Interesting Physics Problems

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1 Current carrying wires with springs

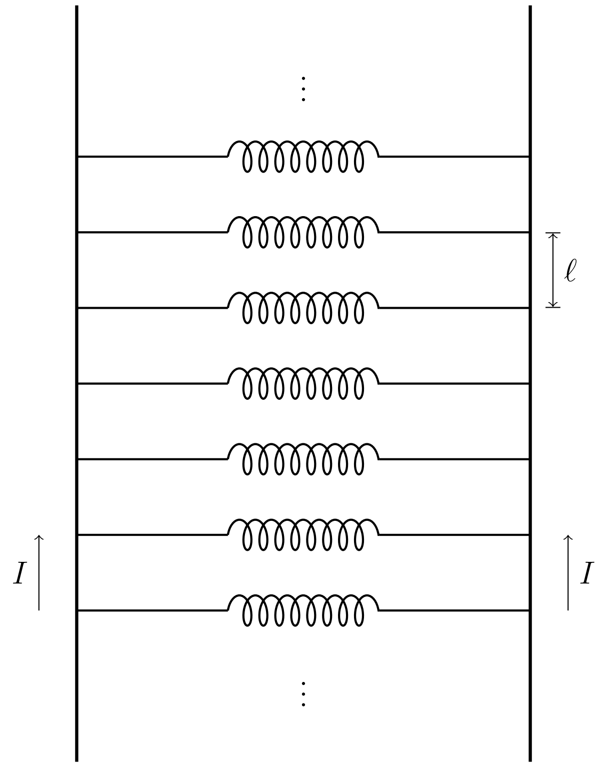

Figure 1: Schematic of current carrying wire and springs.

Consider two current carrying wires as in Figure 1 with springs (evenly spaced, ℓ apart, all with spring constant k) in between. If the uncompressed spring has equilibrium length x_0, what is the position of the two wires when the force from the spring balances the force due to the magnetic field? (That is what is the distance between the two wires at equilibrium). If you get more than one answer explain which answers are physical and which are unphysical (and why), and which are stable and which are unstable.

Solution:

The force applied to the left (right) per unit length on the right (left) wire is

$$\mathcal{F} = \mathbf{I} \times \mathbf{B} = IB = I\frac{\mu_0 I}{2\pi x} \tag{1.1}$$

where x is the distance between the two wires. So the force will be

$$F = \frac{\mu_0 I^2 \ell}{2\pi x} \tag{1.2}$$

The spring will have a force

$$F = k(x - x_0) \tag{1.3}$$

Now let a force to the right (looking at the right wire) be positive. Then we have

$$-\frac{\mu_0 I^2 \ell}{2\pi x} + k(x_0 - x) = 0 \tag{1.4}$$

$$k(x_0 - x) = \frac{\mu_0 I^2 \ell}{2\pi x} \tag{1.5}$$

$$(x x_0 - x^2) = \frac{\mu_0 I^2 \ell}{2\pi k} \tag{1.6}$$

$$x^2 - x x_0 + \frac{\mu_0 I^2 \ell}{2\pi k} = 0 \tag{1.7}$$

$$\Rightarrow$$

$$x = \frac{x_0 \pm \sqrt{x_0^2 - \frac{2\mu_0 I^2 \ell}{\pi k}}}{2} \equiv x_{\pm \text{eq.}} \tag{1.8}$$

Now we need to determine if either of these solutions are stable. If they are stable, then when we put in a small deviation, the force should act against the change.

So let's put in a δx to equilibrium $x_{\pm \text{eq.}}$.

$$-\frac{\mu_0 I^2 \ell}{2\pi (x_{\pm \text{eq.}} + \delta x)} + k(x_0 - (x_{\pm \text{eq.}} + \delta x)) \tag{1.9}$$

$$\approx -\frac{\mu_0 I^2 \ell}{2\pi \left(x_{\pm \text{eq.}}\right)} \left(1 - \frac{\delta x}{x_{\pm \text{eq.}}}\right) + k(x_0 - x_{\pm \text{eq.}}) - k\delta x \tag{1.10}$$

$$\approx \left(\frac{\mu_0 I^2 \ell}{2\pi \left(x_{\pm \text{eq.}}\right)^2} - k\right) \delta x + \underbrace{-\frac{\mu_0 I^2 \ell}{2\pi x_{\pm \text{eq.}}} + k(x_0 - x_{\pm \text{eq.}})}_{=0} \tag{1.11}$$

$$\approx \left(\frac{\mu_0 I^2 \ell}{2\pi \left(x_{\pm \text{eq.}}\right)^2} - k\right) \delta x \tag{1.12}$$

$$\approx \left(\frac{2\mu_0 I^2 \ell}{\pi \left(x_0 \pm \sqrt{x_0^2 - \frac{2\mu_0 I^2 \ell}{\pi k}}\right)^2} - k\right) \delta x \tag{1.13}$$

$$\approx \left(\frac{\frac{2\mu_0 I^2 \ell}{\pi} - k\left(x_0 \pm \sqrt{x_0^2 - \frac{2\mu_0 I^2 \ell}{\pi k}}\right)^2}{\left(x_0 \pm \sqrt{x_0^2 - \frac{2\mu_0 I^2 \ell}{\pi k}}\right)^2}\right) \delta x \tag{1.14}$$

$$\tag{1.15}$$

Now we must have $x_0^2 > \frac{2\mu_0 I^2 \ell}{\pi k}$, so that

$$\frac{2\mu_0 I^2 \ell}{\pi} < k x_0^2 \tag{1.16}$$

and so for $x_{+\text{eq.}}$ we have

$$k\left(x_0+\sqrt{x_0^2-\frac{2\mu_0 I^2\ell}{\pi k}}\right)^2 > kx_0^2 \tag{1.17}$$

$$\Rightarrow$$

$$\frac{2\mu_0 I^2\ell}{\pi} - k\left(x_0+\sqrt{x_0^2-\frac{2\mu_0 I^2\ell}{\pi k}}\right)^2 < 0 \tag{1.18}$$

and so for $\delta x > 0$ we have a negative force, and for $\delta x < 0$ we have a positive force. This means it is a stable equilibrium.

For $x_{-\text{eq.}}$, it must be unstable (as there are only two roots, and so if one is stable, the other one must be unstable, as can be seen in Figure 2).

(Alternatively, we can plot $-\dfrac{\mu_0 I^2\ell}{2\pi x} + k(x_0 - x)$ as a function of x and look at its roots. We see in Figure 2 that the root closest to zero is unstable (if you move a little to the left on it, then the force pushes to the left, and if you move a little to the right, the force pushes to the right), while the larger root is stable (if you move a little to the left on it, then the force pushes to the right, and if you move a little to the right, the force pushes to the left), just as our previous analysis showed.)

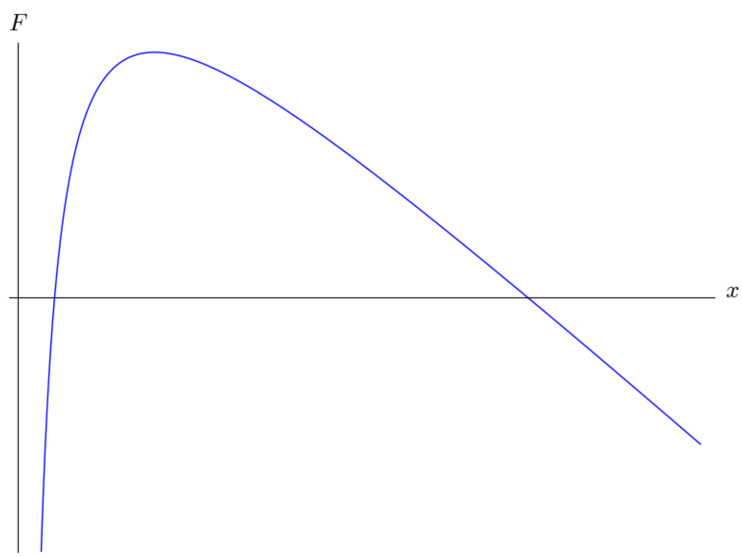

Figure 2: Qualitative behavior of force as function of distance, x.

So we see both answers are physical, although the unstable equilibrium will collapse the spring if x gets any smaller, and the wires will hit each other.

As an example, given $x_0 = 2$ cm, $I = 100$ A, $\ell = 10$ cm, and $k = 6$ N/m, we would find

$$x_{\pm\text{eq.}} = 0.018 \text{ m}, 0.0018 \text{ m}$$
$$x_{+\text{eq.}} = 1.8 \text{ cm}$$
$$x_{-\text{eq.}} = 1.8 \text{ mm}$$

2 Coin on Card on Glass

You've undoubtedly heard of the experiment where you place a playing card on top of a glass with a coin on top of the playing card. If you pull quickly, you learn that the coin falls into the glass, while if you pull slowly the coin comes off with the card. (Assume for the sake of simplicity that you always pull the card out at a constant velocity.) Why does this occur?

Solution:

An often used answer is friction; that is the friction changes depending on the velocity, but this turns out to be incorrect. Experimentally, one finds that friction between two solids is independent of velocity (at least in the Newtonian picture). The answer one might say is inertia, but this is a qualitative answer.

The answer is to look at the forces on the coin. There is of course gravity, but we're only concerned with motion along the horizontal direction. In this direction there is only a force from the friction of the coin and the card. Call the coefficient of friction μ and the mass of the coin m. Then we see that the force on the coin must be $F = \mu mg$.

Now this acceleration is constant for as long as the card is in contact with the coin. The velocity of the coin after t seconds in contact with the card will be

$$\int_0^t \frac{\mathrm{d}v}{\mathrm{d}t'}\,\mathrm{d}t' = \int_0^t \mu g\,\mathrm{d}t' \Rightarrow v(t) = \mu g t \tag{2.1}$$

and the distance covered (calling the initial coin location $x_0 = d/2$ with d the diameter, so the coin was in the center of the glass) will be

$$x(t) = \frac{\mu g t^2}{2} + \frac{d}{2} \quad . \tag{2.2}$$

Now if we pull the card out at a velocity V then the maximum time the coin could be in contact with the card is the half the diameter of the glass from $d/2$ to d. Then the maximum time is $\frac{d}{2V} = T$. The coin must not move farther than $d/2$ in this time, hence

$$\frac{\mu g T^2}{2} + \frac{d}{2} = \frac{\mu g d^2}{8V^2} + \frac{d}{2} \leq d \Rightarrow \frac{\mu g d}{4V^2} \leq 1. \tag{2.3}$$

If you prefer the radius of the glass ($r = d/2$) we have

$$\frac{\mu g r}{2V^2} \leq 1 \quad . \tag{2.4}$$

This makes things much clearer. The amount of time of contact between the coin and the card is what matters, and while friction does matter, the less time of contact, the less the card can affect the coin.

3 Relativistic Rocket at Constant Acceleration

A spaceship is going to be sent to a distant star, which is a distance D away from Earth (measured by Earth). It uses a magic engine that causes it to accelerate at a' (in its frame) without losing any appreciable amount of its mass until it reaches the midpoint of the trip and then the spaceship magically decelerates at a' until it reaches its destination.

Both in the Earth frame and in the rocket frame answer: What time does it take for the spaceship to get from Earth to the star (it starts at $x = 0$ and $v = 0$ at $t = 0$)? How long does it take to go to the star and return via the same flight plan ignoring the time spent at the star?

For Alpha-Centauri at a distance of $D = 4.3$ light years $= 4.3$ lyr and an acceleration of $a' = 1.0\ \mathrm{m/s^2}$, what are the values of these times?

[The Earth frame is the "stationary frame" and the rocket frame is the the frame that the astronauts on the spaceship consider themselves to be in. That is, for example, the astronauts measure a certain time for a process in the rocket frame.]

Solution:

Following the lead of a' let values in the rocket frame be primed and measurements in the Earth frame be unprimed.

Then as, always, we have the Lorentz transformation between the rocket frame and the Earth frame as

$$\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \beta\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} \tag{3.1}$$

Or

$$c\gamma t - \beta\gamma x = ct' \tag{3.2}$$
$$-c\beta\gamma t + \gamma x = x' \tag{3.3}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$ where v is the instanteous velocity of the rocket relative to Earth.

Let $u = \frac{\mathrm{d}x}{\mathrm{d}t}$ the velocity of the rocket in the Earth frame and $u' = \frac{\mathrm{d}x'}{\mathrm{d}t'}$ be the velocity of the rocket in the rocket frame.

$$c\gamma\,\mathrm{d}t - \beta\gamma\,\mathrm{d}x = c\,\mathrm{d}t' \tag{3.4}$$
$$\gamma\,\mathrm{d}t - \frac{\beta\gamma\,\mathrm{d}x}{c} = \mathrm{d}t' \tag{3.5}$$
$$-c\beta\gamma\,\mathrm{d}t + \gamma\,\mathrm{d}x = \mathrm{d}x' \tag{3.6}$$
$$\frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{-c\beta\gamma\,\mathrm{d}t + \gamma\,\mathrm{d}x}{\gamma\,\mathrm{d}t - \beta\frac{\gamma}{c}\,\mathrm{d}x} = \frac{-c\beta\gamma + \gamma\frac{\mathrm{d}x}{\mathrm{d}t}}{\gamma - \gamma\frac{\beta}{c}\frac{\mathrm{d}x}{\mathrm{d}t}} \tag{3.7}$$

$$u' = \frac{-c\beta\gamma + \gamma u}{\gamma - \gamma\frac{\beta}{c}u} \tag{3.8}$$

$$u' = \frac{-c\beta + u}{1 - \frac{\beta}{c}u} \tag{3.9}$$

and so using this for a small change in acceleration we find

$$\mathrm{d}u' = \frac{\left(1 - \frac{\beta}{c}u\right)(\mathrm{d}u) - (-c\beta + u)\left(-\frac{\beta}{c}\,\mathrm{d}u\right)}{\left(1 - \frac{\beta}{c}u\right)^2} \tag{3.10}$$

$$= \frac{\mathrm{d}u\left(1 - \frac{\beta}{c}u - \beta^2 + \frac{\beta}{c}u\right)}{(1 - \frac{\beta}{c}u)^2} \tag{3.11}$$

$$= \frac{\mathrm{d}u\,(1 - \beta^2)}{(1 - \frac{\beta}{c}u)^2} = \mathrm{d}u\frac{\gamma^{-2}}{(1 - \frac{\beta}{c}u)^2} \tag{3.12}$$

now as $u = v$ then

$$\mathrm{d}u' = \mathrm{d}u\gamma^{-2}\gamma^4 = \mathrm{d}u\gamma^2 \tag{3.13}$$

$$\frac{\mathrm{d}u'}{\mathrm{d}t'} = \frac{\mathrm{d}u}{\mathrm{d}t'}\gamma^2 = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'}\gamma^2 = \frac{\mathrm{d}u}{\mathrm{d}t}\gamma^3 \tag{3.14}$$

$$a' = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\gamma^3 \tag{3.15}$$

with $\gamma = \left(1 - \frac{1}{c^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\right)^{-1/2}$.

We can notice that the right hand side can be written

$$\frac{\mathrm{d}(\gamma v)}{\mathrm{d}t} = v\frac{\mathrm{d}\gamma}{\mathrm{d}t} + \gamma\frac{\mathrm{d}v}{\mathrm{d}t} \tag{3.16}$$

$$= v\frac{-\frac{2\beta}{c}a}{-2\left(1 - \beta^2\right)^{3/2}} + \gamma a \tag{3.17}$$

$$= (\beta^2\gamma^2 + 1)a\gamma = \left(\frac{\beta^2 + 1 - 1}{1 - \beta^2} + 1\right)a\gamma \tag{3.18}$$

$$\frac{\mathrm{d}(\gamma v)}{\mathrm{d}t} = \left(-1 + \frac{1}{1 - \beta^2} + 1\right)a\gamma = \gamma^3 a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\gamma^3 \quad . \tag{3.19}$$

So we have that

$$a' = \gamma^3 a = \frac{\mathrm{d}(\gamma v)}{\mathrm{d}t} \tag{3.20}$$

$$\int_0^t a'\,\mathrm{d}t = \int_0^t \frac{\mathrm{d}(\gamma v)}{\mathrm{d}t}\,\mathrm{d}t \tag{3.21}$$

$$a't = \gamma v \tag{3.22}$$

$$\frac{v}{\sqrt{1 - v^2/c^2}} = a't \tag{3.23}$$

$$v^2 = \left(1 - \frac{v^2}{c^2}\right) a'^2 t^2 \tag{3.24}$$

$$v^2 \left(1 + \frac{a'^2 t^2}{c^2}\right) = a'^2 t^2 \tag{3.25}$$

$$v = \frac{a't}{\sqrt{1 + \frac{a'^2 t^2}{c^2}}} = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{3.26}$$

and so with $u = a't/c$ and $\mathrm{d}u = a'\,\mathrm{d}t/c$

$$\int_0^t \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_0^t \frac{a't}{\sqrt{1 + \frac{a'^2 t^2}{c^2}}}\,\mathrm{d}t \tag{3.27}$$

$$x = \frac{c^2}{a'} \int_0^{a't/c} \frac{u\,\mathrm{d}u}{\sqrt{1 + u^2}} \tag{3.28}$$

and using $y = \sqrt{1 + u^2}$ so $\mathrm{d}y = (1 + u^2)^{-1/2}\, u\,\mathrm{d}u$ we find

$$x = \frac{c^2}{a'} \int_1^{\sqrt{1 + a'^2 t^2/c^2}} \mathrm{d}y \tag{3.29}$$

$$x = \frac{c^2}{a'} \left(\sqrt{1 + a'^2 t^2/c^2} - 1\right) \quad . \tag{3.30}$$

Now we need to employ some symmetry in the situation. If the rocket is accelerating constantly to the midpoint and then decelerating constantly to the end, it should take the same amount of time to go from the beginning to the midpoint as from the midpoint to the end.

If this is unclear to you, I suggest you ask yourself if you start at some position, accelerate away at a constant rate for some time t and then turn around and accelerate in the opposite direction with the same rate of acceleration as before (just a different direction). How long will it take until you stop? It should take a time t as that is the time it took to accelerate to the current velocity.

So we only need to find the time it takes to get halfway from Earth to the star. We then muliply by two to find the total time to take the trip from Earth to the star. We then only need to multiply this last result by 2 again to find the total time to go from the Earth to the star and back, as the return trip should take the same amount of time as trip there.

So using that $x = D/2$ we find

$$\frac{D}{2} = \frac{c^2}{a'}\sqrt{1 + a'^2 t^2/c^2} - \frac{c^2}{a'} \tag{3.31}$$

$$\frac{a'D}{2c^2} = \sqrt{1 + a'^2 t^2/c^2} - 1 \tag{3.32}$$

$$\left(\frac{a'D}{2c^2} + 1\right)^2 = 1 + \frac{a'^2 t^2}{c^2} \tag{3.33}$$

$$\frac{a'^2 t^2}{c^2} = \left(\frac{a'D}{2c^2} + 1\right)^2 - 1 \tag{3.34}$$

$$t^2 = \frac{c^2}{a'^2}\left(\frac{a'D}{2c^2}+1\right)^2 - \frac{c^2}{a'^2} \tag{3.35}$$

$$t^2 = \frac{c^2}{a'^2}\left(\frac{a'^2D^2}{4c^4}+\frac{a'D}{c^2}+1-1\right) \tag{3.36}$$

$$t^2 = \left(\frac{D^2}{4c^2}+\frac{D}{a'}\right) \tag{3.37}$$

$$t = \sqrt{\frac{D^2}{4c^2}+\frac{D}{a'}} \tag{3.38}$$

Now we need t'. Let's use that $\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{1}{\gamma}$ so that

$$a' = \gamma^3\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \gamma^3\frac{\mathrm{d}v}{\mathrm{d}t} = \gamma^3\frac{\mathrm{d}v}{\mathrm{d}t'}\frac{\mathrm{d}t'}{\mathrm{d}t} \tag{3.39}$$

$$= \gamma^3\frac{\mathrm{d}v}{\mathrm{d}t'}\frac{1}{\gamma} \tag{3.40}$$

$$= \gamma^2\frac{\mathrm{d}v}{\mathrm{d}t'} \tag{3.41}$$

$$\tag{3.42}$$

and so we see that we have (letting v_{mid} be the velocity of the spaceship at the midpoint relative to Earth)

$$\int_0^{t'} a'\,\mathrm{d}t' = \int_0^{v_{\text{mid}}} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,\mathrm{d}v \tag{3.43}$$

$$a't' = \int_0^{v_{\text{mid}}} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,\mathrm{d}v \tag{3.44}$$

$$a't' = c\tanh^{-1}\left(\frac{v_{\text{mid}}}{c}\right) \tag{3.45}$$

$$t' = \frac{c}{a'}\tanh^{-1}\left(\frac{v_{\text{mid}}}{c}\right) \quad . \tag{3.46}$$

And we know from 3.26 that

$$v_{\text{mid}} = \frac{a't}{\sqrt{1+\frac{a'^2t^2}{c^2}}} \tag{3.47}$$

$$v_{\text{mid}} = a'\sqrt{\frac{D^2}{4c^2}+\frac{D}{a'}}\left(1+\frac{a'^2}{c^2}\left(\frac{D^2}{4c^2}+\frac{D}{a'}\right)\right)^{-1/2} \tag{3.48}$$

Note we can use

$$\begin{aligned} v_{\text{mid}}^2 &= c^2\frac{a'^2t^2/c^2}{1+a'^2t^2/c^2} = c^2\frac{(1+\frac{a'D}{2c^2})^2-1}{1+(1+\frac{a'D}{2c^2})^2-1} \\ &= c^2\frac{(1+\frac{a'D}{2c^2})^2-1}{(1+\frac{a'D}{2c^2})^2} = c^2\left(1-\frac{1}{\left(1+\frac{a'D}{2c^2}\right)^2}\right) \end{aligned} \tag{3.49}$$

Summarizing, the time in the Earth frame to get from Earth to the star is

$$t_{\text{trip}} = 2t = 2\sqrt{\frac{D^2}{4c^2} + \frac{D}{a'}} \tag{3.50}$$

the total time for there and the return trip is

$$t_{\text{return}} = 4t \tag{3.51}$$

and for the people in the rocket frame the time is

$$t'_{\text{trip}} = 2t' = 2\frac{c}{a'}\tanh^{-1}\left(\frac{v_{\text{mid}}}{c}\right) \tag{3.52}$$

and the total time for there and the return trip is

$$t'_{\text{return}} = 4t' \tag{3.53}$$

with

$$v_{\text{mid}} = \frac{a't}{\sqrt{1+\frac{a'^2t^2}{c^2}}} = a'\sqrt{\frac{D^2}{4c^2} + \frac{D}{a'}}\left(1 + \frac{a'^2}{c^2}\left(\frac{D^2}{4c^2} + \frac{D}{a'}\right)\right)^{-1/2} \tag{3.54}$$

$$\frac{v_{\text{mid}}}{c^2} = 1 - \frac{1}{\left(1+\frac{a'D}{2c^2}\right)^2} \quad . \tag{3.55}$$

For the specific case of Alpha-Centauri, we then find

$$t_{\text{trip}} = 2\sqrt{\frac{(4.3\text{ lyr})^2}{4c^2} + \frac{(4.3\text{ lyr})}{(1.0\text{ m/s}^2)}} = 2\sqrt{(4.3)^2\text{ yr}^2 + \frac{4.3\text{ yr}}{1.0\text{ m/s}^2}\frac{(299792458\text{ m/s})(3.156\times10^7\text{ s/yr})}{(3.156\times10^7\text{ s/yr})^2}} \tag{3.56}$$

$$= 2\sqrt{\frac{(4.3)^2\text{ yr}^2}{4} + \frac{4.3\text{ yr}}{1.0\text{ m/s}^2}\frac{(299792458\text{ m/yr})}{(3.156\times10^7\text{ s/yr})}} \tag{3.57}$$

$$\approx 2(6.743\text{ yr}) = 13.486\text{ yr} \tag{3.58}$$

and so

$$t_{\text{return}} = 4(6.743\text{ yr}) = 26.972\text{ yr} \quad . \tag{3.59}$$

We have

$$\beta_{\text{mid}} = \frac{v_{\text{mid}}}{c} = \frac{a't}{c\sqrt{1+\frac{a'^2t^2}{c^2}}} \tag{3.60}$$

$$\approx \frac{(1.0\text{ m/s})(6.743\text{ yr})}{\sqrt{c^2 + (1.0\text{ m/s}^2)(6.743\text{ yr})^2}} \tag{3.61}$$

$$\approx \frac{(1.0\text{ m/s})(6.743\text{ yr})(3.156\times10^7\text{ s/yr})}{\sqrt{(299792458\text{ m/s})^2 - (1.0\text{ m/s}^2)(6.743\text{ yr})^2(3.156\times10^7\text{ s/yr})^2}} \tag{3.62}$$

$$\beta_{\text{mid}} \approx .579 \tag{3.63}$$

and so

$$t'_{\text{trip}} = 2t' = 2\frac{c}{a'}\tanh^{-1}(\beta_{\text{mid}}) \tag{3.64}$$

$$\approx 2\frac{299792458 \text{ m/s}}{1.0 \text{ m/s}^2(3.156 \times 10^7 \text{ s/yr})}\tanh^{-1}(.579) \tag{3.65}$$

$$t'_{\text{trip}} = 2(6.279) = 12.56 \text{ yr} \tag{3.66}$$

and

$$t'_{\text{return}} = 4t' = 4(6.279) = 25.12 \text{ yr} \quad . \tag{3.67}$$

4 Spinning Meter Stick

Consider a meter stick with one end marked in red. In the lab frame, the stick moves in the z direction at a velocity v, which may be relativistic. Suppose that in the rest frame of the meter stick, that angular distribution of the meter stick is uniform (i.e., that the red part is equally likely to point in any direction). Find the probability $P(\theta)\,\mathrm{d}\Omega$ that the stick is pointing in the solid angle Ω about θ where θ measures the angle from the z axis.

Solution:

Let's align the z axis with our z axis in spherical coordinates. Then clearly, the biggest difference between the rest frame and the laboratory frame is that the z direction in the lab frame is length contracted. For clarity, call the pion frame the primed frame. Then $P(\theta')$ is simply $1/4\pi$ for a flat probability distribution. Then we need the conversion from $\mathrm{d}\Omega' \to \mathrm{d}\Omega$. We have

$$\mathrm{d}\Omega' = \sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi' \tag{4.1}$$

In addition, from length contraction, we have for a vector of length R' in the pion frame $\gamma^2 = 1/(1-v^2/c^2)$, so $z = z'/\gamma$. Thus (with $r^2 = x^2 + y^2 = x'^2 + y'^2$)

$$\tan\theta = \frac{r}{z} = \frac{\gamma r}{z'} = \gamma\tan\theta' \tag{4.2}$$

$$\frac{\tan^2\theta}{\gamma^2} = \sec^2\theta' - 1 \tag{4.3}$$

$$\sec^2\theta' = \frac{\tan^2\theta}{\gamma^2} + 1 \tag{4.4}$$

Thus

$$\sec^2\theta\,\mathrm{d}\theta = \gamma\sec^2\theta'\,\mathrm{d}\theta' = \gamma\left[\frac{\tan^2\theta}{\gamma^2} + 1\right]\mathrm{d}\theta' \tag{4.5}$$

So then using $\sin\theta' = \sqrt{1-\cos^2\theta'}$ we find

$$\sin\theta'\,\mathrm{d}\theta' = \frac{\sqrt{1-\cos^2\theta'}\cos^2\theta'}{\gamma\cos^2\theta}\,\mathrm{d}\theta \tag{4.6}$$

We have

$$\cos^2\theta'\sqrt{1-\cos^2\theta'} = \frac{1}{\frac{\tan^2\theta}{\gamma^2}+1}\sqrt{1-\frac{1}{\frac{\tan^2\theta}{\gamma^2}+1}} = \frac{\gamma^2}{\tan^2\theta+\gamma^2}\sqrt{\frac{\frac{\tan^2\theta}{\gamma^2}}{1+\frac{\tan^2\theta}{\gamma^2}}} \tag{4.7}$$

$$= \frac{\gamma^2}{\tan^2\theta+\gamma^2}\sqrt{\frac{\tan^2\theta}{\gamma^2+\tan^2\theta}} = \frac{\gamma^2\tan\theta}{(\tan^2\theta+\gamma^2)^{3/2}} \tag{4.8}$$

and so

$$P(\theta')\,\mathrm{d}\Omega' = \frac{1}{4\pi}\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi' = \frac{1}{4\pi}\frac{\gamma^2\tan\theta}{\gamma\cos^2\theta\left(\gamma^2+\tan^2\theta\right)^{3/2}}\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{1}{4\pi}\frac{\gamma^2}{\gamma\cos^3\theta\left(\gamma^2+\tan^2\theta\right)^{3/2}}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{4.9}$$

$$= \frac{1}{4\pi}\frac{\gamma^2}{\gamma\cos^3\theta\left(\gamma^2+\tan^2\theta\right)^{3/2}}\,\mathrm{d}\Omega = \frac{\gamma}{4\pi\left(\gamma^2\cos^2\theta+\sin^2\theta\right)^{3/2}}\,\mathrm{d}\Omega = \frac{\gamma}{4\pi\left(1+\frac{v^2}{\gamma^2c^2}\cos^2\theta\right)^{3/2}}\,\mathrm{d}\Omega \tag{4.10}$$

(4.11)

So that

$$P(\theta) = \frac{\gamma}{4\pi\left(1+\frac{v^2}{\gamma^2c^2}\cos^2\theta\right)^{3/2}} = \frac{\gamma}{4\pi\left(\left[\gamma^2-1\right]\cos^2\theta+1\right)^{3/2}} \tag{4.12}$$

At $\theta = 0$ we find

$$P(0) = \frac{\gamma}{4\pi\left(\gamma^2\right)^{3/2}} = \frac{\gamma}{4\pi\gamma^3} = \frac{1}{4\pi\gamma^2} \tag{4.13}$$

and for $\theta = \pi/2$ we find

$$P(0) = \frac{\gamma}{4\pi\left(1\right)^{3/2}} = \frac{\gamma}{4\pi} \tag{4.14}$$

which makes sense. In the lab frame, as $v \to c$, then $\gamma \ggg 1$ and we are more likely to find the meter stick pointing towards $\theta = \pi/2$ than towards $\theta = 0$ because of the length contraction of the meter stick. We also see that as $v/c \to 0$ that we recover a uniform probability distribution.