha particle, usually denoted α, is a helium-4 nucleus, i.e., a helium nucleus with two protons and two neutrons. 374, 376

analytic continuation A method of continuing a complex holomorphic function beyond its domain. It is useful because for holomorphic functions, this can be done uniquely. 13, 198

annihilator An annihilator is an operator that when applied to a mathematical object always returns zero. As an example, if we have operator $\mathbf{F}$, and a set of functions $f_i(x)$ with $\mathbf{F}(f_i) = 0$, then $\mathbf{F}$ is an annihilator for the $f_i(x)$. 96, 104, 273

asymptology The study of systems in limits, "the art of dealing with applied mathematical systems in limiting cases". This envelops everything under asymptotics. 13, 95

asymptotic expansion These are expansions (a summed series) that when truncated after a finite number of terms, provides an approximation of a given function as the function approaches a certain value (usually infinity). They need not be a convergent series, since we only require a finite number of terms to be a good approximation. 95

asymptotic series See asymptotic expansion. 97

asymptotics See asymptology. 97

atomic mass number See mass number plural. 372

atomic number The number of protons a chemical element has. Sometimes simply called the proton number and is denoted by Z from German *Zahl* meaning number from an element's position on the periodic table. It became atomic number via German *Atomzahl* when it became clear it was associated with nuclear charge plural. 372

ballooning transform This is a specific kind of transform that changes periodic coordinates into coordinates that vary from $-\infty$ to ∞. it is useful for use in approximations that require things to change slowly, such as in eikonal-like solutions. 13, 14, 79

beta minus particle A beta minus particle, usually denoted by β^- is an electron $(-)$. It is usually only called this when resulting from a radiocative decay. 374

beta particle A beta particle, usually denoted by β^+ or β^-, is either a positron $(+)$ or electron $(-)$. It is usually only called this when resulting from a radiocative decay. 374

beta plus particle A beta plus particle, usually denoted by β^+ is a positron $(+)$. It is usually only called this when resulting from a radiocative decay. 374

big O notation Big O notation uses an upper case $\mathcal{O}$ and $f(x) = \mathcal{O}(g(x))$ means for two functions $f(x)$ and $g(x)$ that $\limsup_{x\to a} |f(x)/g(x)| \leq k$ for $g(x) \neq 0$. 22

binormal vector In the Frenet-Serret formulas, it refers to the unit vector pointing normal to the tangent and normal vectors. 163, 168

burning plasma A burning plasma is a plasma with physicists' Q large enough that an appreciable amount of the required plasma heating is coming from fusion reactions. Typically $Q > 5$ for DT reactions is called a burning plasma, but $Q > 1$ is also often considered burning. It is an artifical category and so putting strict limits on where a plasma is burning and where it is not is arbitary. The important thing is that an appreciable amount of plasma heating comes from plasma fusion reactions. 402

calculus of variations This deals with how functionals change. So it is the calculus of "function of functions". It is used to find optimal functions given some sot of constraints. Typically one uses a δ to indicate variation of a funcitonal as $\delta F[f]$. 135

canonical form A form of the magnetic field given by $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \nabla\psi_t \times \nabla\theta + \nabla\psi_p \times \nabla\zeta$ with ψ_t the toroidal flux divided by 2π, ψ_p the poloidal flux divided by 2π, θ the poloidal angle, and ζ the toroidal angle. This form is the canonical form because it has canonical coordinates for the magnetic Hamiltonian. 159

canonically conjugate momentum Given a generalized coordinate q_i and generalized velocity $\dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}t}$, the canonically conjugate momentum is given by $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ where $\mathcal{L}$ is the Lagrangian of the system. We can write this as a vector array as $\mathtt{p} = \partial\mathcal{L}/\partial\mathtt{q}$ for generalized coordiantes $\mathtt{q}$. 153

Cauchy distribution See Lorentzian. 228

Cauchy principal value A Cauchy principal value avoids a singularity in an integrand path in a specific way. For a real line integrand with singularity at m with $a < m < b$ it is given by

$$\fint_a^b \mathrm{d}x\ f(x) \equiv \lim_{\epsilon\to 0^+} \left[\int_a^{m-\epsilon} \mathrm{d}x\ f(x) + \int_{m+\epsilon}^b \mathrm{d}x\ f(x)\right]$$

and for a contour integral Cauchy principal value (so a singularity is on the contour), it means perform the integral as if there were no singularity on the contour.. 17, 194, 228

characteristic function See eigenfunction and characteristic value for an explanation. 83

characteristic mode See eigenmode and characteristic value for an explanation. 83

characteristic value See eigenvalue. Eigenvalue comes from German *eigenwert* (eigen-value). *Eigen* here means "own" or "self", and would more properly be translated into English as "characteristic", however we simply stole the *eigen* and almost everyone calls them eigenvalues now. 83

Christoffel symbols These are used to take derivatives of tensors, including of the basis vectors. Sometimes called connection coefficients or affine connections. See (1.2.286)-(1.2.291). 72

confluent hypergeometric series See Section 1.15.2.. 208

conservative form This is the general form for an equation satisfying a conservation law. See (2.9.59). 338

contravariant A description of vector components that explains that when the scale for the coordinates is multiplied by a factor x, the component are mutliplied by a factor $1/x$. Hence varying against (contra) the change. For V^i a contravariant component in coordinate system ξ^i and $V^{\prime i}$ a contravariant component in a new coordinate system $\xi^{\prime i}$ we have $V^{\prime i} = \sum_{j=1}^{3} \frac{\partial \xi^{\prime i}}{\partial \xi^j} V^j$. 22, 34, 47, 48

coordinate system A set of variables that can be related to Cartesian space in some manner. Then when giving the variables values, one can uniquely specify where in the Cartesian space this assignment points, and vice versa. 13, 34

covariant A description of vector components that explains that when the scale for the coordinates is multiplied by a factor x, the component are mutliplied by a factor x. Hence varying with (co) the change. For V_i a covariant component in coordinate system ξ^i and $V_{\prime i}$ a coavariant component in a new coordinate system $\xi^{\prime i}$ we have $V_{\prime i} = \sum_{j=1}^{3} \frac{\partial \xi^j}{\partial \xi^{\prime i}} V_j$. 22, 34, 47, 48

covariant basis See the tangent basis. 49

critical point A critical point is a point where the first derivative (or gradient) of the function is zero or the function is not differentiable there. That is, given $f(\mathbf{z})$ for some vector array of variables (possibly complex) $\mathbf{z}$, then if $\frac{\partial f}{\partial \mathbf{z}}\big|_{\mathbf{z}=\mathbf{z}_0} = 0$ or $\frac{\partial f}{\partial \mathbf{z}}$ is undefined, then $\mathbf{z}_0$ is a critical point. 98

cross section There are two princple meanings in physics. (1) A cross section is a an intersection of a 2D surface in 3D space. Thus, if we have an angle θ defined in 3D space we can create a cross section at $\theta = c$ which is a 2D surface or slice of the whole space. (2) A nuclear cross section is a a measurement of the likelihood of a nuclear process occurring. It usually refers to the microscopic cross section denoted by σ and measured in units of area (typically in barn, or $10^{28}\,\mathrm{m}^2$. The macroscopic cross section is measured in inverse length given by the target density times the microscopic cross section. 378, 545

curvature In the Frenet-Serret formulas, it refers to the size of the circle that could be fit to the curve at that local point. 164

curvilinear coordinates A coordinate system where the variables used to describe locations change as one moves through space. That is, the variables do not form straight lines through space as they are varied. 13

Darboux frame A formulation of trajectories that uses three vectors defined along a trajectory constrained to some surface (or subspace of the full space) and using three constants, the normal curvature, geodesic curvature and relative torsion, to completely specify a trajectory's path. Related to the Frenet-Serret formulas. 168

de Broglie wavelength This is a wavelength associated with a particle of momentum $p = mv$ and the Planck constant. It is given by $\lambda = h/p$ for λ the de Broglie wavelengh. 384

Debye length Defined by $\lambda_D^{-2} = \sum_s \frac{n_{0s} q_s^2}{\epsilon_0 k_B T_s}$. The electron Debye length is often called the Debye length and so then $\lambda_{D_e} = \sqrt{\frac{\epsilon_0 k_B T_e}{n_{0e} e^2}}$. This is the screening length in a plasma. This means that a charge at distances beyond this from the constitutents of the plasma will only see a small screened potential. This may be contrary to expectations since the plasma is made up of a bunch of separated charges, but remember the particles are interacting. 254

Debye number Given by $N_D = \frac{4}{3}\pi n \lambda_D^3$, and is the number of particles in a Debye sphere. If we want a typical plasma, this number must be very large. 255

Debye sphere Multiple definitions are sometimes used, but if "sphere" is used it is generally assumed to mean $\frac{4}{3}\pi\lambda_D^3$. For a typical plasma, we desire there to be many particles per Debye sphere. 254

DEMO DEMO stands for demonstration power station. Not very creative, but at least very straightforward. It is imagined as the next step after ITER, resulting in net power production and the stepping stone to commercial power. It is not a single well-defined plant, though. It is simply the name for whatever is built after ITER as a demonstration power plant. There are many possible types of DEMO. 427, 429, 446

deuterium This refers to hydrogen atoms which contain nuclei with one proton and one neutron. Usually denoted T or t rather than ^{3}H. See also deuteron. 375

deuteron This refers to a hydrogen nucleus with one proton and one neutron. Usually denoted D or d rather than ^{2}H. Technically, different from deuterium which refers to the whole atom, but in practice is usually used interchangeably with deuterium. 374

differential cross section This refers to the cross section per angle (i.e., per radian) or solid angle (i.e., per steradian) and is denoted $\frac{\partial\sigma}{\partial\theta}$ for per angle or $\frac{\partial\sigma}{\partial\Omega}$ for per solid angle. It is sometimes referred to as simply the cross section, though this should be avoided if one is trying to reduce chances for confusion. In addition for any quantity q that parameterizes trajectories, a differential cross section can be defined via $\frac{\partial\sigma}{\partial q}$. Note that the differential cross section is enforced to be a positive number so that absolute value signs should actually be used. 379

Dirac delta function This is a distribution (so not actually a function) that is often thought of as a function that has an infinite value at only one point. The actual definition is that for Dirac delta function $\delta(x)$ we have

$$\int_a^b \mathrm{d}x\ f(x)\delta(x-c) = \begin{cases} f(c) & \text{if } a \le c \le b \\ 0 & \text{otherwise} \end{cases}$$

. 293

dispersion relation The dispersion relation is usually given in the angular frequency ω and the wavenumber k as $\omega(k)$. This relation explains how the frequency $\omega/(2\pi)$ and the wavelength ($\lambda = 2\pi/k$) are related. The phase velocity is given by ω/k and the group velocity by $\mathrm{d}\omega/\mathrm{d}k$. 333

dominant balance The principle of dominant balance says we can, as a first approximation, put the largest terms equal to each other. I extend this to subdominant terms, and for situations outside of ordinary differential equations. 122

dyad A dyad is a polyadic made of two vectors, and so an order 2 tensor. 60

eigenfunction This is a solution, usually to a linear equation. The eigenfunction of the equation $\mathsf{F}(f(\mathbf{x})) = \lambda f(\mathbf{x})$ is $f(\mathbf{x})$ (with eigenvalue λ) where F is an operator. It means that the operator on the eigenfunction returns the eigenfunction times a constant (scalar). The difference between this and an eigenmode is that the eigenmode is not necessarily a function, as it an eigenmode could be a vector, for example. 83

eigenmode This is a solution, usually to a linear equation. The eigenmode of the equation $\mathsf{F}(\mathbf{f}) = \lambda\mathbf{f}$ is $\mathbf{f}$ where F is an operator (with eigenvalue λ). It means that the operator on the eigenmode returns the eigenmode times a constant (scalar). 83

eigenvalue This is a constant in a solution, usually to a linear equation. The eigenvalue of the equation $\mathsf{F}(\mathbf{f}) = \lambda\mathbf{f}$ is λ where F is an operator on some object $\mathbf{f}$ which is an eigenmode. It means that the operator on the eigenmode returns the eigenvalue times the eigenmode. 83

eikonal An eikonal is an equation of one of the following forms

$$\begin{aligned} \nabla S \cdot \nabla S &= g(\mathbf{x}) \\ \frac{\partial S}{\partial t} + \nabla S \cdot \nabla S &= g(\mathbf{x}) \\ |\nabla S| &= g(\mathbf{x}) \end{aligned}$$

with $g(\mathbf{x})$ a given function and S the unknown with some given boundary conditions. The first equation allows wave propagation in two directions, the last equation allows propagation in one direction, and the middle equation is a time dependent generalization. Eikonal equations are useful because there is a solution method using Hamiltonian mechanics theory, and arises in waves scattering and propagation. 95, 109, 184

Einstein summation notation This convention/notation means that when two indices are in a term, there is an implied summation. So $c_j a_i b_i = \sum_i c_j a_i b_i$. The rules are that non-dummy variables must be matched across an equality and a dummy index can only be repeated exactly two times in each term. It is often called index notation, Einstein notation, or the Einstein summation convention. 15, 56

electron capture A spontaneous nuclear radiation process where a nucleus absorps an electron and then the electron combines with a proton in the nucleus to create a neutron within the nucleus. 376

energy confinement time This is a characteristic time over which a plasma loses its energy. Given a plasma energy loss time and the average total energy of the plasma it is defined as $\tau_E = E_P/P_L$. It can be found experimentally by giving external power P_{ext} until the energy

maintains a constant average total energy. For then E_P can be calculated and $P_L = P_{\text{ext}}$. 396

engineering Q This is gives the ratio of fusion heating power to total heating power supplied. That is the denominator includes contributions from external heating sources including the power required to actually get energy into the plasma. Thus $Q = P_{\text{fus}}/(P_{\text{fus}} + P_{\text{ext,f}})$. Here P_{fus} is the fusion power and $P_{\text{ext,f}}$ is the total power required to get the necessary heating in the plasma. Compare it against physicists' Q, where the total external power is not used. 401

ensemble A thermodynamic ensemble is a large number of configurations with the same macroscopic properties but different microscopic configurations. 295

error function See Section 1.15.3.. 212

Euclidean space A flat space that can be completely described by a Cartesian coordinate system. 34

Eulerian specification Given quantities for a flowing fluid, this specification requires the current position and the time to parameterize variables. Thus $\mathbf{q}_E = \mathbf{q}_L(\mathbf{R}, t)$ where $\mathbf{q}_E$ is any Eulerian quantitiy, $\mathbf{R}$ is the current position of the quantity, and t is time. If one were to drop a cork into a river, and ask where it is later, one would require to give the position and the time because you do not follow the flow. Instead, it as if you are on the bank and asking where the cork is. 341

exponential integral See Section 1.15.7. 219

flux coordinates A coordinate system for magnetic confinement systems that uses a magnetic flux as the radial-like coordinate. These are also called straight (magnetic) field lines. 13, 273

flux function A function that is only a function of the flux (either poloidal or toroidal). 265

flux surface In plasma physics it always refers to magnetic flux surfaces. 259

flux surface averaging This is the average of a quantity over a flux surface. See Section 2.2.3.2. 271

four-tensor A geometric object in spacetime (4D). It can be built up from four-vectors, and has transformation properties similar to four-vectors. An nth order tensor is built from n four-vectors. 15

four-vector A geometric object in spacetime. It has a magnitude and direction and can be represented by set of scalars and basis vectors, but in four dimensions. It is not an array or column or row of numbers. It must rotate and translate as a geometric object so that it is coordinate independent. 15

Fourier series A series approximation of a function that uses sines and cosines. Sometimes it is simplified into a complex exponential format. 13, 14, 79, 87

Fourier transform An integral that changes a continuous function into an analogue of the Fourier series (therefore one function into a different function). Often useful for solving differential

equations. For example it changes a function from using a domain of time to a domain of frequency or vice versa. 13, 79, 87

Frenet-Serret A formulation of trajectories that uses three vectors defined along a trajectory and two constants, the curvature and torsion, to completely specify a trajectory's path. Named after the codiscoverers. 14, 163, 359

frozen flux This refers to magnetic flux being "frozen" into a perfect conductor. That is, if we have a perfect conductor, then the amount of magnetic flux (and so the magnetic field lines) must move with the conductor such that no electric field is produced within the conductor in its rest frame. This is usually proven in Ideal MHD and called the frozen flux theorem. 315

functional This is a "function" that takes in a function and outputs a scalar. It is traditionally written as $F[f]$ with square brackets and F a functionl taking function f. Functionals do not have to be integrals, but in physics situations functionals are almost always represented as integrals. 135

functional derivative The functional derivative is the linear in variation change of a functional. Given $F[f]$ then $\frac{\delta F}{\delta f}$ is the variational derivative given by expanding δF and keeping only δf terms. 136

gamma particle A gamma particle is a photon released in a radioactive decay. Often called gamma ray. 374

gamma ray See gamma particle. 374

Gamow energy An energy used to characterize the probability of scattering of a particle by another particle due to the Coulomb potential. It is usually given as (3.4.1), though one can find other definitions if you are not looking at the probability of change through a surface. 382

Gaussian See Section 1.15.5. 216

Gaussian integral A Gaussian integral is an integral of the form

$$\int_0^\infty \mathrm{d}x\ x^n \exp(-\alpha x^2) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\alpha^{(n+1)/2}}$$

for any $n > 0$ (not necessarily an integer). One can then use even and oddness for m an integer as

$$\int_{-\infty}^\infty \mathrm{d}x\ x^{2m} \exp(-\alpha x^2) = \frac{\Gamma\left(\frac{2m+1}{2}\right)}{\alpha^{(2m+1)/2}}$$
$$\int_{-\infty}^\infty \mathrm{d}x\ x^{2m+1} \exp(-\alpha x^2) = 0$$

. 99

generalized coordinate A generalized coordinate is anything that, along with all other generalized coordinates for the system, uniquely picks out a system's state. The generalized coordinates together are a vector array and not a geometric vector. Typically the gneralized coordinates are given by `q`, sometimes `x`. 153

geodesic A geodesic is the shortest path between two points. In Euclidean space, this is a straight line, but if one restricts oneself to a surface or some other constraint on a trajectory, a geodesic is not necessarily a straight line. For example, on the surface of a sphere a geodesic between two points on the sphere is the great circle connecting those two points. 167

geodesic curvature When using a Darboux frame, this refers to the curvature due to being constrained to a surface that is not "necessary". This is sometimes called the extrinsic curvature because of this, as it is extra movement from the geodesic (or shortest distance) path along the surface between two points. If one is on a geodesic, then the geodesic curvature is zero. 168, 169

geodesic torsion This is the torsion in the Darboux frame when using the mathematical definition. Sometimes it is called the relative torsion. 169

geometric axis This is the Z axis in cylindrical coordinates. It is the axis that goes through the ceneter of the hole of the torus. 265, 286

gyroaverage This is an operation that averages a quantity over a gyroperiod, that is the inverse of the gyrofrequency. It averages out the motion of particles on the scale of a gyroradius or a gyrofrequency. 134

gyrocenter If we imagine a charged particle gyrating in a magnetic field, the local center of the gyroorbit is called the gyrocenter. This can be interpreted as the position of a charged ring instead of using the individual particle. 134

gyrofrequency The frequency at which a charged particle circles a magnetic field line. It is typically denoted Ω or ω_c with $\Omega = |q|B/m$. Some authors allow the frequency to have a sign so that the electron gyrofrequency is negative. 259, 351

gyromotion This refers to charged particles gyrating around magnetic fields. 134

gyrorbit This is orbit of a charged particle around a magnetic field line. 259

half-life Given a substance of a pure single nuclide, the half-life is the amount of time it would take (on average) for half of the original nuclide to radiocatively decay plural. 373

Hamiltonian A Hamiltonian is a function that when time independent represents the total energy. When it is time dependent, it represents an energy-like quantity. It determines the time evolution of a system via $\frac{\partial H}{\partial q} = -\frac{\mathrm{d}p}{\mathrm{d}t}$ and $\frac{\partial H}{\partial p} = \frac{\mathrm{d}q}{\mathrm{d}t}$ with p and q canonical coordinates. 13, 14, 158

heliotron See Section 2.4.2. 288

Helmholtz transport theorem This gives the derivative of a flux integral. It is given by [see (2.7.11)]

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{G} = \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{V}\boldsymbol{\nabla}\cdot\mathbf{G} - \oint_C \mathrm{d}\boldsymbol{\ell}\ \cdot(\mathbf{V}\times\mathbf{G}) + \iint_S \mathrm{d}S\ \hat{\mathbf{n}}\cdot\frac{\partial\mathbf{G}}{\partial t}$$

. 316, 366, 473

Hessian Given a function $f(\mathtt{x})$, with vector array $\mathtt{x}$ having elements x_i, the Hessian matrix is the matrix given by

$$\mathbf{H} \equiv \frac{\partial^2 f}{\partial \mathtt{x} \partial \mathtt{x}} = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial^2 x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial^2 x_n} \end{bmatrix}$$

For a vector array of complex variables $\mathtt{z}$, then one uses $\frac{\partial^2 f}{\partial \mathtt{z} \partial \bar{\mathtt{z}}}$ which for any analytic function will be identically zero, and so not give you much information. 100

Hessian See Hessian matrix. 100

holomorphic A comlpex function that is complex differentiable on every point in its domain. 114, 193, 199

holonomic constraint From Greek meaning "whole law", these are constraints that deal only with position (and not velocities). They are special because they correspond to systems that are integrable, which loosely means that the variables we have give a unique state. That is, there is no path dependence in the state. 143, 150

hypergeometric series See Section 1.15.1.. 203

Ideal MHD Ideal magnetohydrodynamics. A set of equations used to model electrically conducting fluids. 121

ignited plasma An ignited plasma is a plasma that is undergoing fusion reactions at a rate such that it requires no external power to continue the fusion process. 398

induced radioactivity This refers to radioactive processes that have been made more probable by human intervention. It is said it is "human-made" or "artificial" radioactivity. This is because something has been done to what would usually be a stable nuclide to make it an unstable nuclide. 378

integral cross section This refers to sense (2) of cross section. Rarely used, this terminology makes it clear that it is not referring to a differential cross section. 379

isobar For nuclear processes, an a nuclide is an isobar of another nuclide when both nuclides share the same mass number A. In meterology, an isobar is a line of constant pressure on a figure. 377

isotope An isotope of an element is an element with a specified number of neutrons in its nucleus. An element usually has numerous isotopes, meaning that the nuclei have the same proton number but each isotope is associated with a nucleus with a different number of neutrons. Usually isotopes refer to a single element. See also nuclides plural. 372

ITER This used to stand for international thermonuclear experimental reactor. The story I have heard is that thermonuclear is a taboo word, as it causes people anxiety. So it was changed to just be ITER. Then it is usually stated *iter* means "path" in Latin. In fact, it means something more like "journey" in Latin, but so be it. ITER is a large tokamak that will teach us about the physics of burning plasmas. It will not produce net power, but is an experiment. 398, 429

Jacobian determinant This is the determinant of the matrix of partial derivatives. It is indicated via $\mathcal{J} = |\boldsymbol{\mathcal{J}}| = |\frac{\partial(\xi^1,\xi^2,\xi^3)}{\partial(x^1,x^2,x^3)}|$ and is given by (1.2.48). Sometimes it is defined as the inverse of (1.2.48). This is all a matter of convention. 44

Jacobian matrix This is a matrix of partial derivatives. It is indicated via $\boldsymbol{\mathcal{J}} = \frac{\partial(\xi^1,\xi^2,\xi^3)}{\partial(x^1,x^2,x^3)}$ and is given by (1.2.47). Sometimes it is defined as the transpose of (1.2.47), and some use $\boldsymbol{\mathcal{J}} = \frac{\partial(x^1,x^2,x^3)}{\partial(\xi^1,\xi^2,\xi^3)}$ as the Jacobian matrix, rather than its inverse. This is all a matter of convention. 44

Knudsen number The Knudsen number, usually denoted Kn is defined by the ratio of the mean free path λ_{mfp} to a characteristic length in the system L. So $\text{Kn} = \lambda_{\text{mfp}}/L$. Thus, a small Knudsen number implies a very collisional situation on the characteristic length scale, while a very large Knudsen number implies almost no collisions on the characteristic length scale. Thus a large Knudsen number is a collisionless situation. 301

Lagrange multiplier See Lagrange multiplier method. 146, 147, 149, 150

Lagrange multiplier method This is a method of finding an optimal solution when considering separate constraints. See Section 1.8.2 for details. 144

Lagrangian A Lagrangian for a physical system, when used in the Euler-Lagrange equations, yields the equations of motion that dictate the dynamics of the system. It is typically given the symbol $\mathcal{L}$ or L with $\mathcal{L}(\mathsf{q}(t), \dot{\mathsf{q}}(t), t)$ where q is the generalized coordinates, $\dot{\mathsf{q}}$ is the generalized velocities, and t is time. 149, 150

Lagrangian specification Given quantities for a flowing fluid, this specification only requires the initial position and the time to parameterize variables. Thus $\mathbf{q}_L = \mathbf{q}_L(\mathbf{R}_0, t)$ where $\mathbf{q}_L$ is any Lagrangian quantitiy, $\mathbf{R}_0$ is the initial position of the quantity, and t is time. If one were to drop a cork into a river, and ask where it is later, one would simply supply the initial cork position in the river and the later time we want to know. Thus, you follow the flow in time to see where the cork is. 341

Laplace transform Similar to a Fourier transform except that it includes causality by not including contributions. When using with time, it only includes $t > 0$ contributions. 13, 14, 79, 89

Laplace's method This is a method of approximating an integral whose integrand is dominated by contributions in a small region by a controlling factor. That is, one part of the integrand picks out a region as the most important (often an exponential). See section 1.4.1 for more information. 97, 100

Larmor radius See gyroradius. 259, 357

Lawson criteria I use this to mean any sort of figure of merit that serves to show how much energy is being produced by fusion reactions. Some limit this to mean $n\tau_E$ or $nk_BT\tau_E$ for number density n, temperature T, and confinement time τ_E. Others use it only for Lawson's original paper using essentially what I called physicists' Q in the text. I prefer to be overly broad since it does not seem to matter much which one you use, as they all give similar answers for when fusion is economical. 401

Leibniz integral rule One of many rules that explain how to take a derivative of an integral. For three dimensions and various integrands there are new names. See the Reynold's transport Theorem and Helmholtz transport theorem. For a simple 1D integral we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x,t)}^{b(x,t)} \mathrm{d}t\ f(x,t) = f(x,b)\frac{\partial b}{\partial x} - f(x,a)\frac{\partial a}{\partial x} + \int_a^b \mathrm{d}t\ \frac{\partial f}{\partial x}$$

. 265, 472

linear equation A linear equation is an equation where all unknown quantities appears in each term in the equation only once (to the first power) or zero times (does not appear). This extends to differential equations in the same way including derivatives. Note that the dependent variable (t here) can have nonlinear terms and we still have a linear differential equation for the unknown x. 121

linearization A method that takes a non-linear system of equations and changes it into a linear system of equations. This obviously requires approximations, and the specific scheme is to choose parameters $\delta \ll 1$ so that one only needs to keep $\mathcal{O}(\delta)$ contributions from each variable. 13, 14

linearly independent Given a set of vectors $\mathbf{V}^i$, they are linearly independent if there are no coefficients a_k such that $\mathbf{V}^j = \sum_{k\neq j} a_k \mathbf{V}^k$. That is there is no linear combination of the other vectors in the set that equals the vector you are considering in the set. 42

little O notation Little O notation uses a lower case o and $f(x) = o(g(x))$ means for two functions $f(x)$ and $g(x)$ that $\lim_{x\to a} f(x)/g(x) = 0$ for $g(x) \neq 0$. 22, 97

logarithmic derivative A logarithmic derivative of $q(r)$ is given by $\frac{1}{q}\frac{\mathrm{d}q}{\mathrm{d}r} = \frac{\mathrm{d}\ln q}{\mathrm{d}r}$. It is called logarithmic because of the latter form. Note the latter form implicitly requries a reference value q_0 for dimensional consistency, but that the reference value cancels out: $\frac{\mathrm{d}\ln(q/q_0)}{\mathrm{d}r} = \frac{\mathrm{d}\ln q}{\mathrm{d}r} - \cancel{\frac{\mathrm{d}\ln q_0}{\mathrm{d}r}}$. Note the same trick is often used for the denominator of a derivative as well, so $\frac{r}{q}\frac{\mathrm{d}q}{\mathrm{d}r}$ is written as $\frac{\mathrm{d}\ln q}{\mathrm{d}\ln r}$. 270

Lorentzian This is a fat-tailed probability distribution given by

$$f(x;x_0,\gamma) = \left(\pi\gamma\left[1+\frac{(x-x_0)^2}{\gamma^2}\right]\right)^{-1} = \frac{1}{\pi\gamma}\frac{\gamma^2}{(x-x_0)^2+\gamma^2}$$

that has an undefined mean and infinte variance. The x_0 is where the function peaks and 2γ is the full width half max of the distribution. 228

macroscopic cross section This refers to the typical cross section defined in empirical situations for a stationary target, a beam of projectile particles $\Phi_b = n_b v_b$, and reaction rate per volume R, the macroscopic cross section Σ is defined by $R = \Phi_b\Sigma$ and so is related to the microscopic cross section by $\Sigma = n_t\sigma$ for target number density n_t. 381

magnetic axis This is the magnetic flux surface (actually a line) where the poloidal magnetic field vanishes. 265

magnetic curvature See magnetic curvature vector. 283

magnetic curvature vector For a magnetic field line at a particular location, this tells us the closest approximating circle. It is typically defined as $\boldsymbol{\kappa} = \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} = -\hat{\mathbf{b}} \times (\boldsymbol{\nabla} \times \hat{\mathbf{b}})$ for $B\hat{\mathbf{b}} = \mathbf{B}$. 360

magnetic field line Given a magnetic field $\mathbf{B}$, we can define magnetic field lines via $\frac{\partial \mathbf{x}}{\partial \ell} = \mathbf{B}/|\mathbf{B}|$ with ℓ a field-line length label. Then a trajectory defined by that equation is a field line. In practice they are determined from $\dfrac{\mathrm{d}\ell}{B} = \dfrac{\mathrm{d}\xi^i}{\mathbf{B} \cdot \nabla\xi^i}$ for all i. 13, 158

magnetic flux surface A flux surface given by the magnetic field. This is a surface with normal $\hat{\mathbf{n}}$ such that the $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$. That is the magnetic field lines lie entirely on the surface. 91, 159

magnetic shear The magnetic shear is related to the derivative of the safety factor (or sometimes the rotational transform). It measures how much the safety factor changes radially given by $s = \frac{r}{q}\frac{\mathrm{d}q}{\mathrm{d}r} = \frac{\mathrm{d}\ln q}{\mathrm{d}\ln r}$. More generally, if we construct the Darboux frame, with binormal $\hat{\boldsymbol{\eta}} = \hat{\mathbf{n}} \times \hat{\mathbf{b}}$ for $\hat{\mathbf{n}}$ the magnetic surface unit normal and $\hat{\mathbf{b}}$ the unit magnetic field vector, then the shear is given by $\hat{\boldsymbol{\eta}} \cdot (\boldsymbol{\nabla} \times \hat{\boldsymbol{\eta}})$. In this form, its geometric interpretation is most easily seen as describing how the nearby magnetic field lines curve near a particular location. 92, 173, 270

magnetohydrodynamics See MHD. 251

matrix This is an $n \times n$ array of numbers with associated algebraic manipulations from regular matrix algebra. I will always use square brackets around my matrices. 15

metastable A metastable state is a state that is stable to small enough changes in energy, but that can transition to a lower energy (more stable) state if it is given the correct energetic push. Imagine a flat plain with a volcano with a caldera. If you are in the caldera, and you throw a ball, it will roll to the bottom of the caldera. This is metastable. Now suppose you launch the ball with an advanced trebuchet. If it gets past the rim of the caldera, it will roll to the bottom of the volcano which is the acutal stable state. 377

method of stationary phase See Laplace's method or the method of steepest descent, as the method of stationary phase is similar. The idea is still to use a method of approximating an integral, but now we think that the controlling factor is oscillating so rapidly that it averages things out except at ocations where the phase is "stationary" or not rapidly varying. So the integrand is dominated by contributions in a small region by a controlling factor. See section 1.4.1 for more information. 97

method of steepest descent See Laplace's method, as the method of steepest descent is similar. It is more general in that it can consider complex values. The idea is still to use a method of approximating an integral whose integrand is dominated by contributions in a small region by a controlling factor. See section 1.4.1 for more information. 97, 99

microscopic cross section This refers to the typical cross section defined in empirical situations for a stationary target of number density n_t, a beam of projectile particles $\Phi_b = n_b v_b$, and reaction rate per volume R, the microscopic cross section σ is defined by $R = n_t \Phi_b \sigma$. 379, 381

minor radius For a torus, this is the radius of the poloidal cross section (i.e., of the circle created by going around the torus the short way). 260

modified Bessel function See Section 1.15.8.2. 223

moment hierarchy problem When using kinetic theory and taking moments of the equations, each moment depends on a higher order moment. Thus, one needs infinite moments to retain all kinetic information. 301

multiple scale analysis This is a type of perturbation series that incorporates multiple scales by treating them as independent variables. By creating additional independent variables, and hence degrees of freedom, one can put constraints in that prevent "usual" perturbation series from being valid approximations. So-called secular terms, which grow in time unphysically, can be eliminated by the new constraints. So given an equation $\mathbf{F}(f(\mathbf{x},t)) = 0$ for some operator $\mathbf{F}$, we create multiple scales, with an ordering parameter ϵ and say $\tau_0 = t$ and $\tau_1 = \epsilon t$ and treat the τ_0 and τ_1 as independent variables. We try the following solution form

$$f(\mathbf{x},\tau_0,\tau_1) = \sum_j \epsilon^j f_j(\mathbf{x},\tau_0,\tau_1)$$
$$\frac{\mathrm{d}\tau_0}{\mathrm{d}t} = 1$$
$$\frac{\mathrm{d}\tau_1}{\mathrm{d}t} = \epsilon$$

and solve the resulting equations when put into $\mathbf{F}$ order by order in ϵ. 126

mush limit A way of smoothing discrete particles to continuum variables. It involves spatially spreading charge and mass while conserving the total charge and mass. 295

neutron emission A spontaneous nuclear radiation process where a nucleus emits a neutron or neutrons. 376

non-holonomic constraint Any constraints that are not holonomic. This means that the system is no longer integrable unless it is of a special form linearly depending on the velocities. So the given variables do not uniquely give the state of the system. A common example is rolling a ball on the plane. If you only track where the ball center is, then you can move the ball back to its original position (without slipping), but the ball will be rotated, which is not kept track of. If we put a constraint such that it does come back at its orignal angle, then it is a non-holonomic constraint. 144

nonlinear equation A nonlinear equation is an equation where an unknown quantity appears in at least one nonzero-term in the equation to a power not equal to one or zero. If x is an unknown then any equation with a term having x^n for $n \neq 1$ and $n \neq 0$ (after simplification) is a nonlinear equation for x. For differential equations $x(t)$, anytime you have x multiplying itself implicitly (such as through derivatives), you have a nonlinear equation. 121

normal curvature When using a Darboux frame, this refers to the curvature that must be there due to being constrained to a surface. This is sometimes called the intrinsic curvature because of this, and can be viewed as the natural curvature that one would expect given the constraints on the trajectory. One might remember that this is the natural curvature from being on a surface and so the normal curvature since both begin with an "n". 169

normal torsion This is the torsion in the Darboux frame when following magnetic field line trajectories. It differs from the mathematical definition by a minus sign. 172

nuclear fission The process of a nucleus splitting into two or more nuclei. 372

nuclear fusion The process of two nuclei combining to form a single nucleus. 372

nucleon The particles that make up nuclei. That is, protons and neutrons. 371, 372

nucleon mass number See mass number plural. 372

nucleus This is the center of the atom held together by the nuclear strong force. It is made up of protons and neutrons, and does not include the electrons. 371, 372

nuclide A nuclide refers to a nucleus with a specific number of protons and neutrons. When speaking of nuclides, there is no connotation that the nuclides share the same number of protons as the word isotope does plural. 373

order When referring to tensors, it is the number of basis vectors required for the tensor. So a scalar is 0th order, a vector 1st order, etc. Sometimes this is referred to as the rank of a tensor, but rank of a tensor can have other meanings so it is better to just use order. 15

orthogonal It means that for a given defined dot product between two vectors $\mathbf{V}$ and $\mathbf{W}$, that $\mathbf{V}\cdot\mathbf{W}=0$. It is synonymous with normal and perpendicular. 37

orthonormal When applied to a set of vectors, $\mathbf{V}^i$, it implies $\mathbf{V}^i\cdot\mathbf{V}^j=\delta_{ij}$. 34

parity inversion Often used with to explain a symmetry. Parity takes position $\mathbf{x}$ to time $-\mathbf{x}$. 75

perturbation series A perturbation series uses an ordering parameter ϵ and writes the solution to an equation $\mathsf{F}(f(\mathbf{x}))=0$ with operator F via using

$$f(\mathbf{x})=\sum_j \epsilon^j f_j(\mathbf{x})$$

and solving the resulting equations order by order in ϵ. plural. 126

physicists' Q This is gives the ratio of fusion heating power to total heating power into the plasma. That is the denominator includes contributions from external heating sources. Thus $Q=P_{\text{fus}}/(P_{\text{fus}}+P_{\text{ext}})$ for P_{fus} the fusion power and P_{ext} external power put into the plasm. It is a physicists' quantity because P_{ext} is only the power put into the plasma, and not the actual amount of power required to get the energy into the plasma. Engineering Q uses the actual power used rather than the power put into the plasma. 401

plasma coupling factor This is defined as the ratio of electrostatic energy to thermal energy given by $\Gamma=\frac{1}{4\pi n^{2/3}\lambda_D^2}$ and so is proportional to plasma parameter $\Lambda^{-2/3}$. 255

plasma dispersion function See Section **??** .. 213

plasma parameter There are two contradictory definitions, so either define it or avoid its use. Some call $\Lambda=4\pi n\lambda_D^3$ the plasma parameter. In this case, it fulfills the same function as a Debye sphere calculation. Alternatively it is defined as $g=1/(n\lambda_D^3)$ and has the opposite scaling. 254

Plemelj formula A consistent way of assigning values to poles on the real axis for integrals that are along the real axis. See Section 1.13 for details. It is given for a pole at $x=0$ as a

formula below

$$\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}x\ \frac{f(x)}{x\pm i\delta}=\mp i\pi f(0)+\fint_{-\infty}^{\infty}\mathrm{d}x\ \frac{f(x)}{x}$$

. 193, 195, 196, 215

polar scalar Sometimes called true scalar. See polar tensor. 75

polar tensor An nth order tensor $\mathbf{T}$ that under parity inversion satisfies the relation $\mathbf{T}(-\mathbf{x})=(-1)^{n+1}\mathbf{T}(x)$ is a polar tensor (including scalars and vectors). Sometimes they are called true tensors/vectors/scalars. 75

polar vector Sometimes called true vector. See polar tensor. 75

poloidal Refers to a direction. For an angle-like variable, it means it is related to going along a torus the short way. 79, 82

poloidal flux A flux of something (usually magnetic field) through a constant poloidal angle surface. There are two main types, a ribbon flux and disk flux (see Figures 2.2 and 2.3). 158, 259

polyadic A geometric object in Euclidean space built up as a sum of polyads (made up of vectors). It is essentially synonymous with an nth order tensor, as they represent the same geometric objects. 15, 35

proper length of a field line This is the assigned length of field lines on a flux surface. It is not always useful, see (2.2.133). It is only useful if field lines have similar lengths on flux surfaces. 281

proper time This is the time measured by an observer who uses a frame in which they are at rest. Note that the observer need not be an inertial frame to observe their own proper time. 179

pseudoscalar See pseudotensor. 75

pseudotensor An nth order tensor $\mathbf{T}$ that under parity inversion satisfies the relation $\mathbf{T}(-\mathbf{x})=(-1)^{n}\mathbf{T}(x)$ is a pseudotensor (including scalars and vectors). 75

pseudovector Sometimes called axial vector. See pseudotensor. 75

quasineutrality For plasmas, this means that $\boldsymbol{\nabla}\cdot\mathbf{E}\simeq 0$. This is not a statement that $\boldsymbol{\nabla}\cdot\mathbf{E}=0$, but a statement that electrostatic fields are not large contributions to particle dynamics. That is given a force density law, the contribution from $|\rho_q\mathbf{E}|$ (with ρ_q the charge density) is much smaller than other terms in the force density law. 252

radionuclide This is a nuclide that is not stable, and so undergoes radioactive decay. 410

reactivity This is a generalization of a cross section for processes that involve distributions of velocity. Thus for reaction rate per unit volume R_r for process r, we have for two species t and b that $R_r=n_tn_b\left\langle\sigma_r|v_b-v_t|\right\rangle$ for number densities n, velocities v, cross section σ_r. Here $\left\langle\sigma_r|v_b-v_t|\right\rangle$ is the reactivity. Note that if we have the same species $t=b$, then $2R_r=n_tn_t\left\langle\sigma v'\right\rangle$ with v' indicating the relative velocity in a collision. 381

reciprocal basis Given a vector basis $\mathbf{e}^i$, the reciprocal basis $\mathbf{e}_j$ always has $\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j$. 34

reduced de Broglie wavelength The reduced de Broglie wavelength is simply the de Broglie wavelength divided by 2π. So $\lambda_{dB} = \frac{\hbar}{mv}$ for reduced Planck constant $\hbar$, mass of particle m, and velocity of a particle v. 384, 407

relative torsion See geodesic torsion. 169

reserve When talking about naturally occurring materials, a reserve refers to the quantity that is exploitable with current technical and socioeconomic conditions. This just means it is worth it to exploit it. This follows the definition from Vikström[1]. 425

resource When talking about naturally occuring materilas, resources refers to a geologically assured quantity that is available for exploitation. Compare this to reserves, which are both available for exploitation and worth exploiting (economically). This follows the definition from Vikström[1]. 425

Reynolds transport theorem This gives the derivative of a volume integral. It is given by [see (2.9.58)]

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \mathrm{d}^3x\ \mathbf{F} = \iiint_{V(t)} \mathrm{d}^3x\ \left[\boldsymbol{\nabla}\cdot(\mathbf{VF}) + \frac{\partial \mathbf{F}}{\partial t}\right] = \iiint_{V(t)} \mathrm{d}^3x\ \frac{\partial \mathbf{F}}{\partial t} + \oiint_{\partial V(t)} \mathrm{d}S\ \hat{\mathbf{n}}\cdot\mathbf{VF}$$

. 337, 338, 366, 473

rotational transform This refers to two concepts denoted by ι and $\text{ɬ} = \iota/(2\pi)$. Both give the change in the poloidal flux due to a change in toroidal flux. See and . Iota gives the change in θ due to a change of ζ by 2π. Unfortunately both ι and ɬ are called rotational transform. 269, 288

saddle point A saddle point is a critical point that is not an local or global extrema (i.e., it is not a local or global maximum or minimum). One test for a saddle point is to take the Hessian of your function and find the eigenvalues of the Hessian. If some are positive and some are negative you have a saddle point. If the determinant of the Hessian is zero, it is called a degenerate point. If you can determine it is a saddle point, then it is a degenerate saddle point. 99

safety factor This refers to the average change in toroidal flux given a change in poloidal flux on a flux surface. See (2.2.52). Large values are associated with less instability, though flux surfaces with a rational value are associated with instability, as well. 268, 270

Schrödinger equation There are two variants, the time dependent one

$$i\hbar\frac{\partial |\psi\rangle}{\partial t} = H|\psi\rangle$$
$$i\hbar\frac{\partial \psi}{\partial t} = \left[\frac{-\hbar^2}{2m}\nabla^2 + V\right]\psi$$

where the second is the nonrelativistic version for a generic potential V in the position basis. Here H is the Hamiltonian operator and $|\psi\rangle$ is a ket with ψ the wavefunction. The other is

the time independent form

$$E\left|\psi\right\rangle = H\left|\psi\right\rangle$$
$$E\psi = \left[\frac{-\hbar^2}{2m}\nabla^2 + V\right]\psi$$

where the second is the nonrelativistic version for a generic potential V in the position basis. Here E is the energy eigenvalue. 109, 130, 188

sound speed This is the speed of acoustic/sound waves. It is conventionally given by $v_S^2 = \gamma k_B T/m_i$ in MHD, though some include separate contributions from electron and ion temperatures so that v_S is the ion acoustic speed. 336

stationary point The subset of critical points that have their derivatves be zero (that is, they are defined). 98

stellarator See Section 2.4.2. 288

Stirling's approximation This is an approximation for $\Gamma(N+1) = N!$ with Γ the complete gamma function when $N \to \infty$. It is given as an asymptotic expansion via

$$N! \overset{N\to\infty}{\longrightarrow} \sqrt{2\pi N}\frac{N^N}{\exp(N)}\left(1 + \frac{1}{12N}\right) + \mathcal{O}(N^{-2})$$
$$\ln N! \overset{N\to\infty}{\longrightarrow} N\ln N - N + \mathcal{O}(\ln N)$$

where the second approximation comes from ignoring the prefactor $\sqrt{2\pi N}$ as insignificant compared to the exponential terms. 100, 102

straight field line coordinates See flux coordinates. 251

symmetery This says that given a function of a variable $f(x)$, there is some transformation of x g such that $f(g(x)) = f(x)$. For example, one could have $f(-x) = f(x)$, or $f(x+a) = f(x)$. 41

tangent basis Given coordinates ξ^i and position vector $\mathbf{x}$, the tangent basis set is $\mathbf{e}^i = \frac{\partial \xi^i}{\partial \mathbf{x}} = \nabla\xi^i$. Sometimes they are referred to as the covariant basis, though this should be avoided. 34, 38, 49

tangent-reciprocal basis Given the tangent basis set for coordinates ξ^i with position vector $\mathbf{x}$ given by $\mathbf{e}^i = \frac{\partial \xi^i}{\partial \mathbf{x}}$, the tangent-reciprocal basis is given by $\mathbf{e}_j = \frac{\partial \mathbf{x}}{\partial \xi^j}$. Sometimes they are referred to as the contravariant basis, though this should be avoided. 34, 40, 49

Taylor series A series approximation of a function that uses a power series and the function and its derivatives at a single value. Quite powerful when near the approximating value, but often difficult to approximate well over an entire domain. 13, 14, 79, 101, 104, 111, 113, 114, 140, 253, 255, 267

tensor A geometric object in Euclidean space. It can be built up from vectors, and has transformation properties similar to vectors. An nth order tensor is built from n vectors. 13, 15

time reversal Often used with to explain a symmetry that is equivalent to parity inversion for time. Time reversal takes time t to time $-t$. 77

tokamak See Section 2.4.1. 286

toroidal Refers to a direction. For an angle-like variable, it means it is related to going along a torus the long way. 79, 82

toroidal flux A flux of something (usually magnetic field) through a constant toroidal angle surface. 158, 259

torsion In the Frenet-Serret formulas, it refers to how the curve is departing from a planar curve. 164

total cross section Sometimes refers to nuclear cross section or the integral cross section. More often implies the sum of nuclear cross sections of all possible reactions/processes. 379

tritium This refers to hydrogen atoms which contain nuclei with one proton and two neutrons. Usually denoted T or t rather than ^{3}H. See also triton. 375

tritium breeding ratio The tritium breeding ratio, usually shortened to TBR, is the the number tritium atoms bred per tritium atom burned in a fusion reaction. Generically, people care about the net TBR, which may just be called the TBR, which is the tritium atoms recovered per tritium atom burned in a fusion reaction. 428

triton This refers to a hydrogen nucleus with one proton and two neutrons. Usually denoted T or t rather than ^{3}H. Technically, different from tritium which refers to the whole atom, but in practice it is usually used interchangeably with tritium. 374

variational calculus See the calculus of variations. 135, 151

vector A geometric object in Euclidean space. It has a magnitude and direction and can be represented by set of scalars and basis vectors. It is not an array or column or row of numbers. It must rotate and translate as a geometric object so that it is coordinate independent. 13, 15, 34

vector basis A set of vectors that span the space we are interested in. 34

W7X W7X stands for Wendelstein-7X. This is a stellarator with modular coils (so the coils can be switched out and are put together in modules). It is a German-built stellarator. Wendelstein would literally mean coil stone or coil rock, but the name comes from a mountain in Bavaria. This is said to be inspired from Project Matterhorn (a fusion program in the US) being named after a mountain in the Alps. 398

Watson's lemma This lemma generates asymptotic expansions for $\lambda > -1$ and any $0 < T \leq \infty$ with appropriately nice (infinitely differentiable at $t = 0$) functions $t^\lambda g(t)$. It is given by

$$\int_0^T \mathrm{d}t\ t^\lambda g(t) \exp(-xt) \overset{N\to\infty}{\longrightarrow} \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\Gamma(\lambda+n+1)}{n! x^{\lambda+n+1}}$$

. 102

winding number The winding number is informally at a point in the plane a, the number of times you need to turn your head comletely around (full revolutions) when staring directly at a closed curve and following it along one circuit. You consider turning to the left positive and turning to the right negative when counting revolutions. See Section 1.13 for a more mathematical definition. 191

Wirtinger derivatives Derivatives of complex variables that simplify checking the Cauchy-Riemann equations for multiple complex variables. See (1.5.9) and (1.5.10). 115

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