Notes on Mathematical Programming by Steven Vajda

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Contents

Chapter 1

Background

A short chapter with a history up to the 1960's. There was a surprising amount done in the 1940's and 1950's, with "linear programming" being quite important for a variety of military, industrial, and governmental decisions (at least, for optimal use of resources).

Chapter 2

The Algebra of Linear Equalities

This goes over the basics of understanding how to solve a system of equalities and inequalities subject to constraints.

2.1 Definitions

The first equation given seems to lack an i index so that the system should read

$$\begin{aligned} \sum_{i=1}^{n} a_{ij} x_i &\geq b_j \\ x_i &\geq 0 \end{aligned} \tag{2.1.1}$$

for $i = 1, \ldots, n$ for the x_i and $j = 1, \ldots, m$ in the b_j.

We then begin with the fact that we can go from a system of inequalities to a system of equalities and vice versa in equivalent forms. It is pointed out that we can take a system of m inequalities with n variables to a form m equations and with $N = m + n$ total variables. The "extra " variables are called slack or additional variables. The reverse is also clearly possible (take equalities when we have extra variables, and go back to inequalities) by basically doing the steps in reverse. We are generally interested in problems where we require the variables to be non-negative (non-positive is also easily possible by introducing a replacement variable with a negative sign). A solution for the system satisfies the system but is not necessarily satisfying the constraints, while a feasible solution satisfies the non-negative constraints, as well.

Note that basic solutions are solutions where the number of non-zero values in the solution vector[1] is less than or equal to the rank of the matrix of equations (in addition, we require a basic solution to be unique). A basic feasible solution is then a feasible solution that is basic.

[1] I will be referring to "vector arrays" as vectors. Do not confuse these vectors with geometric or Euclidean vectors. These are simply arrays of numbers.

The example given is

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 2 & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{2.1.2}$$

Then if we exclude x_3, x_4, x_5, we have

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{2.1.3}$$

has the solution $x_1 = 2 - 2x_2$ for $x_2 = 1/2$ and so $x_1 = 2 - 2(1/2) = 1$. But we could choose anything for x_2 and find a solution for x_1 so that there is no unique solution and so this is not a basic solution.

The other example is

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.1.4}$$

It's fairly clear that this is only possible if one of the $x_i = 0$ by simple linear algebra rules. If we exclude x_3 we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.1.5}$$

but we see that the coefficient matrix is now singular, and so there is no unique solution again, and so no basic solution exists with x_3 excluded. If we excluded x_2 we find

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.1.6}$$

and the coefficient matrix is now nonsingular and a solution is given by $x_2 = 0$, $x_3 = 0$ and $x_1 = 1$. This is a basic solution and x_1 and x_3 are considered basic (even though $x_3 = 0$). Our basic solution expressed as a vector is $\mathbf{x}_{13} = (1, 0, 0)$. Clearly another basic solution would be $\mathbf{x}_{12} = (0, 1, 0)$. Here I have put the subscript indices for the basic variables on the basic solutions.

We call the variables that are not excluded the basic variables and the excluded variables the nonbasic ones. The columns formed by the basic variable columns are called a basis.

The lemma is confusingly worded. It is simpler to say:

Lemma 2.1.1 *If we can solve $r \leq m$ of m consistent and independent constraints, written as equations, for r variables, then these values form part of a basic solution, i.e. we can find $m - r$ other variables such that the constraints can be solved for all m of them after excluding (setting equal to zero) the $m - r$ other variables.*

This simply says that if we have our matrix of equations of rank m, then if we can solve a subset of rank $r < m$, then we can obtain a part of a basic solution from the smaller rank r submatrix. The book then defines degenerate basic solutions as basic solutions that have fewer non-zero entries in the solution vector than the rank of the full matrix.

2.2 Basic Feasible Solutions

Next, it is shown that if a feasible solution exists, then a basic feasible solution also exists. It also gives some means to get the answer. We can exclude variables by choosing a submatrix that is non-singular and uniquely solvable. Then we can exclude the other variables (holding their values constant except for one) and adjust the value of the one excluded variable to get more zeros until we get a basic solution.

Let's go through the example.

We have

$$\begin{bmatrix} 3 & 4 & 1 & 1 & 0 \\ 1 & 3 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{2.2.1}$$

Suppose we found a solution by a good guess. The book gives us $\mathbf{x}_0 = (1/6, 1/6, 1/6, 2/3, 0)$.

Let's exclude $i = 3, 4, 5$ for the x_i. Then we have

$$\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 - x_3 - x_4 \\ 1 - 2x_3 - x_5 \end{bmatrix} \tag{2.2.2}$$

We note the coefficient determinant is $9 - 4 = 5$ so it is nonsingular. Suppose we choose $x_3 = 0$. Then the above says

$$\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 - 0 - 2/3 \\ 1 - 2(0) - 0 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} \tag{2.2.3}$$

We rewrite the above as

$$\left[\begin{array}{cc|c} 3 & 4 & 4/3 \\ 1 & 3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 4 & 4/3 \\ 3 & 9 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 4 & 4/3 \\ 0 & 5 & 5/3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 4 & 4/3 \\ 0 & 1 & 1/3 \end{array}\right] \tag{2.2.4}$$

$$\rightarrow \left[\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 1 & 1/3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1/3 \end{array}\right] \tag{2.2.5}$$

and so $x_1 = 0$ and $x_2 = 1/3$. Thus our new solution is $\mathbf{x} = (0, 1/3, 0, 2/3, 0)$ which is a basic solution since there are less than 3 nonzero entries in the solution vector, so $\mathbf{x}_{24} = (0, 1/3, 0, 2/3, 0)$ is a basic solution.

Now we could have chosen x_1 and x_3 as the non-excluded and have

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 4x_2 - x_4 \\ 1 - 3x_2 - x_5 \end{bmatrix} \tag{2.2.6}$$

Let's try setting $x_2 = 0$ and then using the original solution we have

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 4(0) - 2/3 \\ 1 - 3(0) - 0 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} \tag{2.2.7}$$

which we transform into

$$\left[\begin{array}{cc|c} 3 & 1 & 4/3 \\ 1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 4/3 \\ 3 & 6 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 4/3 \\ 0 & 5 & 5/3 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 4/3 \\ 0 & 1 & 1/3 \end{array}\right] \tag{2.2.8}$$

$$\to \left[\begin{array}{cc|c} 3 & 0 & 1 \\ 0 & 1 & 1/3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \end{array}\right] \tag{2.2.9}$$

and so the solution is $(1/3, 0, 1/3, 2/3, 0)$ which is not basic since there are 3 nonzero entries. We can simply repeat the analysis with this possible solution. We can make $x_4 = 0$ with this one (keeping x_1 and x_3 so that we have little to redo). Then

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 4x_2 - x_4 \\ 1 - 3x_2 - x_5 \end{bmatrix} = \begin{bmatrix} 2 - 4(0) - 0 \\ 1 - 3(0) - 0 \end{bmatrix} = \begin{bmatrix} 21 \end{bmatrix} \tag{2.2.10}$$

leading to

$$\left[\begin{array}{cc|c} 3 & 1 & 2 \\ 1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 3 & 6 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 0 & 5 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 0 & 1 & 1/5 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 0 & 9/5 \\ 0 & 1 & 1/5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3/5 \\ 0 & 1 & 1/5 \end{array}\right] \tag{2.2.11}$$

and so our basic solution will be $\mathbf{x}_{13} = (3/5, 0, 1/5, 0, 0)$.

Finally, we could start from our original solution and take x_3 and x_4 as the ones to (most likely) keep basic.

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - x_1 - 3x_2 - x_5 \\ 2 - 3x_1 - 4x_2 \end{bmatrix} \tag{2.2.12}$$

Let's adjust with $x_1 = 0$. Then we get

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - (0) - 3(1/6) - (0) \\ 2 - 3(0) - 4(1/6) \end{bmatrix} = \begin{bmatrix} 1/2 \\ 4/3 \end{bmatrix} \tag{2.2.13}$$

which we can see says $x_3 = 1/4$ and so $x_4 = 4/3 - 1/4 = 13/12$. Thus the new solution is $\mathbf{x} = (0, 1/6, 1/4, 13/12, 0)$ which we can now plug in with $x_2 = 0$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - (0) - 3(0) - (0) \\ 2 - 3(0) - 4(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{2.2.14}$$

and so we find $x_3 = 1/2$ and so $x_4 = 2 - 1/2 = 3/2$ and so our solution is then $\mathbf{x}_{34} = (0, 0, 1/2, 3/2, 0)$, which is basic. We then note that we must have

$$\mathbf{x}_0 = a_1\mathbf{x}_{24} + a_2\mathbf{x}_{13} + a_3\mathbf{x}_{34} \tag{2.2.15}$$

for some coefficients a_i which can be solved via another matrix system. As the book says, we find $a_1 = 1/2, a_2 = 5/18, a_3 = 2/9$.

The same ideas apply when we have optimal solutions. We can have a generic one and form a basic optimal solution via the same methods as above.

The book uses maximize

$$B = 3x_1 + 4x_2 + x_3 \tag{2.2.16}$$

subject to the constraints

$$3x_1 + 4x_2 + x_3 + x_4 = 2 \tag{2.2.17}$$
$$x_1 + 3x_2 + 2x_3 + x_5 = 1 \tag{2.2.18}$$

It is pointed out that the maximum of B is 2 via the first constraint and an optimal solution of the form $\mathbf{x}_0 = (1/2, 1/10, 1/10, 0, 0)$. We can then perform the same machinations. Let's adjust x_3 to zero, for example

$$\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{2.2.19}$$

and so

$$\left[\begin{array}{cc|c} 3 & 4 & 2 \\ 1 & 3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 4 & 2 \\ 3 & 9 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 4 & 2 \\ 0 & 5 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 4 & 2 \\ 0 & 1 & 1/5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 0 & 6/5 \\ 0 & 1 & 1/5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2/5 \\ 0 & 1 & 1/5 \end{array}\right] \tag{2.2.20}$$

as one basic solution $\mathbf{x}_{12} = (2/5, 1/5, 0, 0, 0)$ and we could instead have eliminated x_2 so that

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{2.2.21}$$

and so

$$\left[\begin{array}{cc|c} 3 & 1 & 2 \\ 1 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 3 & 6 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 0 & 5 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 0 & 1 & 1/5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 0 & 9/5 \\ 0 & 1 & 1/5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3/5 \\ 0 & 1 & 1/5 \end{array}\right] \tag{2.2.22}$$

which gives a basic solution of $\mathbf{x}_2 = (3/5, 0, 1/5, 0, 0)$

Note that as the book emphasizes, this does not exhaust the possible basic feasible solutions, but simply that we can decompose any feasible solution into basic feasible solutions.

This is really saying in linear algebra terms that the over the space of optimal solutions, the basic feasible solutions form a basis.

2.3 Geometric Transformations

Now we consider the systems in matrix form. The book writes every detail down, even though some of the ideas are more easily expressed abstractly. For example, we could write our matrices M_{u_i} and D_{u_i} more abstractly using vectors. We could then write that column vector is $\mathbf{a}_{u_ij}$ and row vector is $\mathbf{a}^\intercal_{u_ij}$ for short. So

$$M = \begin{bmatrix} \mathbf{b} & \mathbf{a}_1 & \cdots & \mathbf{a}_N & 0 \\ 0 & -\mathbf{c}^\intercal & \cdots & \cdots & 1 \end{bmatrix} \tag{2.3.1}$$
$$M_{u_i} = \begin{bmatrix} \mathbf{a}_{u_i} & \cdots & \mathbf{a}_{u_m} & 0 \\ -\mathbf{c}^\intercal & \cdots & \cdots & 1 \end{bmatrix} \tag{2.3.2}$$
$$D_{u_i} = \begin{bmatrix} \mathbf{d}_{u_i} & \cdots & \mathbf{d}_{u_m} & 0 \\ \mathbf{d}^\intercal & \cdots & \cdots & 1 \end{bmatrix} \tag{2.3.3}$$

where $\mathbf{a}_i$ is the coefficients on the x_1 for each equation (in a set order). Then since by construction $D_{u_i} = M_{u_i}^{-1}$, we must have the relations stated in (V-2-1) through (V-2-3). These can then more easily be expressed as

$$\mathbf{a}_s \cdot \mathbf{d}_t = \delta_{st} \tag{2.3.4}$$
$$(\mathbf{d})_s = \mathbf{c} \cdot \mathbf{d}_s \tag{2.3.5}$$

where $(\mathbf{d})_s$ is the sth component of $\mathbf{d}$. The (V-2-4) and (V-2-5) then come from an application of the inverse properties given above written component by component.

The rest of the chapter is a fairly straightforward use of the results with definitions for better notation.

2.4 Geometric Representation

The proof that basic feasible solutions are vertices is essentially one by contradiction. Assume that a basic feasible solution is a vertex and that it is halfway between two (distinct) feasible solutions and show that those feasible solutions are actually not possible.

It is then pointed out in the example that one can determine the number of vertices/basic solutions via combinatorics. Given the number of variables n and number of constraints m the number of basic solutions must be $\binom{m}{n}$ or m choose n. However, there is no guarantee that all of these basic solutions will actually be feasible. In fact, usually some of them will not be feasible.

We can consider the transportation problem. Remember that we defined the number of ships to sail from P_i to Q_j as x_{ij} with a_i ships available at port P_i and b_j ships required at Q_j then the equations are

$$\sum_i x_{ij} = b_j \tag{2.4.1}$$
$$\sum_j x_{ij} = a_i \tag{2.4.2}$$
$$\sum_i a_i = \sum_j b_j \tag{2.4.3}$$

when i ranges from 1 to 2 and j from 1 to 3 the equations are then

$$x_{11} + x_{12} + x_{13} = a_1 \tag{2.4.4}$$
$$x_{21} + x_{22} + x_{23} = a_2 \tag{2.4.5}$$
$$x_{11} + x_{21} = b_1 \tag{2.4.6}$$
$$x_{12} + x_{22} = b_2 \tag{2.4.7}$$
$$x_{13} + x_{23} = b_3 \tag{2.4.8}$$
$$a_1 + a_2 = b_1 + b_2 + b_3 \tag{2.4.9}$$

Let's just assume that we give a_i and b_i so that the last equation is satisfied. Then the previous equations can be written as

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 1 & 1 & 1 & a_2 \\ 1 & 0 & 0 & 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & 0 & 0 & 1 & b_3 \end{array}\right] \tag{2.4.10}$$

Since there are five rows and six columns, we know that these cannot all be linearly independent. The book instead uses clever reasoning to reduce the size of the problem. It uses that $x_{21} \leq b_1$ and $x_{13} \leq b_3$. Then with

$$x_{11} + x_{21} = b_1 \tag{2.4.11}$$
$$x_{11} + x_{12} + x_{13} = a_1 \tag{2.4.12}$$
$$b_1 - a_1 = \cancel{x_{11}} + x_{21} - \cancel{x_{11}} - x_{12} - x_{13} \tag{2.4.13}$$
$$b_1 - a_1 = x_{21} - x_{12} - x_{13} \tag{2.4.14}$$
$$b_1 - a_1 + x_{12} = x_{21} - x_{13} \tag{2.4.15}$$

which says that $x_{21} - x_{12}$ must be greater than or equal to $b_1 - a_1$. Then

$$x_{21} + x_{22} + x_{23} = a_2 \tag{2.4.16}$$
$$x_{13} + x_{23} = b_3 \tag{2.4.17}$$
$$a_2 - b_3 = x_{21} + x_{22} + \cancel{x_{23}} - x_{13} - \cancel{x_{23}} \tag{2.4.18}$$
$$a_2 - b_3 = x_{21} + x_{22} - x_{13} \tag{2.4.19}$$
$$a_2 - b_3 = x_{21} - x_{13} + b_2 - x_{12} \tag{2.4.20}$$
$$a_2 - b_3 - b_2 + x_{12} = x_{21} - x_{13} \tag{2.4.21}$$

which says that the most $x_{21} - x_{12}$ could be is when x_{12} is at its largest possible value of b_2 meaning $x_{21} - x_{13} \leq a_2 - b_3$ as stated in the book.

Using our previous rule with $m + n - 1$ variables and mn equations we'd expect $\binom{mn}{m+n-1} = \frac{(mn)!}{(m+n-1)!(mn-m-n+1)!} = \frac{(mn)!}{(m+n-1)!([m-1][n-1])!}$ basic solutions. For our case $\binom{2(3)}{2+3-1} = \binom{6}{4} = \frac{6!}{4!2!} = \frac{6(5)}{2} = 15$ possible solutions. But in fact the general rule is $m^{n-1}n^{m-1}$ which in our case yields $2^{3-1}3^{2-1} = 2^2 3 = 12$ basic solutions. This means that our combinatorial rule is not always accurate. In any case, I cannot find a proof that it is $m^{n-1}n^{m-1}$ that is not behind a paywall.

The rest of the geometric interpretations are somewhat straightforward. The Example (V-2-12) is using that we can form a polyhedronal cone and then only consider the values that satisfy our constraints. From these we choose the value that maximizes or minimizes our constraint. This method only gives us the maximal value, and not the x_i that give us such a value, which can be more complicated to find.

In any case the idea is that if you have the vector of points

$$P_1 = (a_{11}, a_{12}, c_1) \tag{2.4.22}$$
$$P_2 = (a_{21}, a_{22}, c_2) \tag{2.4.23}$$
$$P_3 = (a_{31}, a_{32}, c_3) \tag{2.4.24}$$

then we find the values

$$t_1a_{11} + t_2a_{21} + t_3a_{31} = b_1 \tag{2.4.25}$$
$$t_1a_{12} + t_2a_{22} + t_3a_{32} = b_2 \tag{2.4.26}$$

and use that to find the maximal solution for

$$t_1c_1 + t_2c_2 + t_3c_3 = B \tag{2.4.27}$$

In fact, this is simply what we have done before. Solve the constraints and put them into our objective function. The difference is in the geometrical construction idea.

In the example, we have

$$P_1 = (3, 1, 3) \tag{2.4.28}$$
$$P_2 = (4, 3, 6) \tag{2.4.29}$$
$$P_3 = (1, 2, 2) \tag{2.4.30}$$
$$\left[\begin{array}{ccc|c} 3 & 4 & 1 & 2 \\ 1 & 3 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 4 & 1 & 2 \\ 3 & 9 & 6 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 4 & 1 & 2 \\ 0 & 5 & 5 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 4 & 1 & 2 \\ 0 & 1 & 1 & 1/5 \end{array}\right] \tag{2.4.31}$$
$$\to \left[\begin{array}{ccc|c} 3 & 0 & -3 & 6/5 \\ 0 & 1 & 1 & 1/5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2/5 \\ 0 & 1 & 1 & 1/5 \end{array}\right] \tag{2.4.32}$$

which means our constraint becomes

$$t_1 = 2/5 + t_3 \tag{2.4.33}$$
$$t_2 = 1/5 - t_3 \tag{2.4.34}$$

and so enforcing our constraint we require

$$B = 3t_1 + 6t_2 + 2t_3 = 3\left(\frac{2}{5} + t_3\right) + 6\,(1/5 - t_3) + 2t_3 = \frac{12}{5} - 3t_3 \tag{2.4.35}$$

so $t_3 = 0$ is the optimal point yielding $B = 12/5$.

2.5 Exercises

2.5.1 Problem 1

Find the basic solutions of the following problem:

$$x_1 + x_2 + x_3 = 4 \tag{2.5.1}$$
$$x_4 + x_5 + x_6 = 5 \tag{2.5.2}$$
$$x_1 + x_4 = 3 \tag{2.5.3}$$
$$x_2 + x_5 = 3 \tag{2.5.4}$$

Indicate which solutions are feasible.

Solution:

We immediately see 6 variables and 4 equations. Naively we would then expect there to be 15 possible basic solutions. (It turns out this is a transport problem so that there are fewer basic solutions, $3(2^2) = 12$ so 3 of the combinations must be impossible.)

We then write out the matrix to see what possible combinations could form basic solutions.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.5.5}$$

We then have the unenviable position of having to evaluate all the possibilities. The options are

$$\begin{pmatrix} 1234 & 1246 & 1356 & 2356 \\ 1235 & 1256 & 1456 & 2456 \\ 1236 & 1345 & 2345 & 3456 \\ 1245 & 1346 & 2346 & \end{pmatrix} \tag{2.5.6}$$

where 1234 means take x_1, x_2, x_3, x_4. The determinants for all these possibilities are given by

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & \end{pmatrix} \tag{2.5.7}$$

which means that three are missing (singular zero determinants, as expected, with 1245, 1346 and 2356 being the singular ones). We then just can find the solutions by substitution. We find

$$\mathbf{x}_{1234} = \begin{pmatrix} -2 & 3 & 3 & 5 & 0 & 0 \end{pmatrix} \tag{2.5.8}$$
$$\mathbf{x}_{1235} = \begin{pmatrix} 3 & -2 & 3 & 0 & 5 & 0 \end{pmatrix} \tag{2.5.9}$$
$$\mathbf{x}_{1236} = \begin{pmatrix} 3 & 3 & -2 & 0 & 0 & 5 \end{pmatrix} \tag{2.5.10}$$
$$\mathbf{x}_{1246} = \begin{pmatrix} 1 & 3 & 0 & 2 & 0 & 3 \end{pmatrix} \tag{2.5.11}$$
$$\mathbf{x}_{1256} = \begin{pmatrix} 3 & 1 & 0 & 0 & 2 & 3 \end{pmatrix} \tag{2.5.12}$$
$$\mathbf{x}_{1345} = \begin{pmatrix} 1 & 0 & 3 & 2 & 3 & 0 \end{pmatrix} \tag{2.5.13}$$
$$\mathbf{x}_{1356} = \begin{pmatrix} 3 & 0 & 1 & 0 & 3 & 2 \end{pmatrix} \tag{2.5.14}$$
$$\mathbf{x}_{1456} = \begin{pmatrix} 4 & 0 & 0 & -1 & 3 & 3 \end{pmatrix} \tag{2.5.15}$$
$$\mathbf{x}_{2345} = \begin{pmatrix} 0 & 1 & 3 & 3 & 2 & 0 \end{pmatrix} \tag{2.5.16}$$
$$\mathbf{x}_{2346} = \begin{pmatrix} 0 & 3 & 1 & 3 & 0 & 2 \end{pmatrix} \tag{2.5.17}$$
$$\mathbf{x}_{2456} = \begin{pmatrix} 0 & 4 & 0 & 3 & -1 & 3 \end{pmatrix} \tag{2.5.18}$$
$$\mathbf{x}_{3456} = \begin{pmatrix} 0 & 0 & 4 & 3 & 3 & -1 \end{pmatrix} \tag{2.5.19}$$

Clearly the only basic feasible solutions are those without negative values.

It is also clear here how there will be quite a few of the solutions are singular every time we have a transportation problem.

2.5.2 Problem 2

Find the basic solutions of

$$x_1 + x_2 + x_3 = 1 \tag{2.5.20}$$
$$3x_1 + 2x_2 - x_4 = 6 \tag{2.5.21}$$

Solution:

We have 4 variables and 2 equations leading to $C_2^4 \equiv \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$ possible basic solutions (with no singularities).

In matrix form we have

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 2 & 0 & -1 \end{bmatrix} \tag{2.5.22}$$

Now we have 12, 13, 14, 23, 24, 34 as the possibilities with determinants (respectively) of -1, -3, -1, -2, -1, -1. Thus there are no singularities.

$$\mathbf{x}_{12} = \begin{pmatrix} 4 & -3 & 0 & 0 \end{pmatrix} \tag{2.5.23}$$
$$\mathbf{x}_{13} = \begin{pmatrix} 2 & 0 & -1 & 0 \end{pmatrix} \tag{2.5.24}$$
$$\mathbf{x}_{14} = \begin{pmatrix} 1 & 0 & 0 & -3 \end{pmatrix} \tag{2.5.25}$$
$$\mathbf{x}_{23} = \begin{pmatrix} 0 & 3 & -2 & 0 \end{pmatrix} \tag{2.5.26}$$
$$\mathbf{x}_{24} = \begin{pmatrix} 0 & 1 & 0 & -4 \end{pmatrix} \tag{2.5.27}$$
$$\mathbf{x}_{34} = \begin{pmatrix} 0 & 0 & 1 & -6 \end{pmatrix} \tag{2.5.28}$$

This means there are no feasible solutions. This could also be seen visually, as our problem is $x_1 + x_2 \leq 1$ and $3x_1 + 2x_2 \geq 6$. If we plot the regions we find that it is the region below the line $x_2 = 1 - x_1$ in the (x_1, x_2) plane, and the region above $x_2 = 3 - \frac{3}{2}x_1$. I have filled in the regions where the inequalities are actually satisfied in Figure 2.1. These regions do not overlap when x_1 and x_2 are greater than zero.

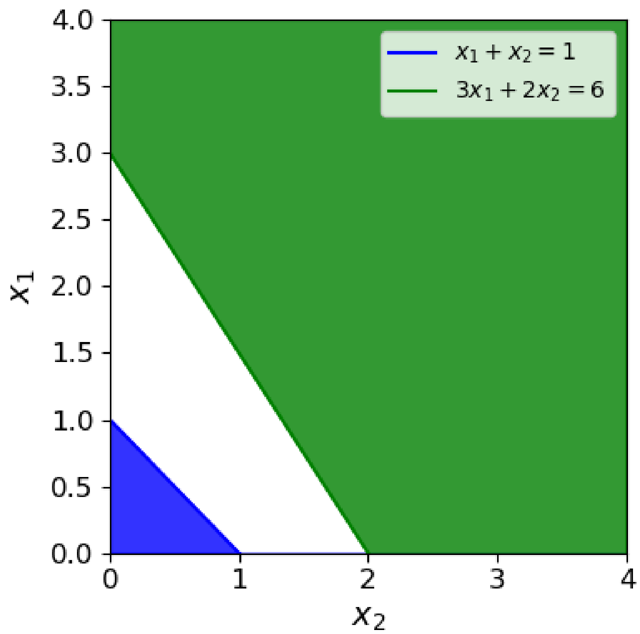

Figure 2.1: This shows the graphical solution method from the two inequalities. The regions where the inequalities is shaded, and we see there is no overlap, and hence no feasible solutions.

2.5.3 Problem 3

Now change the second equation from the previous problem. Find the basic solutions of

$$x_1 + x_2 + x_3 = 1 \tag{2.5.29}$$
$$3x_1 + 2x_2 - x_4 + x_5 = 6 \tag{2.5.30}$$

Solution:

Now we have 5 variables and 2 equations yielding $C_2^5 = 10$ possible bases. In matrix form we have

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & -4 & 1 \end{bmatrix} \tag{2.5.31}$$

We have 12, 13, 14, 15, 23, 24, 25, 34, 35, 45 as our possibilities with (respectively) determinants -1, -3, -4, 1, -2, -4, 1, -4, 1, 0. This means the 45 cannot be a basis, but all others are possibilities. We then find

$$\mathbf{x}_{12} = \begin{pmatrix} 4 & -3 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.32}$$
$$\mathbf{x}_{13} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \end{pmatrix} \tag{2.5.33}$$
$$\mathbf{x}_{14} = \begin{pmatrix} 1 & 0 & 0 & -3 & 0 \end{pmatrix} \tag{2.5.34}$$
$$\mathbf{x}_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \end{pmatrix} \tag{2.5.35}$$
$$\mathbf{x}_{23} = \begin{pmatrix} 0 & 3 & -2 & 0 & 0 \end{pmatrix} \tag{2.5.36}$$
$$\mathbf{x}_{24} = \begin{pmatrix} 0 & 1 & 0 & -4 & 0 \end{pmatrix} \tag{2.5.37}$$
$$\mathbf{x}_{25} = \begin{pmatrix} 0 & 1 & 0 & 0 & 4 \end{pmatrix} \tag{2.5.38}$$
$$\mathbf{x}_{34} = \begin{pmatrix} 0 & 0 & 1 & -6 & 0 \end{pmatrix} \tag{2.5.39}$$
$$\mathbf{x}_{35} = \begin{pmatrix} 0 & 0 & 1 & 0 & 6 \end{pmatrix} \tag{2.5.40}$$

Here we can see that $\mathbf{x}_{15}$, $\mathbf{x}_{25}$ and $\mathbf{x}_{35}$ are now basic feasible solutions. This makes sense since swapping x_4 and x_5 simply changes the sign.

Essentially what has happened is that we have allowed $3x_1 + 2x_2 \leq 6$ with the x_5 variable, and so the shaded area is the entire plane. This leaves us with overlap as seen in Figure 2.2.

2.5.4 Problem 4

Find the basic solutions of a.

$$2x_1 + x_2 = 6 \tag{2.5.41}$$
$$4x_1 + 2x_2 = 12 \tag{2.5.42}$$

and

$$2x_1 + x_2 + z_1 = 6 \tag{2.5.43}$$
$$4x_1 + 2x_2 + z_2 = 12 \tag{2.5.44}$$

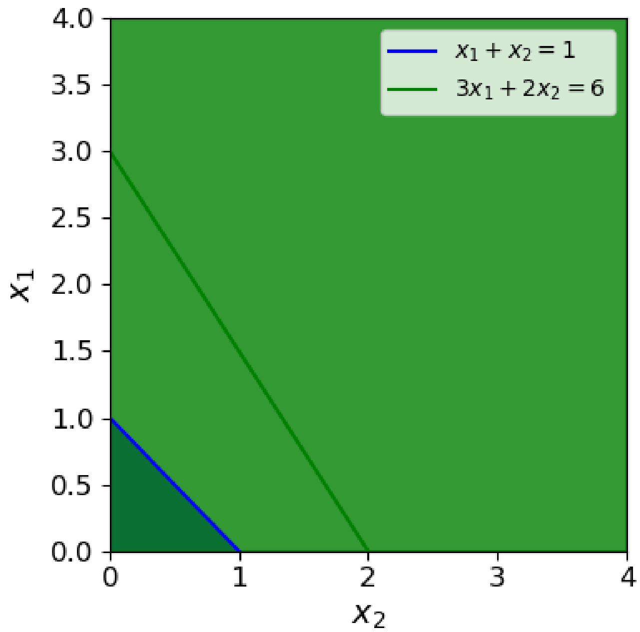

Figure 2.2: This shows the graphical solution method from the two inequalities. The regions where the inequalities are satisfied are shaded, and we see there is overlap in the lower left corner, and hence feasible solutions.

Solution:

For a. we note that the second equation is simply two times the first, thus this is a degenerate problem with an infinite number of solutions. Thus we find $x_1 = 3$ and $x_2 = 0$ is a "basic" solution and $x_1 = 0$ and $x_2 = 6$ is the other "basic" solution. I think this is a really strange result since it isn't actually solving the problem, and taking one variable at a time seems like it should not really be called a solution, but it is consistent with the definition we used for basic solution in the book. We still clearly have that any numbers whatsoever for x_1 and x_2 that satisfy one equation will satisfy the other. Another way of saying this is that the matrix formed from the coefficients has a determinant equal to zero.

For b. we have a real problem. In this case the matrix is given by

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{bmatrix} \tag{2.5.45}$$

I will rename $z_1 = x_3$ and $z_2 = x_4$ for convenience. We have $C^4_2 = 6$ possible bases (though we know one is impossible from part a.). Then the possible bases are 12, 13, 14, 23, 24, 34 with determinants 0, -4, 2, -2, 1, 1. Thus all but the 12 basis are fine.

$$\mathbf{x}_{13} = \begin{pmatrix} 3 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.46}$$
$$\mathbf{x}_{14} = \begin{pmatrix} 3 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.47}$$
$$\mathbf{x}_{23} = \begin{pmatrix} 0 & 6 & 0 & 0 \end{pmatrix} \tag{2.5.48}$$
$$\mathbf{x}_{24} = \begin{pmatrix} 0 & 6 & 0 & 0 \end{pmatrix} \tag{2.5.49}$$
$$\mathbf{x}_{34} = \begin{pmatrix} 0 & 0 & 6 & 12 \end{pmatrix} \tag{2.5.50}$$

and so we see that some of the bases are degenerate, however all of them are feasible.

Graphically, we have

$$2x_1 + x_2 \geq 6 \tag{2.5.51}$$
$$4x_1 + 2x_2 \geq 12 \tag{2.5.52}$$

which is of course the same region, but clearly has boundaries unlike the previous case where the lines simply coincide.

2.5.5 Problem 5

Given

$$x_1 + 2x_2 + x_3 + x_4 = 2 \tag{2.5.53}$$
$$x_1 + 2x_2 + \frac{1}{3}x_3 + x_5 = 2 \tag{2.5.54}$$
$$x_1 + x_2 + x_3 + x_6 = 2 \tag{2.5.55}$$

with $x_i \geq 0$. Is $(1, 1/2, 0, 0, 0, 1/2)$ a vertex? If not, express it as a combination of two vertices.

Solution:

We have 6 variables and 3 equations for $C_3^6 = 20$ possible vertices. We also know that a basis will have 3 non-zero entries. Thus the vector given is potentially a vertex. Let's test it in each of the equations.

$$1 + 2(1/2) = 1 + 1 = 2\checkmark \tag{2.5.56}$$
$$1 + 2(1/2) = 1 + 1 = 2\checkmark \tag{2.5.57}$$
$$1 + 1/2 + 1/2 = 1 + 1 = 2\checkmark \tag{2.5.58}$$

Thus, it appears to be a vertex. We could worry that it is interior to our region however. We see that we need to test if this is a degenerate solution. Taking columns 1, 2, and 6 of the assocatiated matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1/3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{2.5.59}$$

yields for us

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{2.5.60}$$

since the first two rows are identical. Thus, it is not a vertex. Suppose we use 1,2,5, then we have

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 \tag{2.5.61}$$

and so this is fine. For 1,5,6 we'd find

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1 \tag{2.5.62}$$

Also 1,4,5 is a possibility

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -1 \tag{2.5.63}$$

Unfortunately all of these yield the same basis vector. The vector $\mathbf{x}_1 = (2, 0, 0, 0, 0, 0)$.

Let's try 2,3,6 and

$$\begin{vmatrix} 2 & 1 & 0 \\ 2 & 1/3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 2/3 - 2 = -4/3 \tag{2.5.64}$$

and so

$$2x_2 + x_3 = 2 \tag{2.5.65}$$

$$2x_2 + (1/3)x_3 = 2 \tag{2.5.66}$$

$$x_1 + x_3 + x_6 = 2 \tag{2.5.67}$$

clearly $x_3 = 0$ is required and we then have $x_2 = 1$ and $x_6 = 1$ for vector $\mathbf{x}_2 = (0, 1, 0, 0, 0, 1)$.

Then

$$t(2, 0, 0, 0, 0, 0) + s(0, 1, 0, 0, 0, 1) = (1, 1/2, 0, 0, 0, 1/2) \tag{2.5.68}$$

implies $t = 1/2$ and $s = 1/2$ for our solution being $(\mathbf{x}_1 + \mathbf{x}_2)/2$.

Chapter 3

The Algebra of Duality

Duality here is the "mirror" image of maximization and minimization problems.

3.1 Definitions

The second duality is simply recognizing that we can put minus signs in the objective function to swap minimization and maximization.

3.2 Homogeneous Systems

This section has a proof that is not explained very clearly. The induction proof for Theorem (V-3-1) is not clear at all without any explanations of reindexing or how we are supposed to even see that there is a "solution" to something.

Let's rewrite things with index notation so that it is not so crazily messy.

We have for $i = 1, \ldots, n$ that

$$a_{ij}y_j \geq 0 \tag{3.2.1}$$

and for $j = 1, \ldots, m$ that

$$a_{ij}x_i = 0 \tag{3.2.2}$$

with $x_i \geq 0$ solutions with

$$x_i + a_{ij}y_j > 0 \tag{3.2.3}$$

for all $i = 1, \ldots, n$.

When $n = 1$ we have only x_1. Thus, if $a_{ij} = 0$ completely we choose $y_j = 0$ and $x_1 = 1$ as a solution. In other cases we choose $y_j = a_{1j}$ and $x_1 = 0$ as a solution. Thus a solution clearly exists satisfying our constraints.

We assume that the statement is true, so that a solution exists of the form given when i ranges to n.

Let's simply call such a solution for the case $i = 1, \ldots, n$

$$a_{ij} y_j \geq 0 \tag{3.2.4}$$

and for $j = 1, \ldots, m$ that

$$a_{ij} x_i = 0 \tag{3.2.5}$$

with $x_i \geq 0$ the solutions satisfying

$$x_i + a_{ij} y_j > 0 \tag{3.2.6}$$

x_i^0 and y_j^0. That is x_i^0 and y_j^0 satisfies

$$a_{ij} y_j^0 \geq 0 \tag{3.2.7}$$
$$a_{ij} x_i^0 = 0 \tag{3.2.8}$$
$$x_i^0 + a_{ij} y_j^0 > 0 \tag{3.2.9}$$

For the $(n+1)$ case, our system can be written in terms of our previous n case as

$$(a_{ij} + a_{n+1,j}) y_j \geq 0 \tag{3.2.10}$$
$$a_{ij} x_i + a_{n+1,j} x_{n+1} = 0 \tag{3.2.11}$$

The final condition is then (for $i = 1, \ldots, n+1$)

$$x_i + a_{ij} y_j > 0 \tag{3.2.12}$$

which we remember includes the "new" condition

$$x_{n+1} + a_{n+1,j} y_j > 0 \tag{3.2.13}$$

Clearly if we have $x_{n+1} = 0$ and $a_{n+1,j} y_j^0 > 0$ then[1] the solution to the above can directly incorporate our y_j^0 and x_i^0 solutions. Indeed, then all the previous conditions are fine, and we have the extra condition

$$\overbrace{x_{n+1}}^{=0} + \overbrace{a_{n+1,j} y_j^0}^{>0} > 0 \tag{3.2.14}$$

In the case $a_{n+1,j} y_j^0 = 0$ we cannot easily get a solution unless $a_{n+1,j} = 0$ in which case we can then set $x_{n+1} = 1$.

In the case $a_{n+1,j} y_j^0 < 0$ then we can consider a system with $i = 1, \ldots, n$ and $j = 1, \ldots, m$ given by

$$(a_{ij} + r_i a_{n+1,j}) y_j \geq 0 \tag{3.2.15}$$
$$(a_{ij} + r_i a_{n+1,j}) x_i = 0 \tag{3.2.16}$$
$$r_i = \frac{a_{ij} y_j^0}{-a_{n+1,j} y_j^0} > 0 \tag{3.2.17}$$

[1]The book gets this wrong since it uses $\geq$, but if we have equality then $x_{n+1} + a_{n+1,j} y_j = 0 + 0 \not> 0$. Such a case implies something is wrong with our system of equations.

which must have a solution satisfying

$$x'_i + (a_{ij} + r_i a_{n+1,j})y'_j > 0 \tag{3.2.18}$$

We can see that if we define

$$x^{(1)}_{n+1} = \overbrace{r_i}^{>0} \overbrace{x'_i}^{>0} > 0 \tag{3.2.19}$$

then we have in (3.2.16) that

$$a_{ij}x'_i + a_{n+1,j}x^{(1)}_{n+1} = 0 \tag{3.2.20}$$

We can find our solution using a trick that the book does not explain well. We can use

$$x'_i(a_{ij} + a_{n+1,j}r_i)y'_j \geq 0 \tag{3.2.21}$$
$$0 + x_{n+1}a_{n+1,j}y'_j \geq 0 \tag{3.2.22}$$
$$a_{n+1,j}y'_j \geq 0 \tag{3.2.23}$$

which means $r = \frac{a_{n+1,j}y'_j}{-a_{n+1,j}y'_0} \geq 0$. Note that this implies

$$a_{ij}y'_j + \overbrace{r_i}^{>0} \overbrace{a_{n+1,j}y'_j}^{\geq 0} \geq 0 \tag{3.2.24}$$
$$a_{ij}y'_j + \frac{a_{ij}y^0_j}{-a_{n+1,j}y^0_j}a_{n+1,j}y'_j \geq 0 \tag{3.2.25}$$
$$a_{ij}y'_j + \frac{a_{n+1,j}y'_j}{-a_{n+1,j}y^0_j}a_{ij}y^0_j \geq 0 \tag{3.2.26}$$
$$a_{ij}y'_j + ra_{ij}y^0_j \geq 0 \tag{3.2.27}$$

Then we can consider

$$a_{ij}(y'_j + ry^0_j) > 0 \tag{3.2.28}$$

which is clearly true for $i < n+1$. Then we find

$$a_{n+1,j}(y'_j + ry^0_j) = a_{n+1,j}y'_j + \frac{a_{n+1,j}y'_j}{-a_{n+1,j}y^0_j}a_{n+1,j}y^0_j = a_{n+1,j}y'_j - a_{n+1,j}y'_j = 0 \tag{3.2.29}$$

which means that we then have a system that works for the $n+1$. That is using

$$x_i = x'_i \tag{3.2.30}$$
$$x_{n+1} = r_i x'_i = \frac{a_{ij}y^0_j}{-a_{n+1,j}y^0_j}x'_i \tag{3.2.31}$$
$$y_j = y'_j \tag{3.2.32}$$

allows us to extend our solutions from the $i \leq n$ system to the $i \leq n+1$ system.

We could more simply write when

$$\mathbf{A}\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \geq 0 \tag{3.2.33}$$
$$\mathbf{y}^\mathsf{T}\mathbf{A} \geq \mathbf{0} \tag{3.2.34}$$

we have

$$\mathbf{x}^\mathsf{T} + \mathbf{y}^\mathsf{T}\mathbf{A} > \mathbf{0} \tag{3.2.35}$$

For $\mathbf{x} = \mathbf{0}$, then this is obvious. We can use for any other case that that $\mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x} = \mathbf{0}$. If $\mathbf{x} > \mathbf{0}$ then $\mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{0}$ for this to be true and the equality will still hold. This method doesn't give a constructive proof, but satisfies the theorem.

The next theorems are fairly straightforward uses of our previous Theorem (V-3-1).

Farkas' theorem is also stated incorrectly. We set up the system ($i = 1, \ldots, n$ and $y = 1, \ldots, m$)

$$-a_{0j}y_j \geq 0 \tag{3.2.36}$$
$$a_{ij}y_j \geq 0 \tag{3.2.37}$$
$$-a_{0j}x_0 + a_{ij}x_i = 0 \tag{3.2.38}$$

with $x_0, x_i \geq 0$. Then Theorem (V-3-1) states it has a solution satisfying

$$x_0 - a_{0j}y_j > 0 \tag{3.2.39}$$
$$x_i + a_{ij}y_j > 0 \tag{3.2.40}$$

When $x_0 = 0$ this implies

$$-a_{0j}y_j > 0 \Rightarrow a_{0j}y_j < 0 \tag{3.2.41}$$
$$a_{ij}x_i = 0 \tag{3.2.42}$$

The book non-sensically gets the exact opposite conclusion, which is clearly wrong. There is no way to assert $a_{0j}y_j > 0$ as a consequence of $x_0 = 0$. The point is that if we considered another system removing the minus signs then we'd have

$$a_{0j}y_j \geq 0 \tag{3.2.43}$$
$$a_{ij}y_j \geq 0 \tag{3.2.44}$$
$$a_{0j}x_0 + a_{ij}x_i = 0 \tag{3.2.45}$$

the solution would be

$$x_0 + a_{0j}y_j > 0 \tag{3.2.46}$$
$$x_i + a_{ij}y_j > 0 \tag{3.2.47}$$

Then with $x_0 = 0$, then we must have the same solution as the previous set of relations. This means that we'd require $a_{0j}y_j > 0$ and $a_{0j}y_j < 0$ which is impossible. Thus $x_0 = 0$ is not a possibility and so $x_0 > 0$. This means we can divide through by x_0 the equation

$$-a_{0j}x_0 + a_{ij}x_i = 0 \tag{3.2.48}$$
$$-a_{0j} + a_{ij}\frac{x_i}{x_0} = 0 \tag{3.2.49}$$
$$a_{0j} = a_{ij}\frac{x_i}{x_0} \tag{3.2.50}$$

Then Theorem (V-3-4) follows from skew-symmetric matrices satisifying $A_{ij} = -A_{ji}$. Therefore $w_iA_{ij}w_j = -w_iA_{ji}w_j = -w_jA_{ji}w_i \overset{i\leftrightarrow j}{=} -w_iA_{ij}w_j$.

3.3 Polarity

This is again easiest to see in vector language. We denote the m dimensional vectors as $\mathbf{a}_i$ with $i = 1, \ldots, n$. We can then summarize them as A in a $n \times m$ matrix. We have our $\mathbf{x} > 0$ as vectors with n rows. We then are interested in $\mathsf{A}^\intercal\mathbf{x}$ which forms the polyhedral cone. We then define the polar polyhedral cone as the cone formed by the vectors $\mathbf{y}$ such that $\mathbf{y}\mathsf{A}^\intercal\mathbf{x} \geq 0$. Geometrically, that is the points $\mathbf{y} \cdot \mathbf{x}^* \geq 0$ which implies the "angle" is non-obtuse (since $\mathbf{y} \cdot \mathbf{x}^* = |\mathbf{y}||\mathbf{x}| \cos(\theta)$ and so to be non-negative requires the angle to be less than or equal to $\pi/2$ radians) with $\mathbf{x}^* = \mathsf{A}^\intercal\mathbf{x}$. If we denote the polyhedral cone set as A and the polar as A^*, then we would like to show that $A^{**} = A$.

First, consider A^{**}. This is the set

$$\mathbf{z} \cdot \mathbf{y} \geq 0 \tag{3.3.1}$$

because A^* is defined by the points $\mathbf{y}$. We can use

$$\mathbf{z} = \mathsf{A}^\intercal\mathbf{x} \tag{3.3.2}$$

which will clearly satisfy the inequality. This means that A^{**} is defined by the points $\mathbf{A}^\intercal\mathbf{x}$, which is the original cone A.

Next, we consider A. It clearly is of the form $\mathsf{A}^\intercal\mathbf{x}$. Farkas theorem tells us that if $\mathbf{y}\mathsf{A}^\intercal \geq 0$ and so $\mathbf{y}\mathsf{A}^\intercal\mathbf{x} \geq 0$, which we can rewrite as $\mathbf{c} \cdot \mathbf{y} \geq 0$, then $\mathbf{c} = \mathsf{A}^\intercal\mathbf{t}$ with $\mathbf{t} > 0$. But then $\mathbf{c}$, which defines A^{**} must be the same as $\mathsf{A}^\intercal\mathbf{x}$ and hence the same as A.

3.4 Inhomogeneous Inequalities. Duality Theorem. Existence Theorem

The beginning statement is simply applying our previous theorems.

The statement

$$b_j y_j \leq c_i x_i \tag{3.4.1}$$

is from simply looking at the dual problem at the beginning of the chapter.

The statement of the problem also requires $b_j y_j \geq c_i x_i$ and so our particular solution must satisfy this restraint as well. If $t^0 = 0$ however, we get a contradiction as indicated in the text.

The Lagrange multiplier method is nice to show that there is a way of arriving at this problem with calculus.

3.5 Orthogonality

This is a straightforward section. The orthogonality follows very clearly from the coefficient being dot producted as vectors.

3.6 Exercises

3.6.1 Problem 1

Find the solution of

$$x_1 - 3x_2 + x_3 = 0 \tag{3.6.1}$$
$$x_1 - x_2 - x_3 = 0 \tag{3.6.2}$$
$$x_1, x_2, x_3 \geq 0 \tag{3.6.3}$$
$$y_1 + y_2 \geq 0 \tag{3.6.4}$$
$$-3y_1 - y_2 \geq 0 \tag{3.6.5}$$
$$y_1 - y_2 \geq 0 \tag{3.6.6}$$

which also satisfies

$$x_1 + y_1 + y_2 > 0 \tag{3.6.7}$$
$$x_2 - 3y_1 - y_2 > 0 \tag{3.6.8}$$
$$x_3 + y_1 - y_2 > 0 \tag{3.6.9}$$

Solution:

Let us see if this forms a matrix system. We need

$$\mathbf{A}^\mathsf{T}\mathbf{x} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3.6.10}$$

Thus we want $\mathbf{A}\mathbf{y} > 0$ which would require

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{3.6.11}$$

Thus, we do have $\mathbf{A}^\mathsf{T}\mathbf{x} = 0$ and $\mathbf{A}\mathbf{y} \geq 0$ with solutions that must satisfy $\mathbf{x} + \mathbf{A}\mathbf{y} > 0$. Thus, a solution clearly exists that satisfies the constraints. We can rewrite the first x_i equation eliminating x_1 as

$$x_2 + x_3 - 3x_2 + x_3 = 0 \tag{3.6.12}$$
$$2(-x_2 + x_3) = 0 \tag{3.6.13}$$

So that $x_2 = x_3$ and $x_1 = x_2 + x_3 = 2x_2$.

Now we require $3y_1 \leq y_2$ and $y_1 \geq y_2$ which is impossible unless $y_1 = y_2 = 0$. Another way of seeing this is that $y_1 + y_2 \geq 0$ and $y_1 - y_2 \geq 0$ imply $2y_1 \geq 0$ or $y_1 \geq 0$. Whereas if we add $-3y_1 - y_2 \geq 0$ to $y_1 + y_2 \geq 0$ we find $-2y_1 \geq 0$.

Since $y_1 = y_2 = 0$ are the only solutions we can choose pretty much anything with $x_2 = x_3 > 0$ So $x_2 = x_3 = 1$ and $x_1 = 2$ is a solution.

3.6.2 Problem 2

Find a solution of

$$3x_1 + x_2 = 0 \tag{3.6.14}$$
$$-2x_1 + 2x_2 = 0 \tag{3.6.15}$$
$$x_1, x_2 \geq 0 \tag{3.6.16}$$

$$3y_1 - 2y_2 \geq 0 \tag{3.6.17}$$
$$y_1 + 2y_2 \geq 0 \tag{3.6.18}$$

which satisfies also

$$x_1 + 3y_1 - 2y_2 > 0 \tag{3.6.19}$$
$$x_2 + y_1 + 2y_2 > 0 \tag{3.6.20}$$

Solution:

The matrix is clearly given by

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \tag{3.6.21}$$

Thus such solutions should exist.

We see

$$6x_1 + 2x_2 - 2x_1 + 2x_2 = 0 \tag{3.6.22}$$
$$4x_1 = 0 \tag{3.6.23}$$
$$x_1 = 0 \tag{3.6.24}$$

and so $x_2 = 0$ as well. Then we simply choose $3y_1 - 2y_2 \geq 0$ or $3y_1 \geq 2y_2$ so $y_1 = y_2 = 1$ would work.

3.6.3 Problem 3

Show that all solutions of

$$3x_1 + 2x_2 - x_3 \geq 0 \tag{3.6.25}$$
$$5x_1 - 3x_2 + x_3 \geq 0 \tag{3.6.26}$$
$$4x_1 - x_2 + 5x_3 \geq 0 \tag{3.6.27}$$

satisfy also $22x_1 + x_2 + 8x_3 \geq 0$, and hence find the non-negative coefficients mentioned in Farkas' theorem.

Solution:

We can rewrite these as

$$3x_1 + 2x_2 - x_3 + x_4 = 0 \tag{3.6.28}$$
$$5x_1 - 3x_2 + x_3 + x_5 = 0 \tag{3.6.29}$$
$$4x_1 - x_2 + 5x_3 + x_6 = 0 \tag{3.6.30}$$

and so solve

$$\begin{bmatrix} 3 & 2 & -1 & 1 & 0 & 0 \\ 5 & -3 & 1 & 0 & 1 & 0 \\ 4 & -1 & 5 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 15 & 10 & -5 & 5 & 0 & 0 \\ 15 & -9 & 3 & 0 & 3 & 0 \\ 4 & -1 & 5 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 12 & 8 & -4 & 4 & 0 & 0 \\ 0 & 19 & -8 & 5 & -3 & 0 \\ 12 & -3 & 15 & 0 & 0 & 3 \end{bmatrix} \tag{3.6.31}$$

$$\to \begin{bmatrix} 3 & 2 & -1 & 1 & 0 & 0 \\ 0 & 19 & -8 & 5 & -3 & 0 \\ 0 & 11 & -19 & 4 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{8}{19} & \frac{5}{19} & -\frac{3}{19} & 0 \\ 0 & 1 & -\frac{19}{11} & \frac{4}{11} & 0 & \frac{-3}{11} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{2}{13} & \frac{9}{91} & \frac{1}{91} \\ 0 & 1 & 0 & \frac{3}{13} & -\frac{19}{91} & \frac{8}{91} \\ 0 & 0 & 1 & -\frac{1}{13} & -\frac{11}{91} & \frac{19}{91} \end{bmatrix} \tag{3.6.32}$$

Thus,

$$22x_1 = \frac{44}{13}x_4 + \frac{198}{19}x_5 + \frac{22}{91}x_6 \tag{3.6.33}$$
$$x_2 = \frac{3}{13}x_4 - \frac{19}{91}x_5 + \frac{8}{91}x_6 \tag{3.6.34}$$
$$8x_3 = -\frac{8}{13}x_4 + -\frac{88}{91}x_5 + \frac{152}{91}x_6 \tag{3.6.35}$$

So

$$22x_1 + x_2 + 8x_3 = \frac{39}{13}x_4 + x_5 + 2x_6 > 0 \tag{3.6.36}$$

This is essentially just finding a solution to

$$\begin{bmatrix} 3 & 5 & 4 \\ 2 & -3 & -1 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 1 \\ 8 \end{bmatrix} \tag{3.6.37}$$

Thus

$$\to \left[\begin{array}{ccc|c} 6 & 10 & 8 & 44 \\ 6 & -9 & -3 & 3 \\ 3 & -3 & -15 & 24 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 5 & 4 & 22 \\ 0 & -19 & -11 & -41 \\ 0 & -8 & -19 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 5 & 4 & \\ 0 & 1 & \frac{11}{19} & \frac{41}{19} \\ 0 & 1 & \frac{19}{8} & \frac{-1}{4} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \tag{3.6.38}$$

which means we have $t_1 = 3$, $t_2 = 1$ and $t_3 = 2$.

Note that we could simply have proceeded from the second part. If such a solution exists, because Farkas' theorem works in reverse we know that there is an inequality that must be satisfied by the variables.

3.6.4 Problem 4

Prove that not all solutions of the first three inequalities in Problem 3 satisfy $12x_1+7x_2+6x_3 \geq 0$.

Solution:

We have two choices. The easiest way given our previous work is to use x_1, x_2, x_3 in terms of x_4, x_5, and x_6 and show that the inequality given isn't satisfied. But in general we'd simply row reduce the matrix

$$\left[\begin{array}{ccc|c} 3 & 5 & 4 & 12 \\ 2 & -3 & -1 & 7 \\ -1 & 1 & 5 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \tag{3.6.39}$$

which means that $t_2 < 0$ and it fails. The other way yields

$$12x_1 = \frac{24}{13}x_4 + \frac{108}{19}x_5 + \frac{12}{91}x_6 \tag{3.6.40}$$

$$7x_2 = \frac{21}{13}x_4 - \frac{133}{91}x_5 + \frac{56}{91}x_6 \tag{3.6.41}$$

$$6x_3 = -\frac{6}{13}x_4 + -\frac{66}{91}x_5 + \frac{114}{91}x_6 \tag{3.6.42}$$

So

$$12x_1 + 7x_2 + 6x_3 = \frac{39}{13}x_4 - x_5 + 2x_6 \tag{3.6.43}$$

which is not guaranteed to be greater than zero since $x_5 \geq 0$.

Chapter 4

Theory of Graphs and Combinatorial Theory

The title says it all.

4.1 Definitions

Definitions of planar graph and *a*-*b* graph. The definitions are poor. It's not at all clear what he's talking about with an *a*-*d* graph. Because of the really bad definition, it's not even clear what he means by the graphs being equivalent.

4.2 Shortest Path

This section is actually fairly straightforward. Just follow the directions and clearly you will form the shortest path.

4.3 Maximal Flow. Minimum Cut

The definitions here are again not exactly great. Residual flow and chain flows would be better explained by examples. A chain flow is the number associated with a chain (a set of nodes from a source to a sink) which is less than or equal to the maximum number possible along any arc in the chain. If the sum of all chain flows through an arc is equal to the maximum, we say that it is saturated. If it is not saturated, then the number that could be added to it to achieve saturation is the residual flow.

The directions for labeling the graph are not very clear. The idea is to form a chain, add the max possible of extra capacity to all the arcs in the chain and continue this process until all arcs are saturated. When all are saturated, we do the relabeling process to ensure that we actually have the best maximum possible. Thus one unsaturated chain at the beginning is $\oplus$ to a to b to $\ominus$. The smallest residual flow is 2, so add two to each arc. Next we could try $\oplus$ to c to $\ominus$ with 1 the residual flow. Next we could try a chain such as $\oplus$ to b to a to $\ominus$. The max residual flow is clearly

3, and this diverges from the book's treatment. The book takes a different method, but ignores this possible chain, and so doesn't really use the algorithm correctly.

Essentially, we can keep doing this labeling until $\ominus$ has only saturated connections.

The tabular method is even more poorly explained. It is better to look for modern coverage with Ford-Fulkerson and Edmond-Karp algorithms.

4.4 Dual Graphs

Not a particularly effective explanation of how to make a dual graph. A picture really is worth a thousand words. But the idea is to convert chains into nodes and vice versa.

4.5 Directed Network of the Transportation Problem

The example doesn't label a,b,c,d,e,f so that it isn't clear how the example could be helpful.

4.6 Trees. Triangularity

Finally, a good section that actually explains things fairly well.

4.7 Dantzig Property. Unimodular Property

An all right section. The proof with the Dantzig property has the typo in the substitution of

$$a_{1j}\left(x_1^0 + c_{m+1,1}x_{m+1}^0 + \cdots + a_{N,1}x_N^0\right) + \cdots \tag{4.7.1}$$

when it should be

$$a_{1j}\left(x_1^0 + c_{m+1,1}x_{m+1}^0 + \cdots + c_{N,1}x_N^0\right) + \cdots \tag{4.7.2}$$

which could be more conveniently written

$$a_{1j}(x_1^0 + c_{i',1}x_{i'}^0) \tag{4.7.3}$$

with i' ranging from $m+1$ to N.

The rest of the theorems are presented without proof and refer to the literature, which is a little bit annoying, but at least understandable. The example with the chart is not very helpful since the chart is never explained.

We can see that (0,-6,9),(6,0,3) , and (9,3,0) are the basic solutions. This of course means that the only feasible solutions are the latter two.

4.8 Systems of Distinct Representatives. Related Theorems

The proof is fairly good except for the extremely annoying fact that u_i and v_j are never defined and so the dual problem is not actually explained. You cannot just make up new variables without explanation. It turns out that they show up in a later chapter.

We have a problem of the form

$$\mathbf{A}\mathbf{y} \leq \mathbf{c} \tag{4.8.1}$$

with $\mathbf{y} \geq 0$ and maximizing $\mathbf{b} \cdot \mathbf{y}$. This means we must have $a_{ij}x_{ij} = b_j y_j$. Then the dual would be

$$\mathbf{A}^{\intercal}\mathbf{x} \geq \mathbf{b} \tag{4.8.2}$$

with $\mathbf{x} \geq 0$ and minimizing $\mathbf{c} \cdot \mathbf{x}$.

Thus, the book cheats, as we should have a really long matrix rather than creating x_{ij} it should just be an $x_{i'}$ with i' going over all possible values.

4.9 Exercises

4.9.1 Problem 1

Solve Example 4-4 by the tabular method of labeling.

Solution:

Considering that the tabular form is never actually explained, this is not easy to do

I can create the initial table, though.

–	$\oplus$	a	b	c	d	e	f	$\ominus$	–
$\oplus$	0	36	15	47	0	0	0	0	–
a	36	0	19	0	0	0	0	40	–
b	15	19	0	0	28	0	0	0	–
c	47	0	0	0	20	34	0	0	–
d	0	0	28	20	0	0	21	0	–
e	0	0	0	34	0	0	14	38	–
f	0	0	0	0	21	14	0	24	–
$\ominus$	0	40	0	0	0	38	24	0	–

(4.9.1)

We can start by going form $\oplus$ to a to $\ominus$. The max possible is 36, and so we subtract 36 from each entry along this path and find.

$-$	$\oplus$	a	b	c	d	e	f	$\ominus$	$-$
$\oplus$	0	0	15	47	0	0	0	0	$-$
a	72	0	19	0	0	0	0	4	$-$
b	15	19	0	0	28	0	0	0	$-$
c	47	0	0	0	20	34	0	0	$-$
d	0	0	28	20	0	0	21	0	$-$
e	0	0	0	34	0	0	14	38	$-$
f	0	0	0	0	21	14	0	24	$-$
$\ominus$	0	76	0	0	0	38	24	0	$-$

(4.9.2)

Next we go $\oplus$ to b to d to f to $\ominus$. In this case 15 is the maximum and so

$-$	$\oplus$	a	b	c	d	e	f	$\ominus$	$-$
$\oplus$	0	0	0	47	0	0	0	0	$-$
a	72	0	19	0	0	0	0	4	$-$
b	30	19	0	0	13	0	0	0	$-$
c	47	0	0	0	20	34	0	0	$-$
d	0	0	43	20	0	0	6	0	$-$
e	0	0	0	34	0	0	14	38	$-$
f	0	0	0	0	36	14	0	9	$-$
$\ominus$	0	76	0	0	0	38	39	0	$-$

(4.9.3)

Finally, we take $\oplus$ to c to e to $\ominus$ with the max 34 yielding

$-$	$\oplus$	a	b	c	d	e	f	$\ominus$	$-$
$\oplus$	0	0	0	13	0	0	0	0	$-$
a	72	0	19	0	0	0	0	4	$-$
b	30	19	0	0	13	0	0	0	$-$
c	81	0	0	0	20	0	0	0	$-$
d	0	0	43	20	0	0	6	0	$-$
e	0	0	0	68	0	0	14	4	$-$
f	0	0	0	0	36	14	0	9	$-$
$\ominus$	0	76	0	0	0	72	39	0	$-$

(4.9.4)

We can now go $\oplus$ to c to d to f to $\ominus$. The maximum possible here is 6 so that we get

$-$	$\oplus$	a	b	c	d	e	f	$\ominus$	$-$
$\oplus$	0	0	0	7	0	0	0	0	$-$
a	72	0	19	0	0	0	0	4	$-$
b	30	19	0	0	13	0	0	0	$-$
c	94	0	0	0	14	0	0	0	$-$
d	0	0	43	26	0	0	0	0	$-$
e	0	0	0	68	0	0	14	4	$-$
f	0	0	0	0	42	14	0	3	$-$
$\ominus$	0	76	0	0	0	72	45	0	$-$

(4.9.5)

Now every chain has been exhausted and we only need to worry about relabelings.

–	$\oplus$	a	b	c	d	e	f	$\ominus$	–
$\oplus$	0	0	0	7	0	0	0	0	–
a	72	0	19	0	0	0	0	4	$(19, b)$
b	30	19	0	0	13	0	0	0	$(13, d)$
c	87	0	0	0	14	0	0	0	$(7, \oplus)$
d	0	0	43	26	0	0	0	0	$(14, c)$
e	0	0	0	68	0	0	14	4	$(-14, f)$
f	0	0	0	0	42	14	0	3	$(14, e)$
$\ominus$	0	76	0	0	0	72	45	0	$(4, a)$

(4.9.6)

Clearly the chain $\ominus$ to a to b to d to c to $\oplus$ exists where we can take 4 off.

–	$\oplus$	a	b	c	d	e	f	$\ominus$	–
$\oplus$	0	0	0	3	0	0	0	0	–
a	72	0	23	0	0	0	0	0	–
b	26	15	0	0	17	0	0	0	–
c	97	0	0	0	10	0	0	0	$(3, \oplus)$
d	0	0	39	30	0	0	0	0	$(10, c)$
e	0	0	0	68	0	0	14	4	–
f	0	0	0	0	42	14	0	3	–
$\ominus$	0	80	0	0	0	72	45	0	–

(4.9.7)

There appear to be no more possibilities to relabel $\ominus$. If we subtract the top half entries from the last chart from the first we find

–	$\oplus$	a	b	c	d	e	f	$\ominus$	–
$\oplus$	0	36	15	44	0	0	0	0	–
a	–	0	4	0	0	0	0	40	–
b	–	–	0	0	11	0	0	0	–
c	–	–	–	0	9	34	0	0	–
d	–	–	–	–	0	0	21	0	–
e	–	–	–	–	–	0	0	34	–
f	–	–	–	–	–	–	0	21	–
$\ominus$	–	–	–	–	–	–	–	0	–

(4.9.8)

Note that this has the same maximal flow, but has different entries than the graphical method presented.

4.9.2 Problem 2

Find a set of distinct representatives of the following sets

(a) 1,2,3,4,5

(b) 2,3,4

(c) 1,3,4,5

(d) 1,3,4

(e) 3,4

(f) 2,4

(g) 1,4,6,9

(h) 2,5,7,8,9

if this is not possible, which extension of one of the sets would make it possible?

Solution:

Well, we need to see if we choose k entries from k of them, and see if among the entirety of those chosen k sets that there are k distinct elements. Let's try $k = 2$. Any one set has two distinct elements, so we're good. For $k = 3$ we see that it is still true since the two element sets when combined with any other set will have a third element. For $k = 4$ try (b), (e), (f), which containt distinctly (2,3,4). This does not contain 4 distinct elements and so it is impossible to do so.

We can write out the matrix

$$\left[\begin{array}{c|ccccccccc}
- & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
a & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
b & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
d & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
f & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
g & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
h & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{array}\right] \tag{4.9.9}$$

Since there are nine numbers and only 8 sets, we can at most get $k = 8$.

Let's star the ones that can be independent 1s. Clearly h must star the 8 since that is the only one with an 8. Then g can be either 6 or 9. The book decides to omit 9, but this isn't strictly speaking necessary. We could

$$\left[\begin{array}{c|ccccccccc}
- & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
a & 1^* & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
b & 0 & 1^* & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 1 & 0 & 1 & 1 & 1^* & 0 & 0 & 0 & 0 \\
d & 1 & 0 & 1 & 1^* & 0 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 1^* & 1 & 0 & 0 & 0 & 0 & 0 \\
f & 0 & 1 & 0 & 1 & 0 & 0 & 0^* & 0 & 0 \\
g & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1^* \\
h & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1^* & 1
\end{array}\right] \tag{4.9.10}$$

or we could choose g with 6, but either way adding 7 to f gives us a distinct representative set.

4.9.3 Problem 3

In the graph of Fig. 4-10, the short lines have "length" 1, and the long ones "length" 2. Find the shortest route from A to B. Draw the dual graph and determine the maximum flow through it.

Solution:

If we simply take the shortest route from A to B we see that there is a route with length 1 along each way. Thus 7 is the smallest possible length. To actually prove this, we could go through the algorithm presented. Drawing the dual is possible, but the instructions from the text are nearly worthless and I don't want to try and create it as a computer graphic. Clearly the maximum flow is 7 by all of our duality theorems.

4.9.4 Problem 4

Find all sets of independent "1's" in the following table. (Note that all the row sums and all the column sums are 3).

$$\begin{matrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{matrix} \tag{4.9.11}$$

Solution:

We simply need to find a distinct representative set from the above. We can use

$$\begin{matrix} 0 & 1^* & 1 & 1 & 0 \\ 1^* & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1^* & 0 \\ 0 & 1 & 0 & 1 & 1^* \\ 1 & 0 & 1^* & 0 & 1 \end{matrix} \tag{4.9.12}$$

which gives 2,1,4,5,3 meaning we could form it from a permutation matrix like

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{matrix} \tag{4.9.13}$$

But there are other possibilities. For example

$$\begin{matrix} 0 & 1 & 1^* & 1 & 0 \\ 1^* & 0 & 1 & 0 & 1 \\ 1 & 1^* & 0 & 1 & 0 \\ 0 & 1 & 0 & 1^* & 1 \\ 1 & 0 & 1 & 0 & 1^* \end{matrix} \tag{4.9.14}$$

for 3,1,2,4,5. Also

$$\begin{matrix} 0 & 1 & 1 & 1^* & 0 \\ 1 & 0 & 1^* & 0 & 1 \\ 1 & 1^* & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1^* \\ 1^* & 0 & 1 & 0 & 1 \end{matrix} \tag{4.9.15}$$

for 4,3,2,5,1. Such switches give us essentially everything we need.

This then turns into a tedious exercise of enumerating all the possibilities.

Clearly there are $n!$ permutation matrices in general given a $n \times n$ size. Thus $6! = 720$ possibilities. The book lists the 12 that are actually possible. It uses that if we expand the array out as a matrix, we the independent ones will be the nonzero entries in the determinant. So if we do the determinant on the first row, there will be no 1* numbers since the first entry is a zero. This is essentially doing what we have done above, but perhaps (?) more systematically.

Our way would be more systematic if we started with the first row, chose one of the columns to be a representative, and crossed out terms in that column. Then went through all the possibilities in the second row, etc. Repeating as needed.

So clearly 2,1,4,5,3 then 2,3,1,4,5 then 2,3,4,5,1 then 2,5,1,4,3 then 3,1,2,4,5 then 3,1,4,2,5 then 3,5,2,4,1 then 3,5,4,2,1 then 4,1,2,5,3 then 4,3,1,2,5 then 4,3,2,5,1 then 4,5,1,2,3.

Chapter 5

General Algorithms

Linear algebra for the first "quadrant".

5.1 Simplex Method

The tableau formation is simply a convenient representation. It uses the transformation rules that were derived earlier.

The most complicated part is the

$$z_{st} = z_{st} - z_{rt} z_{sh} / z_{rh} \tag{5.1.1}$$

which is not using summation notation. Instead, it is saying find the value in the pivot column and pivot row, multiply them and divide by the pivot value. An example below will illustrate.

Let's look at the example. We convert

$$3x_1 + 4x_2 + x_3 \leq 2 \tag{5.1.2}$$

$$x_1 + 3x_2 + 2x_3 \leq 1 \tag{5.1.3}$$

$$x_1, x_2, x_3 \geq 0 \tag{5.1.4}$$

and maximize $B = 3x_1 + 6x_2 + 2x_3$ with extra variables to make equations

$$3x_1 + 4x_2 + x_3 + x_4 = 2 \tag{5.1.5}$$

$$x_1 + 3x_2 + 2x_3 + x_5 = 1 \tag{5.1.6}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \tag{5.1.7}$$

Note a simple (trivial) solution is $x_1 = x_2 = x_3 = 0$ and then $x_4 = 2$ and $x_5 = 1$ must be true.

The tableau is

$$\begin{array}{c|ccccc} & & 3 & 6 & 2 & \\ \hline & & x_1 & x_2 & x_3 & 2 \\ 0 & x_4 & 3 & 4 & 1 & 2 \\ 0 & x_5 & 1 & 3 & 2 & 1 \\ \hline & B & -3 & -6 & -2 & 0 \end{array} \tag{5.1.8}$$

Now the ways to decide on what to pivot is by convention, and then by deciding upon ratios that will get our objective function to a better value.

In our case clearly we want x_2 to increase, so let's "pivot" between x_2 and one of the other variables. We have a choice of x_4 and x_5 with the values in those columns being 4 and 3. We consider the ratio of these values to the "equality" values (this is essentially considering how much we can scale each one). Thus $2/4$ and $1/3$ are the ratios to consider. Clearly $1/3$ is smaller, (and this will lead to a larger B) so we choose to switch x_5 and x_2.

We then follow the rules of dividing the pivot row and column values by the pivot, and the other rule above and find

$$\begin{array}{c|ccccc}
 & & 3 & 0 & 2 & \\
\hline
 & & x_1 & x_5 & x_3 & 2 \\
0 & x_4 & 3-(4/3) & -4/3 & 1-8/3 & 2-4/3 \\
6 & x_2 & 1/3 & 1/3 & 2/3 & 1/3 \\
\hline
 & B & -3-(-6/3) & 6/3 & -2-(-12/3) & 0-(-6/3)
\end{array} \tag{5.1.9}$$

$$= \begin{array}{c|ccccc}
 & & 3 & 0 & 2 & \\
\hline
 & & x_1 & x_5 & x_3 & 2 \\
0 & x_4 & 5/3 & -4/3 & -5/3 & 2/3 \\
6 & x_2 & 1/3 & 1/3 & 2/3 & 1/3 \\
\hline
 & B & -1 & 2 & 2 & 2
\end{array} \tag{5.1.10}$$

We can use the same reasoning as before. Clearly the x_1 column is the most negative so that is the pivot column. Then we compare $(1/3)/(1/3) = 1$ and $(2/3)/(5/3) = 2/5$. Then $2/5$ is the smaller value and so we choose to pivot x_4 and x_1. This yields

$$\begin{array}{c|ccccc}
 & & 0 & 0 & 2 & \\
\hline
 & & x_4 & x_5 & x_3 & 2 \\
3 & x_1 & 3/5 & -4/5 & -1 & 2/5 \\
6 & x_2 & -1/5 & 1/3-(-4/15) & 2/3-(-1/3) & 1/3-2/15 \\
\hline
 & B & 3/5 & 2-(4/5) & 2-(1) & 2-(-2/5)
\end{array} = \tag{5.1.11}$$

$$\begin{array}{c|ccccc}
 & & 0 & 0 & 2 & \\
\hline
 & & x_4 & x_5 & x_3 & 2 \\
3 & x_1 & 3/5 & -4/5 & -1 & 2/5 \\
6 & x_2 & -1/5 & 3/5 & 1 & 1/5 \\
\hline
 & B & 3/5 & 6/5 & 1 & 12/5
\end{array} \tag{5.1.12}$$

To keep a check (when doing these yourself) it is prudent to add a second objective function that can do so. In this problem, a check is given by $\sum_i x_i$ (just adding all the x_i). If we add this, it is clear that the coefficients are invariant and so the final row is given by $\sum_i z_{ij} + z_{0j} - 1$ (the final -1 coming from our new constraint). This is invariant because of the previous transformation rules.

5.2 Simplex Method. Finding a first feasible solution

Now we see how we can find a feasible solution by adding additional variables with a constraint that is not very restrictive as we solve. Basically, it is equivalent to doing a Lagrange multiplier problem. This is sometimes called the M method since M, a large number in absolute value is multiplied by a constraint on the additional variables to create a new constraint.

The two-phase method uses special extra variables and constraints to ensure a quick solution.

5.3 Simplex Method. Degeneracy

Degeneracy occurs when one of the basic variables is zero. We may then have problems pivoting(though in actuality, it is easy to treat as mentioned in the book), and it means that, in fact, we must have a problem where there are actually fewer basic variables than we initially thought. (We have a degenerate matrix, hence some of the equations are not linearly independent.) The bigger problem is that the pivot does not lead to a change in the max/min function and so we may not be certain we will hit the optimal value in a finite number of pivots. Note that degeneracy means that when we compare ratios, two of the rows have the same ratio. Then a tiebreaker is needed if we want our algorithm to terminate in a finite time for certain.

The book states that in practice, cycles don't occur, but can theoretically and gives a good example. One theoretical way out is to slightly perturb the problem and solve that simultaneously so that we can choose between ties using the perturbed values.

5.4 The Inverse Matrix Method

This is simply rewriting the simplex algorithm so that it can be done easily in matrix algebra using the inverse matrices we discussed in 2.

5.5 Constructive Proof of the Duality and Existence Theorems

A proof that is actually fairly straightforward with no sudden new definitions.

5.6 Dual Simplex Method

This is essentially using that we could instead of solving with the simplex method but changing some signs to negative and then pivoting based on rows rather than columns. Note that this is essentially the same as solving the dual problem, but it allows us to start with some of our basic variables having negative values. We then go in the "opposite" direction until all variables are non-negative and can proceed by the usual simplex method.

Thus we compare variables in a row to their values in the objective function for determining the pivot.

5.7 Dual Simplex Method. Cycling

Good discussion of cycling.

5.8 Bounded Variables

A special case of the simplex method or dual simplex method where we have special constraints.

5.9 Multiplex Method

Similar to the simplex method, but uses projections on to planes to find the answer.

5.10 The Cross-Section Method

This involves starting from a lower dimensional problem and finding answers by putting in the constraints one by one to higher dimensions.

5.11 The Primal-Dual Algorithm

This uses duality to help one get the optimal answer. It is somewhat complicated by constructing new problems using previous methods, and then iterating on there.

5.12 Relaxation Method

This just takes arbitrary points in the region and uses some method to let them iterate to a good solution.

5.13 Exercises

5.13.1 Problem 1

Solve the following problems by the Simplex method.

$$x_1 + x_2 \leq 3 \tag{5.13.1}$$
$$x_1 - 2x_2 \leq 1 \tag{5.13.2}$$
$$-2x_1 + x_2 \leq 2 \tag{5.13.3}$$
$$x_i \geq 0 \tag{5.13.4}$$

(a) Minimize $x_1 - x_2$. (b) Maximize $x_1 - x_2$.

Solution:

We proceed with the tableau form for (a) first.

We see that we can initially rewrite the problem as

$$x_1 + x_2 + x_3 = 3 \tag{5.13.5}$$
$$x_1 - 2x_2 + x_4 = 1 \tag{5.13.6}$$
$$-2x_1 + x_2 + x_5 = 2 \tag{5.13.7}$$

Then the solution $x_1 = x_2 = 0$ and $x_3 = 3$, $x_4 = 1$, and $x_5 = 2$ is one trivial solution

$$\begin{array}{c|cccc} & & 1 & -1 & \\ \hline & & x_1 & x_2 & \\ 0 & x_3 & 1 & 1 & 3 \\ 0 & x_4 & 1 & -2 & 1 \\ 0 & x_5 & -2 & 1 & 2 \\ \hline & B & -1 & 1 & 0 \\ & Ck & -2 & 0 & 5 \end{array} \tag{5.13.8}$$

As we are undergoing minimization, we choose values from the B row such that they are the most positive. Hence we choose the x_2 column. The ratios are then $3/1 = 3$, $1/-2 = -0.5$ and $2/1 = 2$. Thus we'd like to pivot on the x_5 row as it has the smallest value. We cannot choose negative value rows, as this does not guarantee positivity for future values (remember we take reciprocals, etc.). Indeed, if all values were negative in the column, then there is a bounding problem.

We then perform the tableau rules with column x_2 and row x_5

$$\begin{array}{c|cccc} & & 1 & 0 & \\ \hline & & x_1 & x_5 & \\ 0 & x_3 & 1-(-2)(1) & -1 & 3-(1)(2) \\ 0 & x_4 & 1-(-2)(-2) & 2 & 1-(-2)(2) \\ -1 & x_2 & -2 & 1 & 2 \\ \hline & B & -1-(-2)(1) & -1 & 0-2 \end{array} \tag{5.13.9}$$

$$= \begin{array}{c|cccc} & & 1 & 0 & \\ \hline & & x_1 & x_5 & \\ 0 & x_3 & 3 & -1 & 1 \\ 0 & x_4 & -3 & 2 & 5 \\ -1 & x_2 & -2 & 1 & 2 \\ \hline & B & 1 & -1 & -2 \\ & Ck & -2 & 0 & 5 \end{array} \tag{5.13.10}$$

Clearly column x_1 should now be chosen. The ratios are then $1/3$, and the others are negative. Thus the x_3 row is the pivot row.

$$\begin{array}{c|cccc} & & 0 & 0 & \\ \hline & & x_3 & x_5 & \\ 1 & x_1 & 1/3 & -1/3 & 1/3 \\ 0 & x_4 & 1 & 2-(-1)(-3)/3 & 5-(-3)(1)/3 \\ -1 & x_2 & 2/3 & 1-(-2)(-1)/3 & 2-(-2)(1)/3 \\ \hline & B & -1/3 & -1-(1)(-1)/3 & -2-(1)(1)/3 \end{array} \tag{5.13.11}$$

$$= \begin{array}{c|cccc} & & 0 & 0 & \\ \hline & & x_3 & x_5 & \\ 1 & x_1 & 1/3 & -1/3 & 1/3 \\ 0 & x_4 & 1 & 1 & 6 \\ -1 & x_2 & 2/3 & 1/3 & 8/3 \\ \hline & B & -1/3 & -2/3 & -7/3 \\ & Ck & 2/3 & -2/3 & 16/3 \end{array} \tag{5.13.12}$$

which shows that the check row is not always so nice. Only when we pivot on a zero row.

We have now all negative coefficients on the B row and so have hit the minimization.

Thus our answer is $x_1 = 1/3$, $x_2 = 8/3$, $x_3 = 0$, $x_4 = 6$, $x_5 = 0$ with $B = -7/3$. We can check this by looking at our initial requirements and see that it does.

Now for (b). This time we proceed by choosing the most negative coefficient in the B row for the pivot column. Then we switch with x_4 since $1/1 < 3/1$.

$$
\begin{array}{c|cccc}
 & & 1 & -1 & \\
\hline
 & & x_1 & x_2 & \\
0 & x_3 & 1 & 1 & 3 \\
0 & x_4 & 1 & -2 & 1 \\
0 & x_5 & -2 & 1 & 2 \\
\hline
 & B & -1 & 1 & 0
\end{array}
\tag{5.13.13}
$$

$$
= \begin{array}{c|cccc}
 & & 0 & -1 & \\
\hline
 & & x_4 & x_2 & \\
0 & x_3 & -1 & 1-(1)(-2) & 3-(1)(1) \\
1 & x_1 & 1 & -2 & 1 \\
0 & x_5 & 2 & 1-(-2)(-2) & 2-(-2)(1) \\
\hline
 & B & 1 & 1-(-1)(-2) & 0-(-1)(1)
\end{array}
\tag{5.13.14}
$$

$$
= \begin{array}{c|cccc}
 & & 0 & -1 & \\
\hline
 & & x_4 & x_2 & \\
0 & x_3 & -1 & 3 & 2 \\
1 & x_1 & 1 & -2 & 1 \\
0 & x_5 & 2 & -3 & 4 \\
\hline
 & B & 1 & -1 & 1
\end{array}
\tag{5.13.15}
$$

We then pivot on the x_2 column. the x_3 row is the only possiblity and so

$$
\begin{array}{c|cccc}
 & & 0 & 0 & \\
\hline
 & & x_4 & x_3 & \\
-1 & x_2 & -1/3 & 1/3 & 2/3 \\
1 & x_1 & 1-(-2)(-1)/3 & 2/3 & 1-(2)(-2)/3 \\
0 & x_5 & 2-(-1)(-3)/3 & 1 & 4-(-3)(2)/3 \\
\hline
 & B & 1-(-1)(-1)/3 & 1/3 & 1-(2)(-1)/3
\end{array}
\tag{5.13.16}
$$

$$
= \begin{array}{c|cccc}
 & & 0 & 0 & \\
\hline
 & & x_4 & x_3 & \\
-1 & x_2 & -1/3 & 1/3 & 2/3 \\
1 & x_1 & 1/3 & 2/3 & 7/3 \\
0 & x_5 & 1 & 1 & 6 \\
\hline
 & B & 2/3 & 1/3 & 5/3
\end{array}
\tag{5.13.17}
$$

So that the solution is $x_1 = 7/3$, $x_2 = 2/3$, $x_3 = x_4 = 0$, $x_5 = 6$ and $B = 5/3$, which can be checked.

5.13.2 Problem 2

Show that by omitting one constraint in Problem 1 both (a) and (b) have an infinite solution.

Solution:

Suppose we eliminate $x_1 + x_2 \leq 3$. Then the first row in the tableau is missing. And so there is no positive number to pivot for the second time, but B still has a positive 1. Similarly, for the maximization problem.

5.13.3 Problem 3

Indicate the limits of t in the objective function $tx_1 - x_2$ beyond which a finite minimum or finite maximum exists when those constraints hold that were considered to remain in Problem 2.

Solution:

This is asking if we replace the objective function, what t are possible. We can see that we have $-t - (-2)(1)$ as our way of determining this for the minimization problem. In that case, we want this to be less than or equal to zero and so $-t + 2 \leq 0$ means that $t \geq 2$ gives us finite solutions.

For the maximization problem, we have $1-(-t)(-2) \geq 0$ for finite solutions. This means $1-2t \geq 0$ or $t \leq 1/2$.

5.13.4 Problem 4

Solve the following problems.

$$-x_1 + 2x_2 - x_3 = 1 \tag{5.13.18}$$
$$-x_1 - x_2 + 2x_3 = 1 \tag{5.13.19}$$
$$x_i \geq 0 \tag{5.13.20}$$

(a) Maximize $2x_1 - x_2 - x_3$. (b) Minimize $2x_1 - x_2 - x_3$.

Solution:

This is the same as our previous problems in form, and we can use the tableau to find the answer again.

We do require an initial solution however. I'll simply add two more slack variables to find this with $x_4 = 1$ and $x_5 = 1$ (other $x_i = 0$) for the system

$$-x_1 + 2x_2 - x_3 + x_4 = 1 \tag{5.13.21}$$
$$-x_1 - x_2 + 2x_3 + x_5 = 1 \tag{5.13.22}$$
$$x_i \geq 0 \tag{5.13.23}$$

and change the maximization/minimization problem to $2x_1 - x_2 - x_3 \pm M(x_4 + x_5)$

For (a) (maximization) we use $-M$ and find

$$
\begin{array}{r|ccccc}
 & & 2 & -1 & -1 & \\
\hline
 & & x_1 & x_2 & x_3 & \\
-M & x_4 & -1 & 2 & -1 & 1 \\
-M & x_5 & -1 & -1 & 2 & 1 \\
\hline
 & B & -2 & 1 & 1 & 0 \\
 & M & 2 & -1 & -1 & -2
\end{array}
\tag{5.13.24}
$$

We can then pivot off of either x_2 or x_3. These are entirely equivalent and so we'll choose x_2 We then must pivot the row off of x_4. This means we find

$$
\begin{array}{r|ccccc}
 & & 2 & -M & -1 & \\
\hline
 & & x_1 & x_4 & x_3 & \\
-1 & x_2 & -1/2 & 1/2 & -1/2 & 1/2 \\
-M & x_5 & -1-(-1)(-1)/2 & 1/2 & 2-(-1)(-1)/2 & 1-(-1)(1)/2 \\
\hline
 & B & -2-(-1)(1)/2 & -1/2 & 1-(-1)(1)/2 & 0-(1)(1)/2 \\
 & M & 2-(-1)(-1)/2 & 1/2 & -1-(-1)(-1)/2 & -2-(1)(-1)/2
\end{array}
\tag{5.13.25}
$$

$$
= \begin{array}{r|ccccc}
 & & 2 & -M & -1 & \\
\hline
 & & x_1 & x_4 & x_3 & \\
-1 & x_2 & -1/2 & 1/2 & -1/2 & 1/2 \\
-M & x_5 & -3/2 & 1/2 & 3/2 & 3/2 \\
\hline
 & B & -3/2 & -1/2 & 3/2 & -1/2 \\
 & M & 3/2 & 1/2 & -3/2 & -3/2
\end{array}
\tag{5.13.26}
$$

We can simply eliminate the x_4 column for convenience now, as it is an extra variable we do not care about.

$$
\begin{array}{r|cccc}
 & & 2 & -1 & \\
\hline
 & & x_1 & x_3 & \\
-1 & x_2 & -1/2 & -1/2 & 1/2 \\
-M & x_5 & -3/2 & 3/2 & 3/2 \\
\hline
 & B & -3/2 & 3/2 & -1/2 \\
 & M & 3/2 & -3/2 & -3/2
\end{array}
\tag{5.13.27}
$$

We can then pivot off of x_3 and x_5.

$$
\begin{array}{r|cccc}
 & & 2 & -M & \\
\hline
 & & x_1 & x_5 & \\
-1 & x_2 & -1/2-(-1/2)(-3/2)(2/3) & 1/3 & 1/2-(-1/2)(3/2)(2/3) \\
-1 & x_3 & -1 & 2/3 & 1 \\
\hline
 & B & -3/2-(-3/2)(3/2)(2/3) & -1 & -1/2-(3/2)(3/2)(2/3) \\
 & M & 3/2-(-3/2)(-3/2)(2/3) & 1 & -3/2-(-3/2)(3/2)(2/3)
\end{array}
\tag{5.13.28}
$$

$$
= \begin{array}{r|cccc}
 & & 2 & -M & \\
\hline
 & & x_1 & x_5 & \\
-1 & x_2 & -1 & 1/3 & 1 \\
-1 & x_3 & -1 & 2/3 & 1 \\
\hline
 & B & 0 & -1 & -2 \\
 & M & 0 & 1 & 0
\end{array}
\tag{5.13.29}
$$

which yields a solution of $x_1 = 0$, $x_2 = 1$, $x_3 = 1$. with $B = -2$.

For the minimization problem we choose $+M$.

		2	-1	-1	
		x_1	x_2	x_3	
M	x_4	-1	2	-1	1
M	x_5	-1	-1	2	1
	B	-2	1	1	0
	M	-2	1	1	2

(5.13.30)

We can then pivot off of either x_2 or x_3. These are entirely equivalent and so we'll choose x_2 We then must pivot the row off of x_4. This means we find

		2	M	-1	
		x_1	x_4	x_3	
-1	x_2	$-1/2$	$1/2$	$-1/2$	$1/2$
M	x_5	$-1-(-1)(-1)/2$	$1/2$	$2-(-1)(-1)/2$	$1-(-1)(1)/2$
	B	$-2-(-1)(1)/2$	$-1/2$	$1-(-1)(1)/2$	$0-(1)(1)/2$
	M	$-2-(-1)(-1)/2$	$-1/2$	$1-(-1)(-1)/2$	$2-(1)(-1)/2$

(5.13.31)

$=$

		2	M	-1	
		x_1	x_4	x_3	
-1	x_2	$-1/2$	$1/2$	$-1/2$	$1/2$
M	x_5	$-3/2$	$1/2$	$3/2$	$3/2$
	B	$-3/2$	$-1/2$	$3/2$	$-1/2$
	M	$-5/2$	$-1/2$	$1/2$	$5/2$

(5.13.32)

We can simply eliminate the x_4 column for conveninence now, as it is an extra variable we do not care about.

		2	-1	
		x_1	x_3	
-1	x_2	$-1/2$	$-1/2$	$1/2$
M	x_5	$-3/2$	$3/2$	$3/2$
	B	$-3/2$	$3/2$	$-1/2$
	M	$-5/2$	$1/2$	$5/2$

(5.13.33)

We can then pivot off of x_3 and x_5.

		2	$-M$	
		x_1	x_5	
-1	x_2	$-1/2-(-1/2)(-3/2)(2/3)$	$1/3$	$1/2-(-1/2)(3/2)(2/3)$
-1	x_3	-1	$2/3$	1
	B	$-3/2-(-3/2)(3/2)(2/3)$	-1	$-1/2-(3/2)(3/2)(2/3)$
	M	$-5/2-(-3/2)(-3/2)(2/3)$	$-1/3$	$5/2-(-3/2)(3/2)(2/3)$

(5.13.34)

=

		2	$-M$	
		x_1	x_5	
-1	x_2	-1	$1/3$	1
-1	x_3	-1	$2/3$	1
	B	0	-1	-2
	M	-4	$-1/3$	4

(5.13.35)

which means we find the same solution for the minimization as well!

5.13.5 Problem 5

Solve the following problem by the Simplex method

$$-0.5x + 1.3y \leq 0.8 \tag{5.13.36}$$
$$4x + y \leq 10.7 \tag{5.13.37}$$
$$6x + y \leq 15.4 \tag{5.13.38}$$
$$6x - y \leq 13.4 \tag{5.13.39}$$
$$4x - y \leq 8.7 \tag{5.13.40}$$
$$5x - 3y \leq 10.0 \tag{5.13.41}$$

Maximize $11x + 10y$ for $x, y \geq 0$.

Solution:

We introduce a bunch of slack variables and so the tableau becomes

		11	10	
		x	y	
0	x_1	-0.5	1.3	0.8
0	x_2	4	1	10.7
0	x_3	6	1	15.4
0	x_4	6	-1	13.4
0	x_5	4	-1	8.7
0	x_6	$5*$	-3	10.0
	B	-11	-10	0

(5.13.42)

Clearly we pivot off of column x (I have asterisked the pivot). The best ratio is then 10/5=2 on row x_6. Thus

		0	10	
		x_6	y	
0	x_1	$0.5/5$	$1.3-(-0.5)(-3)/5$	$0.8-(-0.5)(10)/5$
0	x_2	$-4/5$	$1-(4)(-3)/5$	$10.7-(4)(10)/5$
0	x_3	$-6/5$	$1-(6)(-3)/5$	$15.4-(6)(10)/5$
0	x_4	$-6/5$	$-1-(6)(-3)/5$	$13.4-(6)(10)/5$
0	x_5	$-4/5$	$-1-(4)(-3)/5$	$8.7-(4)(10)/5$
11	x	$1/5$	$-3/5$	2
	B	$11/5$	$-10-(-11)(-3)/5$	$0-(-11)(10)/5$

(5.13.43)

=

		0	10	
		x_6	y	
0	x_1	1/10	1	1.8
0	x_2	−4/5	3.4	2.7
0	x_3	−6/5	4.6	3.4
0	x_4	−6/5	2.6	1.4
0	x_5	−4/5	1.4∗	0.7
11	x	1/5	−3/5	2
	B	11/5	−16.6	−22

(5.13.44)

We then clearly should choose the y pivot. The optimal row is x_5 with ratio 0.5. Thus

		0	0	
		x_6	x_5	
0	x_1	1/10 − (1)(−4/5)(10/14)	−10/14	1.8 − (1)(0.7)(10/14)
0	x_2	−4/5 − (3.4)(−4/5)(10/14)	−34/14	2.7 − (3.4)(0.7)(10/14)
0	x_3	−6/5 − (4.6)(−4/5)(10/14)	−46/14	3.4 − (4.6)(0.7)(10/14)
0	x_4	−6/5 − (2.6)(−4/5)(10/14)	−26/14	1.4 − (2.6)(0.7)(10/14)
10	y	−8/14	10/14	7/14
11	x	1/5 − (−3/5)(−4/5)(10/14)	6/14	2 − (−3/5)(0.7)(10/14)
	B	11/5 − (−16.6)(−4/5)/(10/14)	166/14	22 − (−16.6)(0.7)(10/14)

(5.13.45)

=

		0	0	
		x_6	x_5	
0	x_1	33/70	−10/14	1.3
0	x_2	8/7	−34/14	1
0	x_3	10/7	−46/14	1.1
0	x_4	2/7∗	−26/14	0.1
10	y	−1/2	10/14	7/14
11	x	−1/7	6/14	2.3
	B	−51/7	166/14	30.3

(5.13.46)

We can then pivot on x_6. Row x_4 provides the smallest ratio. We thus find

		0	0	
		x_4	x_5	
0	x_1	−33/20	−10/14 − (33/70)(−26/14)(7/2)	1.3 − (33/70)(0.1)(7/2)
0	x_2	−8/2	−34/14 − (8/7)(−26/14)(7/2)	1 − (8/7)(0.1)(7/2)
0	x_3	−10/2	−46/14 − (10/7)(−26/14)(7/2)	1.1 − (10/7)(0.1)(7/2)
0	x_6	7/2	−26/4	7/20
10	y	1/2	10/14 − (−1/2)(−26/14)(7/2)	1/2 − (−1/2)(0.1)(7/2)
11	x	1/2	6/14 − (−1/7)(−26/14)(7/2)	2.3 − (−1/7)(0.1)(7/2)
	B	51/2	166/14 − (−51/7)(−26/14)(7/2)	30.3 − (−51/7)(0.1)(7/2)

(5.13.47)

This process continues. The book gives the final solution, but it requires a couple of more tableaus. It turns out that had we chosen a pivot column initially on y the problem solves much more quickly.

In that case, the best ratio is from row x_1 at $8/13 = 0.61$.

		11	10	
		x	y	
0	x_1	-0.5	$1.3*$	0.8
0	x_2	4	1	10.7
0	x_3	6	1	15.4
0	x_4	6	-1	13.4
0	x_5	4	-1	8.7
0	x_6	5	-3	10.0
	B	-11	-10	0

(5.13.48)

=

		11	0	
		x	x_1	
10	y	$-5/13$	$10/13$	$8/13$
0	x_2	$4-(-0.5)(1)(10/13)$	$-10/13$	$10.7-(0.8)(1)(10/13)$
0	x_3	$6-(-0.5)(1)(10/13)$	$-10/13$	$15.4-(0.8)(1)(10/13)$
0	x_4	$6-(-0.5)(-1)(10/13)$	$10/13$	$13.4-(0.8)(-1)(10/13)$
0	x_5	$4-(-0.5)(-1)(10/13)$	$10/13$	$8.7-(0.8)(-1)(10/13)$
0	x_6	$5-(-0.5)(-3)(10/13)$	$30/13$	$10.0-(0.8)(-3)(10/13)$
	B	$-11(-0.5)(-10)(10/13)$	$100/13$	$0-(0.8)(-10)(10/13)$

(5.13.49)

=

		11	0	
		x	x_1	
10	y	$-5/13$	$10/13$	$8/13$
0	x_2	$57/13*$	$-10/13$	$1311/130$
0	x_3	$83/13$	$-10/13$	$961/65$
0	x_4	$73/13$	$10/13$	$911/65$
0	x_5	$47/13$	$10/13$	$1211/130$
0	x_6	$50/13$	$30/13$	$154/13$
	B	$-193/13$	$100/13$	$80/13$

(5.13.50)

(5.13.51)

We then pivot on column x choosing row x_2 with smallest ratio of 2.3. Then

=

		11	0	
		x	x_1	
10	y	$5/57$	$10/13-(-5/13)(-10/13)(13/57)$	$8/13-(-5/13)(1311/130)(13/57)$
0	x_2	$13/57*$	$-10/57$	$23/10$
0	x_3	$-83/57$	$-10/13-(83/13)(-10/13)(13/57)$	$961/65-(83/13)(1311/130)(13/57)$
0	x_4	$-73/57$	$10/13-(73/13)(-10/13)(13/57)$	$911/65-(73/13)(1311/130)(13/57)$
0	x_5	$-47/57$	$10/13-(47/13)(-10/13)(13/57)$	$1211/130-(47/13)(1311/130)(13/57)$
0	x_6	$-50/57$	$30/13-(50/13)(-10/13)(13/57)$	$154/13-(50/13)(1311/130)(13/57)$
	B	$193/57$	$100/13-(-193/13)(-10/13)(13/57)$	$80/13-(-193/13/13)(1311/130)(13/57)$

(5.13.52)

$$= \begin{array}{c|c|cccc} & & 11 & 0 & \\ \hline & & x & x_1 & \\ 10 & y & 5/57 & 40/57 & 3/2 \\ 0 & x_2 & 13/57 & -10/57 & 23/10 \\ 0 & x_3 & -83/57 & 20/57 & 1/10 \\ 0 & x_4 & -73/57 & 100/57 & 11/10 \\ 0 & x_5 & -47/57 & 80/57 & 1 \\ 0 & x_6 & -50/57 & 170/57 & 3 \\ \hline & B & 193/57 & 290/57 & 403/10 \end{array} \tag{5.13.53}$$

which is the correct solution.

This is much better by hand, but would not matter much when we program a computer.

5.13.6 Problem 6

Solve the following problem.

$$2x_1 + x_2 + x_3 = 10 \tag{5.13.54}$$
$$-44x_1 - 42x_2 + x_4 - x_8 = -183 \tag{5.13.55}$$
$$36x_1 - 102x_2 + x_5 - x_8 = 17 \tag{5.13.56}$$
$$-164x_1 + 298x_2 + x_6 - x_8 = 1517 \tag{5.13.57}$$
$$-12x_1 - 6x_2 + x_7 - x_8 = -79 \tag{5.13.58}$$
$$x_i \geq 0 \qquad \text{for } i = 1,2,3,4,5,6,7 \tag{5.13.59}$$

with x_8 unrestricted in sign, minimizing x_8.

Solution:

We can rewrite $x_8 = y_1 - y_2$ with $y_1, y_2 \geq 0$ minimizing $y_1 - y_2$. We can then also introduce new variables $z_i \geq 0$ for the five equations such that they gives us the correct answer. Thus the system becomes

$$2x_1 + x_2 + x_3 + z_1 = 10 \tag{5.13.60}$$
$$-44x_1 - 42x_2 + x_4 - y_1 + y_2 - z_2 = -183 \tag{5.13.61}$$
$$36x_1 - 102x_2 + x_5 - y_1 + y_2 + z_3 = 17 \tag{5.13.62}$$
$$-164x_1 + 298x_2 + x_6 - y_1 + y_2 + z_4 = 1517 \tag{5.13.63}$$
$$-12x_1 - 6x_2 + x_7 - y_1 + y_2 - z_5 = -79 \tag{5.13.64}$$

Then the new problem tableau is then

$$\begin{array}{c|c|cccccccccc} & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \\ \hline & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & y_1 & y_2 & \\ 0 & z_1 & 2 & 1 & 1* & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & z_2 & 44 & 42 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 183 \\ 0 & z_3 & 36 & -102 & 0 & 0 & 1* & 0 & 0 & -1 & 1 & 17 \\ 0 & z_4 & -164 & 298 & 0 & 0 & 0 & 1* & 0 & -1 & 1 & 1517 \\ 0 & z_5 & 12 & 6 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 79 \\ \hline & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{array} \tag{5.13.65}$$

However, to do the method correctly, we'd need to change B with the M, and so put an M onto all the z_i. This then requires us to calculate z_i in terms of the x_i and find

		0	0	0	0	0	0	0	−1	1	
		x_1	x_2	x_3	x_4	x_5	x_6	x_7	y_1	y_2	
M	z_1	2	1	1∗	0	0	0	0	0	0	10
M	z_2	44	42	0	−1	0	0	0	1	−1	183
M	z_3	36	−102	0	0	1∗	0	0	−1	1	17
M	z_4	−164	298	0	0	0	1∗	0	−1	1	1517
M	z_5	12	6	0	0	0	0	−1	1	−1	79
	B	0	0	0	0	0	0	0	−1	1	0
	M	−70	245	1	−1	1	1	−1	0	0	1806

(5.13.66)

I'd like to eliminate some of the variables quickly, and so we'll swap the starred. Only the M line will be modified. We'll start with x_3 and z_1 first

		0	0	M	0	0	0	0	−1	1	
		x_1	x_2	z_1	x_4	x_5	x_6	x_7	y_1	y_2	
M	x_3	2	1	1∗	0	0	0	0	0	0	10
M	z_2	44	42	0	−1	0	0	0	1	−1	183
M	z_3	36	−102	0	0	1	0	0	−1	1	17
M	z_4	−164	298	0	0	0	1	0	−1	1	1517
M	z_5	12	6	0	0	0	0	−1	1	−1	79
	B	0	0	0	0	0	0	0	−1	1	0
	M	−70 − (2)(1)	245 − (1)(1)	−1	−1	1	1	−1	0	0	1806 − (183)(1)

(5.13.67)

=

		0	0	M	0	0	0	0	−1	1	
		x_1	x_2	z_1	x_4	x_5	x_6	x_7	y_1	y_2	
M	x_3	2	1	1∗	0	0	0	0	0	0	10
0	z_2	44	42	0	−1	0	0	0	1	−1	183
M	z_3	36	−102	0	0	1	0	0	−1	1	17
M	z_4	−164	298	0	0	0	1	0	−1	1	1517
M	z_5	12	6	0	0	0	0	−1	1	−1	79
	B	0	0	0	0	0	0	0	−1	1	0
	M	−72	244	−1	−1	1	1	−1	0	0	1796

(5.13.68)

We can then drop z_1 and swap x_5 and z_3

		0	0	0	M	0	0	-1	1	
		x_1	x_2	x_4	z_3	x_6	x_7	y_1	y_2	
0	x_3	2	1	0	0	0	0	0	0	10
M	z_2	44	42	-1	0	0	0	1	-1	183
0	x_5	36	-102	0	$1*$	0	0	-1	1	17
M	z_4	-164	298	0	0	1	0	-1	1	1517
M	z_5	12	6	0	0	0	-1	1	-1	79
	B	0	0	0	0	0	0	-1	1	0
	M	$-72-(36)(1)$	$244-(-102)(1)$	-1	-1	1	-1	$0-(1)(-1)$	$0-(1)(1)$	$1796-(17)(1)$

(5.13.69)

=

		0	0	0	M	0	0	-1	1	
		x_1	x_2	x_4	z_3	x_6	x_7	y_1	y_2	
0	x_3	2	1	0	0	0	0	0	0	10
M	z_2	44	42	-1	0	0	0	1	-1	183
0	x_5	36	-102	0	$1*$	0	0	-1	1	17
M	z_4	-164	298	0	0	1	0	-1	1	1517
M	z_5	12	6	0	0	0	-1	1	-1	79
	B	0	0	0	0	0	0	-1	1	0
	M	-108	346	-1	-1	1	-1	1	-1	1779

(5.13.70)

We can then eliminate the z_3 column and swap x_6 and z_4

		0	0	0	M	0	-1	1	
		x_1	x_2	x_4	z_4	x_7	y_1	y_2	
0	x_3	2	1	0	0	0	0	0	10
M	z_2	44	42	-1	0	0	1	-1	183
0	x_5	36	-102	0	0	0	-1	1	17
0	x_6	-164	298	0	$1*$	0	-1	1	1517
M	z_5	12	6	0	0	-1	1	-1	79
	B	0	0	0	0	0	-1	1	0
	M	$-108-(-164)(1)$	$346-(298)(1)$	-1	-1	-1	$1-(-1)(1)$	$-1-(1)(1)$	$1779-1517(1)$

(5.13.71)

=

		0	0	0	0	-1	1	
		x_1	x_2	x_4	x_7	y_1	y_2	
0	x_3	2	1	0	0	0	0	10
M	z_2	44	42	-1	0	1	-1	183
0	x_5	$36*$	-102	0	0	-1	1	17
0	x_6	-164	298	0	0	-1	1	1517
M	z_5	12	6	0	-1	1	-1	79
	B	0	0	0	0	-1	1	0
	M	56	48	-1	-1	2	-2	262

(5.13.72)

There are no other easy pivots, so let's start with column x_1 and so pivot on row x_5. This gives

		0	0	0	0	−1	1	
		x_5	x_2	x_4	x_7	y_1	y_2	
0	x_3	−1/18	20/3	0	0	1/18	−1/18	163/18
M	z_2	−11/9	500/3∗	−1	0	20/9	−20/9	1460/9
0	x_1	1/36	−17/6	0	0	−1/36	1/36	17/36
0	x_6	41/9	−500/3	0	0	−50/9	50/9	14350/9
M	z_5	−1/3	40	0	−1	4/3	−4/3	220/3
	B	0	0	0	0	−1	1	0
	M	−14/9	620/3	−1	−1	32/9	−32/9	2120/9

(5.13.73)

Clearly the next pivot can be on x_2 which gives z_2 as the corresponding row.

		0	M	0	0	−1	1	
		x_5	z_2	x_4	x_7	y_1	y_2	
0	x_3	−1/150	−1/25	1/25	0	−1/30	1/30	77/30
0	x_2	−11/1500	3/500∗	−3/500	0	1/75	−1/75	73/75
0	x_1	7/1000	17/1000	−17/1000	0	1/100	−1/100	323/100
0	x_6	10/3	1	−1	0	−10/3	10/3	5270/3
M	z_5	−1/25	6/25	6/25	−1	4/5	−4/5	172/5
	B	0	0	0	0	−1	1	0
	M	−1/25	−31/25	6/25	−1	4/5	−4/5	172/5

(5.13.74)

We can then eliminate z_2 and we have

		0	0	0	−1	1	
		x_5	x_4	x_7	y_1	y_2	
0	x_3	−1/150	1/25	0	−1/30	1/30	77/30
0	x_2	−11/1500	−3/500	0	1/75	−1/75	73/75
0	x_1	7/1000	−17/1000	0	1/100	−1/100	323/100
0	x_6	10/3	−1	0	−10/3	10/3	5270/3
M	z_5	−1/25	6/25	−1	4/5∗	−4/5	172/5
	B	0	0	0	−1	1	0
	M	−1/25	6/25	−1	4/5	−4/5	172/5

(5.13.75)

Next let's choose column y_1 so that we can eliminate z_5.

		0	0	0	M	1	
		x_5	x_4	x_7	z_5	y_2	
0	x_3	−1/120	1/20	−1/24	1/24	0	4
0	x_2	−1/150	−1/100	1/60	−1/60	0	2/5
0	x_1	3/400	−1/50	1/80	−1/80	0	14/5
0	x_6	19/6	0	−25/6	25/6	0	1900
−1	y_1	−1/20	3/10	−5/4	5/4∗	−1	43
	B	−1/20	3/10	−5/4	5/4	0	43
	M	0	0	0	−1	0	0

(5.13.76)

We can then elminiate the z_5 column.

		0	0	0	1	
		x_5	x_4	x_7	y_2	
0	x_3	$-1/120$	$1/20*$	$-1/24$	0	4
0	x_2	$-1/150$	$-1/100$	$-1/60$	0	$2/5$
0	x_1	$3/400$	$-1/50$	$-1/80$	0	$14/5$
0	x_6	$19/6$	0	$-25/6$	0	1900
-1	y_1	$-1/20$	$3/10$	$-5/4$	-1	43
	B	$-1/20$	$3/10$	$-5/4$	0	43
	M	0	0	0	0	0

(5.13.77)

There is no need to keep the M line now.

Let's choose column x_4 and so row x_3.

		0	0	0	1	
		x_5	x_3	x_7	y_2	
0	x_4	$-1/6$	20	$-5/6$	0	80
0	x_2	$-1/120$	$1/5$	$1/120$	0	$6/5$
0	x_1	$1/240$	$2/5$	$-1/240$	0	$22/5$
0	x_6	$19/6$	0	$-25/6$	0	1900
-1	y_1	0	-6	-1	-1	19
	B	0	-6	-1	0	19

(5.13.78)

The hardest part is getting all of the arithmetic right. The book's method has fewer pivot operations and hence is better when doing these by hand, as they are annoying.

My solution is $x_1 = 22/5 = 4.4$, $x_2 = 6/5 = 1.2$, $x_3 = 0$, $x_4 = 80$, $x_5 = 0$, $x_6 = 1900$, $x_7 = 0$, and $x_8 = 19$. Compare the book's solution of $x_1 = 19/10 = 1.9$, $x_2 = 31/5 = 6.2$, $x_3 = 0$, $x_4 = 180$, $x_5 = 600$, $x_6 = x_7 = 0$ and $x_8 = 19$.

5.13.7 Problem 7

Solve the following problem by the Two-phase method

$$\text{Maximize } 0.98n + 0.06x_1 + 0.15x_2 + 0.3x_3 \tag{5.13.79}$$

where n is a constant, subject to

$$x_1 \le n, \qquad x_2 - n_2 \le 0, \qquad x_3 - n_3 \le 0 \tag{5.13.80}$$

$$n_2 + 0.3x_1 = 0.6n, \qquad n_3 + 0.18x_1 + 0.3x_2 = 0.36n \tag{5.13.81}$$

where n_2 and n_3 are unknowns.

Solution:

Our new constraints are interesting mostly because we require $x_i \le n_i$ for each i. This means we have a further restriction beyond being positive.

Suppose we simply add our constraint equations after putting in some slack variables.

$$x_1 - n + z_1 = 0 \tag{5.13.82}$$
$$x_2 - n_2 + z_2 = 0 \tag{5.13.83}$$
$$x_3 - n_3 + z_3 = 0 \tag{5.13.84}$$
$$n_2 + 0.3x_1 - 0.6n + y_1 = 0 \tag{5.13.85}$$
$$n_3 + 0.18x_1 + 0.3x_2 - 0.36n + y_2 = 0 \tag{5.13.86}$$

Note that if we now add these together we get a constraint saying

$$1.48x_1 + 1.3x_2 + x_3 - 1.96n + z_1 + z_2 + z_3 + y_1 + y_2 = 0 \tag{5.13.87}$$

Note that we can view the first phase as ignoring the first three constraints for our objective function. Thus, the new objective function is

$$C = n_2 + n_3 = -0.48x_1 - 0.3x_2 + 0.96n - y_1 - y_2 \tag{5.13.88}$$

Thus, we can create a tableau

		0	0	0	0	0	0	
		x_1	x_2	x_3	n_2	n_3	n	
0	z_1	$1*$	0	0	0	0	-1	0
0	z_2	0	1	0	-1	0	0	0
0	z_3	0	0	1	0	-1	0	0
1	y_1	0.3	0	0	1	0	-0.6	0
1	y_2	0.18	0.3	0	0	1	-0.36	0
	B	-0.06	-0.15	-0.3	0	0	-0.98	0
	C	0.48	0.3	0	1	1	-0.96	0

(5.13.89)

We want to minimize C and so we want to make all the C values negative. Let's just start with the column x_1. Note that since we have all zeros in the far right row, we should choose a different column for checking values. Let's do the n column for consistency. Then row z_1 is the correct choice. We get

		0	0	0	0	0	0	
		z_1	x_2	x_3	n_2	n_3	n	
0	x_1	1	0	0	0	0	-1	0
0	z_2	0	$1*$	0	-1	0	0	0
0	z_3	0	0	1	0	-1	0	0
1	y_1	-0.3	0	0	1	0	-0.3	0
1	y_2	-0.18	0.3	0	0	1	-0.18	0
	B	0.06	-0.15	-0.3	0	0	-1.04	0
	C	-0.48	0.3	0	1	1	-0.48	0

(5.13.90)

We see the next choice forced on us is x_2 and z_2 yielding

		0	0	0	0	0	0	
		z_1	z_2	x_3	n_2	n_3	n	
0	x_1	1	0	0	0	0	−1	0
0	x_2	0	1	0	−1	0	0	0
0	z_3	0	0	1	0	−1	0	0
1	y_1	−0.3	0	0	1∗	0	−0.3	0
1	y_2	−0.18	−0.3	0	0.3	1	−0.18	0
	B	0.06	0.15	−0.3	−0.15	0	−1.04	0
	C	−0.48	−0.3	0	1.3	1.0	−0.48	0

(5.13.91)

So we now choose column n_2 which means we choose row y_1 (the lowest non-negative ratio) and we find

		0	0	0	1	0	0	
		z_1	z_2	x_3	y_1	n_3	n	
0	x_1	1	0	0	0	0	−1	0
0	x_2	−0.3	1	0	1	0	−0.3	0
0	z_3	0	0	1	0	−1	0	0
0	n_2	−0.3	0	0	1	0	−0.3	0
1	y_2	−0.09	−0.3	0	−0.3	1.0∗	−0.09	0
	B	0.015	0.15	−0.3	0.15	0	−1.085	0
	C	−0.09	−0.3	0	−1.3	1	−0.09	0

(5.13.92)

We are then forced to column n_3 and row y_2.

		0	0	0	1	1	0	
		z_1	z_2	x_3	y_1	y_2	n	
0	x_1	1	0	0	0	0	−1	0
0	x_2	−0.3	1	0	1	0	−0.3	0
0	z_3	−0.09	−0.3	1	−0.3	1	−0.09	0
0	n_2	−0.3	0	0	1	0	−0.3	0
0	n_3	−0.09	−0.3	0	−0.3	1	−0.09	0
	B	0.015	0.15	−0.3	0.15	0	−1.085	0
	C	0	0	0	−1	−1	0	0

(5.13.93)

This means we have minimized C. Then we can enter the second phase. We eliminate the y_i variables. Now we maximize B so try to get positive coefficients on the bottom row. We can start with the x_3 column enforcing a z_3 row as the only viable pivot.

		0	0	0.3	0.98	
		z_1	z_2	x_3	n	
0.06	x_1	1	0	0	−1	0
0.15	x_2	−0.3	1	0	−0.3	0
0	z_3	−0.09	−0.3	1∗	−0.09	0
0	n_2	−0.3	0	0	−0.3	0
0	n_3	−0.09	−0.3	0	−0.09	0
	B	0.015	0.15	−0.3	−1.085	0

(5.13.94)

yielding

		0	0	0	0.98	
		z_1	z_2	z_3	n	
0.06	x_1	$1*$	0	0	-1	0
0.15	x_2	-0.3	1	0	-0.3	0
0.3	x_3	-0.09	-0.3	1	-0.09	0
0	n_2	-0.3	0	0	-0.3	0
0	n_3	-0.09	-0.3	0	-0.09	0
	B	-0.012	0.06	0.3	-1.112	0

(5.13.95)

I'll choose the z_1 column since it is the only one that is possible with row x_1 giving us

		0	0	0	0.98	
		x_1	z_2	z_3	n	
0.06	z_1	$1*$	0	0	-1	0
0.15	x_2	0.3	1	0	-0.6	0
0.3	x_3	0.09	-0.3	1	-0.18	0
0	n_2	0.3	0	0	-0.6	0
0	n_3	0.09	-0.3	0	-0.18	0
	B	0.012	0.06	0.3	-1.124	0

(5.13.96)

This is the maximum possible. Thus our answer is $x_1 = 0$, $x_2 = 0.6n$, $x_3 = 0.18n$, $z_1 = n$, $z_2 = z_3 = 0$, $n_2 = 0.6n$, $n_3 = 0.18n$.

We of course don't actually care about z_1 since that only gives us the inequality. The maximum is thus $1.124n$.

5.13.8 Problem 8

Solve the following problem

$$x_1 - x_2 + x_3 - x_4 = 2 \tag{5.13.97}$$

$$2x_1 - 2x_2 - x_2 + x_4 = 1 \tag{5.13.98}$$

$$4x_1 - 4x_2 + x_3 - x_4 = 5 \tag{5.13.99}$$

minimizing $x_1 + x_2 + x_3 + x_4$ (one of the equations is redundant), $x_i \geq 0$.

Solution:

We put in extra variables z_1, z_2 and z_3 with the solution $x_i = 0$ $z_1 = 2$, $z_2 = 1$, and $z_3 = 5$. With minimization we then need to add an objective M function to it.

We create the tableau

		1	1	1	1	
		x_1	x_2	x_3	x_4	
M	z_1	1	-1	1	-1	2
M	z_2	$2*$	-2	-1	1	1
M	z_3	4	-4	1	-1	5
	B	-1	-1	-1	-1	0
	M	7	-7	1	-1	8

(5.13.100)

We can choose x_1 and z_2 which yields

		M	1	1	1	
		z_2	x_2	x_3	x_4	
M	z_1	-0.5	0	1.5	-1.5	1.5
1	x_1	0.5	-1	-0.5	0.5	0.5
M	z_3	-2	0	$3*$	-3	3
	B	0.5	-2	-1.5	-0.5	0.5
	M	-3.5	0	4.5	-4.5	4.5

(5.13.101)

We use the x_3 column and we could choose either row z_3 or z_1. I'll choose z_3.

		M	1	M	1	
		z_2	x_2	z_3	x_4	
M	z_1	0.5	0	-0.5	0	0
1	x_1	$1/6$	-1	0	$1/6$	1
1	x_3	$-2/3$	0	$1/3$	-1	1
	B	-0.5	-2	0.5	-2	2
	M	-0.5	0	-1.5	0	0

(5.13.102)

We can then see that we get a solution here of $x_2 = x_4 = 0$ and $x_1 = 1$, $x_3 = 1$, as a possibility.

Now let's try the other possibility of a pivot on z_1 giving

		M	1	M	1	
		z_2	x_2	z_1	x_4	
1	x_3	-1	0	-2	0	0
1	x_1	$1/3$	-1	$1/3$	0	1
M	z_3	$-1/3$	0	$2/3$	-1	1
	B	0	-2	1	-2	2
	M	-2	0	-3	0	0

(5.13.103)

This means we require $x_1 = 1$, $z_3 = 1$, but this is not a valid solution.

5.13.9 Problem 9

Which constraints, and which objective function, are implied in the following tableau? Find the minimum of the objective function.

		4	22	-1	-1	-1	-1	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_7	2	1	0	0	0	0	10
0	x_8	16	-12	-1	0	0	2	44
0	x_9	-12	34	0	-1	0	1	42
0	x_{10}	0	0	0	0	-1	1	0
	B	4	22	-1	-1	-1	4	86

(5.13.104)

Solution:

The constraints are

$$2x_1 + x_2 \le 10 \tag{5.13.105}$$
$$16x_1 - 12x_2 - x_3 + 2x_6 \le 44 \tag{5.13.106}$$
$$-12x_1 + 34x_2 - x_4 + x_6 \le 42 \tag{5.13.107}$$
$$-x_5 + x_6 \le 0 \tag{5.13.108}$$

or

$$2x_1 + x_2 + x_7 = 10 \tag{5.13.109}$$
$$16x_1 - 12x_2 - x_3 + 2x_6 + x_8 = 44 \tag{5.13.110}$$
$$-12x_1 + 34x_2 - x_4 + x_6 + x_9 = 42 \tag{5.13.111}$$
$$-x_5 + x_6 + x_{10} = 0 \tag{5.13.112}$$

with objective function

$$B = -4x_1 - 22x_2 + x_3 + x_4 + x_5 - 4x_6 + 86 \tag{5.13.113}$$

To minimize we choose the x_2 column which means we need the x_9 row.

(5.13.114)

		4	22	−1	−1	−1	−1	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_7	2	1	0	0	0	0	10
0	x_8	16	−12	−1	0	0	2	44
0	x_9	−12	34*	0	−1	0	1	42
0	x_{10}	0	0	0	0	−1	1	0
	B	4	22	−1	−1	−1	4	86

yielding

(5.13.115)

		4	0	−1	−1	−1	−1	
		x_1	x_9	x_3	x_4	x_5	x_6	
0	x_7	40/17*	−1/34	0	1/34	0	−1/34	149/17
0	x_8	200/17	6/17	−1	−6/17	0	40/17	1000/17
22	x_2	−6/17	1/34	0	−1/34	0	1/34	21/17
0	x_{10}	0	0	0	0	−1	1	0
	B	200/17	−11/17	−1	−6/17	−1	57/17	1000/17

By convention, we choose column x_1 which implies row x_7 with the favorable ratio.

(5.13.116)

		0	0	−1	−1	−1	−1	
		x_7	x_9	x_3	x_4	x_5	x_6	
4	x_1	17/40	−1/80	0	1/80	0	−1/80	149/40
0	x_8	−5	1/2	−1	−1/2	0	5/2	15
22	x_2	3/20	1/40	0	−1/40	0	1/40	51/20
0	x_{10}	0	0	0	0	−1	1*	0
	B	−5	−1/2	−1	−1/2	−1	7/2	15

and finally we go off of column x_6 and row x_{10} leaving us

$$\begin{array}{c|cccccccc} & & 0 & 0 & -1 & -1 & -1 & 0 & \\ \hline & & x_7 & x_9 & x_3 & x_4 & x_5 & x_{10} & \\ 4 & x_1 & 17/40 & -1/80 & 0 & 1/80 & -1/80 & 1/80 & 149/40 \\ 0 & x_8 & -5 & 1/2 & -1 & -1/2 & 5/2* & -5/2 & 15 \\ 22 & x_2 & 3/20 & 1/40 & 0 & -1/40 & 1/40 & -1/40 & 51/20 \\ -1 & x_6 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ \hline & B & -5 & -1/2 & -1 & -1/2 & 5/2 & -7/2 & 15 \end{array} \tag{5.13.117}$$

and we are forced onto column x_5 and row x_8

$$\begin{array}{c|cccccccc} & & 0 & 0 & -1 & -1 & 0 & 0 & \\ \hline & & x_7 & x_9 & x_3 & x_4 & x_8 & x_{10} & \\ 4 & x_1 & 2/5 & -1/100 & -1/200 & 1/100 & 1/200 & 0 & 19/5 \\ -1 & x_5 & -2 & 1/5 & -2/5 & -1/5 & 2/5 & -1 & 6 \\ 22 & x_2 & 1/5 & 1/50 & 1/100 & -1/50 & -1/100 & 0 & 12/5 \\ -1 & x_6 & -2 & 1/5 & -2/5 & -1/5 & 2/5 & 0 & 6 \\ \hline & B & 0 & -1 & 0 & 0 & -1 & -1 & 0 \end{array} \tag{5.13.118}$$

This means the solution is $x_1 = 19/5 = 3.8$, $x_2 = 12/5 = 2.4$, $x_3 = 0$, $x_4 = 0$, $x_5 = 6$, $x_6 = 6$, and all other $x_i = 0$. The Basis function is clearly minimized at 0.

Looking at the original equations we find

$$2\frac{19}{5} + \frac{12}{5} = 10 \tag{5.13.119}$$

$$16\frac{19}{5} - 12\frac{12}{5} - 0 + 2\frac{6}{1} = 44 \tag{5.13.120}$$

$$-12\frac{19}{5} + 34\frac{12}{5} - 0 + 6 = 42 \tag{5.13.121}$$

$$-6 + 6 = 0 \tag{5.13.122}$$

with objective function

$$-4\frac{19}{5} - 22\frac{12}{5} + 0 + 0 + 6 - 4(6) + 86 = 0 \tag{5.13.123}$$

5.13.10 Problem 10

Find all basic optimal solutions of

$$2x_1 + x_2 + z_1 = 6 \tag{5.13.124}$$

$$4x_1 + 2x_2 + z_2 = 12 \tag{5.13.125}$$

Minimize $z_1 + z_2$.

Solution:

Phew, a simple problem that isn't large in size. The tableau is

		0	0	
		x_1	x_2	
1	z_1	2*	1	6
1	z_2	4	2	12
	B	6	3	18

(5.13.126)

which is a simple solution. There is a degeneracy here. We first choose x_1 z_1 and find

		1	0	
		z_1	x_2	
0	x_1	1/2	1/2	3
1	z_2	−2	0	0
	B	−3	0	0

(5.13.127)

And so $x_1 = 3$, $x_2 = 0$, $z_1 = z_2 = 0$. I'll write this $[3, 0, 0, 0]$.

We can pivot on x_1 and x_2 to get

		1	0	
		z_1	x_1	
0	x_2	1	2	6
1	z_2	−2	0	0
	B	−3	0	0

(5.13.128)

for $[0, 6, 0, 0]$. These must be all of the optimal solutions since any other changes in z_i would lead to an increase.

So two solutions $[3, 0, 0, 0]$ and $[0, 6, 0, 0]$.

5.13.11 Problem 11

Solve the following problem (a) by the M-method and (b) by the Dual Simplex method.

$$y_3 + y_4 - 2y_5 - y_1 = 1 \tag{5.13.129}$$

$$y_3 - 2y_4 + y_5 - y_2 = -1 \tag{5.13.130}$$

Minimize $3y_3 + y_4 + 2y_5$.

Solution:

(a) Let's introduce z_1 by adding z_1 to the first equation left hand side. Then we penalize the z_1 with an M constraint line (there's only one so we only need it from the first equation)

		3	1	2	0	
		y_3	y_4	y_5	y_1	
M	z_1	1*	1	−2	−1	1
0	y_2	1	−2	1	0	1
	B	−3	−1	−2	0	0
	M	1	1	−2	−1	1

(5.13.131)

Let's pivot off of z_1 and y_3 giving

		M	1	2	0	
		z_1	y_4	y_5	y_1	
3	y_3	1	$1*$	-2	-1	1
0	y_2	-1	-3	3	1	0
	B	3	2	-8	-3	3
	M	-1	0	0	0	0

(5.13.132)

We can then drop the M equation line and ignore pivoting off of z_1 as we want it to be zero. Then we pivot off of row y_3 and column y_4

		M	3	2	0	
		z_1	y_3	y_5	y_1	
1	y_4	1	1	-2	-1	1
0	y_2	2	3	-3	-2	3
	B	1	-2	-4	-1	1
	M	-1	0	0	0	0

(5.13.133)

which is the solution we desired. So $y_1 = 0$, $y_2 = 3$, $y_3 = 0$, $y_4 = 1$ and $y_5 = 0$. We can test it with

$$0 + 1 - 2(0) - 0 = 1 \tag{5.13.134}$$

$$0 - 2(1) + 0 - (3) = -5 \tag{5.13.135}$$

and so doesn't work.

It appears the degeneracy just breaks the problem as given. Instead, we can take equation 1 minus equation 2 to get

$$3y_4 - 3y_5 - y_1 + y_2 = 2 \tag{5.13.136}$$

to replace the second equation giving us

		3	1	2	0	
		y_3	y_4	y_5	y_1	
M	z_1	$1*$	1	-2	-1	1
0	y_2	0	3	-3	-1	2
	B	-3	-1	-2	0	0
	M	1	1	-2	-1	1

(5.13.137)

pivoting on z_1 and y_3 gives

		M	1	2	0	
		z_1	y_4	y_5	y_1	
3	y_3	1	1	-2	-1	1
0	y_2	0	$3*$	-3	-1	2
	B	3	2	-8	-3	3
	M	-1	0	0	0	0

(5.13.138)

We can then ignore the M row and the z_1 column to pivot on row y_2 and column y_4.

		M	0	2	0	
		z_1	y_2	y_5	y_1	
3	y_3	1	$-1/3$	-1	$-2/3$	$1/3$
1	y_4	0	$1/3$	-1	$-1/3$	$2/3$
	B	3	$-2/3$	-6	$-7/3$	$5/3$
	M	-1	0	0	0	0

(5.13.139)

yielding $y_3 = 1/3$ and $y_4 = 2/3$ as the correct values which do check out.

(b)

The dual method is not very well explained. Essentially, we let the values be the negatives in the coefficient table. We then choose a pivot column based on the smallest absolute value (considering only negative values) along the column. We do this for each basic variable that is negative (thus $|-1/-1|$ is smaller than $|-3/-1|$ and we choose the y_4 column.

		3	1	2	
		y_3	y_4	y_5	
0	y_1	-1	$-1*$	2	-1
0	y_2	-1	2	-1	1
	B	-3	-1	-2	0

(5.13.140)

So we pivot on y_1 and y_4 (one could do y_3 but it doesn't really matter) giving

		3	0	2	
		y_3	y_1	y_5	
1	y_4	1	-1	-2	1
0	y_2	$-3*$	2	3	-1
	B	-2	-1	-4	1

(5.13.141)

Now we choose y_2 and y_3 and find

		3	0	2	
		y_2	y_1	y_5	
1	y_4	1	$-1/3$	-1	$2/3$
3	y_3	$-1/3$	$-2/3$	-1	$1/3$
	B	$-2/3$	$-7/3$	-6	$5/3$

(5.13.142)

Yielding $y_3 = 1/3$ and $y_4 = 2/3$ with $B = 5/3$ as the correct solution again.

5.13.12 Problem 12

(a) Find all optimal basic feasible solutions of

$$\begin{array}{rrrrrl} 12x_1+ & 7x_2+ & 14x_3+ & 5x_4+ & 16x_5 \leq & 1 \\ 7x_1+ & 14x_2+ & 5x_3+ & 16x_4+ & 3x_5 \leq & 1 \\ 14x_1+ & 5x_2+ & 16x_3+ & 3x_4+ & 18x_5 \leq & 1 \\ 5x_1+ & 16x_2+ & 3x_3+ & 18x_4+ & x_5 \leq & 1 \\ 16x_1+ & 3x_2+ & 18x_3+ & 1x_4+ & 20x_5 \leq & 1 \end{array} \tag{5.13.143}$$

Maximize $x_1 + x_2 + x_3 + x_4 + x_5$ using the Inverse Matrix Method.

(b) Discuss the solutions to the dual problem.

(c) Discuss the problem with initial matrices M

$$\begin{bmatrix} 1 & 12 & 7 & 14 & 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 14 & 5 & 16 & 0 & 1 & 0 & 0 & 0 \\ 1 & 14 & 5 & 16 & 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & 5 & 16 & 3 & 18 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.13.144}$$

$$\begin{bmatrix} 1 & 12 & 7 & 14 & 1 & 0 & 0 & 0 \\ 1 & 7 & 14 & 5 & 0 & 1 & 0 & 0 \\ 1 & 14 & 5 & 16 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.13.145}$$

$$\begin{bmatrix} 1 & 12 & 7 & 1 & 0 & 0 \\ 1 & 7 & 14 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{5.13.146}$$

Solution:

(a) No way I am going to do this step by step for such a large problem. It is the same as the tableau method but keeping track of basis changes explicitly, which may be useful for accuracy sometimes, but is hardly worth it by hand. It's just matrix algebra all over the place that doesn't teach me anything new.

(b)

The matrix is symmetric and so the dual solution must be the same as this solution.

(c)

There aren't any problems, per se, they just remove variables successively from the problem above. So we can find optimal solutions by using the basic optimal solutions for the larger problem where those variables are actually zero.

Chapter 6

Special Algorithms

Faster than simplex algorithm for some specific types of problems.

6.1 Transportation Problem

This explains some solution methods for the Transportation problem, where we have sources and destinations and wish to transport from source to destination most efficiently (shortest/cheapest route).

When we get to the "northwest" corner tabulation method, the algorithm as described makes no sense since it never explains what to do with the subtracted quantity. Essentially you go along from the northwest corner putting in the smallest value you can so that as you go right or down you are less than or equal to the values on the topmost column or leftmost row. Basically, we start by exhausting a column or row (making the sum equal to the column or row) and then head in the other direction until we can no longer do so.

So the example they give

$$\begin{array}{c|cccc} & 1 & 6 & 2 & 6 \\ \hline 5 & & & & \\ 5 & & & & \\ 5 & & & & \end{array} \tag{6.1.1}$$

we see that we can exhaust the first 1 column by putting a 1 there. Then we can exhaust the row along the first 5 by putting a 4 to the right of the one giving

$$\begin{array}{c|cccc} & 1 & 6 & 2 & 6 \\ \hline 5 & 1 & 4 & & \\ 5 & & & & \\ 5 & & & & \end{array} \to \begin{array}{c|cccc} & 1 & 6 & 2 & 6 \\ \hline 5 & 1 & 4 & & \\ 5 & & 2 & & \\ 5 & & & & \end{array} \to \begin{array}{c|cccc} & 1 & 6 & 2 & 6 \\ \hline 5 & 1 & 4 & & \\ 5 & & 2 & 3? & \\ 5 & & & & \end{array} \tag{6.1.2}$$

Note that 3 is too large to exhaus the 52 row, and so we must use a 2 instead

$$\begin{array}{c|cccc} & 1 & 6 & 2 & 6 \\ \hline 5 & 1 & 4 & & \\ 5 & & 2 & 2 & 1 \\ 5 & & & & \end{array} \to \begin{array}{c|cccc} & 1 & 6 & 2 & 6 \\ \hline 5 & 1 & 4 & & \\ 5 & & 2 & 2 & 1 \\ 5 & & & & 5 \end{array} \tag{6.1.3}$$

which ends the algorithm. The book then discusses what to do if you end early. Basically, just go southeast one and start over. Add zeros in some of the rows and columns if you need more basic variables.

Note that we then can read off the values of the basic variables via

$$\begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & x_{11} & x_{12} & x_{13} & x_{14} \\ a_2 & x_{21} & x_{22} & x_{23} & x_{24} \\ a_3 & x_{23} & x_{32} & x_{33} & x_{34} \end{array} = \begin{array}{c|cccc} & 1 & 6 & 2 & 6 \\ \hline 5 & 1 & 4 & 0 & 0 \\ 5 & 0 & 2 & 2 & 1 \\ 5 & 0 & 0 & 0 & 5 \end{array} \tag{6.1.4}$$

The rest of the chapter is dealing with how to get to an optimal solution using the dual problem and other methods. It explains the sense in which

$$\sum_j x_{ij} = a_i \qquad \sum_i x_{ij} = b_j, x_{ij} \geq 0 \tag{6.1.5}$$

$$\text{Minimize } \sum_{i,j} x_{ij} c_{ij} \tag{6.1.6}$$

is dual to

$$u_i + v_j \leq c_{ij} \tag{6.1.7}$$

$$\text{Maximize } \sum_i a_i u_i + \sum_j b_j v_j \tag{6.1.8}$$

where the u_i correspond to the $i$the equation from $\sum_j x_{ij} = a_i$ and v_j corresponds to the jth equation from $\sum_i x_{ij} = b_j$. The book notes that u_i and v_j don't actually have a sign restriction because our original set is equations and not inequalities.

6.2 Transshipment

The book illogically decides to write the matrix in what seems to be a purposefully asymmetric manner. It's much easier to read as

	D_1	D_2	D_3	D_4	S_1	S_2	S_3
D_1	0	1	3	2	5	9	9
D_2	3	0	2	3	4	6	7
D_3	2	3	0	1	3	4	9
D_4	4	1	2	0	4	7	3
S_1	5	4	3	2	0	2	1
S_2	10	8	4	7	1	0	4
S_3	9	9	8	4	3	2	0

(6.2.1)

which makes the failures of symmetry in shipping much more apparent.

Then if we play the northwest corner game with $L = 100$ on the diagonal and the following coefficients, we find

	101	106	102	106	100	100	100
100	100						
100		100					
100			100				
100				100			
105					100		
105						100	
105							100

(6.2.2)

Then the 105 row is a good northewest corner.

	101	106	102	106	100	100	100
100	100						
100		100					
100			100				
100				100			
105	1	4			100		
105		2	2	1		100	
105				5			100

(6.2.3)

Using the shadow costs of u_i and v_j we could then find a better solution as given in the book by

	101	106	102	106	100	100	100
100	100						
100		100					
100			100				
100				100			
105	1	3		1	100		
105		3	2			100	
105				5			100

(6.2.4)

Now, we use the shadow costs again and when $u_i + v_j = c_{ij}$ and there is a zero in the entries above, we can transfer to it while improving our objective function. This then leads to the solution proposed in the book.

A proof that is ok for proving optimality follows the solution.

6.3 Transportation With Capacity Restrictions

Discussion on how to incorporate restrictions on the traffic. Restricts possible solutions and may cause there to be no solution, but usually changed into a transshipment problem.

6.4 Network Flow Method

A not remarkably clear presentation of the Ford-Fulkerson method. I recommend online resources which make the method much easier to understand. The basic idea is to look for the best pipelines to route through rather than go through all possible pipelines when doing the extremizing.

6.5 The Assignment Problem

This is basically a simpler version of the network flow method, but we use the primal dual (so essentially working with the dual problem) because of possible degeneracies in the problem.

A not remarkably clear presentation of the Ford-Fulkerson method. I recommend online resources which make the method much easier to understand. The basic idea is to look for the best pipelines to route through rather than go through all possible pipelines when doing the extremizing.

6.6 The Assignment Problem. The Hungarian Method

An alternative way of solving the problem.

6.7 The Bottleneck Assignment Problem

Just what the book describes

6.8 Multiple Distribution

The book doesn't explain the notation for its cube and so the example is more of a puzzle than an aid. The book really needs someone to go through and say, "explain a bit more here what this is showing".

6.9 Exercises

I'm not sure I'll do them, because the book has such awful explanations, and so probably not worth the effort since it basically becomes learning how to actually do the problems by experimenting for hours and looking at other sources.

Chapter 7

Uses of Duality. Economic Interpretation

Good explanation of why duals are often useful constructs to solve problems.

7.1 Shadow Prices

The shadow price is simply the coefficients on the objective function using the non-basic z variables (multiplied by the corresponding original right-hand sides). It tells us how much we should value increasing the amount we buy in those non-basic variables.

Thus in the problem

$$3x_1 + 4x_2 + x_3 \leq 20 \tag{7.1.1}$$
$$x_1 + 3x_2 + 2x_3 \leq 10 \tag{7.1.2}$$
$$x_1, x_2, x_3 \geq 0 \tag{7.1.3}$$

and maximize $B = 3x_1 + 6x_2 + 2x_3$ (we can add slack variables to make equalities) we get a final tableau of

		0	0	2	
		x_4	x_5	x_3	2
3	x_1	0.6	-0.8	-1	4
6	x_2	-0.2	0.6	1	2
	B	0.6	1.2	1	24

(7.1.4)

So the shadow prices are 0.6 for x_1 and 1.2 for x_2. Note that $20(0.6) + 1.2(10) = 12 + 12 = 24$ the same as $3(4) + 6(2) = 12$, as it should.

Then if we increased the amount available (keeping the basic variables the same) this will lead to a corresponding increase in profit (or loss, or whatever you are calculating). So in our case, if we originally had a constraint of 21 instead of 20, we know that the profit would increase by 0.6 to 24.6. This is why the bottom z entries are called marginal or shadow prices. Sometimes they are called opportunity costs or multipliers. Note that this argument really only works if the basic variables remain the same, but for very small perturbations, this is true enough and so they are used more generically as marginal/shadow prices.

The duality relations let us find the dual y values easily under such changes.

7.2 Efficient Points

This uses the tableau and is essentially just looking at the problem such that we keep an objective function (to be maximized) at zero. Thus, we need to think about what variables can be changed such that the objective function remains zero.

7.3 Input-Output Analysis

Pretty straightforward.

7.4 Exercises

7.4.1 Problem 1

How does the minimum of $x_1 - x_2$ change when the right-hand side of the second inequality in Exercise 5-1 is changed (a) to 2, (b) to 4?

Solution:

Exercise 5-1 is given by

$$x_1 + x_2 \leq 3 \tag{7.4.1}$$
$$x_1 - 2x_2 \leq 1 \tag{7.4.2}$$
$$-2x_1 + x_2 \leq 2 \tag{7.4.3}$$
$$x_i \geq 0 \tag{7.4.4}$$

(a) Minimize $x_1 - x_2$. (b) Maximize $x_1 - x_2$.

With final tableau for 5-1(a) given by

$$\begin{array}{c|cccc}
 & & 0 & 0 & \\
\hline
 & & x_3 & x_5 & \\
1 & x_1 & 1/3 & -1/3 & 1/3 \\
0 & x_4 & 1 & 1 & 6 \\
-1 & x_2 & 2/3 & 1/3 & 8/3 \\
\hline
 & B & -1/3 & -2/3 & -7/3 \\
 & Ck & 2/3 & -2/3 & 16/3
\end{array} \tag{7.4.5}$$

Of course, we don't know if by changing the second row (a priori) whether it will cause different basic variables to arise. Thus, we must redo the tableau for each problem.

(a) We replace 1 with 2 and so get

		-1	1	
		x_1	x_2	
0	x_3	1	1	3
0	x_4	1	-2	2
0	x_5	-2	1*	2
	B	-1	1	0

(7.4.6)

yielding

		-1	0	
		x_1	x_5	
0	x_3	3*	-1	1
0	x_4	-3	2	6
1	x_2	-2	1	2
	B	1	-1	-2

(7.4.7)

and

		0	0	
		x_3	x_5	
-1	x_1	1/3	-1/3	1/3
0	x_4	1	1	7
1	x_2	2/3	1/3	8/3
	B	-1/3	-2/3	-7/3

(7.4.8)

Thus, no change in the price.

(b) We start with

		-1	1	
		x_1	x_2	
0	x_3	1	1	3
0	x_4	1	-2	2
0	x_5	-2	1*	2
	B	-1	1	0

(7.4.9)

to get

		-1	0	
		x_1	x_5	
0	x_3	3*	-1	1
0	x_4	-3	2	8
1	x_2	-2	1	2
	B	1	-1	-2

(7.4.10)

		0	0	
		x_3	x_5	
-1	x_1	1/3	-1/3	1/3
0	x_4	1	1	9
1	x_2	2/3	1/3	8/3
	B	-1/3	-2/3	-7/3

(7.4.11)

which also makes no difference to the basis and doesn't change the cost function.

The book lies, and says they do change, because they consider what happens to the maximum of $x_1 - x_2$, not the minimum, which is unaffected... You'd think the book would realize the difference between minimum and maximum. This book is so awful at times.

If we do the maximum, then we can see that it might make a difference because the pivot operations actually are affected by things in the second row.

The original final tableau yields

		0	0	
		x_4	x_3	
-1	x_2	$-1/3$	$1/3$	$2/3$
1	x_1	$1/3$	$2/3$	$7/3$
0	x_5	1	1	6
	B	$2/3$	$1/3$	$5/3$

(7.4.12)

so that if no basis change happened, we'd expect an increase by 2/3 to 7/3.

We can easily see this

		-1	1	
		x_1	x_2	
0	x_3	1	1	3
0	x_4	$1*$	-2	2
0	x_5	-2	1	2
	B	-1	1	0

(7.4.13)

and so

		0	1	
		x_4	x_2	
0	x_3	-1	$3*$	1
-1	x_1	1	-2	2
0	x_5	2	-3	6
	B	1	-1	2

(7.4.14)

and

		0	0	
		x_4	x_3	
1	x_2	$-1/3$	$1/3$	$1/3$
-1	x_1	$1/3$	$2/3$	$8/3$
0	x_5	1	1	7
	B	$2/3$	$1/3$	$7/3$

(7.4.15)

For the other case, we can already see we will have to use a different basis.

		-1	1	
		x_1	x_2	
0	x_3	$1*$	1	3
0	x_4	1	-2	4
0	x_5	-2	1	2
	B	-1	1	0

(7.4.16)

		0	1	
		x_3	x_2	
-1	x_1	1	1	3
0	x_4	-1	-3	1
0	x_5	2	3	8
	B	1	2	3

(7.4.17)

And we end, so that now the new maximum is just 3.

7.4.2 Problem 2

Consider the numerical example at the end of Chapter 1. How much should we be prepared to pay for another ten yards of (a) red wool, (b) green wool?

Solution:

This problem was

$$3x_1 + 4x_2 + x_3 \leq 20 \tag{7.4.18}$$

$$x_1 + 3x_2 + 2x_3 \leq 10 \tag{7.4.19}$$

$$x_i \geq 0 \tag{7.4.20}$$

Maximize $3x_1 + 6x_2 + 2x_3$.

We have to check for each case if the final tableau uses different basic variables. Thus we start with (a).

The tableau is

		3	6	2	
		x_1	x_2	x_3	
0	x_4	3	4	1	30
0	x_5	1	$3*$	2	10
	B	-3	-6	-2	0

(7.4.21)

We choose column row x_5 and column x_2.

		3	0	2	
		x_1	x_5	x_3	
0	x_4	$5/3*$	$-4/3$	$-5/3$	$50/3$
6	x_2	$1/3$	$1/3$	$2/3$	$10/3$
	B	-1	2	2	20

(7.4.22)

This leaves pivoting on row x_4 and column x_1 (though there is now a degeneracy) giving

		0	0	2	
		x_4	x_5	x_3	
3	x_1	$3/5$	$-4/5$	-1	10
6	x_2	$-1/5$	$3/5$	1	0
	B	$3/5$	$6/5$	1	30

(7.4.23)

This is simply the shadow price outwards, of course. That is, if we did ten more yards we increase the profit by $3/5(10) = 6$.

If we increase the number of green yards available we then have

		3	6	2	
		x_1	x_2	x_3	
0	x_4	3	$4*$	1	20
0	x_5	1	3	2	20
	B	-3	-6	-2	0

(7.4.24)

giving

		3	0	2	
		x_1	x_4	x_3	
6	x_2	$3/4$	$1/4$	$1/4$	5
0	x_5	$-5/4$	$-3/4$	$5/4*$	5
	B	$3/2$	$3/2$	$-1/2$	30

(7.4.25)

and

		3	0	2	
		x_1	x_4	x_3	
6	x_2	1	$2/5$	$-1/5$	4
0	x_5	-1	$-3/5$	$4/5$	4
	B	1	$6/5$	$2/5$	32

(7.4.26)

and so the we now have an increase to 32.

Note that we could have constructed a non-optimal tableau from the original final tableau, which involves just changing the final column values and then iterated from there.

7.4.3 Problem 3

A unit of product A, B, or C sells at 5, 3, or 4 respectively. The following table shows how much of raw materials a or b is required for one unit of A, B, or C, and also how much of a and b is altogether available.

	A	B	C	Available
a	3	2	3	12
b	4	1	2	15

(7.4.27)

Impute costs to the raw materials such that the total cost of all material equals the total price obtained for the finished commodities, nd such that no commodity contains material of less imputed cost than the selling price of that commodity.

Solution:

Let x_i correspond to A, B, and C via $i = 1, 2, 3$. Then we'd like to maximize $5x_1 + 3x_2 + 4x_3$. Thus

		5	3	2	
		x_1	x_2	x_3	
0	x_4	3	2	3	12
0	x_5	$4*$	1	2	15
	B	-5	-3	-2	0

(7.4.28)

pivoting on row x_5 and column x_1 yields

		0	3	2	
		x_5	x_2	x_3	
0	x_4	$-3/4$	$5/4*$	$3/2$	$3/4$
5	x_1	$1/4$	$1/4$	$1/2$	$15/4$
	B	$5/4$	$-7/4$	$1/2$	$75/4$

(7.4.29)

Then the next pivot is row x_4 and column x_2 giving

		0	0	2	
		x_5	x_4	x_3	
3	x_2	$-3/5$	$4/5$	$6/5$	$3/5$
5	x_1	$2/5$	$-1/5$	$1/5$	$18/5$
	B	$1/5$	$7/5$	$13/5$	$99/5$

(7.4.30)

Then we must have 0.2 for the cost of a and 1.4 for the cost of b. The thing to note is that we were asked to maximize the profit in a very ugly way. (Maximizing the profit is the same as minimizing the costs, and so we could have minimized the dual problem.)

Chapter 8

Selected Applications

No comment. They are worked out examples.

Chapter 9

Parametric Linear Programming

This is simply the same methods and ideas, but using variables in the objective function. There is nothing really new about this idea.

9.1 Exercises

9.1.1 Problem 1

Find the various basic sets and the values of the objective function corresponding to the values of the parameter t in the following problem.

$$x_1 + x_2 + x_3 = 3 \tag{9.1.1}$$
$$x_1 - 2x_2 + x_4 = 1 \tag{9.1.2}$$
$$-2x_1 + x_2 + x_5 = 2 \tag{9.1.3}$$

Maximize $(1 + t)x_1 - (1 - t)x_2$.

Solution:

We start with the tableau

		$(1+t)$	$(t-1)$	
		x_1	x_2	
0	x_3	1	1	3
0	x_4	$1*$	-2	1
0	x_5	-2	1	2
	B	$-1-t$	$1-t$	0

(9.1.4)

If $t \leq -1$, then this is an optimal tableau as the B row is completely non-negative. For $t > -1$, then we clearly have the first column x_1 as the most negative, and so we pivot on row x_4 to get

		0	$(t-1)$		
		x_4	x_2		
0	x_3	-1	$3*$	2	
$1+t$	x_1	1	-2	1	(9.1.5)
0	x_5	2	-3	4	
	B	$1+t$	$-1-3t$	$1+t$	

Clearly in this case if $-1 < t < -1/3$ we have an optimal tableau, but if $t > -1/3$ then the second column should be pivoted upon with the only possible row x_3.

		0	0		
		x_4	x_3		
$t-1$	x_2	$-1/3$	$1/3$	$2/3$	
$1+t$	x_1	$1/3$	$2/3$	$7/3$	(9.1.6)
0	x_5	1	1	6	
	B	$2/3$	$1/3+t$	$5/3+3t$	

To summarize

t	B	i,j,k	x_i	x_j	x_k
$t \le -1$	0	3,4,5	3	1	2
$-1 < t \le -\frac{1}{3}$	$1+t$	3,1,5	2	1	3
$t \ge -\frac{1}{3}$	$\frac{5}{3}+3t$	2,1,5	$\frac{2}{3}$	$\frac{7}{3}$	6

Table 9.1: This shows the basis (i,j,k) for x_i, x_j, x_k, the values of the x's, the basis function B, for various parameter values t.

9.1.2 Problem 2

Find the various basic sets and the values of the objective function corresponding to the values of the parameters u and v in the following problem.

$$x_1 + x_2 \le 3 \tag{9.1.7}$$

$$2x_1 - x_2 \le 2 \tag{9.1.8}$$

$$x_1, x_2 \ge 0 \tag{9.1.9}$$

Maximize $(2+2u+v)x_1 + (1+u-v)x_2$.

Solution:

We set up the tableau

		$2+2u+v$	$1+u-v$		
		x_1	x_2		
0	x_3	1	1	3	(9.1.10)
0	x_4	$2*$	-1	2	
	B	$-2-2u-v$	$v-1-u$	0	

With two parameters, we have a much larger number of cases to consider. We see that if $2u+v < -2$ with $v > 1+u$ then this is an optimal solution. In any other case we pivot on row x_4 and column x_1 to

		0	$1+u-v$	
		x_4	x_2	
0	x_3	$-1/2$	$3/2*$	2
$2+2u+v$	x_1	$1/2$	$-1/2$	1
	B	$1+u+v/2$	$-2u+v/2-2$	$2+2u+v$

(9.1.11)

This is clearly good for $2+2u+v \geq 0$ and $-2u+v/2-2 \geq 0$. In the case that the second column is negative we find

		0	0	
		x_4	x_3	
$1+u-v$	x_2	$-1/3$	$2/3$	$4/3$
$2+2u+v$	x_1	$1/3*$	$1/3$	$5/3$
	B	$(u+2v+1)/3$	$(4u+4-v)/3$	$(14u+v+14)/3$

(9.1.12)

Which is good for when the bottom leftmost two B are greater than or equal to zero.

It is not obvious, but the next one to consider is the first column. If we collected our conditions, then it's essentially because that is the only region in uv space that we have not actually covered.

		$2+2u+v$	0	
		x_1	x_3	
$1+u-v$	x_2	1	1	3
0	x_4	3	1	5
	B	$2v-u-1$	$1+u-v$	$3+3u-3v$

(9.1.13)

Because the book is not very good, they never actually explain how you would figure this out. Plotting would be the most accessible way, but such a thing would never work in more complicated cases. In those cases, you would have to actually have to go through all sorts of cases. It would make more sense to consider things one parameter at a time, rather than two parameters at a time.

9.1.3 Problem 3

Solve the following problem for all values of t.

$$3x_1 + x_4 + 2x_5 = 12 \tag{9.1.14}$$
$$3x_2 - x_4 + x_5 + y = 3 \tag{9.1.15}$$
$$x_3 + x_4 + x_5 = 9 \tag{9.1.16}$$

Minimize $x_2 - x_1 + ty$.

Solution:

The initial solution is easy with x_1 and x_2 and x_3 appearing in each equation only once. We need $x_2 - x_1$ in terms of x_3, x_4, x_5 and y for the objective function. So

$$x_2 - x_1 + ty = \frac{3 - x_5 + x_4 - y}{3} - \frac{12 - 2x_5 - x_4}{3} + \frac{3ty}{3} \tag{9.1.17}$$

$$= \frac{-9 + x_5 + (3t-1)y + 2x_4}{3} = \frac{2x_4}{3} + \frac{x_5}{3} + \left(t - \frac{1}{3}\right) y - 3 \tag{9.1.18}$$

Thus our new objective function is the above and we write (divide the coefficients by 3 in the first two equations so that there is no trickiness about the solutions)

		0	0	0	
		x_4	x_5	y	
-1	x_1	$1/3$	$2/3$	0	4
1	x_2	$-1/3$	$1/3$	$1/3*$	1
0	x_3	1	1	0	9
	B	$-2/3$	$-1/3$	$(1/3 - t)$	-3

(9.1.19)

Clearly this is a minimum if $t \geq 1/3$. Otherwise we pivot on column y and row x_2

		0	0	0	
		x_4	x_5	y	
-1	x_1	$1/3$	$2/3$	0	4
1	x_2	-1	1	3	3
0	x_3	$1*$	1	0	9
	B	$-t - 1/3$	$t - 2/3$	$3t - 1$	$3t - 4$

(9.1.20)

For $t < 1/3$ we see that this is a minimum only if $t > -1/3$ (then the x_4 column no longer is negative). Thus we pivot on row x_3 and column x_4.

		0	0	0	
		x_3	x_5	y	
-1	x_1	$-1/3$	$1/3$	0	1
1	x_2	1	2	3	12
0	x_4	1	1	0	9
	B	$t + 1/3$	$2t - 1/3$	$3t - 1$	$12t - 1$

(9.1.21)

In this case, we see that we have a minimum for $t < -1/3$ which exhausts all possible cases.

9.1.4 Problem 4

Solve the following problem for all values of t.

$$-2x_1 + 2x_2 + x_3 \leq 12 \tag{9.1.22}$$

$$3x_1 - 18x_2 - 4x_3 \leq 24 \tag{9.1.23}$$

$$x_1 + 2x_2 + 4x_3 \leq 24 \tag{9.1.24}$$

Minimize $C = (-1 + t)x_1 + (2 + t)x_2 + (2 + t)x_3$.

Solution:

We can introduce slack variables to find some of the solutions. This means that we need to solve the objective function in terms of these new variables.

The tableau is then

		$-1+t$	$2+t$	$2+t$	
		x_1	x_2	x_3	
0	x_4	-2	2	1	12
0	x_5	$3*$	-18	-4	24
0	x_6	1	2	4	24
	B	$1-t$	$-2-t$	$-2-t$	0

(9.1.25)

This will be an optimal tableau for $t \geq 1$. Otherwise the first column will be positive and so we pivot off of x_5 and x_1.

		0	$2+t$	$2+t$	
		x_5	x_2	x_3	
0	x_4	$2/3$	-10	$-5/3$	28
$-1+t$	x_1	$1/3$	-6	$-4/3$	8
0	x_6	$-1/3$	$8*$	$16/3$	16
	B	$t/3-1/3$	$4-7t$	$(-7t-2)/3$	$8t-8$

(9.1.26)

For $t < 1$ the first column is clearly negative. Then $t = 4/7$ or $t = -2/7$ is the next change point We see that for $1 > t \geq 4/7$ that all the columns are negative, and so row x_6 and column x_2 is the next pivot.

		0	0	$2+t$	
		x_5	x_6	x_3	
0	x_4	$1/4$	$5/4$	5	48
$-1+t$	x_1	$1/12$	$3/4$	$8/3$	20
$2+t$	x_2	$-1/24$	$1/8$	$2/3*$	2
	B	$t/24-1/6$	$7t/8-1/2$	$(7t-10)/3$	$22t-16$

(9.1.27)

Which is going to be negative for $t < 4/7$ on every column. Thus we have found all of our possible solutions.

Chapter 10

Discrete Linear Programming

This just means integer linear programming and mixed integer linear programming.

10.1 Travelling Salesman

This does not give a solution to the problem, just discusses it and how it is difficult to find integer solutions.

10.2 Allocation Problem

This is given a radius for a radio tower, and a bunch of towers, how can frequencies be distributed.

10.3 Logical Relations

This is simply the idea that we can write logical relationships using binary encodings with integers/real numbers.

10.4 Fixed Charge Problem

This simply says that if we minimize a problem without considering an on switch for extra costs, then we get the minimum even when considering the on switch. This only works if the on switch turns at the same level for every variable. The proof in the book is pretty clear.

10.5 Discrete Linear Programming Algorithms

This is a clever idea. It uses "cuts" where we consider the fractional parts of a solution and add a constraint based on that to the original problem to get us to integer solutions. The only real big problem is that we might be forced to continually keep making cuts.

10.6 Discrete Linear Programming Algorithms. Mixed Case

This shows a method to use the Simplex method when we want only some values to be integer. It is more involved than the "cut" algorithm, but allows a mix of integers and real numbers in the solution for the best objective function.

10.7 Exercises

10.7.1 Problem 1

Derive the constraints which make the region of Figure 10-6 feasible.

Solution:

We need $x_1 \leq 2$ and $x_2 \leq 2$ with $x_1 \leq 1$ or $x_2 \leq 1$ or both $x_1, x_2 \leq 1$ as our feasible region constraints.

The book explains one could rewrite this as a logical relation (using the binary encoded integers). Then

$$1 - d \leq x_1 \tag{10.7.1}$$
$$2 - d \leq x_2 \tag{10.7.2}$$

with $d = 0$ or $d = 1$ as our sets of constraints.

10.7.2 Problem 2

Consider Example 10-3 and (a) starting by considering the variable x_5' (b) consider x_1.

Solution:

Example 10-3 is given

$$-x_1 + 10x_2 \leq 40 \tag{10.7.3}$$
$$x_1 + x_2 \leq 20 \tag{10.7.4}$$

Maximize $-10x_1 + 111x_2 = B$.

The tableau goes to

(10.7.5)

		0	0	
		x_3	x_4	
-10	x_1	$-1/11$	$10/11$	$160/11$
111	x_2	$1/11$	$1/11$	$60/11$
	B	11	1	460

The variable x_5' comes from looking at row x_1. We see that $-1 + 10/11 = -1/11$ and $160/11 = 14 + 6/11$ so that the row to add is $[x_5, -10/11, -10/11, -6/11]$ to the tableau for

		0	0	
		x_3	x_4	
-10	x_1	$-1/11$	$10/11$	$160/11$
111	x_2	$1/11$	$1/11$	$60/11$
0	x_5	$-10/11$	$-10/11*$	$-6/11$
	B	11	1	460

(10.7.6)

We can use the Dual simplex algorithm. We choose between $1/(10/11) = 11/10$ and $11/(10/11) = 121/10$. Clearly column x_4 has the smaller value and so we choose that.

		0	0	
		x_3	x_5	
-10	x_1	-1	1	14
111	x_2	0	$1/10$	$27/5$
0	x_4	1	$-11/10$	$3/5$
	B	10	$11/10$	$2297/5$

(10.7.7)

We are now forced to consider adding another column to eliminate the fractions. We can consider one for x_2 or for x_4. Since x_2 was our original one, we'll do that one first. Clearly the new row is $[x_6, 0, -1/10, -2/5]$ for

		0	0	
		x_3	x_5	
-10	x_1	-1	1	14
111	x_2	0	$1/10$	$27/5$
0	x_4	1	$-11/10$	$3/5$
0	x_6	0	$-1/10*$	$-2/5$
	B	10	$11/10$	$2297/5$

(10.7.8)

and so we get

		0	0	
		x_3	x_6	
-10	x_1	-1	10	10
111	x_2	0	1	5
0	x_4	1	-11	5
0	x_5	0	-10	4
	B	10	11	455

(10.7.9)

which if we allow x_5 as a variable gives us a final solution.

Note that had we chosen to eliminate x_4 as a fraction, we'd add the row $[x_6, 0, -(20-11)/10, -3/5] =$

$[x_6, 0, -9/10, -3/5]$.

$$\begin{array}{r|cccc} & & 0 & 0 & \\ \hline & & x_3 & x_5 & \\ -10 & x_1 & -1 & 1 & 14 \\ 111 & x_2 & 0 & 1/10 & 27/5 \\ 0 & x_4 & 1 & -11/10 & 3/5 \\ 0 & x_6 & 0 & -9/10* & -3/5 \\ \hline & B & 10 & 11/10 & 2297/5 \end{array} \tag{10.7.10}$$

to

$$\begin{array}{r|cccc} & & 0 & 0 & \\ \hline & & x_3 & x_6 & \\ -10 & x_1 & -1 & 10/9 & 40/3 \\ 111 & x_2 & 0 & 1/9 & 16/3 \\ 0 & x_4 & 1 & -11/9 & 4/3 \\ 0 & x_5 & 0 & -10/9 & 2/3 \\ \hline & B & 10 & 11/9 & 1376/3 \end{array} \tag{10.7.11}$$

We then need to add another line. Since we've already done x_1 and x_4, then x_2 is the only one left. So our new line is $[x_7, 0, -1/9, -1/3]$ and we get

$$\begin{array}{r|cccc} & & 0 & 0 & \\ \hline & & x_3 & x_6 & \\ -10 & x_1 & -1 & 10/9 & 40/3 \\ 111 & x_2 & 0 & 1/9 & 16/3 \\ 0 & x_4 & 1 & -11/9 & 4/3 \\ 0 & x_5 & 0 & -10/9 & 2/3 \\ 0 & x_7 & 0 & -1/9* & -1/3 \\ \hline & B & 10 & 11/9 & 1376/3 \end{array} \tag{10.7.12}$$

$$= \begin{array}{r|cccc} & & 0 & 0 & \\ \hline & & x_3 & x_6 & \\ -10 & x_1 & -1 & 10 & 10 \\ 111 & x_2 & 0 & 1 & 5 \\ 0 & x_4 & 1 & -11 & 5 \\ 0 & x_5 & 0 & -10 & 4 \\ 0 & x_7 & 0 & -9 & 3 \\ \hline & B & 10 & 11 & 455 \end{array} \tag{10.7.13}$$

which is another possible solution.

(b) We start by considering x_1. But this is just the same as considering x_5 and so we do the exact same things.

10.7.3 Problem 3

Consider those values of u and v in Exercise 9-2 for which

$$u + 2v \geq -1 \Rightarrow u + 2v + 1 \geq 0 \tag{10.7.14}$$

$$-4u + v \leq 4 \Rightarrow 4u + 4 - v \geq 0 \tag{10.7.15}$$

subject to the further condition that all variables have integer values.

We need to look back at the tableau given by

$$\begin{array}{c|cccc} & & 0 & 0 & \\ \hline & & x_4 & x_3 & \\ 1+u-v & x_2 & -1/3 & 2/3 & 4/3 \\ 2+2u+v & x_1 & 1/3* & 1/3 & 5/3 \\ \hline & B & (u+2v+1)/3 & (4u+4-v)/3 & (14u+v+14)/3 \end{array} \tag{10.7.16}$$

which satisfies the given constraints. Now if we want integers, we can consider changing x_1. Then we need to add a new row. Unfortunately, we don't know which column is larger, so we will just consider each case. First $(u+2v+1) > 4u+4-v$. Then

$$\begin{array}{c|cccc} & & 0 & 0 & \\ \hline & & x_4 & x_3 & \\ 1+u-v & x_2 & -1/3 & 2/3 & 4/3 \\ 2+2u+v & x_1 & 1/3 & 1/3 & 5/3 \\ 0 & x_5 & -1/3 & -1/3* & -2/3 \\ \hline & B & (u+2v+1)/3 & (4u+4-v)/3 & (14u+v+14)/3 \end{array} \tag{10.7.17}$$

$$= \begin{array}{c|cccc} & & 0 & 0 & \\ \hline & & x_4 & x_5 & \\ 1+u-v & x_2 & -1 & 2 & 0 \\ 2+2u+v & x_1 & 0 & 1 & 1 \\ 0 & x_3 & 1 & -3 & 2 \\ \hline & B & -u+v-1 & 4u-v+4 & 2u+v+2 \end{array} \tag{10.7.18}$$

Because we are using the dual simplex method, we don't have to worry about the bottom row being positive (though we can certainly check if we desire). $u+2v+1 > 4u+4-v \Rightarrow -3u+3v-3 > 0 \Rightarrow v-u-1 > 0$ and so the first column is fine. In any case, we should now consider the other case where $4u+4-v > u+2v+1$ and so we pivot on the other column

$$\begin{array}{c|cccc} & & 0 & 0 & \\ \hline & & x_4 & x_3 & \\ 1+u-v & x_2 & -1/3 & 2/3 & 4/3 \\ 2+2u+v & x_1 & 1/3 & 1/3 & 5/3 \\ 0 & x_5 & -1/3* & -1/3 & -2/3 \\ \hline & B & (u+2v+1)/3 & (4u+4-v)/3 & (14u+v+14)/3 \end{array} \tag{10.7.19}$$

$$= \begin{array}{c|cccc} & & 0 & 0 & \\ \hline & & x_5 & x_3 & \\ 1+u-v & x_2 & -1 & 1 & 2 \\ 2+2u+v & x_1 & 1 & 0 & 1 \\ 0 & x_4 & -3 & 1 & 2 \\ \hline & B & u+2v+1 & u-v+1 & 4u-v+4 \end{array} \tag{10.7.20}$$

which is the other possible solution.

10.7.4 Problem 4

Let a triangle ABC have sides of length

$$BC = 3 \quad AC = 4 \quad AB = 5 \tag{10.7.21}$$

Lay pipes along its sides with capacities such that

1. the pipes leading out of A have a total capacity of 3.
2. the pipes leading out of B or of C have total capacities of either 2 or 3.
3. no pipe between any two vertices must have a capacity exceeding 2.

Only pipes of an integer number of capacity units are available, and the cost of a pipe is proportional to its capacity and to its length. It is required to determine the cheapest system of piping.

Solution:

This is a confusingly worded problem. A picture is worth a thousand words here... Anyway it is saying that $x_{AB} + x_{AC} = 3$ with $2 \le x_{BC} + x_{AB} \le 3$ and $2 \le x_{AC} + x_{BC} \le 3$ with $0 \le x_{ij} \le 2$. We'd like to minimize $5x_{AB} + 4x_{AC} + 3x_{BC}$.

In tableau form we write for

$$x_{AB} + x_{AC} - x_1 = 3 \Rightarrow -x_{AB} - x_{AC} + x_1 = -3 \tag{10.7.22}$$
$$x_{AB} + x_{BC} + x_2 = 3 \tag{10.7.23}$$
$$x_{AC} + x_{BC} + x_3 = 3 \tag{10.7.24}$$
$$x_{AB} + x_{BC} - x_4 = 2 \Rightarrow -x_{AB} - x_{BC} + x_4 = -2 \tag{10.7.25}$$
$$x_{AC} + x_{BC} - x_5 = 2 \Rightarrow -x_{AC} - x_{BC} + x_5 = -2 \tag{10.7.26}$$

then we use the dual simplex algorithm

(10.7.27)

		5	4	3	
		x_{AB}	x_{AC}	x_{BC}	
$1+u-v$	x_1	-1	-1	0	-3
$2+2u+v$	x_2	1	0	1	3
0	x_3	0	1	1	3
0	x_4	-1	0	$-1*$	-2
0	x_5	0	-1	-1	-2
	B	-5	-4	-3	0

$=$ (10.7.28)

		5	4	3	
		x_{AB}	x_{AC}	x_4	
$1+u-v$	x_1	$-1*$	-1	0	-3
$2+2u+v$	x_2	0	0	1	1
0	x_3	-1	1	1	1
0	x_{BC}	1	0	-1	2
0	x_5	1	-1	-1	0
	B	-2	-4	-3	6

Note that had we not used the dual $x_{AB} + x_{AC} - x_1 = 3$, but $x_{AB} + x_{AC} + x_1 = 3$, we would have to think carefully about how to continue on to an optimum. We use the dual simplex algorithm to flip x_1 and x_{AB}.

$$
= \begin{array}{c|cccccc}
 & & 5 & 4 & 3 & \\
\hline
 & & x_1 & x_{AC} & x_4 & \\
1+u-v & x_{AB} & -1 & 1 & 0 & 3 \\
2+2u+v & x_2 & 0 & 0 & 1 & 1 \\
0 & x_3 & -1 & 2 & 1 & 4 \\
0 & x_{BC} & 1 & -1 & -1 & -1 \\
0 & x_5 & 1 & -2* & -1 & -3 \\
\hline
 & B & -2 & -2 & -3 & 12
\end{array} \tag{10.7.29}
$$

An application of the dual simplex again on x_5 and x_{AC} yields

$$
= \begin{array}{c|cccccc}
 & & 5 & 4 & 3 & \\
\hline
 & & x_1 & x_5 & x_4 & \\
1+u-v & x_{AB} & -1/2 & 1/2 & -1/2 & 3/2 \\
2+2u+v & x_2 & 0 & 0 & 1 & 1 \\
0 & x_3 & 0 & 1 & 0 & 1 \\
0 & x_{BC} & 1/2 & -1/2 & -1/2 & 1/2 \\
0 & x_{AC} & -1/2 & -1/2 & 1/2 & 3/2 \\
\hline
 & B & -3 & -1 & -2 & 15
\end{array} \tag{10.7.30}
$$

Let's now remove the fractions from x_{AB} with row $[x_6, -1/2, -1/2, -1/2, -1/2]$

$$
= \begin{array}{c|cccccc}
 & & 5 & 4 & 3 & \\
\hline
 & & x_1 & x_5 & x_4 & \\
1+u-v & x_{AB} & -1/2 & 1/2 & -1/2 & 3/2 \\
2+2u+v & x_2 & 0 & 0 & 1 & 1 \\
0 & x_3 & 0 & 1 & 0 & 1 \\
0 & x_{BC} & 1/2 & -1/2 & -1/2 & 1/2 \\
0 & x_{AC} & -1/2 & -1/2 & 1/2 & 3/2 \\
0 & x_6 & -1/2 & -1/2* & -1/2 & -1/2 \\
\hline
 & B & -3 & -1 & -2 & 15
\end{array} \tag{10.7.31}
$$

yielding

$$
= \begin{array}{c|cccccc}
 & & 5 & 4 & 3 & \\
\hline
 & & x_1 & x_6 & x_4 & \\
1+u-v & x_{AB} & -1 & 1 & -1 & 1 \\
2+2u+v & x_2 & 0 & 0 & 1 & 1 \\
0 & x_3 & -1 & 2 & -1 & 0 \\
0 & x_{BC} & 1 & -1 & 0 & 1 \\
0 & x_{AC} & 0 & -1 & 1 & 2 \\
0 & x_5 & 1 & -2 & 1 & 1 \\
\hline
 & B & -2 & -2 & -1 & 16
\end{array} \tag{10.7.32}
$$

which yields a solution of $x_{AB} = 1$, $x_{BC} = 1$ and $x_{AC} = 2$ with cost 16.

This is the same solution as the book, though they used a different method.

10.7.5 Problem 5

In Exercise 5-7 let the value of n be 15, and solve the problem for integer, non-negative values of the variables.

Solution:

With $n = 15$ the problem becomes maximize $14.7 + 0.06x_1 + 0.15x_2 + 0.3x_3$ subject to

$$x_1 \leq 15 \tag{10.7.33}$$

$$x_2 - n_2 \leq 0 \tag{10.7.34}$$

$$x_3 - n_3 \leq 0 \tag{10.7.35}$$

$$n_2 + 0.3x_1 = 9 \tag{10.7.36}$$

$$n_3 + 0.18x_1 + 0.3x_2 = 5.4 \tag{10.7.37}$$

We can then use our previous tableau with $n = 15$ in it

		0.06	0	0	
		x_1	z_2	z_3	
0	z_4	1	0	0	15
0.15	x_2	3/10	1	0	9
0.3	x_3	9/100	−3/10	1	27/10
0	n_2	3/10	0	0	9
0	n_3	9/100	−3/10	0	27/10
	B	12/1000	6/100	3/10	1686/100

(10.7.38)

We can then apply our fraction method to x_3 with a row $[y_1, -9/100, -7/10, 0, -7/10]$

		0.06	0	0	
		x_1	z_2	z_3	
0	z_4	1	0	0	15
0.15	x_2	3/10	1	0	9
0.3	x_3	9/100	−3/10	1	27/10
0	n_2	3/10	0	0	9
0	n_3	9/100	−3/10	0	27/10
0	y_1	−9/100	−7/10∗	0	−7/10
	B	12/1000	6/100	3/10	1686/100

(10.7.39)

Then dual simplex on row y_1 and column z_2 yields

		0.06	0	0	
		x_1	y_1	z_3	
0	z_4	1	0	0	15
0.15	x_2	6/35	10/7	0	8
0.3	x_3	9/70	−3/7	1	3
0	n_2	3/10	0	0	9
0	n_3	9/70	−3/7	0	3
0	z_2	9/70	−10/7	0	1
	B	3/700	3/35	3/10	84/5

(10.7.40)

which yields a solution of $x_1 = 0$, $x_2 = 8$, $x_3 = 3$, $n_2 = 9$ and $n_3 = 3$.

10.7.6 Problem 6

Solve the problem

$$3x_1 + 4x_2 + x_3 \le 2 \tag{10.7.41}$$
$$x_1 + 3x_2 + 2x_3 \le 1 \tag{10.7.42}$$

Maximize $3x_1 + 6x_2 + 2x_3$ with x_1, x_2 non-negative integers, and x_3 any non-negative value.

Solution:

We have a mixed problem. Let's first solve the problem without worrying about integer values.

$$\begin{array}{r|lrrrr} & & 3 & 6 & 2 & \\ \hline & & x_1 & x_2 & x_3 & \\ = 0 & x_4 & 3 & 4 & 1 & 2 \\ 0 & x_5 & 1 & 3* & 2 & 1 \\ \hline & B & -3 & -6 & -2 & 0 \end{array} \tag{10.7.43}$$

pivoting on x_5 and x_2 gives

$$\begin{array}{r|lrrrr} & & 3 & 0 & 2 & \\ \hline & & x_1 & x_5 & x_3 & \\ = 0 & x_4 & 5/3* & -4/3 & -5/3 & 2/3 \\ 6 & x_2 & 1/3 & 1/3 & 2/3 & 1/3 \\ \hline & B & -1 & 2 & 2 & 2 \end{array} \tag{10.7.44}$$

Now a pivot on x_4 and x_1 yields

$$\begin{array}{r|lrrrr} & & 0 & 0 & 2 & \\ \hline & & x_4 & x_5 & x_3 & \\ = 3 & x_1 & 3/5 & -4/5 & -1 & 2/5 \\ 6 & x_2 & -1/5 & 3/5 & 1 & 1/5 \\ \hline & B & 3/5 & 6/5 & 1 & 12/5 \end{array} \tag{10.7.45}$$

One would usually now use the tricks. Essentially, we see what the nearby integers do to our objective function. First let's consider $x_1 = 1$. There can be no feasible solution since this implies $x_2 = x_3 = 0$ which does not satisfy our initial constraints. $x_2 = 1$ is also fairly obviously not suitable. Thus, we are left with $x_1 = 0$ or $x_2 = 0$ as possibilities.

In this case, we need not use the tricks since we see that of the available integers, only the zeros will do. $x_1 = x_2 = 0$ is the only possibility, which is $x_3 = 1/2$ is the maximum and $B = 1$.

Chapter 11

Stochastic Linear Programming

The idea is how should we approach problems where we are unsure that the actual coefficients for our problem. Say we know they're within a certain range, though, and we can make some general statements.

11.1 Range of Values

The easy way to understand these is simply from looking at the chain of inequalities. They are pretty clear how they work out. In this case, the naive thought that using the smallest coefficients leads to a value that is smallest, and the largest coefficients lead to the largest possible value. This of course requires thinking about the constraints. I'll reproduce the coefficient requirements

$$\begin{array}{c} \min \sum_i c_i^- x_i \\ \sum_i a_{ij}^+ x_i \geq b_j^- \end{array} \leq \begin{array}{c} \min \sum_i c_i x_i \\ \sum_i a_{ij} x_i \geq b_j \end{array} \leq \begin{array}{c} \min \sum_i c_i^+ x_i \\ \sum_i a_{ij} x_i^- \geq b_j^+ \end{array} \tag{11.1.1}$$

$$\begin{array}{c} \max \sum_i c_i^- x_i \\ \sum_i a_{ij}^+ x_i \geq b_j^- \end{array} \leq \begin{array}{c} \max \sum_i c_i x_i \\ \sum_i a_{ij} x_i \geq b_j \end{array} \leq \begin{array}{c} \max \sum_i c_i^+ x_i \\ \sum_i a_{ij} x_i^- \geq b_j^+ \end{array} \tag{11.1.2}$$

11.2 Distribution Problems and Expected Value Problems

Essentially how to look at things when we use distribution functions for the values rather than just pure ranges.

When we get to page 213, there is bad notation, we should have

$$M_0 = \sum_j y_j^0 b_j = E\left(\sum_j y_j^0 [b_j + b_{jt}]\right) \leq E(M_{0t}) \tag{11.2.1}$$

Where M_0 is the minimum for the problem with $b_{jt} = 0$.

11.3 Exercises

11.3.1 Problem 1

Use the following example to illustrate $M_0 \leq \sum_t p_t M_{0t}$.

$$2x_1 + x_2 - x_3 = 1 + b_{1t} \tag{11.3.1}$$
$$x_1 + 2x_2 - x_3 = 1 + b_{2t} \tag{11.3.2}$$

Minimize $x_1 + x_2 + x_3$ where $b_{1t} = -2, 0, 2$ and $b_{2t} = -2, 0, 2$.

Solution:

Since p_t is not given, it's not exactly obvious what the correct choice would be. So let's find M_0. I'll just add two new slack variables to make it easy. Then we'll penalize them to get rid of them.

$$\begin{array}{c|ccccc}
 & & 1 & 1 & 1 & \\
\hline
 & & x_1 & x_2 & x_3 & \\
M & y_1 & 2 & 1 & -1 & 1 \\
M & y_2 & 1 & 2* & -1 & 1 \\
\hline
 & B & -1 & -1 & -1 & 0 \\
 & M & 3 & 3 & -2 & 2
\end{array} \tag{11.3.3}$$

$$\begin{array}{c|ccccc}
 & & 1 & M & 1 & \\
\hline
 & & x_1 & y_2 & x_3 & \\
M & y_1 & 3/2* & -1/2 & -1/2 & 1/2 \\
1 & x_2 & 1/2 & 1/2 & -1/2 & 1/2 \\
\hline
 & B & -1/2 & 1/2 & -3/2 & 1/2 \\
 & M & 3/2 & -3/2 & -1/2 & 1/2
\end{array} \tag{11.3.4}$$

$$\begin{array}{c|ccccc}
 & & M & M & 1 & \\
\hline
 & & y_1 & y_2 & x_3 & \\
1 & x_1 & 2/3 & -1/3 & -1/3 & 1/3 \\
1 & x_2 & -1/3 & 2/3 & -1/3 & 1/3 \\
\hline
 & B & 1/3 & 1/3 & -5/3 & 2/3 \\
 & M & -1 & -1 & 0 & 0
\end{array} \tag{11.3.5}$$

$$\tag{11.3.6}$$

So that $M_0 = 2/3$.

Now we repeat with M_{01} where $b_{11} = b_{21} = -2$. We could perform the same functions, though we now need to use the dual simplex algorithm.

$$\begin{array}{c|ccccc}
 & & 1 & 1 & 1 & \\
\hline
 & & x_1 & x_2 & x_3 & \\
M & y_1 & 2 & 1 & -1 & -1 \\
M & y_2 & 1 & 2 & -1* & -1 \\
\hline
 & B & -1 & -1 & -1 & 0 \\
 & M & 3 & 3 & -2 & -2
\end{array} \tag{11.3.7}$$

		1	1	M	
		x_1	x_2	y_2	
M	y_1	$1*$	-1	-1	0
1	x_3	-1	-2	-1	1
	B	-2	-3	-1	1
	M	1	-1	-2	0

(11.3.8)

		M	1	M	
		y_1	x_2	y_2	
1	x_1	1	-1	-1	0
1	x_3	1	-3	-2	1
	B	2	-5	-3	1
	M	-1	0	-1	0

(11.3.9)

		1	
		x_2	
1	x_1	-1	0
1	x_3	-3	1
	B	-5	1

(11.3.10)

Which tells us this is an optimum. Note that $x_1 = x_2 = 0$ and $x_3 = 1$ is a minimum has a degeneracy but $M_{01} = 1$.

M_{02} is clearly the same as $M_0 = 2/3$. Finally M_{03} with $b_{13} = b_{23} = 2$. Then this is similar to our original problem (no dual simplex necessary).

		1	1	1	
		x_1	x_2	x_3	
M	y_1	2	1	-1	3
M	y_2	1	$2*$	-1	3
	B	-1	-1	-1	0
	M	3	3	-2	2

(11.3.11)

		1	M	1	
		x_1	y_2	x_3	
M	y_1	$3/2*$	$-1/2$	$-1/2$	$3/2$
1	x_2	$1/2$	$1/2$	$-1/2$	$3/2$
	B	$-1/2$	$1/2$	$-3/2$	$3/2$
	M	$3/2$	$-3/2$	$-1/2$	$-5/2$

(11.3.12)

		M	M	1	
		y_1	y_2	x_3	
1	x_1	$2/3$	$-1/3$	$-1/3$	1
1	x_2	$-1/3$	$2/3$	$-1/3$	1
	B	$1/3$	$1/3$	$-5/3$	2
	M	-1	-1	0	-4

(11.3.13)

(11.3.14)

Which means $x_1 = x_2 = 1$ and $x_3 = 0$ giving $M_{03} = 2$.

Thus we have

$$M_0 = 2/3 \le p_1 M_{01} + p_2 M_{02} + p_3 M_{03} = p_1(1) + p_2(2/3) + p_3(2) \tag{11.3.15}$$

with $p_1 + p_2 + p_3 = 1$. Clearly the smallest the right hand side can ever be is $2/3$ when $p_2 = 1$.

The book, because it is unclear, apparently mean that you can choose b_{1t} independently of b_{2t} so that $b_{10} = -2$ and $b_{20} = 2$ is a possibility. In any case, it would then pay to consider this problem in its most general form (the parametric form) and figure out all the possible basic variables. We can then just form a table of the possible values. In any case, we would find that $M_0 = 2/3$ is the smallest possible value.

11.3.2 Problem 2

Use the following example to illustrate the relationship between M_0, M_1, and M_2.

$$2x_1 + x_2 - x_3 \le 1 + b_{1t} \tag{11.3.16}$$

$$x_1 + 2x_2 - x_3 \le 1 + b_{12} \tag{11.3.17}$$

Minimize $x_1 + x_2 + x_3 + Ef_1(1 + b_{1t} - 2x_1 - x_2 + x_3) + Ef_2(1 + b_{2t} - x_1 - 2x_2 + x_3)$. Take $f_1 = f_2 = 1$.

Solution:

Since b_{it} is not given, I guess we are just supposed to assume it is the same as the previous problem. $E(b_{it}) = 0$ so we are minimizing

$$x_1 + x_2 + x_3 + (1 - 2x_1 - x_2 + x_3) + (1 - x_1 - 2x_2 + x_3) = 2 - 2x_1 - 2x_2 + 3x_3 \tag{11.3.18}$$

Previously, we showed $M_0 = 2/3$ where $f_j = 0$. To find M_1, we use the minimum values possible for b_{it} which from the previous problem is -2. Thus, including the penalty and using the dual simplex algorithm we get

		-2	-2	3	
		x_1	x_2	x_3	
M	y_1	2	1	-1	-1
M	y_2	1	2	$-1*$	-1
	B	2	2	-3	2
	M	3	3	-2	-2

(11.3.19)

		-2	-2	M	
		x_1	x_2	y_2	
M	y_1	$1*$	-1	-1	0
2	x_3	-1	-2	-1	1
	B	-1	-4	-3	5
	M	1	-1	-2	4

(11.3.20)

		M	-2	M	
		y_1	x_2	y_2	
-2	x_1	1	-1	-1	0
2	x_3	1	-3	-2	1
	B	1	-5	-4	5
	M	-1	0	-1	4

(11.3.21)

Which tells us $x_1 = x_2 = 0$ and $x_3 = 1$ gives $M_1 = 5$ and $M_1 \geq M_0$ clearly.

Now if we didn't put in the b_{it} at all and went through the above we would get M_2 yielding

$$\begin{array}{c|ccccc} & & -2 & -2 & 3 & \\ \hline & & x_1 & x_2 & x_3 & \\ M & y_1 & 2 & 1 & -1 & 1 \\ M & y_2 & 1 & 2* & -1 & 1 \\ \hline & B & 2 & 2 & -3 & 2 \\ & M & 3 & 3 & -2 & -2 \end{array} \tag{11.3.22}$$

$$\begin{array}{c|ccccc} & & -2 & M & 3 & \\ \hline & & x_1 & y_2 & x_3 & \\ M & y_1 & 3/2* & -1/2 & -1/2 & 1/2 \\ -2 & x_2 & 1/2 & 1/2 & -1/2 & 1/2 \\ \hline & B & 1 & -1 & -2 & 1 \\ & M & 3/2 & -3/2 & -1/2 & 1/2 \end{array} \tag{11.3.23}$$

$$\begin{array}{c|ccccc} & & M & M & 3 & \\ \hline & & y_1 & y_2 & x_3 & \\ -2 & x_1 & 2/3 & -1/3 & -1/3 & 1/3 \\ -2 & x_2 & -1/3 & 2/3 & -1/3 & 1/3 \\ \hline & B & -2/3 & -2/3 & -5/3 & 2/3 \\ & M & -1 & -1 & 0 & 0 \end{array} \tag{11.3.24}$$

which indicates $x_1 = x_2 = 1/3$, $x_3 = 0$ with $M_2 = 2/3$ is a solution. Since $M_0 = 2/3$ they (M_2 and M_0) are equal in this instance.

Chapter 12

Nonlinear Programming

We have methods for some nonlinear cases, though they have more restrictions for use.

12.1 Definitions

Just the definition of a convex function.

12.2 Approximations

This is the use of basis functions and minimizing of some error function (objective function). Basically, this is the basis for numerical approximation schemes like finite elements. Note that the constraints are still limited to linear, but the objective function can be nonlinear.

12.3 Quadratic Programming

This is with a quadratic objective functions. Most of the ways are like those for linear, but with some caveats. Such as needing to use derivative information.

12.4 Quadratic Duality

The duality relies on semidefinite forms of the objective function.

12.5 Exercises

12.5.1 Problem 1

Minimize

$$4t_0^2 + 4t_0t_1 + 6t_1^2 + 4t_0 - 56t_1 + 180 \tag{12.5.1}$$

subject to $t_0 \geq -3$, and $t_1 \leq 6$.

Solution:

Well, let's first find the derivatives calling the objective function C.

$$\frac{\partial C}{\partial t_0} = 8t_0 + 4t_1 + 4 \tag{12.5.2}$$
$$\frac{\partial C}{\partial t_1} = 4t_0 + 12t_1 - 56 \tag{12.5.3}$$

Should we set both to zero we'd find

$$8t_0 + 4t_1 + 4 = 0 \tag{12.5.4}$$
$$4t_0 + 12t_1 - 56 = 0 \tag{12.5.5}$$
$$8t_0 + 24t_1 - 112 = 0 \tag{12.5.6}$$
$$20t_1 + (-112 - 4) = 0 \tag{12.5.7}$$
$$t_1 = \frac{108}{20} t_0 = -\frac{16}{5} \tag{12.5.8}$$

Which is the total minimum. Unfortuantely, $t_0 \leq -3$, and so it is not an acceptable solution.

I like Lagrange multipliers, so let's add a constraint of the form $t_0 + 3 = 0$. To force t_0 to be at its smallest allowed value. Then we look at

$$C_1 = C + \lambda(t_0 + 3) \tag{12.5.9}$$

then we have

$$\frac{\partial C_1}{\partial t_0} = 8t_0 + 4t_1 + 4 + \lambda \frac{\partial C_1}{\partial t_1} = 4t_0 + 12t_1 - 56 \frac{\partial C_1}{\partial \lambda} = t_0 + 3 \tag{12.5.10}$$

Then $t_0 = -3$, and

$$4t_0 + 12t_1 - 56 = 0 \Rightarrow t_1 = \frac{56 - 4t_0}{12} = \frac{68}{12} = \frac{17}{3} \tag{12.5.11}$$

If we looked at the derivative values at this position we see

$$\left(\frac{\partial C}{\partial t_0}\right)_{t_i} = 2.66 \tag{12.5.12}$$
$$\left(\frac{\partial C}{\partial t_1}\right)_{t_i} = 0 \tag{12.5.13}$$

This is good because it means we can't move any direction for t_1 to increase the value, which is exactly what we wanted.

12.5.2 Problem 2

Minimize

$$x_1^2 + x_2^2 - 4x_1 - 2x_2 + 5 \tag{12.5.14}$$

subject to $x_1 + x_2 \leq 4$ with $x_i \geq 0$.

Solution:

Let's try and make some perfect squares.

$$x_1^2 + x_2^2 - 4x_1 - 2x_2 + 5 = (x_1 - 2)^2 + x_2^2 - 2x_2 + 1 = (x_1 - 2)^2 + (x_2 - 1)^2 \tag{12.5.15}$$

Clearly the minimum would be $x_1 = 2$ and $x_2 = 1$. This is clearly the answer since $x_1 + x_2 \leq 4$ and this is the global minimum.

12.5.3 Problem 3

Which of the following quadratic forms are positive definite, positive semidefinite, or neither? Give examples to illustrate your answers.

1. $5x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 + 4x_2x_3$
2. $5x_1^2 + 3x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 + 4x_2x_3$
3. $5x_1^2 + x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 + 4x_2x_3$

Solution:

For the 1., we can write the matrix and find the eigenvalues of the matrix. This is usually faster than trying to figure out a possible factorization. If the eigenvalues are all non negative we have a positive semidefinite form, if they're all positive, then it is positive definite. We use that **xMx** is the same as our quadratic and so we must put halves on some of the entries.

$$M = \begin{bmatrix} 5 & -2/2 & -2/2 \\ -2/2 & 2 & 4/2 \\ -2/2 & 4/2 & 2 \end{bmatrix} \tag{12.5.16}$$

$$M - \lambda \mathbb{1} = \begin{bmatrix} 5-\lambda & -1 & -1 \\ -1 & 2-\lambda & 2 \\ -1 & 2 & 2-\lambda \end{bmatrix} = -\lambda^3 + 9\lambda^2 - 18\lambda \tag{12.5.17}$$

The eigenvalues are thus $\lambda_1 = 6$, $\lambda_2 = 3$, and $\lambda_3 = 0$. Thus it is positive semidefinite. For example, we can get zero by $x_1 = 0$, $x_2 = -1$ and $x_3 = 1$.

$$5(0)^2 + 2(-1)^2 + 2(1)^2 + 4(-1)(1) = 2 + 2 - 4 = 0 \tag{12.5.18}$$

For 2., we perform the same operation

$$M = \begin{bmatrix} 5 & -2/2 & -2/2 \\ -2/2 & 3 & 4/2 \\ -2/2 & 4/2 & 2 \end{bmatrix} \tag{12.5.19}$$

$$M - \lambda \mathbb{1} = \begin{bmatrix} 5-\lambda & -1 & -1 \\ -1 & 3-\lambda & 2 \\ -1 & 2 & 2-\lambda \end{bmatrix} = -\lambda^3 + 10\lambda^2 - 25\lambda + 9 \tag{12.5.20}$$

The eigenvalues are thus $\lambda_1 \approx 6.2$, $\lambda_2 \approx 3.4$, and $\lambda_3 \approx 0.43$. Thus it is positive definite.

We can thus write it as a sum of perfect squares.

For 3., we can use $x_1 = 0$, $x_2 = -1$, $x_3 = 1$ to find

$$5(0)^2 + (-1)^2 + 2(1)^2 - 2(0)(-1) - 2(0)(1) + 4(-1)(0) = 1 + 2 - 4 = -1 < 0 \tag{12.5.21}$$

where as $x_2 = 1$ would yield $1 + 2 + 4 = 7 > 0$ and so it is not definite.

12.5.4 Problem 4

Construct and solve the problem which is dual to Example 12-1.

Solution:

Example 12-1 is of the form Minimize

$$183 - 44x_1 - 42x_2 + 8x_1^2 - 12x_1x_2 + 17x_2^2 \tag{12.5.22}$$

with constraints $2x_1 + x_2 + x_3 = 10$ and $x_i \geq 0$.

Thus the dual is maximize

$$D = -8u_1^2 + 12u_1u_2 - 17u_2^2 + 10y \tag{12.5.23}$$

subject to

$$2y - 16u_1 + 12u_2 \leq -44 \tag{12.5.24}$$

$$y + 12u_1 - 34u_2 \leq -42 \tag{12.5.25}$$

We can easily create a tableau for this pseudolinear problem. However, we see that $y \leq 0$, and we'd prefer the opposite, so we introduce a new $y = -y$ and get

Maximize

$$D = -8u_1^2 + 12u_1u_2 - 17u_2^2 - 10y \tag{12.5.26}$$

subject to

$$2y + 16u_1 - 12u_2 \geq 44 \tag{12.5.27}$$

$$y - 12u_1 + 34u_2 \geq 42 \tag{12.5.28}$$

where $y \geq 0$.

Let's find the general maximum

$$\frac{\partial D}{\partial u_1} = -16u_1 + 12u_2 \tag{12.5.29}$$

$$\frac{\partial D}{\partial u_2} = 12u_1 - 34u_2 \tag{12.5.30}$$

$$\frac{\partial D}{\partial y} = -10 \tag{12.5.31}$$

This means that the larger y is the smaller the value of our objective function. This indicates that we'd like y to be as small as possible while also satisfying the constraints. Let's then rewrite the constraints as

$$2y \geq 44 + 12u_2 - 16u_1 \tag{12.5.32}$$
$$y \geq 42 + 12u_1 + 34u_2 \tag{12.5.33}$$

Let's add these constraints as $2y - 44 - 12u_2 + 16u_1$ and $y - 42 - 12u_1 - 34u_2$. Then our new function is

$$\begin{aligned} D_1 = &-8u_1^2 + 12u_1u_2 - 17u_2^2 - 10y + \lambda_1 (2y - 44 - 12u_2 + 16u_1) \\ &+ \lambda_2 (y - 42 - 12u_1 + 34u_2) \end{aligned} \tag{12.5.34}$$

And so

$$\frac{\partial D_1}{\partial u_1} = -16u_1 + 12u_2 + 16\lambda_1 - 12\lambda_2 \tag{12.5.35}$$
$$\frac{\partial D_1}{\partial u_2} = 12u_1 - 34u_2 - 12\lambda_1 + 34\lambda_2 \tag{12.5.36}$$
$$\frac{\partial D_2}{\partial y} = -10 + 2\lambda_1 + \lambda_2 \tag{12.5.37}$$

in addition to the constraint equations. Thus our matrix is (with columns giving coefficients of $[u_1, u_2, y, \lambda_1, \lambda_2]$)

$$\begin{bmatrix} -16 & 12 & 0 & 16 & -12 & 0 \\ 12 & -34 & 0 & -12 & 34 & 0 \\ 0 & 0 & 0 & 2 & 1 & 10 \\ 16 & -12 & 2 & 0 & 0 & 44 \\ -12 & 34 & 1 & 0 & 0 & 42 \end{bmatrix} \tag{12.5.38}$$

If we assume all constraints are active so that λ_1 and $\lambda_2 \geq 0$ then $y < 0$ then we find $u_1 = 19/5 = 3.8$, $u_2 = 12/5 = 2.4$, and $y = 6$. This is a good solution. But we need to check other possibilities for answers.

So let's try $\lambda_2 = 0$ and remove that constraint. This yields $u_1 = 5$, $u_2 = 0$, and $y = -18$ which is excluded.

Finally $\lambda_1 = 0$ gives $u_1 = 0$, $u_2 = 10$, and $y = -298$ which is also an excluded possibility.

Thus we find that both constraints being active gives us our maximum.

This seems to indicate that Lagrangian multipliers won't work, which is wrong. Something must be wrong with my formulation. If we simply took u_1 and u_2 added slack variables and solved our constraint equations for u_1 and u_2 in terms of y and the slack variables, then we could put that in our constraint and find values. This is what the book does.

Chapter 13

Dynamic Programming

This includes "time" as a factor. Mostly, it just means we can do some iterations.

13.1 Principle of Optimality

This is a statement that is not at all obviously true for most problems. It says that if we want an optimal procedure at the end of a long chain (over the entire process), then we should choose the optimal procedure along each link in the chain to the end. This is clearly not always true. Sometimes you can get more out by choosing a non-optimal way early on that allows a huge boon later on. But for linear programs, this principle often applies, and makes it much easier to find solutions.

The convexity of functions makes this a more likely to be true statement, in that it makes only the edges the most optimal.

13.2 Functional Equations

Here we delve briefly into equations that define functions, and so we have functional equations.

The proof is not very satisfying since it is essentially a proof by verification. How one would guess $(a/k)^k$ is not at all obvious, but then it can easily be shown to give the correct value for $k > 1$. For we can take a derivative to find the maximum.

13.3 Exercises

13.3.1 Problem 1

Solve Example 13-1 for the following numerical data

$$N = 3, \qquad a = \frac{1}{3}, \qquad b = \frac{2}{3}, \qquad g(x) = 3x, \qquad h(y) = 2.5y \tag{13.3.1}$$

Solution:

The fact that a and b are constants show that the book makes no sense. This would say that no matter how many machines we initially had, they actually magically turn into $1/3$ or $2/3$ of a machine. Presumably they mean that $a(x) = x/3$ and $b(x) = 2x/3$ or else this problem literally makes no sense.

We follow the principle of optimality and so we find

$$\begin{aligned} f_1(n) &= \max_{0 \le x \le n} (g(x) + h(n-x)) = \max_{0 \le x \le n} (3x + 2.5(n-x)) \\ &= \max_{0 \le x \le n} (0.5x + 2.5n) \end{aligned} \tag{13.3.2}$$

This is clearly at its maximum value when $n = x$ giving $f_1(n) = 3n$.

We then apply

$$\begin{aligned} f_2 &= \max_{0 \le x \le n} (g(x) + h(n-x) + f_1(a+b)) = \max_{0 \le x \le n} \left(0.5x + 2.5n + f_1\left(\frac{x}{3} + \frac{2(n-x)}{3}\right)\right) \\ &= \max_{0 \le x \le n} (0.5x + 2.5n + 2n - x) = \max_{0 \le x \le n} (-0.5x + 4.5n) \overset{x=0}{=} 4.5n \end{aligned} \tag{13.3.3}$$

for $f_2(n) = 4.5n$ Finally,

$$\begin{aligned} f_3 &= \max_{0 \le x \le n} (g(x) + h(n-x) + f_2(a+b)) = \max_{0 \le x \le n} \left(0.5x + 2.5n + f_2\left(\frac{x}{3} + \frac{2(n-x)}{3}\right)\right) \\ &= \max_{0 \le x \le n} \left(0.5x + 2.5n + \frac{45}{10}\frac{2n-x}{3}\right) = \max_{0 \le x \le n} (0.5x + 2.5n + 3n - 1.5x) \\ &= \max_{0 \le x \le n} (-x + 5.5n) \overset{x=0}{=} 5.5n \end{aligned} \tag{13.3.4}$$

We have thus found $x_3 = n$ and $x_2 = x_1 = 0$ (because we go backwards with the value for f_3 corresponding to x_1, etc.). The chart is given by

Stage	First Job	Second Job	Results	Remaining Machines
1	0	n	$5n/2$	$2n/3$
2	0	$2n/3$	$5n/3$	$4n/9$
3	$4n/9$	0	$4n/3$	
			Total	
			$11n/2$	

Table 13.1: This shows the results from our calculations in tabular form.

13.3.2 Problem 2

A merchant can buy a number of items at 10 each, and resells them at 25 each. He can return those which he has not sold after one month and recover 5 for each, and he can also buy new items, again at 10 each. At the end of two months he can return unsold items for 2 each. Determine the best policy: How many should he buy to begin with, or after one month? How may should he return after one month?

Assume the following probability distribution of demands during the first two months: First month: required number of items

$$\text{Probability} \quad \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1/8 & 1/8 & 1/4 & 1/4 & 1/8 & 1/8 \end{matrix} \tag{13.3.5}$$

Second month: required number of items:

$$\text{Probability} \quad \begin{matrix} 0 & 1 & 2 \\ 1/4 & 1/2 & 1/4 \end{matrix} \tag{13.3.6}$$

It is required to optimize the expected profit.

Solution:

As with the previous problem, we work backwards. We start with possibilities after the first month has ended and find what all the possibilities are, then use those possibilities with the possibilities in the first month to look at all the potential ways of getting a profit.

There is nothing clever about it, as there are just a lot of steps of doing arithmetic with the given probabilities (expected cost if buying 7 selling 3, for example). The only other thing required is that there is no reason to every have more than 7 bought since that is the most it is possible to sell in this scenario.

13.3.3 Problem 3

Solve Example 4-1 by an application of the Principle of Optimality.

Solution:

Example 4-1 is a find the shortest route problem. This would entail us starting at the end, finding the shortest route backwards We can note that to get to H from any of the nodes Y, S, or D there is clearly a shortest route. Y, S, and D all connect variously to L and B. So we need to check which values are optimal from L to H and B to H via all various paths. Clearly L to S to H is fastest from L and B to D to H is fastest from B. Then from M we have to choose between the M to L to S to H giving 95 or from M to B to D to H giving 98 and so choose M to L to S to H.

13.3.4 Problem 4

A man who can swim at a speed c wants to cross a river. At a distance x from the opposite bank the speed of the river's flow is v_x. It is required to determine in which direction to swim at any given stage so as to make the deviation downstream from the point opposite to the start as small as possible. Formulate the functional equation for solving the problem, assuming that v_x is constant for all values above $x - 1$ up to and including x. Solve the functional equation for $v_x = x$ (a constant larger than c).

Solution:

The book has a crucial typo again. It meant for $v_x = v > c$ since v_x cannot both be a constant and vary with x at the same time. If the book just could make clear questions, it would be possible to solve these without so many headaches. The book is not very good at making clear, unambiguous questions. What is stated is very confusing, and I honestly don't know what it means by assume that the velocity is constant for values above $x - 1$ up to including x. Especially since the very next line says to assume v_x simply is a constant. I think it just means assume v_x is a constant across the river.

In that case, we can make an angle α from simply pointing yourself at the opposite bank and swimming. Then during any time t across the river, the deviation from going to the opposite shore will be (if your speed is c) $(-c\sin\alpha + v_x)t$. And, the time can be found fairly easily. If we cover a distance d, then we must have $t = d/(c\cos\alpha)$. Thus the deviation will be

$$\frac{-c\sin\alpha + v_x}{c\cos\alpha} d = d\left(\frac{v_x}{c\cos\alpha} - \tan\alpha\right) \tag{13.3.7}$$

Now we can make the distance $d = 1$ so that the deviation will be per unit across the river for convenience (which is what I think the book is trying to say with its odd $x - 1$ and x sentences). Then we must have (calling the deviation $F(x)$ for location x) that

$$F(x) = \left(\frac{v_x}{c\cos\alpha} - \tan\alpha\right) + F(x-1) \tag{13.3.8}$$

$$F(x) = d\left(\frac{v_x}{c\cos\alpha} - \tan\alpha\right) + F(x-d) \tag{13.3.9}$$

where the latter equation is the more general form that makes sense for a constant velocity river. We want to find one that minimizes this, so we look for

$$F(x) = \min\left[d\left(\frac{v_x}{c\cos\alpha} - \tan\alpha\right) + F(x-d)\right] \tag{13.3.10}$$

We can try what looks like a possible recursive solution via

$$f(x) = x\left(\frac{v_x}{c\cos\alpha} - \tan\alpha\right) \tag{13.3.11}$$

which yields

$$d\left(\frac{v_x}{c\cos\alpha} - \tan\alpha\right) + (x-d)\left(\frac{v_x}{c\cos\alpha} - \tan\alpha\right) \tag{13.3.12}$$

$$= x\left(\frac{v_x}{c\cos\alpha} - \tan\alpha\right) \tag{13.3.13}$$

and so is a solution to the above functional relationship. We then just need to minimize it (with the idea that this is the most generic function that satisfies the relationship). This involves taking a derivative with respect to α and setting it equal to zero to find the smallest deviation as a function of α.

$$\frac{\mathrm{d}f}{\mathrm{d}\alpha} = x\left(-\frac{v_x(-\sin\alpha)}{c\cos^2\alpha} - \sec^2\alpha\right) = x\left(\frac{v_x\tan\alpha}{c\cos\alpha} - \sec^2\alpha\right) = 0 \tag{13.3.14}$$

$$\frac{v_x\tan\alpha}{c\cos\alpha} = \frac{1}{\cos^2\alpha} \tag{13.3.15}$$

$$\frac{v_x\sin\alpha}{c} = 1 \tag{13.3.16}$$

$$\alpha = \sin^{-1}\left(\frac{c}{v_x}\right) \tag{13.3.17}$$

Thus, this is the optimal α with which to minimize the deviation from directly across the river.