University of Chicago Graduate Problems in Physics with Solutions Notes

Kyle J. Bunkers

January 8, 2018

Contents

Chapter 1

Mathematical Physics

1.1 Strange Birth Control

A community practices birth control in the following peculiar fashion. Each set of parents continues having children until a son is born; then they stop. What is the ratio of boys to girls in this community if, in the absence of birth control, 51% of the babies born are male.

Solution:

There is no change. We still have 51% of babies born as male, and 49% as female. Every baby still has a 51% chance of being a baby boy, and stopping does not change this ratio, so that on average we still have 51/49 split.

Another way to think about is consider two child families (GB) that would have gone on to have three children. Given GB, there was a 49% chance for GBG that was eliminated , but note also that there was a 51% chance that the family would have had GBB that was also eliminated. So it didn't change the sex ratio of two child families on average because it eliminated the would-be three child families equally.

1.2 Painting Cubes/Dice

A die consists of a cube which has a different color on each of 6 faces. (a) How many distinguishably different kinds of dice can be made? (b) How many different ways are there to make a pair of dice?

Solution:

(a)

There are clearly 6! dice if we disregard rotation. So we need to see how many possible rotation states are the same. Choose one face of the 6 faces to be on the bottom. Then the top is fixed. Then choose one of the 4 remaining faces to be facing you. Thus, there are $6 \cdot 4 = 24$ equivalent dice through rotations. We thus find $6!/(6 \cdot 4) = 5 \cdot 3 \cdot 2 = 30$ different dice.

(b)

Assuming that both dice have the same six colors, we must be careful about whether die 1 or die 2 are swappable. That is if we swap the labels for die 1 and die 2, is it equivalent to another entry in our list of all permutations. Clearly, we have 30 choices for the first die. Let the two dice be designated AB. We use that AB=BA for any two set of dice so that we divide our $30 \cdot 30$ choices in half. However, this overcounts the corrections because if A and B happen to be the same dice, it shouldn't be divided out twice. We need to be careful of the 30 cases of the dice being identical. The simplest way to see this is consider there are N dice. Then we can construct a matrix

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & \cdots & (1,N) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,N) \\ \vdots & (1,2) & (1,3) & \cdots & (3,N) \\ & & & \ddots & \\ (N,1) & (N,2) & (N,3) & \cdots & (N,N) \end{bmatrix} \tag{1.1}$$

We then see that there are N^2 total elements, but that the upper and lower half are identical since we cannot distinguish $(1,2)$ from $(2,1)$ for example. We must keep the main diagonal though. So we have $N + (N^2 - N)/2 = \frac{2N+N^2-N}{2} = \frac{N^2+N}{2} = \frac{N(N+1)}{2}$ distinct sets. Thus, for $N = 30$ we have $15 \cdot 31 = 450 + 15 = 465$ possibilities.

1.3 Painting an Octahedron

Each face of a regular octahedron is to be given a different color. If eight different colors are available, how many distinguishable octahedra can be made?

Solution:

(An octahedron is made purely of triangle faces along a square "base" in the center). We have 8! ways of painting the octahedron, but we have to remove symmetries. Choose one of the 6 vertices as pointing up, and then choose one of the four top faces to be pointing towards us. Thus there are $6 \cdot 4 = 24$ degeneracies and we thus have

$$\frac{8!}{24} = 8 \cdot 7 \cdot 5 \cdot 3 \cdot 2 = 56 \cdot 30 = 1680 \tag{1.2}$$

distinct octahedra.

1.4 Card Dealing

In dealing 52 cards, consisting of 4 suits of 13 cards each among 2 teams (each team containing two partners), what is the probability that a particular pair of partners obtains at least one complete suit between them?

Solution:

Each team will get 26 cards. There are 52 choose 26 possible ways of putting a 52 card deck into two 26 card hands. Pick a specific suit. Then the opposite pair must choose their 26 cards from

39 non spades which has 39 choose 26 possibilities. Now there are 4 ways of choosing the suit. So the probability for this would be

$$4\frac{\binom{39}{26}}{\binom{52}{26}} \tag{1.3}$$

However, note we have certainly over-counted (in fact double counted) because the top number includes cases where we have two suits that are the same counted twice (if we chose spades as the specific suit above, then there is a hand where we have all spades and hearts, so we can't just multiply by four because that counts this case again in the hearts specific suit case). So we need to subtract off the number of ways of getting exactly two specific suites. How many ways are there of dealing two specific suites to each side? There are 6 or 4 choose 2 ways (choose the two suits to be given). So the probability actually is

$$\frac{4\binom{39}{26}-6}{\binom{52}{26}} = \frac{4\frac{39!}{26!13!}-6}{\frac{52!}{26!26!}} = \frac{4\frac{39!26!}{13!}-6(26!26!)}{52!} \approx 0.0000655 = 0.00655\% \tag{1.4}$$

1.5 Probability Process

A certain process has the property that, regardless of what has transpired in an interval $[0,t]$, the probability that an event will take place in the interval $[t,(t+h)]$ is λh. Assume that the probability of more than one event is of higher order in h. Determine the probability that at a time t, n events have taken place, passing to the limit of h going to zero. Evaluate the average value of n and the average value of n^2 for the distribution function.

Solution:

We have $(\lambda h)^n(1-\lambda h)^{t/h-n}$ as the probability of a specific string of n events and $t/h-n$ non-events taking place. There are $t/h \equiv N$ choose n ways of getting n events so that the probability is

$$P = \frac{N!}{n!(N-n)!}\left(\frac{\lambda t}{N}\right)^n\left(1-\frac{\lambda t}{N}\right)^{N-n} \tag{1.5}$$

In the limit $h \to 0$ then $N \to \infty$ so that for the factorials we can use Stirling's formula which says

$$\lim_{N\to\infty} N! \to \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \sim N^N \tag{1.6}$$

We also know that the top factorial will dominate so that

$$P_{h\to 0} = \frac{N^N}{n!N^{N-n}}(\lambda t)^n N^{-n}\left(1-\frac{\lambda t}{N}\right)^{N-n} = \frac{(\lambda t)^n}{n!}\left(1-\frac{\lambda t}{N}\right)^{-n}\left(1-\frac{\lambda t}{N}\right)^N \tag{1.7}$$

and $(1-\frac{\lambda t}{N})^N \to e^{-\lambda t}$. Therefore

$$P_{h\to 0} = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \tag{1.8}$$

To find the average for n we use

$$\langle n \rangle = \sum_{n=0}^{N} nP = \sum_{n=0}^{N} n \frac{N!}{n!(N-n)!} \left(\frac{\lambda t}{N}\right)^n \left(1 - \frac{\lambda t}{N}\right)^{N-n} \tag{1.9}$$

For $N \to \infty$

$$\langle n \rangle = \sum_{n=0}^{\infty} nP_h = \sum_{n=0}^{\infty} n \frac{(\lambda t)^n}{n!} e^{-\lambda t} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{n(\lambda t)^n}{n!} \tag{1.10}$$

We remember that

$$e^{\alpha x} = \sum_{n=0}^{\infty} \frac{(\alpha x)^n}{n!} \tag{1.11}$$

So using

$$\frac{\mathrm{d}e^{\alpha x}}{\mathrm{d}\alpha} = xe^{\alpha x} = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{(\alpha x)^n}{n!} = \sum_{n=0}^{\infty} \frac{nx(\alpha x)^{n-1}}{n!} = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{n(\alpha x)^n}{n!} \tag{1.12}$$

$$\sum_{n=0}^{\infty} \frac{n(\lambda t)^{n-1}}{n!} = \lambda t e^{\lambda t} \tag{1.13}$$

Thus the average becomes

$$\langle n \rangle = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{n(\lambda t)^n}{n!} = e^{-\lambda t} \lambda t e^{\lambda t} = \lambda t \tag{1.14}$$

Next we find

$$\langle n^2 \rangle = \sum_{n=0}^{\infty} n^2 P_h = \sum_{n=0}^{\infty} n^2 \frac{(\lambda t)^n}{n!} e^{-\lambda t} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{n^2(\lambda t)^n}{n!} \tag{1.15}$$

We use a similar trick to last time.

$$\frac{\mathrm{d}^2 e^{\alpha x}}{\mathrm{d}\alpha^2} = x^2 e^{\alpha x} = \sum_{n=0}^{\infty} \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \frac{(\alpha x)^n}{n!} = \sum_{n=0}^{\infty} \frac{n(n-1)x^2(\alpha x)^{n-2}}{n!} = \frac{1}{\alpha^2} \sum_{n=0}^{\infty} \frac{(n^2-n)(\alpha x)^n}{n!} \tag{1.16}$$

$$= \frac{1}{\alpha^2} \sum_{n=0}^{\infty} \frac{n^2(\alpha x)^n}{n!} - \frac{1}{\alpha^2} \underbrace{\sum_{n=0}^{\infty} \frac{n(\alpha x)^n}{n!}}_{=\alpha x e^{\alpha x}} \tag{1.17}$$

$$\sum_{n=0}^{\infty} \frac{n^2(\alpha x)^n}{n!} = (\alpha^2 x^2 + \alpha x)e^{\alpha x} \tag{1.18}$$

and so

$$\langle n^2 \rangle = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{n^2(\lambda t)^n}{n!} = e^{-\lambda t}(\lambda^2 t^2 - \lambda t)e^{\lambda t} = \lambda^2 t^2 + \lambda t \tag{1.19}$$

1.6 Star Distribution

there are about 6500 stars visible to the naked eye. Sometimes two stars appear very close together, though upon careful examination no physical connection is found between them. Such a pair is called an optical double star. (a) Assuming the stars to be distributed at random on the celestial sphere, compute the expectation value of the number of optical double stars with a separation of no more than $1'$ of arc. (b) What is the probability that there are precisely two optical double stars? (c) Estimate roughly the probability of an optical triple star.

Solution:

(a)

The celestial sphere has 4π steradians of surface. The solid angle carved out by a $\theta = 1' = \frac{1}{3600}^{\circ} \approx 2.9 \times 10^{-4}$ rad angle is given by (Ω is solid angle and in the following formula θ is in radians) $\Omega = 2\pi(1 - \cos\theta)$. Because θ is so small this is essentially $\Omega = \pi\theta^2$. Since all solid angle is equally likely for the first star we have no weighting for that one (i.e., it's one), and for the next star we see that the weighting factor p for likelihood of a star being in the solid angle around the first star

$$p \approx \frac{\pi\theta^2}{4\pi} \approx \frac{\theta^2}{4} \approx 2.1 \times 10^{-8} \tag{1.20}$$

as this is the likelihood of the second star being in that bit of solid angle.

If we want only double stars (that is, we do not count triple stars and above as double stars) we have the probability as $p(1-p)^{N-2}$ so that no other stars are in this area, with N the number of stars [here $N = 6500$] . Because $Np \ll 1$ $[(1-p)^{N-2} \approx 1 - (N-2)p + \cdots \approx 1]$ we can ignore this additional factor. The number of independent pairs (that is, we count each double star once rather than each star in a double star twice) is $\frac{N(N-1)}{2}$ pairings, so the average number of double stars would be

$$\frac{N(N-1)}{2}p \approx 0.45 \tag{1.21}$$

(b)

For there to be exactly two optical double stars, we require the above to happen exactly twice. For a specific two pairs that are distinct, the first star can be anywhere, the second star must be within the radius so p, the next star can be anywhere but this area so $(1-p)$ and its pair must be at this same location so probability p. The rest of the stars have increasingly less area to choose from

$$P_{double} = p(1-p)p(1-2p)(1-3p) \cdot (1-(N-3)p) \tag{1.22}$$

We now need to find the number of ways that this could be done (call this number W). We first must choose 4 stars from the N, and then choose 2 of the 4 to form a pair giving $\binom{N}{4}\binom{4}{2}$ ways of choosing these stars. We must also divide by 2 because we don't care the order of the stars in each pairing. Thus

$$W = \frac{1}{2}\frac{N!}{4!(N-4)!}\frac{4!}{2!2!} = \frac{1}{2}\frac{N!}{2!2!(N-4)!} \sim \frac{1}{2^3}\frac{N^N}{N^{N-4}} \sim \frac{1}{2}\left(\frac{N^2}{2}\right)^2 \tag{1.23}$$

If we use a trick that for $jp \ll 1$

$$\log X = \log \sum_{n=1}^{N-3} [(1 - jp)] \approx \log \sum_{n=1}^{N-3} (-jp) = \frac{-p(N-3)(N-2)}{3} \sim \frac{-pN^2}{2} \tag{1.24}$$

which is nearly the same as the average number of double stars before with the same scaling. If we call $pN^2/2 = \lambda$. We see that in the appropriate limit this is basically

$$WP_{double} \sim \frac{1}{2}\left(\frac{N^2p}{2}\right)^2 X = \frac{1}{2}\lambda^2 e^{-\lambda} \approx 0.063 \tag{1.25}$$

(c)

To find a triple optical star let's look at the number of ways to get a specific three stars first. The first star can be anywhere, the second must be in the correct solid angle, so its there with probability p, and so must the third star, giving another p. If we want the rest to not be in the triplet, then we tack on the necessary $(1-p)^{N-3}$. So we have p^2 if it is a triplet or higher, and $p^2(1-p)^{N-3}$ if it must be a triplet.

Now how many ways are there to form this triplet? There are N choose 3 ways. Thus the probability is

$$p^2 \frac{N!}{3!(N-3)!} \sim \frac{p^2 N^3}{6} \approx 2 \times 10^{-5} \tag{1.26}$$

1.7 Eigenvalues and Eigenvectors

Find eigenvalues and normalized eigenvectors of the matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{1.27}$$

Solution:

There are four eigenvalues. We note that this matrix is the inverse of itself, thus it has four eigenvalues with two eigenvalues being inverses of two others. We also see that it is Hermitian, and so its eigenvalues have absolute value 1. We also note the trace is zero. Thus

$$\lambda_1 + \frac{1}{\lambda_1} + \lambda_2 + \frac{1}{\lambda_2} = 0 \tag{1.28}$$

$$\frac{\lambda_1^2 - 1}{\lambda_1} = \frac{1 - \lambda_2^2}{\lambda_2} \tag{1.29}$$

Clearly $\lambda_1 = 1$ and $\lambda_2 = -1$ will satisfy these equations. We see

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{1.30}$$

So for $\mu = 1$ we see we require $x_1 = x_4$ and $x_2 = x_3$. For $\mu = -1$ we require $x_1 = -x_4$ and $x_2 = -x_3$.

Thus two eigenvectors are

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\\0\end{bmatrix} \tag{1.31}$$

1.8 Trace and Square Trace

Let λ_i (for $i = 1, 2, 3$) be the eigenvalues of the matrix

$$H = \begin{bmatrix}2 & -1 & -3\\-1 & 1 & 2\\-3 & 2 & 3\end{bmatrix} \tag{1.32}$$

Calculate the sums (a) $\sum_{i=1}^{3} \lambda_i$ and (b) $\sum_{i=1}^{3} \lambda_i^2$.

Solution:

The trace doesn't change when diagonalizing a matrix. Thus

$$\sum_{i=1}^{3} \lambda_i = 2 + 1 + 3 = 6 \tag{1.33}$$

One way to see this is to write matrix $\mathbf{M}$ as M_{ij}. Then the trace (with the Einstein sum notation) is $\mathrm{Tr}[\mathbf{M}] = M_{ii}$. When diagonalizing a matrix, we have the matrix $\mathbf{M} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P}$ with $\mathbf{D}$ the eigenvalue matrix. We use

$$\mathrm{Tr}[\mathbf{AB}] = A_{ij}B_{ji} = B_{ji}A_{ij} = B_{ij}A_{ji} = \mathrm{Tr}[\mathbf{BA}] \tag{1.34}$$

$$\mathrm{Tr}[\mathbf{ABC}] = A_{ij}B_{jk}C_{ki} = B_{jk}C_{ki}A_{ij} = B_{ij}C_{jk}A_{ki} = \mathrm{Tr}[\mathbf{BCA}] \tag{1.35}$$

So that $\mathrm{Tr}\,\mathbf{M} = \mathrm{Tr}[\mathbf{P}^{-1}\mathbf{D}\mathbf{P}] = \mathrm{Tr}[\mathbf{D}\mathbf{P}^{-1}\mathbf{P}] = \mathrm{Tr}[\mathbf{D}]$.

We can find the sum of the squares either by squaring the matrix and taking the trace, or by finding the eigenvalues. We can also note that the matrix is symmetric. I prefer matrix multiplication to eigenvalue solving (esp. since we need only calculate the diagonals), so

$$H^2 = \begin{bmatrix}2 & -1 & -3\\-1 & 1 & 2\\-3 & 2 & 3\end{bmatrix}\begin{bmatrix}2 & -1 & -3\\-1 & 1 & 2\\-3 & 2 & 3\end{bmatrix} = \begin{bmatrix}4+1+9 & * & *\\ * & 1+1+4 & *\\ * & * & 9+4+9\end{bmatrix} = \begin{bmatrix}14 & * & *\\ * & 6 & *\\ * & * & 22\end{bmatrix} \tag{1.36}$$

Thus the sum of the squares of the eigenvalues is $14 + 6 + 22 = 42$. Even more simply, since the matrix is symmetric we could use

$$\mathrm{Tr}[H^2] = \mathrm{Tr}[HH^T] = H_{ij}H_{ij} = \sum_{i,j=1}^{3} H_{ij}^2 = 4+1+9+1+1+4+9+4+9 = 14+6+22 = 42 \tag{1.37}$$

1.9 Pauli Matrices

Calculate $T = \text{Tr}[e^{i\boldsymbol{\sigma}\cdot\mathbf{a}}e^{i\boldsymbol{\sigma}\cdot\mathbf{b}}]$, where the components of $\boldsymbol{\sigma}$ are the three standard Pauli matrices σ_i for spin 1/2.

Solution:

The Pauli matrices are as below

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.38}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{1.39}$$

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.40}$$

We easily see that $\sigma_1\sigma_1 = \sigma_2\sigma_2 = \sigma_3\sigma_3 = \mathbb{1}$, and $\sigma_a\sigma_b = \delta_{ab}\mathbb{1} + i\epsilon_{abc}\sigma_c$. Thus

$$a_a b_b \sigma_a \sigma_b = a_a b_b \delta_{ab}\mathbb{1} + i\epsilon_{abc}a_a b_b \sigma_c \tag{1.41}$$

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\mathbf{b})\mathbb{1} + i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma} \tag{1.42}$$

So for $\mathbf{a} = \mathbf{b}$ we find

$$(\mathbf{a}\cdot\boldsymbol{\sigma})^2 = a^2\mathbb{1} \tag{1.43}$$

So using that $a\hat{\mathbf{n}}_a = \mathbf{a}$ and $b\hat{\mathbf{n}}_b = \mathbf{b}$ we get

$$e^{i\boldsymbol{\sigma}\cdot\mathbf{a}} = \mathbb{1}\cos a + i(\hat{\mathbf{n}}_a\cdot\boldsymbol{\sigma})\sin a \tag{1.44}$$

$$e^{i\boldsymbol{\sigma}\cdot\mathbf{b}} = \mathbb{1}\cos b + i(\hat{\mathbf{n}}_b\cdot\boldsymbol{\sigma})\sin b \tag{1.45}$$

$$\tag{1.46}$$

We use

$$e^{i\boldsymbol{\sigma}\cdot\mathbf{a}} = \mathbb{1} + i\mathbf{a}\cdot\boldsymbol{\sigma} + \frac{i^2}{2!}(\mathbf{a}\cdot\boldsymbol{\sigma})^2 + \frac{i^3}{3!}(\mathbf{a}\cdot\boldsymbol{\sigma})^3 + \ldots \tag{1.47}$$

$$= \mathbb{1} + i(\mathbf{a}\cdot\boldsymbol{\sigma}) + \frac{i^2a^2}{2!}\mathbb{1} + \frac{i^3a^3}{3!}(\mathbf{a}\cdot\boldsymbol{\sigma}) + \cdots \tag{1.48}$$

$$= \mathbb{1}(1 + \frac{i^2a^2}{2!} + \frac{i^4a^4}{4!} + \frac{i^6a^6}{6!} + \cdots) + i(\mathbf{a}\cdot\boldsymbol{\sigma})\left(\frac{a}{1!} + \frac{a^3}{3!} + \frac{a^5}{5!} + \cdots\right) \tag{1.49}$$

$$= \mathbb{1}(1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \cdots) + i(\mathbf{a}\cdot\boldsymbol{\sigma})\left(\frac{a}{1!} + \frac{a^3}{3!} + \frac{a^5}{5!} + \cdots\right) \tag{1.50}$$

$$= \mathbb{1}\cos a + i(\mathbf{a}\cdot\boldsymbol{\sigma})\sin a \tag{1.51}$$

Thus,

$$e^{i\boldsymbol{\sigma}\cdot\mathbf{a}}e^{i\boldsymbol{\sigma}\cdot\mathbf{b}} = (\mathbb{1}\cos a + i(\hat{\mathbf{n}}_a\cdot\boldsymbol{\sigma})\sin a)(\mathbb{1}\cos b + i(\hat{\mathbf{n}}_b\cdot\boldsymbol{\sigma})\sin b) \tag{1.52}$$

$$= \mathbb{1}\cos a\cos b + i(\hat{\mathbf{n}}_b\cdot\boldsymbol{\sigma})\cos a\sin b + i(\hat{\mathbf{n}}_a\cdot\boldsymbol{\sigma})\sin a\cos b - (\hat{\mathbf{n}}_a\cdot\boldsymbol{\sigma})(\hat{\mathbf{n}}_b\cdot\boldsymbol{\sigma})\sin a\sin b \tag{1.53}$$

We can use $\text{Tr}[\sigma_a] = 0$ and $\text{Tr}[\sigma_a \sigma_b] = 2\delta_{ab}$. Thus, the two central terms are zero when we take the trace. So we then need

$$\text{Tr}[(\hat{\mathbf{n}}_a \cdot \boldsymbol{\sigma})(\hat{\mathbf{n}}_b \cdot \boldsymbol{\sigma})] = \text{Tr}[n_{a,a}\sigma_a n_{b,b}\sigma_b] = 2n_{a,a}n_{b,b}\delta_{ab} = 2\hat{\mathbf{n}}_a \cdot \hat{\mathbf{n}}_b \tag{1.54}$$

So we get altogether

$$T = 2\cos a \cos b - 2(\hat{\mathbf{n}}_a \cdot \hat{\mathbf{n}}_b)\sin a \sin b = 2\left(\cos a \cos b - \mathbf{a} \cdot \mathbf{b} \sin a \sin b\right) \tag{1.55}$$

1.10 Symmetric Second Rank Tensor

Consider a symmetric second-rank tensor $\overleftrightarrow{\mathbf{T}}$ with components T_{ik} $(i, k = 1, 2, 3)$. (a) Show that there exist three invariants, say I_0, I_1, I_2, with respect to coordinate transformations, associated with $\overleftrightarrow{\mathbf{T}}$. (b) Associate a surface $1 = \sum_{i,k} T_{ik}X_iX_k$ (X_j are Cartesian coordinates) with $\overleftrightarrow{\mathbf{T}}$. Give interpretations of the three invariants in terms of properties of the surface.

Solution:

(a) A coordinate transformation can be written as $\mathbf{PTP}^\intercal$. Where $\mathbf{PP}^\intercal = \mathbb{1}$ with eigenvalues of ± 1. This leaves distances unchanged, for example. This property also ensures that the trace is left unchanged, as it is equivalent to finding the eigenvalues of $\mathbf{T}$.

To see, this we use that

$$\det(\mathbf{T} - \lambda\mathbb{1}) = 0 \tag{1.56}$$

must remain unchanged by transformation. This implies that the characteristic polynomial generated must be unchanged. Because the powers of λ are independent, then we also must have their coefficients be independent. Writing these out yields

$$\begin{aligned}\det(\mathbf{T} - \lambda\mathbb{1}) = {}& \lambda^3 + \lambda^2(T_{11} + T_{22} + T_{33}) \\ &+ \lambda\left(T_{12}T_{21} - T_{11}T_{22} + T_{13}T_{31} + T_{23}T_{32} - T_{11}T_{33} - T_{22}T_{33}\right) \\ &+ \left(-T_{13}T_{22}T_{31} + T_{12}T_{23}T_{31} + T_{13}T_{21}T_{32} - T_{11}T_{23}T_{32} - T_{12}T_{21}T_{33} + T_{11}T_{22}T_{33}\right)\end{aligned} \tag{1.57}$$

We note the λ^0 coefficient is the determinant of $\mathbf{T}$ itself.

In other words, the invariants are

$$I_0 = \det(\mathbf{T}) \tag{1.58}$$

$$I_1 = T_{12}T_{21} - T_{11}T_{22} + T_{13}T_{31} + T_{23}T_{32} - T_{11}T_{33} - T_{22}T_{33} \tag{1.59}$$

$$I_2 = \text{Tr}(\mathbf{T}) = T_{11} + T_{22} + T_{33} \tag{1.60}$$

(b)

If we go in a system where are axes are aligned with the eigenvectors, we can more easily explain our three values. Let the three diagonal entries be $T_{11} = a^{-2}$, $T_{22} = b^{-2}$, and $T_{33} = c^{-2}$. Then $1 = T_{ik}X_iX_k$ is an ellipsoid surface. The volume of the ellipsoid is related to the determinant, I_0.

Then if we slice ellipses along the ellipsoid, I_1 is related to the areas of these ellipses. Finally, I_2 is simply saying that an ellipsoid is the sum of the reciprocal squares of the lengths of the intercepts form the center of the ellipse.

1.11 Residues of Simple Functions

What are the residues of the following functions at the points indicated? (a) e^{az}/z^5 at $z = 0$. (b) $1/\sin^3(z)$ at $z = 0$.

Solution:

(a) We remember the residue comes from finding $\oint \frac{C\mathrm{d}z}{z}$, where C is the residue. In other words, in the Laurent series of a function $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, it is the coefficient a_{-1}. All other powers of z will vanish in the contour integral. Thus, writing out the power series, we see

$$\frac{1}{z^5}e^{az} = \frac{1}{z^5}\left(1 + az + \frac{(az)^2}{2!} + \frac{(az)^3}{3!} + \frac{(az)^4}{4!} + \frac{(az)^5}{5!} + \cdots\right) \tag{1.61}$$

$$= \frac{1}{z^5} + \frac{a}{z^4} + \frac{a^2}{2!z^3} + \frac{a^3}{3!z^2} + \frac{a^4}{4!z} + \cdots \tag{1.62}$$

Thus the residue is $a^4/4! = \frac{a^4}{24}$.

(b) The power series may be written as

$$\begin{aligned}\frac{1}{\sin^3(z)} &= \frac{1}{(z - \frac{z^3}{3!} + \frac{z^5}{5!})^3 + \cdots} = \frac{1}{z^3}\frac{1}{\left(1 - \frac{z^2}{3!} + \cdots\right)} = \frac{1}{z^3}\frac{1}{1 - 3\frac{z^3}{3!} + \cdots} = \\ &= \frac{1}{z^3}\left(1 + 3\frac{z^2}{3!} + \cdots\right) = \frac{1}{z^3} + \frac{1}{2z} + \cdots\end{aligned} \tag{1.63}$$

and so the residue is $\frac{1}{2}$.

1.12 Integral 1

Calculate

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{(k^2 - a^2 - i\epsilon)^3}\,,\ a > 0 \tag{1.64}$$

Solution:

We note that we can extend the integral into a contour integral because if we parameterize $k = Re^{i\theta}$ for this part of the contour the integrand will rapidly approach zero if we let $R \to \infty$ due to a scaling of $\frac{1}{R^5}$ for the integrand. We note that we have poles here as the denominator goes to zero at $k^2 = a^2 + i\epsilon$ or $k = \pm\sqrt{a + i\epsilon}$. Note that these poles are both order 3, because of the cube in the denominator. Thus,

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{(k^2 - a^2 - i\epsilon)^3} + \underbrace{\lim_{R\to\infty}\int_{\pi}^{0}\frac{iRe^{i\theta}\,\mathrm{d}\theta}{\left(R^2e^{-2i\theta} - a^2 - i\epsilon\right)^3}}_{\to 0} = \lim_{\epsilon\to 0^+}\oint\frac{\mathrm{d}k}{(k^2 - a^2 - i\epsilon)^3} \tag{1.65}$$

Here, I will close the contour in the upper half-plane so the only residue is at $\sqrt{a^2+i\epsilon} \equiv z_0$. We need to find the integral, and using the sum of the residues, we see

$$\oint \frac{\mathrm{d}k}{(k^2-a^2-i\epsilon)^3} = \frac{2\pi i}{(3-1)!}\frac{\mathrm{d}^2}{\mathrm{d}k^2}\left[(z-z_0)^3 \frac{1}{(z-z_0)^3(z+\sqrt{a^2+i\epsilon})^3}\right]_{z=z_0} = i\pi \frac{\mathrm{d}}{\mathrm{d}k}\left[\frac{-3}{(z+\sqrt{a^2+i\epsilon})^4}\right]_{z=z_0} \tag{1.66}$$

$$= i\pi\left[\frac{12}{(z+\sqrt{a^2+i\epsilon})^5}\right]_{z=z_0} = 12i\pi\frac{1}{2^5(a^2+i\epsilon)^{5/2}} = \frac{12i\pi}{32}(a^2+i\epsilon)^{-5/2} = i\frac{3\pi}{8}(a^2+i\epsilon)^{-5/2} \tag{1.67}$$

Thus, as $\epsilon \to 0^+$, we find

$$\lim_{\epsilon\to 0^+}\oint \frac{\mathrm{d}k}{(k^2-a^2-i\epsilon)^3} = \frac{3\pi i}{8a^5} \tag{1.68}$$

Hence,

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{(k^2-a^2-i\epsilon)^3} = \frac{3\pi i}{8a^5} \tag{1.69}$$

1.13 Integral 2

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,\mathrm{d}x \tag{1.70}$$

Solution:

This integral would normally require the use of the Plemlej formula, but we are lucky that the possible residue at $z=0$ is completely removable and so adds nothing to the contour integral.

We can change this into a contour integral if we extend the integral into the upper plane with a giant half circle, and make a tiny half-circle dip around the origin (to miss $z=0$), that we will get a proper answer. For the upper half plane, we will show that choosing this appropriately will make this part go to zero. Call this contour C_1. If we make the small half-circle dip near the origin contour C_2 and let it get smaller and smaller so that we get back to our original integral in the limit. The entire contour (contour C) is then

$$\oint_C \frac{\sin^3 z}{z^3}\,\mathrm{d}z = \lim_{\epsilon\to 0^+}\underbrace{\left[\int_{-\infty}^{-\epsilon}\frac{\sin^3 z}{z^3}\,\mathrm{d}z + \int_{\epsilon}^{\infty}\frac{\sin^3 z}{z^3}\,\mathrm{d}z\right]}_{\to\int_{-\infty}^{\infty}\mathrm{d}x\ \frac{\sin^3 x}{x^3}} + \underbrace{\int_{C_1}\frac{\sin^3 z}{z^3}\,\mathrm{d}z}_{\overset{?}{\to}0} + \lim_{\epsilon\to 0^+}\int_{\pi}^{2\pi}\frac{i\epsilon\theta e^{i\theta}\sin^3(\epsilon e^{i\theta})\,\mathrm{d}\theta}{\epsilon^3 e^{3i\theta}} \tag{1.71}$$

The C_1 part seems like it would go to zero because it goes as $R^{-3}\to 0$ as $R\to\infty$, but remember that for imaginary z, that $\sin(z)$ can be greater than 1. We will remedy this later. We note that the final term in C_2 still goes to zero because as $\epsilon\to 0^+$ then $\sin(\epsilon e^{i\theta})^3 \to \epsilon^3 e^{3i\theta} + \mathcal{O}(\epsilon^4)$ so that the integrand approaches zero. As we will soon see, the $\oint_c \mathrm{d}z\ \sin^3 z/z^3$ won't actually be what we want on the left hand side.

Note, that because $\sin(z) \to 0$ as $z \to 0$, we need to be careful about the application of Cauchy's theorem. That is, note that

$$\frac{\sin^3 z}{z^3} = \frac{(z - \frac{z^3}{3!} + \cdots)^3}{z^3} = 1 - \frac{3z^5}{z^3} + \cdots \tag{1.72}$$

which would seem to imply that we do not have any poles for us to calculate over. So we use instead that $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ and so $\sin^3 z = \frac{e^{i3z}+3e^{iz}e^{-i2z}-3e^{-iz}e^{2iz}-e^{-3iz}}{(2i)^3} = \frac{e^{3iz}+3e^{-iz}-3e^{iz}-e^{-3iz}}{(2i)^3}$ Then it is clear that we have some poles, as

$$\frac{e^{3iz}}{z^3} = \frac{1}{z^3} + \frac{3iz}{z^3} + \frac{(3iz)^2}{2!z^3} + \cdots \tag{1.73}$$

for example.

Now the exponentials with positive imaginary exponent are closed in the upper half plane so that $i\Im[z] < 0$. Those with negative imaginary exponent will be closed in the lower half plane so that $-i\Im[z] < 0$. The lower half-plane calculations do not contribute as there are no residues surrounded at $z = 0$. That is we can write

$$\frac{\sin^3 z}{z} = \frac{e^{i3z} - 3e^{iz} + 3e^{-iz} - e^{-3iz}}{(2iz)^3} \tag{1.74}$$

And we close the contour in the upper half with our indent around the origin for

$$\frac{e^{3iz} - 3e^{iz}}{(2iz)^3} \tag{1.75}$$

and close for the lower half with the indent excluding the origin for

$$\frac{e^{-3iz} + 3e^{-iz}}{(2iz)^3} \tag{1.76}$$

Because the arcs into the upper half plane and lower half plane are zero for both of these, we can add them together to form the contour for $\sin^3(z)/z^3$.

Thus

$$\int_{-\infty}^{\infty} \frac{\sin^3 z}{z^3}\,\mathrm{d}z = \oint_C \frac{e^{3iz}}{(2i)^3 z^3} - 3\oint_C \frac{e^{iz}}{(2i)^3 z^3} \tag{1.77}$$

$$= \frac{2\pi i}{2!(2i)^3}[(3i)^2 e^{3iz}]_{z=0} - \frac{2\pi i}{2!(2i)^3}[3(i^2)e^{iz}]_{z=0} \tag{1.78}$$

$$= \pi\left[\frac{3^2}{2^3} - \frac{3}{2^3}\right] = \frac{6\pi}{8} = \frac{3\pi}{4} \tag{1.79}$$

Alternatively, we could use that for real x we have

$$\frac{\sin^3(x)}{x^3} = \Im\left[\frac{-e^{3ix} + 3e^{ix}}{4x^3}\right] \tag{1.80}$$

And so we could just use $(-e^{3ix}+3e^{ix})/x^3$ and take the imaginary part at the end. This would yield

$$-\oint_C \frac{e^{3iz}}{4z^3}+3\oint_C \frac{e^{iz}}{4z^3} = -\frac{2\pi i}{2!4}[(3i)^2e^{3iz}]_{z=0}+\frac{2\pi i}{2!4}[3(i^2)e^{iz}]_{z=0} \tag{1.81}$$

$$= i\pi\left[\frac{3^2}{4}-\frac{3}{4}\right] = i\frac{6\pi}{4} = i\frac{3\pi}{2} \tag{1.82}$$

and so

$$\int_{-\infty}^{\infty}\frac{\sin^3 z}{z^3}\,\mathrm{d}z = \Im\left[-\oint_C \frac{e^{3iz}}{4z^3}+3\oint_C \frac{e^{iz}}{4z^3}\right] = \frac{3\pi}{4} \tag{1.83}$$

1.14 Integral 3

Calculate (a)

$$I_1 = \int_0^\infty \frac{x\,\mathrm{d}x}{e^x-1} \tag{1.84}$$

(b)

$$I_3 = \int_0^\infty \frac{x^3\,\mathrm{d}x}{e^x-1} \tag{1.85}$$

Solution:

(a)

Consider extending the integral into a contour where we go up to $i\pi$ (with $z=x+iy$). So the contour goes from the origin to inf, up to $y=i\pi$ back to $x=0$ and down to the origin. There are no singularities for this integrand in this region, so this contour is zero. Thus we have for $n\geq 1$ (Note that the contour integral going up to $y=i\pi$ at infinity vanishes because $1/(e^{\infty+iy}-1)$ makes the integrand so small).

$$0=\oint\frac{x^n\,\mathrm{d}x}{e^x-1} = \int_0^\infty \frac{x^n\,\mathrm{d}x}{e^x-1}+\int_\infty^0\frac{(x+i\pi)^n\,\mathrm{d}x}{e^{x+i\pi}-1}+\int_\pi^0\frac{(iy)^n i\,\mathrm{d}y}{e^{iy}-1} \tag{1.86}$$

$$\int_0^\infty \frac{x^n\,\mathrm{d}x}{e^x-1} = \int_0^\infty\frac{(x+i\pi)^n\,\mathrm{d}x}{-(e^x+1)}+i^{n+1}\int_0^\pi\frac{y^n\,\mathrm{d}y}{e^{iy}-1} \tag{1.87}$$

We use (for n odd)

$$2\Re\left[i^{n+1}\frac{y^n}{e^{iy}-1}\right] = \frac{i^{n+1}y^n}{e^{iy}-1}+\frac{(-i)^{n+1}y^n}{e^{-iy}-1} = i^{n+1}\frac{y^n(e^{-iy}-1+[e^{iy}-1])}{(e^{iy}-1)(e^{-iy}-1)} = y^n i^{n+1}\frac{e^{-iy}+e^{iy}-2}{2-e^{-iy}-e^{iy}} = -y^n i^{n+1} \tag{1.88}$$

And so taking the real part of the integral, we find

$$\int_0^\infty\frac{x^n\,\mathrm{d}x}{e^x-1} = -\int_0^\infty\frac{\Re[(x+i\pi)^n]\,\mathrm{d}x}{e^x+1}-i^{n+1}\int_0^\pi\frac{y^n}{2}\,\mathrm{d}y \tag{1.89}$$

Consider $n = 1$ now. Then

$$\int_0^\infty \frac{x\,\mathrm{d}x}{e^x-1} = -\int_0^\infty \frac{x\,\mathrm{d}x}{e^x+1} + \int_0^\pi \frac{y}{2}\,\mathrm{d}y \tag{1.90}$$

We can then use that

$$\frac{1}{e^x-1} - \frac{1}{e^x+1} = \frac{2}{e^{2x}-1} \tag{1.91}$$

So, with $u = 2x$ and $\mathrm{d}u = 2\,\mathrm{d}x$ we see

$$\int_0^\infty \mathrm{d}x \left[\frac{x^n}{e^x-1} - \frac{x^n}{e^x+1}\right] = \int_0^\infty \mathrm{d}x\,\frac{2x^n}{e^{2x}-1} = \int_0^\infty \mathrm{d}u\,\frac{u^n}{2^n(e^u-1)} = 2^{-n}\int_0^\infty \mathrm{d}x\,\frac{x^n}{e^x-1} \tag{1.92}$$

$$(1-2^{-n})\int_0^\infty \mathrm{d}x\,\frac{x^n}{e^x-1} = \int_0^\infty \mathrm{d}x\,\frac{x^n}{e^x+1} \tag{1.93}$$

Therefore,

$$\int_0^\infty \frac{x\,\mathrm{d}x}{e^x-1} = -\frac{1}{2}\int_0^\infty \mathrm{d}x\,\frac{x}{e^x-1} + \frac{\pi^2}{4} \tag{1.94}$$

$$\frac{3}{2}\int_0^\infty \frac{x\,\mathrm{d}x}{e^x-1} = \frac{\pi^2}{4} \tag{1.95}$$

$$\int_0^\infty \frac{x\,\mathrm{d}x}{e^x-1} = \frac{\pi^2}{6} \tag{1.96}$$

Then for $n = 3$, we see

$$\int_0^\infty \frac{x^3\,\mathrm{d}x}{e^x-1} = -\int_0^\infty \frac{(x^3-3\pi^2 x)\,\mathrm{d}x}{e^x+1} - \int_0^\pi \frac{y^3}{2}\,\mathrm{d}y \tag{1.97}$$

$$\int_0^\infty \frac{x^3\,\mathrm{d}x}{e^x-1} = -\int_0^\infty \frac{x^3\,\mathrm{d}x}{e^x+1} + 3\pi^2 \underbrace{\int_0^\infty \mathrm{d}x\,\frac{x}{e^x+1}}_{\frac{1}{2}\frac{\pi^2}{6}} - \frac{\pi^4}{8} \tag{1.98}$$

$$\frac{15}{8}\int_0^\infty \frac{x^3\,\mathrm{d}x}{e^x-1} = \frac{\pi^4}{4} - \frac{\pi^4}{8} = \frac{\pi^4}{8} \tag{1.99}$$

$$\int_0^\infty \frac{x^3\,\mathrm{d}x}{e^x-1} = \frac{8}{15}\frac{\pi^4}{8} = \frac{\pi^4}{15} \tag{1.100}$$

1.15 Fourier Integral

Develop $f(x) = \cos(x^2)$ in a Fourier integral.

Solution:

I think this means, find the Fourier transform of $\cos(x^2)$. Then with

$$f(x) = \int_{-\infty}^\infty \mathrm{d}k\; e^{ikx}\widehat{f}(k) \tag{1.101}$$

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{d}x\; e^{-ikx}\cos(x^2) \tag{1.102}$$

$$\cos(x^2) = \frac{e^{ix^2}+e^{-ix^2}}{2} \tag{1.103}$$

Thus,

$$2\widehat{f}(k) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \mathrm{d}x\ e^{-ikx}(e^{ix^2} + e^{-ix^2})\right] \tag{1.104}$$

We can complete the square to find

$$i(x^2 - kx) = i[(x - \frac{k}{2})^2 - \frac{k^2}{4}] \tag{1.105}$$

$$-i(x^2 + kx) = -i[(x + \frac{k}{2})^2 - \frac{k^2}{4}] \tag{1.106}$$

and so

$$\widehat{f}(k) = \frac{1}{4\pi}\left[\int_{-\infty}^{\infty} \mathrm{d}x\ e^{-ikx}e^{ix^2} + \int_{-\infty}^{\infty} \mathrm{d}x\ e^{-ikx}e^{-ix^2}\right] = \frac{1}{4\pi}\left[\int_{-\infty}^{\infty} \mathrm{d}x\ e^{i(x-\frac{k}{2})^2}e^{-ik^2/4} + \int_{-\infty}^{\infty} \mathrm{d}x\ e^{-i(x+\frac{k}{2})^2}e^{ik^2/4}\right] \tag{1.107}$$

Then with $y = (x \pm k/2)$ we see we have

$$\widehat{f}(k) = \frac{1}{4\pi}\left[e^{-ik^2/4}\int_{-\infty}^{\infty} \mathrm{d}y\ e^{-iy^2} + \text{c.c.}\right] = \frac{1}{2\pi}\Re\left[e^{-ik^2/4}\int_{-\infty}^{\infty} \mathrm{d}y\ e^{-iy^2}\right] \tag{1.108}$$

We can use $x = (-i)^{1/2}y = (e^{-i\pi}e^{i\pi/2})^{1/2}y = e^{-i\pi/4}y$, thus

$$\int_{-\infty-\frac{i}{\sqrt{2}}}^{\infty-\frac{i}{\sqrt{2}}} \mathrm{d}x\ e^{-i\pi/4}e^{-x^2} = e^{-i\pi/4}\sqrt{\pi} = (1-i)\sqrt{\frac{\pi}{2}} \tag{1.109}$$

where the $-\frac{i}{\sqrt{2}}$ indicates we are actually not on the real axis with our integration. To show this is legitimate, make y the complex z and consider the parametrization $z = \frac{t-ti}{\sqrt{2}}$ so that $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1-i}{\sqrt{2}}$. This is a line starting from the upper left quadrant and going to the bottom right quadrant as t increases.

$$\int_C e^{-iz^2}\,\mathrm{d}z = \int_{-\infty}^{\infty} e^{-i(t-ti)^2/2}\frac{1-i}{\sqrt{2}}\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{-it^2(1-2i-1)/2}e^{-i\pi/4}\,\mathrm{d}t = e^{-i\pi/4}\int_{-\infty}^{\infty} e^{-t^2}\,\mathrm{d}t \tag{1.110}$$

Clearly we can form contours for each half of the real plane, where we create a contour that goes from the origin out to $\pm\inf$ along the parameterization, arcs back to the real axis and returns to the origin. The arc will not contribute to the integral, and so the integral along the real axis and this parametrization must be equal.

Hence,

$$\widehat{f}(k) = \frac{1}{2\pi}\Re\left[e^{-ik^2/4}\int_{-\infty}^{\infty} \mathrm{d}y\ e^{-iy^2}\right] = \frac{\sqrt{\pi}}{\sqrt{2}2\pi}\Re\left[\left(\cos(k^2/4) + i\sin(k^2/4)\right)(1-i)\right] \tag{1.111}$$

$$= \frac{1}{\sqrt{8\pi}}\left[\cos(k^2/4) + \sin(k^2/4)\right] \tag{1.112}$$

1.16 Laplace Transform

Find $f(t)$ by inverting the Laplace transform

$$\frac{a^2}{p^2+a^2} = \int_0^\infty \mathrm{d}t\ e^{-pt} f(t) \tag{1.113}$$

Solution:

To invert a Laplace transform, we must use (γ is chosen so that it is above all the singularities of $f(t)$ in the complex plane)

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}p\ e^{pt} \int_0^\infty \mathrm{d}t'\ e^{-pt'} f(t') = \frac{1}{2\pi i}\int_0^\infty \mathrm{d}t' \int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}p\ e^{pt} e^{-pt'} f(t') \tag{1.114}$$

and use $q = p - \gamma$ and we find

$$= \frac{1}{2\pi i}\int_0^\infty \mathrm{d}t' \int_{-i\infty}^{+i\infty} \mathrm{d}q\ e^{(q+\gamma)(t-t')} f(t') = \frac{e^{\gamma t}}{2\pi i}\int_0^\infty \mathrm{d}t' \int_{-i\infty}^{+i\infty} \mathrm{d}q\ e^{q(t-t')} e^{-\gamma t'} f(t') \tag{1.115}$$

and we use that $\int_{-\infty}^{\infty} \mathrm{d}k\ e^{-ik(t-t')} = 2\pi\delta(t-t')$ so with $z = iq$, we find $\int_{-i\infty}^{i\infty} \mathrm{d}q\ e^{q(t-t')} = -\int_{\infty}^{-\infty} \mathrm{d}z\ ie^{-iz(t-t')} = 2\pi i\delta(t-t')$

$$= \frac{2\pi i e^{\gamma t}}{2\pi i}\int_0^\infty \mathrm{d}t'\ e^{-\gamma t'} i\delta(t-t') f(t') = e^{\gamma t-\gamma t} f(t) = f(t) \tag{1.116}$$

assuming that $t > 0$. Otherwise we get zero, as we should for a Laplace transform. Thus

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}p\ e^{pt} \frac{a^2}{p^2+a^2} \tag{1.117}$$

For this we see that we must find the poles and choose γ such that it contains all the poles when we hook around for our contour integration. We see we have

$$p^2 + a^2 = 0 \Rightarrow p = \pm ia \tag{1.118}$$

and so we need only choose $\gamma > 0$. Both are order one poles, so that the residue is simply

$$\frac{e^{iat}a^2}{ia+ia} + \frac{e^{-iat}a^2}{-ia-ia} = a\frac{e^{iat}-e^{-iat}}{2i} = a\sin(at) \tag{1.119}$$

and so

$$f(t) = \frac{1}{2\pi i} 2\pi i(a\sin(at)) = a\sin(at) \tag{1.120}$$

1.17 Integral 4

Calculate

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi}{\alpha + \cos\varphi} \tag{1.121}$$

(a) when $\alpha > 1$, (b) when $\alpha = \alpha_0 + i\epsilon$, α_0 and ϵ real, $\epsilon > 0$ and $0 < \alpha_0 < 1$ as $\epsilon \to 0$, (c) when $\alpha = -1$.

Solution:

Consider making this a closed circular contour (around the unit circle) in the complex plane with z being parameterized around this point. Then

$$\oint f(z)\mathrm{d}z \;= \int_0^{2\pi} f(e^{i\phi})ie^{i\phi}\mathrm{d}\varphi \tag{1.122}$$

while

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi}{\alpha + \frac{e^{i\varphi}+e^{-i\varphi}}{2}} = \oint \frac{-ie^{-i\varphi}\,\mathrm{d}z}{\alpha + \frac{z+z^*}{2}} = \oint \frac{-i\,\mathrm{d}z}{z\alpha + z\frac{z+z^*}{2}} \tag{1.123}$$

which becomes ($zz^* = 1$)

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi}{\alpha + \cos\varphi} = \frac{2}{i}\oint \frac{\mathrm{d}z}{z^2 + 2z\alpha + 1} = \frac{2}{i}\oint \frac{\mathrm{d}z}{[z-(-\alpha+\sqrt{\alpha^2-1})][z-(-\alpha-\sqrt{\alpha^2-1})]} \tag{1.124}$$

The roots are at $z = -\alpha \pm \sqrt{\alpha^2 - 1}$.

(a)

For $\alpha > 1$ then z is purely real and we see that in order for a root to be inside the unit circle we require $-\alpha \pm \sqrt{\alpha^2+1} < 1$. Clearly $-\alpha > 1$ implies that the $\alpha - \sqrt{\alpha^2-1}$ is outside the unit circle. We can also see that at $\alpha = 1$ both roots are on the unit circle, so the $+$ sign will go into the unit circle and is our only pole.

Thus, as this is an order one pole

$$\oint \frac{\mathrm{d}z}{[z-(-\alpha+\sqrt{\alpha^2-1})][z-(-\alpha-\sqrt{\alpha^2-1})]} = 2\pi i \frac{1}{-\alpha+\sqrt{\alpha^2-1}-(-\alpha-\sqrt{\alpha^2-1})} = \frac{2\pi i}{2\sqrt{\alpha^2-1}} \tag{1.125}$$

so for $\alpha > 1$ we find

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi}{\alpha+\cos\varphi} = \frac{2}{i}\frac{\pi i}{\sqrt{\alpha^2-1}} = \frac{2\pi}{\sqrt{\alpha^2-1}} \tag{1.126}$$

(b)

$$\oint \frac{\mathrm{d}z}{[z-(-\alpha+\sqrt{\alpha_0^2+2i\alpha_0\epsilon-\epsilon^2-1})][z-(-\alpha-\sqrt{\alpha_0^2+2i\alpha_0\epsilon-\epsilon^2-1})]} \tag{1.127}$$

We see that as $\epsilon \to 0$ that we will get a small deflection up or down on the imaginary axis of about $i\sqrt{1-\alpha_0^2}$. If we include first order ϵ terms, then we have

$$z = -\alpha_0 - i\epsilon \pm i\sqrt{1-\alpha_0^2 - 2i\alpha_0\epsilon} = -\alpha_0 - i\epsilon \pm i\sqrt{1-\alpha_0^2} \mp \frac{2ii\alpha_0\epsilon}{2\sqrt{1-\alpha_0^2}} \tag{1.128}$$

$$= -\alpha_0 - i\epsilon \pm i\sqrt{1-\alpha_0^2} \pm \frac{\alpha_0\epsilon}{\sqrt{1-\alpha_0^2}} \tag{1.129}$$

thus

$$|z_\pm|^2 = (-\alpha_0 \pm \frac{\alpha_0\epsilon}{\sqrt{1-\alpha_0^2}})^2 + (-\epsilon \pm \sqrt{1-\alpha_0^2})^2 = \alpha_0^2(1 \mp \frac{2\epsilon}{\sqrt{1-\alpha_0^2}}) + 1 - \alpha_0^2 \mp 2\epsilon\sqrt{1-\alpha_0^2} \tag{1.130}$$

$$= 1 \mp \left(\frac{2\epsilon\alpha_0^2}{\sqrt{1-\alpha_0^2}} + 2\epsilon\sqrt{1-\alpha_0^2}\right) \tag{1.131}$$

and so we see that only the + sign's residue will contribute as the − root is ever so slightly outside the unit circle as $\epsilon \to 0$. Thus

$$\oint \frac{dz}{[z-(-\alpha+i\sqrt{1-\alpha_0^2})][z-(-\alpha-i\sqrt{1-\alpha_0^2})]} \tag{1.132}$$

$$= 2\pi i \left[\frac{1}{[-\alpha+i\sqrt{1-\alpha_0^2}-(-\alpha-i\sqrt{1-\alpha_0^2})]}\right] \tag{1.133}$$

$$= 2\pi i \left[\frac{1}{2i\sqrt{1-\alpha_0^2}}\right] = \frac{\pi}{\sqrt{1-\alpha_0^2}} \tag{1.134}$$

and so

$$\int_0^{2\pi} \frac{d\varphi}{\alpha + \cos\varphi} = \frac{2}{i}\oint \frac{dz}{z^2 + 2z\alpha + 1} = \frac{-2\pi i}{\sqrt{1-\alpha_0^2}} \tag{1.135}$$

(c)

For this case, we see that the integral diverges. One way to see this is to use $1+\cos(2x) = \sin^2(x)$ and so the integral becomes $\csc^2(\theta)$.

1.18 Integral 5

Evaluate

$$I_1 = \int_{-\infty}^{+\infty} \frac{dx}{\cosh x} \tag{1.136}$$

$$I_3 = \int_{-\infty}^{+\infty} \frac{dx}{\cosh^3 x} \tag{1.137}$$

Solution:

Use $u = \sinh(x)$ so $\mathrm{d}u = \cosh(x)\,\mathrm{d}x = \sqrt{1+u^2}\,\mathrm{d}x$. Then

$$I_1 = \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{\sqrt{1+u^2}\sqrt{1+u^2}} = \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{1+u^2} \tag{1.138}$$

There are poles at $u = \pm i$. So we hook the contour in the upper (one could do the lower if desired) half plane, and it's clear that the denominator will vanish as $R \to \infty$ for $u = z = Re^{i\theta}$, so the residue from the $+i$ root is

$$I_1 = \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{1+u^2} = 2\pi i \frac{1}{i-(-i)} = \frac{2\pi i}{2i} = \pi \tag{1.139}$$

Note for I_3 that we get

$$I_3 = \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{(1+u^2)^2} = 2\pi i \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{1}{(z+i)^2}\right]_{z=i} = 2\pi i \frac{-2}{(i+i)^3} = \frac{-4\pi i}{-8i} = \frac{\pi}{2} \tag{1.140}$$

1.19 Integral 6

Evaluate

$$I = \int_0^{2\pi} \mathrm{d}\phi\, \frac{b + a\cos\phi}{a^2 + b^2 + 2ab\cos\phi} \tag{1.141}$$

with $|a| \neq |b|$.

Solution:

If we write $z = e^{i\phi}$ $\mathrm{d}z = iz\,\mathrm{d}\phi$ then we see the integral becomes

$$I = \oint \mathrm{d}z\, \frac{b + a\frac{z+\frac{1}{z}}{2}}{iz(a^2 + b^2 + 2ab\frac{z+z^*}{2})} = \oint \mathrm{d}z\, \frac{2bz + az^2 + a}{2iz(abz^2 + z(a^2+b^2) + ab)} \tag{1.142}$$

$$= \frac{1}{2iab}\oint \frac{2bz + a(z^2+1)}{z(z^2 + 2z\alpha + 1)} \tag{1.143}$$

where $\alpha = \frac{a^2+b^2}{2ab}$. It's clear we have poles at $z = \beta_\pm = -\alpha \pm \sqrt{\alpha^2 - 1} = \beta_+$ and $z = 0$. Call

$$I_a = \oint \frac{2bz + a(z^2+1)}{z(z^2 + 2z\alpha + 1)} = \frac{2bz + a(z^2+1)}{z(z-\beta_+)(z-\beta_-)} \tag{1.144}$$

We can just calculate the $\beta_\pm$

$$\alpha^2 - 1 = \frac{(a^2+b^2)^2 - 4a^2b^2}{4a^2b^2} = \frac{(a^2-b^2)^2}{4a^2b^2} \tag{1.145}$$

$$\beta_+ = -\frac{a^2+b^2}{2ab} + \frac{a^2-b^2}{2ab} = -\frac{b}{a} \tag{1.146}$$

$$\beta_- = -\frac{a^2+b^2}{2ab} - \frac{a^2-b^2}{2ab} = -\frac{a}{b} \tag{1.147}$$

First let's choose $|a| > |b|$ so that β_+ is the pole inside the unit circle (with $z = 0$). Then

$$I_a = 2\pi i \left[\frac{2b\beta_+ + a(\beta_+^2 + 1)}{\beta_+(\beta_+ - \beta_-)} + \frac{a}{\beta_+\beta_-}\right] \tag{1.148}$$

$$\beta_+\beta_- = 1 \tag{1.149}$$

$$\beta_+ - \beta_- = 2\sqrt{\alpha^2 - 1} \tag{1.150}$$

$$\begin{aligned} I_a &= 2\pi i \left[\frac{2b\frac{-b}{a} + a(\frac{b^2}{a^2} + 1)}{\frac{-b}{a}(\frac{a^2-b^2}{ab})} + a\right] = 2\pi i \left[\frac{a - \frac{b^2}{a}}{\frac{b^2}{a^2} - 1} + a\right] \\ &= 2\pi i \left[a\frac{1 - \frac{b^2}{a^2}}{\frac{b^2}{a^2} - 1} + a\right] = 0 \end{aligned} \tag{1.151}$$

and

$$I = \frac{1}{2iab}(0) = 0 \tag{1.152}$$

So then choose $|b| > |a|$ so that β_- and 0 are the poles, and we find

$$\begin{aligned} I_a &= 2\pi i \left[\frac{2b\beta_- + a(\beta_-^2 + 1)}{\beta_-(\beta_- - \beta_+)} + \frac{a}{\beta_+\beta_-}\right] = 2\pi i \left[\frac{2b\frac{-a}{b} + a(\frac{a^2}{b^2} + 1)}{\frac{-a}{b}\frac{b^2-a^2}{ab}} + a\right] \\ &= 2\pi i \left[\frac{-2a + \frac{a^3}{b^2} + a}{\frac{a^2-b^2}{b^2}} + a\right] = 2\pi i \left[a\frac{\frac{a^2}{b^2} - 1}{\frac{a^2}{b^2} - 1} + a\right] \\ &= 4\pi i a \end{aligned} \tag{1.153}$$

and so

$$I = \frac{I_a}{2iab} = \frac{4\pi i a}{2iab} = \frac{2\pi}{b} \tag{1.154}$$

1.20 Gamma Function

The gamma function is defined by

$$\Gamma(x) = \int_0^\infty \mathrm{d}t\ t^{x-1}e^{-t}\,,\ \Re(x) > 0 \tag{1.155}$$

Show that for $0 < x < 1$,

$$\int_0^\infty \mathrm{d}t\ t^{x-1}\cos t = \Gamma(x)\cos\left(\frac{\pi x}{2}\right) \tag{1.156}$$

$$\int_0^\infty \mathrm{d}t\ t^{x-1}\sin t = \Gamma(x)\sin\left(\frac{\pi x}{2}\right) \tag{1.157}$$

Solution:

Consider the integral

$$\int_0^\infty \mathrm{d}t\ t^{x-1}e^{it} \tag{1.158}$$

Create an arc in the upper plane (we should be careful around 0, but we can simply make a small arc around it as well to exclude it). The far arc part will obviously disappear. Go up to $y=\infty$. We then have no poles in our box and so (note $\Im[e^{i\theta}]>0$ for $\pi/2>\theta>0$ so that the exponential is of the form e^{-R}). So

$$\oint \mathrm{d}t\ t^{x-1}e^{it} = \int_0^\infty \mathrm{d}t\ t^{x-1}e^{it} + \cancel{\int_0^{\pi/2} \mathrm{d}\theta\ iRe^{i\theta}(Re^{i\theta})^{x-1}e^{iRe^{i\theta}}} + \int_\infty^0 \mathrm{d}t\ i(it)^{x-1}e^{-t} = 0 \tag{1.159}$$

And so

$$i^x\int_0^\infty \mathrm{d}t\ t^{x-1}e^{-t} = \int_0^\infty \mathrm{d}t\ t^{x-1}e^{it} \tag{1.160}$$

$$i^x\Gamma(x) = \int_0^\infty \mathrm{d}t\ t^{x-1}e^{it} \tag{1.161}$$

We remember

$$i^x = (e^{i\pi/2})^x = e^{i\pi x/2} = \cos\left(\frac{\pi x}{2}\right) + i\sin\left(\frac{\pi x}{2}\right) \tag{1.162}$$

and thus taking first the Real part of the above equation and then the Imaginary part we find

$$\cos\left(\frac{\pi x}{2}\right)\Gamma(x) = \int_0^\infty \mathrm{d}t\ t^{x-1}\cos(t) \tag{1.163}$$

$$\sin\left(\frac{\pi x}{2}\right)\Gamma(x) = \int_0^\infty \mathrm{d}t\ t^{x-1}\sin(t) \tag{1.164}$$

as desired.

1.21 Integral 7

Show that

$$\int_0^\infty \mathrm{d}x\ \frac{\sinh(\alpha x)}{\sinh(\pi x)} = \frac{1}{2}\tan\frac{a}{2} \tag{1.165}$$

for $-\pi<a<\pi$ by integrating $e^{az}/\sinh(\pi z)$ around an appropriate contour.

Solution:

Consider a contour that extends from $-\infty$ to ∞ on the real line, goes up to $y=1$ and returns from $x=\infty$ to $-\infty$. The upward portions clearly vanish as they are too far away and the $\sinh(\pi x)$

goes to ∞ causing the integrand to vanish. Note that

$$\sinh(\pi z) = \frac{e^{\pi z} - e^{-\pi z}}{2} \tag{1.166}$$

$$\sinh(\pi 0) = \frac{1-1}{2} = 0 \tag{1.167}$$

$$\sinh(\pi i) = \frac{e^{i\pi} - e^{-i\pi}}{2} = \frac{-1-(-1)}{2} = 0 \tag{1.168}$$

$$\sinh(\pi(x \pm i)) = \frac{e^{\pm i\pi}e^{\pi x} - e^{\mp i\pi}e^{-\pi x}}{2} = \frac{e^{-\pi x} - e^{\pi x}}{2} = \sinh(-\pi x) = -\sinh(\pi x) \tag{1.169}$$

and so we have two poles on our contours. Thus, we must add πi times the residue of these two poles.

So

$$\oint \mathrm{d}z\, \frac{e^{az}}{\sinh(\pi x)} = \int_{-\infty}^{0} \mathrm{d}x\, \frac{e^{ax}}{\sinh(\pi x)} + \int_{0}^{\infty} \mathrm{d}x\, \frac{e^{ax}}{\sinh(\pi x)} + \int_{\infty}^{0} \mathrm{d}x\, \frac{e^{a(i+x)}}{\sinh(\pi(i+x))} + \int_{0}^{-\infty} \mathrm{d}x\, \frac{e^{a(i+x)}}{\sinh(\pi(i+x))} \tag{1.170}$$

$$= \pi i \mathrm{Res}_{z=0,z=i} = \pi i \lim_{x\to 0} \frac{(x)e^{ax}}{\sinh(\pi x)} + \pi i \lim_{z\to i} \underbrace{\frac{(z-i)e^{ai}}{\sinh(\pi z)}}_{(z-i)e^{zi}/-\sinh(\pi(z-i))} = \pi i \left[\frac{1}{\pi} - \frac{e^{ai}}{\pi}\right] \tag{1.171}$$

And we also have

$$\int_{-\infty}^{0} \mathrm{d}x\, \frac{e^{ax}}{\sinh(\pi x)} = -\int_{0}^{-\infty} \mathrm{d}x\, \frac{e^{ax}}{\sinh(\pi x)} = \int_{0}^{\infty} \mathrm{d}(-x)\, \frac{e^{-a(-x)}}{\sinh(-\pi[-x])} = -\int_{0}^{\infty} \mathrm{d}x\, \frac{e^{-ax}}{\sinh(\pi x)} \tag{1.172}$$

$$\int_{\infty}^{0} \frac{e^{a(i+x)}}{\sinh(\pi(i+x))} = -\int_{0}^{\infty} \mathrm{d}x\, \frac{e^{a(i+x)}}{-\sinh(\pi x)} = \int_{0}^{\infty} \mathrm{d}x\, \frac{e^{a(i+x)}}{\sinh(\pi x)} \tag{1.173}$$

$$\int_{0}^{-\infty} \frac{e^{a(i+x)}}{\sinh(\pi(i+x))} = -\int_{0}^{\infty} \mathrm{d}(-x)\, \frac{e^{a(i-(-x))}}{\sinh(-\pi x)} = -\int_{0}^{\infty} \mathrm{d}x\, \frac{e^{a(i-x)}}{\sinh(\pi x)} \tag{1.174}$$

And so

$$\int_{-\infty}^{0} \mathrm{d}x\, \frac{e^{ax}}{\sinh(\pi x)} + \int_{0}^{\infty} \mathrm{d}x\, \frac{e^{ax}}{\sinh(\pi x)} = \int_{0}^{\infty} \mathrm{d}x\, \frac{e^{ax} - e^{-ax}}{\sinh(\pi x)} = 2\int_{0}^{\infty} \mathrm{d}x\, \frac{\sinh(ax)}{\sinh(\pi x)} \tag{1.175}$$

$$\int_{-\infty}^{0} \mathrm{d}x\, \frac{e^{a(i+x)}}{\sinh(\pi(i+x))} + \int_{0}^{\infty} \mathrm{d}x\, \frac{e^{a(i+x)}}{\sinh(\pi(i+x))} = e^{ai}\int_{0}^{\infty} \mathrm{d}x\, \frac{e^{ax} - e^{-ax}}{\sinh(\pi x)} = 2e^{ai}\int_{0}^{\infty} \mathrm{d}x\, \frac{\sinh(ax)}{\sinh(\pi x)} \tag{1.176}$$

Thus, using

$$\tan(x/2) = \frac{1}{i}\frac{e^{ix/2} - e^{-ix/2}}{e^{ix/2} + e^{-ix/2}} = \frac{e^{-ix/2}}{ie^{-ix/2}}\frac{e^{ix} - 1}{e^{ix} + 1} = \frac{1}{i}\frac{e^{ix} - 1}{e^{ix} + 1} \tag{1.177}$$

and so

$$2(1+e^{ai})\int_0^\infty \mathrm{d}x\ \frac{\sinh(ax)}{\sinh(\pi x)} = i(1-e^{ai}) \tag{1.178}$$

$$\int_0^\infty \mathrm{d}x\ \frac{\sinh(ax)}{\sinh(\pi x)} = -\frac{1}{2i}\frac{1-e^{ai}}{1+e^{ai}} = \frac{1}{2}\tan\left(\frac{a}{2}\right) \tag{1.179}$$

as desired.

1.22 Integral 8

Evaluate by contour integration

$$\int_0^\infty \frac{x^{1/2}\,\mathrm{d}x}{1+x^2} \tag{1.180}$$

Show your contour and all poles and branch cuts in the complex plane.

Solution:

We will take a contour from $-\infty$ to ∞ and close it in the upper half of the complex plane using a trick for what the value is along the $-\infty$ to 0 contour. This is effectively choosing a branch cut. The upper half-circle arc will clearly vanish as the integrand goes as $RR^{1/2}R^{-2} \sim R^{-1/2}$ and so as $R \to \infty$ we will indeed have the arc contribution vanish.

Thus, we have

$$\int_{-\infty}^0 \mathrm{d}x\ \frac{x^{1/2}\,\mathrm{d}x}{1+x^2} + \int_0^\infty \mathrm{d}x\ \frac{x^{1/2}\,\mathrm{d}x}{1+x^2} = 2\pi i\mathrm{Res}_{z=i} \tag{1.181}$$

First

$$\int_\infty^0 \mathrm{d}(-x)\ \frac{[-(-x)]^{1/2}\,\mathrm{d}x}{1+(-[-x])^2} = -\int_0^\infty \mathrm{d}x\ \frac{(-x)^{1/2}}{1+x^2} = -\int_0^\infty \mathrm{d}x\ \frac{ix^{1/2}}{1+x^2} \tag{1.182}$$

and so

$$\int_{-\infty}^0 \mathrm{d}x\ \frac{x^{1/2}\,\mathrm{d}x}{1+x^2} + \int_0^\infty \mathrm{d}x\ \frac{x^{1/2}\,\mathrm{d}x}{1+x^2} = (1+i)\int_0^\infty \frac{x^{1/2}\,\mathrm{d}x}{1+x^2} \tag{1.183}$$

The residue is for a pole of order one, so

$$\left[\frac{x^{1/2}\cancel{(x-i)}}{(x+i)\cancel{(x-i)}}\right]_{x=i} = \frac{i^{1/2}}{2i} = \frac{e^{i\pi/4}}{2e^{i\pi/2}} = \frac{e^{-i\pi/4}}{2} = \frac{1-i}{2\sqrt{2}} \tag{1.184}$$

Thus

$$(1+i)\int_0^\infty \frac{x^{1/2}\,\mathrm{d}x}{1+x^2} = 2\pi i\frac{1-i}{2\sqrt{2}} \tag{1.185}$$

$$\int_0^\infty \frac{x^{1/2}\,\mathrm{d}x}{1+x^2} = i\frac{\pi}{\sqrt{2}}\frac{1-i}{1+i} = \frac{\pi}{\sqrt{2}}\frac{i+1}{1+i} = \frac{\pi}{\sqrt{2}} \tag{1.186}$$

To do this integral through contour integration, one has to actually use a "keyhole" arc. That is we choose the positive real axis as our branch cut, and have a keyhole-like curve. Then the small circle around $z = 0$ will not contribute anything and we have

$$\begin{aligned}\oint \mathrm{d}z\ \frac{z^{1/2}}{1+z^2} &= \lim_{\epsilon\to 0^+}\left[\int_{i\epsilon}^{i\epsilon+\infty} \mathrm{d}z\ \frac{z^{1/2}}{1+z^2} + \int_{-i\epsilon+\infty}^{-i\epsilon} \mathrm{d}z\ \frac{z^{1/2}}{1+z^2}\right] \\ &+ \lim_{\epsilon\to 0^+}\left[\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon} \mathrm{d}\theta\ \frac{iRR^{1/2}e^{i\theta/2}}{1+R^2e^{2i\theta}} + \int_{\epsilon}^{2\pi-\epsilon} \mathrm{d}\theta\ \frac{i\epsilon\epsilon^{1/2}e^{i\theta/2}}{1+\epsilon^2e^{2i\theta}}\right]\end{aligned} \tag{1.187}$$

The last two integrals vanish in the limits. The first integral simply becomes the integral we desire when we take the limit. For the second integral, it would at first appear that we would get the negative of the first integral, but remember that we have a branch cut. We write $z^{1/2} = \exp(\frac{1}{2}\log(z))$. Then in the first quadrant $\log(z) \to \log(|z|)$ so we write $z^{1/2} \to \sqrt{z}$ whereas in the fourth quadrant $\log(z) \to \log(|z|) + 2\pi i$ since $\log(z) = \log(|z|) + i\arg(z)$ and so $z^{1/2} \to \sqrt{z}\exp(i\pi) = -\sqrt{z}$. So then the integral in the fourth quadrant becomes

$$\int_{-i\epsilon+\infty}^{-i\epsilon} \mathrm{d}z\ \frac{z^{1/2}}{1+z^2} \to \int_{\infty}^{0} \mathrm{d}z\ \frac{\exp\left(\frac{1}{2}\log|z| + \frac{2i\pi}{2}\right)}{1+\exp\left(\log|z| + 2\pi i\right)} = \int_{\infty}^{0} \mathrm{d}z\ \frac{-\sqrt{z}}{1+z^2} = \int_{0}^{\infty} \mathrm{d}x\ \frac{\sqrt{x}}{1+x^2} \tag{1.188}$$

and so we get

$$\oint \mathrm{d}z\ \frac{z^{1/2}}{1+z^2} = 2\int_0^\infty \mathrm{d}x\ \frac{\sqrt{x}}{1+x^2} \tag{1.189}$$

We then find the residue as before for $z = \pm i$ [our branch cut makes $-i = \exp(3i\pi/2)$] and so

$$\begin{aligned}\frac{1}{2\pi i}\oint \mathrm{d}z\ \frac{z^{1/2}}{1+z^2} &= \left[\frac{x^{1/2}\cancel{(x-i)}}{(x+i)\cancel{(x-i)}}\right]_{x=i} + \left[\frac{x^{1/2}\cancel{(x+i)}}{\cancel{(x+i)}(x-i)}\right]_{x=-i} = \frac{i^{1/2}}{2i} + \frac{(-i)^{1/2}}{-2i} \\ &= \frac{\exp(i\pi/4)}{2\exp(i\pi/2)} - \frac{\exp(3i\pi/4)}{2\exp(i\pi/2)} = \frac{\exp(-i\pi/4)}{2} - \frac{\exp(i\pi/4)}{2} \\ &= \frac{1-i}{2\sqrt{2}} - \frac{1+i}{2\sqrt{2}} = -\frac{2i}{\sqrt{2}}\end{aligned} \tag{1.190}$$

and so

$$\cancel{2}\int_0^\infty \mathrm{d}x\ \frac{\sqrt{x}}{1+x^2} = 2\pi i\frac{-\cancel{2}i}{\sqrt{2}} \tag{1.191}$$

$$\int_0^\infty \mathrm{d}x\ \frac{\sqrt{x}}{1+x^2} = \frac{2\pi}{\sqrt{2}} = \frac{\pi}{\sqrt{2}} \tag{1.192}$$

which agrees with our previous, cleverer substitution method above.

1.23 Evaluate Series Through Contour Integration

Evaluate the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = \frac{-7\pi^4}{720} \tag{1.193}$$

by contour integral techniques. [Hint: Use the fact that the function $\frac{1}{\sin(\pi z)}$ has poles along the real axis at $z = 0, \pm 1, \pm 2, \cdots$.]

Solution:

We can use the fact that if we have a contour enclosing the real axis then

$$\oint \frac{\mathrm{d}z}{\sin(\pi z)} = 2\pi i \ \mathrm{Res}_{z=0,\pm 1,\ldots} \tag{1.194}$$

and that

$$\mathrm{Res}_{z=\pm n} = \frac{(z \mp n)}{(-1)^n \sin(\pi(z \mp n))} = \frac{(-1)^n}{\pi} \tag{1.195}$$

Let's consider an integral around the origin as a circle with the radius increasing to ∞ in all directions. As our integral goes as $1/z^4$ this will clearly vanish. For the poles on the real and negative real axis we can find that for $|n| > 0$ that ($\oint_{+\infty}$ is for counting poles on the positive real axis)

$$\oint_{+\infty} \frac{\mathrm{d}z}{z^4 \sin(\pi z)} = 2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 \pi} \tag{1.196}$$

$$\oint_{-\infty} \frac{\mathrm{d}z}{z^4 \sin(\pi z)} = 2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{(-n)^4 \pi} = 2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 \pi} \tag{1.197}$$

Adding these means that

$$4i \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \tag{1.198}$$

We left out the residue at $n = 0$

Thus we use that the residue of $1/[z^4 \sin(\pi z)]$ at $z = 0$ is given by a_{-1} of its Laurent series. To simplify this, the Taylor series for $1/\sin(z)$ is

$$\frac{1}{\sin(z)} = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \mathcal{O}(z^4) \tag{1.199}$$

so

$$\frac{1}{z^4 \sin(z)} = \frac{1}{z^5} + \frac{1}{6z^3} + \frac{7}{360z} + \mathcal{O}(z^4) \tag{1.200}$$

and so the residue is $7/360$. However we used $\sin(z\pi)$ and so it is given by $7\pi^3/(360)$. And so adding these residues and realizing the contour integral must be zero, we find

$$4i \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} + 2\pi i \frac{7\pi^3}{360} = 0 \tag{1.201}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \frac{-7\pi^4}{720} \tag{1.202}$$

1.24 Riemann Surface Analytic Surface

Consider the analytic function

$$F(z) = \rho(z) \ln \left[1 - \frac{2z}{a}\{1 - \rho(z)\}\right] \tag{1.203}$$

with $\rho(z) = \sqrt{z-a}/\sqrt{z}$. Here a is real and positive. Choose the branch lines for $\rho(z)$ along the real axis from $-\infty$ to 0 and from a to ∞.

(a) Discuss the Riemann surface of $F(z)$. (b) Show that there is one sheet where $F(z)$ may be represented in the form

$$F(z) = F(z_0) + (z - z_0) \int_a^{\infty} \mathrm{d}s \, \frac{W(s)}{(s-z)(s-z_0)} \tag{1.204}$$

and determine $W(s)$.

Solution:

(a) We worry about the multivaluedness of $F(z)$ due to the logarithm and the square root term.

(b) For the logarithm we have

$$\log(z) = \log(re^{i\theta}) = \log|r| + i\theta + 2\pi i p \tag{1.205}$$

for p an integer and $-\pi < \theta \leq \pi$. The principal value is $p = 0$.

For the square root

$$\rho(z) \equiv \sqrt{\frac{z-a}{z}} = \sqrt{1 - \frac{a}{z}} \tag{1.206}$$

if we take $z = x + i\epsilon$ with $\epsilon \to 0$ we see

$$\rho = \sqrt{1 - \frac{a}{x(1 - \frac{i\epsilon}{x})}} = \sqrt{1 - \frac{a}{x}(1 + i\epsilon + (i\epsilon^2) + \cdots)} \tag{1.207}$$

We see as $\epsilon \to 0$ then we get

$$\rho = \sqrt{1 - \frac{a}{x}} \tag{1.208}$$

For $a > 0$ we see that ρ will be purely real if $x < 0$ and if $x > a$. For these cases

$$\rho = \sqrt{1 - \frac{a}{x}} \tag{1.209}$$

However, there is another solution, namely the negative square root. Call the two solutions $\pm$ so

$$\rho_{\pm} = \pm\sqrt{1 - \frac{a}{z}} \tag{1.210}$$

Then we have for consistency the two possible solutions (assuming p has been chosen for the logarithm)

$$F_{\pm} = \pm\sqrt{1-\frac{a}{z}}\ln\left(1-\frac{2z}{a}\left[1\mp\sqrt{1-\frac{a}{z}}\right]\right) \tag{1.211}$$

We then note that for $x<0$ the argument of the ln is positive for both cases. That is

$$1+\frac{2|x|}{a}\left[1\mp\sqrt{1+\frac{a}{|x|}}\right]>0 \tag{1.212}$$

the $+$ sign is obvious, for the $-$ sign note that for $x<0$ and $|x|>a$ it is also clear. For $x<0$ and $|x|<a$ we then have

$$\frac{a^2}{4|x|^2}>0 \tag{1.213}$$

$$1+\frac{a}{|x|}+\frac{a^2}{4|x|^2}>1+\frac{a}{|x|} \tag{1.214}$$

$$(1+\frac{a}{2|x|})^2>1+\frac{a}{|x|} \tag{1.215}$$

Note that both sides are positive, and so taking the positive square root on both sides (and using $a/|x|>0$)

$$1+\frac{a}{2|x|}>\sqrt{1+\frac{a}{|x|}} \tag{1.216}$$

$$1-\sqrt{1+\frac{a}{|x|}}>-\frac{a}{2|x|} \tag{1.217}$$

$$\frac{2|x|}{a}\left(1-\sqrt{1+\frac{a}{|x|}}\right)>-1 \tag{1.218}$$

$$1+\frac{2|x|}{a}\left(1-\sqrt{1+\frac{a}{|x|}}\right)>0 \tag{1.219}$$

Now if $x>a$, we see that

$$1-\frac{2x}{a}\left[1\mp\sqrt{1-\frac{a}{x}}\right]<0 \tag{1.220}$$

This is again obvious for the $+$ sign, for the minus sign we note

$$0<\frac{a^2}{4x^2} \tag{1.221}$$

$$1-\frac{a}{x}<1-\frac{a}{x}+\frac{a^2}{4x^2}=(1-\frac{a}{2x})^2 \tag{1.222}$$

Note that because $a/x < 1$ then $a/(2x) < 1/2 < 1$ and so when we take the positive square root we find

$$\sqrt{1-\frac{a}{x}} < 1-\frac{a}{2x} \tag{1.223}$$

$$\frac{a}{2x} < 1-\sqrt{1-\frac{a}{x}} \tag{1.224}$$

$$1 < \frac{2x}{a}\left(1-\sqrt{1-\frac{a}{x}}\right) \tag{1.225}$$

$$1-\frac{2x}{a}\left(1-\sqrt{1-\frac{a}{x}}\right) < 0 \tag{1.226}$$

as desired.

Then the discontinuity across the real axis from $-\infty$ to 0 will be

$$F_+ - F_- = \sqrt{1-\frac{a}{z}}\ln\left(1-\frac{2z}{a}\left[1-\sqrt{1-\frac{a}{z}}\right]\right) - \left\{-\sqrt{1-\frac{a}{z}}\ln\left(1-\frac{2z}{a}\left[1+\sqrt{1-\frac{a}{z}}\right]\right)\right\} \tag{1.227}$$

$$= \sqrt{1-\frac{a}{z}}\left(\ln\left|\left\{1-\frac{2z}{a}\left[1-\sqrt{1-\frac{a}{z}}\right]\right\}\left\{1-\frac{2z}{a}\left[1+\sqrt{1-\frac{a}{z}}\right]\right\}\right| + 2\pi ip + 2\pi ip\right) \tag{1.228}$$

$$= \sqrt{1-\frac{a}{z}}\left(\ln\left|1-\frac{2z}{a}\left[1-\sqrt{1-\frac{a}{z}}+1+\sqrt{1-\frac{a}{z}}\right]+\frac{4z^2}{a^2}\left[1-(1-\frac{a}{z})\right]\right| + 4\pi ip\right) \tag{1.229}$$

$$= \sqrt{1-\frac{a}{z}}\left(\ln\left|1-\frac{4z}{a}+\frac{4z}{a}\right| + 4\pi ip\right) \tag{1.230}$$

$$= \sqrt{1-\frac{a}{z}}\,(0+4\pi ip) = 4\pi ip\sqrt{1-\frac{a}{z}} \tag{1.231}$$

For the branch line of $x > a$ define the argument of the logarithm as $\omega_\pm$ with $\pm$ matching the $F_\pm$ it is in. We see

$$\omega_\pm = 1-\frac{2z}{a}\left(1\mp\sqrt{1-\frac{a}{z}}\right) \tag{1.232}$$

$$\log\omega_+ = \ln|\omega_+| + i(\pm\pi) + 2\pi ip \tag{1.233}$$

$$\log\omega_- = \ln|\omega_-| + i(\pm\pi) + 2\pi ip \tag{1.234}$$

where these two $\pm i\pi$ indicate uncertainty in which sign to choose for $\pm\pi$. Because of the definition of log it must be π for both so that they coincide for a given p.

So

$$F_+ - F_- = \sqrt{1-\frac{a}{z}}\ln(\omega_+) - \left\{-\sqrt{1-\frac{a}{z}}\ln(\omega_-)\right\} \tag{1.235}$$

$$= \sqrt{1-\frac{a}{z}}[\ln|\omega_+| + \pi i + 2\pi ip] - \left\{-\sqrt{1-\frac{a}{z}}[\ln|\omega_-| + \pi i + 2\pi ip]\right\} \tag{1.236}$$

$$= \sqrt{1 - \frac{a}{z}}\left[\ln|\omega_+| + \ln|\omega_-| + 2\pi i + 4\pi i p\right] \tag{1.237}$$

$$= \sqrt{1 - \frac{a}{z}}\left[2\pi i + 4\pi i p\right] \tag{1.238}$$

(b) We consider the function

$$G(z) = \frac{F(z) - F(z_0)}{z - z_0} \tag{1.239}$$

For this, we take a contour that extends to infinity in all directions but avoids our two branch cuts. As our function is analytic everywhere in this region, the contour integral must be zero. Thus, by Cauchy's integral theorem we have

$$G(z) = \frac{1}{2\pi i}\oint \frac{G(z')\mathrm{d}z'}{z - z'} \tag{1.240}$$

Thus, we only need to calculate the discontinuity across our two branch cuts along our contour to find the answer (note the $F_+(z_0) - F_-(z_0)$ term will cancel in both integrals as z_0 should be in an analytic region.)

$$G(z) = \frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{2\pi i}\left[\int_{-\infty}^{0} \mathrm{d}s\ \frac{F_+(s) - F_-(s)}{(s - z_0)(s - z)} + \int_{a}^{\infty} \mathrm{d}s\ \int_{-\infty}^{0} \mathrm{d}s\ \frac{F_+(s) - F_-(s)}{(s - z_0)(s - z)}\right] \tag{1.241}$$

Note that for $p = 0$ the principal branch $F_+ - F_-$ is zero along $-\infty$ to 0 and so

$$G(z) = \frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{2\pi i}\int_{a}^{\infty} \mathrm{d}s\ \frac{2\pi i\sqrt{1 - \frac{a}{s}}}{(s - z_0)(s - z)} \tag{1.242}$$

$$F(z) = F(z_0) + (z - z_0)\frac{1}{2\pi i}\int_{a}^{\infty} \mathrm{d}s\ \frac{2\pi i\sqrt{1 - \frac{a}{s}}}{(s - z_0)(s - z)} \tag{1.243}$$

and so $W(s) = \sqrt{1 - \frac{a}{z}}$.

1.25 Integral 9

Evaluate

$$\lim_{n\to\infty} \sqrt{n}\int_{-\infty}^{\infty} \mathrm{d}x\ \frac{1}{(1 + x^2)^n} \tag{1.244}$$

where n is a positive integer.

Solution:

Note for

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \frac{1}{(1 + x^2)^n} \tag{1.245}$$

we can close the integral in the upper half plane, and there will be an nth order pole at $z = i$. The arc in the upper half plane will clearly vanish, as $(1+x^2)^n\ R^{2n}$ so the integrand will scale as R^{1-2n} which will vanish as $R \to \infty$ and $n > 1$. Thus,

$$\oint \mathrm{d}z\ \frac{1}{(x-i)^n(x+i)^n} = \int_{-\infty}^{\infty} \mathrm{d}x\ \frac{1}{(1+x^2)^n} + \underbrace{\int_0^{\pi} \mathrm{d}\theta\ \frac{iRe^{i\theta}}{(1+R^2e^{2i\theta})^n}}_{\to 0} \tag{1.246}$$

Thus, $\frac{\mathrm{d}^j}{\mathrm{d}x^j}x^{-n} = (-1)^j\frac{(n+j-1)!}{(n-1)!}x^{-(n+j)}$ for $n > 0$

$$\oint \mathrm{d}z\ \frac{\frac{1}{(z+i)^n}}{(z-i)^n} = \frac{2\pi i}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left[\frac{1}{(z+i)^n}\right]_{z=i} = \frac{2\pi i}{(n-1)!}(-1)^{n-1}\frac{(n+n-1-1)!}{(n-1)!}\frac{1}{(i+i)^{n+n-1}} \tag{1.247}$$

$$= 2\pi i(-1)^{n-1}\frac{(2n-2)!}{[(n-1)!]^2}\frac{1}{(2i)^{2n-1}} = \pi(2)^{2-2n}(-1)^{n-1}i^{2-2n}\frac{(2n-2)!}{[(n-1)!]^2} \tag{1.248}$$

$$= \pi(2)^{2-2n}(-1)^n i^{-2n}\frac{(2n-1)!}{[(n-1)!]^2} = \pi(2)^{2-2n}(-1)^n i^{2n}\frac{(2n-1)!}{[(n-1)!]^2} \tag{1.249}$$

$$= \pi(2)^{2-2n}(-1)^n(-1)^n\frac{(2n-1)!}{[(n-1)!]^2} = \pi 2^{2-2n}\frac{(2n-1)!}{[(n-1)!]^2} \tag{1.250}$$

So putting this in, we find (using Stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$ and ignoring contributions past factors of n)

$$\lim_{n\to\infty}\sqrt{n}\int_{-\infty}^{\infty}\mathrm{d}x\ \frac{1}{(1+x^2)^n} = \lim_{n\to\infty}\pi\sqrt{n}2^{2-2n}\frac{(2n-1)!}{[(n-1)!]^2} = \lim_{n\to\infty}\pi\sqrt{n}2^{-2n}\frac{(2n)!}{[(n)!]^2} \tag{1.251}$$

$$= \lim_{n\to\infty}2^{2-2n}\pi\sqrt{n}\frac{\sqrt{4\pi n}(2n/e)^{2n}}{2\pi n(n/e)^{2n}} = \lim_{n\to\infty}2^{-2n}\sqrt{n}\frac{2^{2n}\pi}{\sqrt{\pi n}} = \lim_{n\to\infty}\frac{\pi}{\sqrt{\pi}} = \sqrt{\pi} \tag{1.252}$$

It is rather interesting to note that this approaches a value independent of n, which is not obvious from the initial expression (although could be guessed since we are evaluating it).

Alternatively, take $x = y/\sqrt{n}$ and then $\sqrt{n}\,\mathrm{d}x = \mathrm{d}y$ and

$$\lim_{n\to\infty}\sqrt{n}\int_{-\infty}^{\infty}\mathrm{d}x\ \frac{1}{(1+x^2)^n} = \lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{(1+\frac{y^2}{n})^n} = \int_{-\infty}^{\infty}\mathrm{d}y\ \lim_{n\to\infty}\frac{1}{(1+\frac{y^2}{n})^n} = \int_{-\infty}^{\infty}\mathrm{d}y\ e^{-y^2} = \sqrt{\pi} \tag{1.253}$$

1.26 Integral 10

Compute

$$f(a,b) = \int_0^{\infty}\mathrm{d}x\ \frac{e^{-ax}-e^{-bx}}{x} \tag{1.254}$$

Solution:

We note that a and b are interchangeable with only a $-$ sign introduced. So take $y = ax$ and we can write

$$\int_0^\infty \frac{e^{-y} - e^{-\frac{a}{b}y}}{y/a} \frac{\mathrm{d}y}{a} = \int_0^\infty \mathrm{d}y \; \frac{e^{-y} - e^{-\frac{a}{b}y}}{y} \tag{1.255}$$

Thus, calling $a/b \equiv c$, we see that we can write

$$f(c) = \int_0^\infty \mathrm{d}x \; \frac{e^{-x} - e^{-cx}}{x} \tag{1.256}$$

$$f'(c) = \int_0^\infty \mathrm{d}x \; e^{-cx} = \frac{e^{-c\infty} - e^{-c0}}{c} = \frac{1}{c} \tag{1.257}$$

Thus

$$\int f'(c)\,\mathrm{d}c = f(c) = \int \frac{\mathrm{d}c}{c} = \ln(c) + C \tag{1.258}$$

for c a constant. For $c = 1$ we have

$$\int_0^\infty \frac{e^{-x} - e^{-x}}{x}\,\mathrm{d}x = \int_0^\infty 0\,\mathrm{d}x = 0 \tag{1.259}$$

and so

$$f(1) = 0 = \ln(1) + C \Rightarrow C = 0 \tag{1.260}$$

Thus,

$$f(c) = f(a,b) = \int_0^\infty \mathrm{d}x \; \frac{e^{-ax} - e^{-bx}}{x} = \ln\left(\frac{a}{b}\right) \tag{1.261}$$

1.27 Summation of Infinite Series

Find the sum of the following infinite series

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots \tag{1.262}$$

for $|x| < 1$.

Solution:

Note we may write this as

$$S(x) = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n \tag{1.263}$$

$$\int_0^x S(x')\,\mathrm{d}x' = \sum_{n=0}^{\infty} x^{n+1} = \sum_{n=1}^{\infty} x^n \tag{1.264}$$

We can use that (for $|x| < 1$)

$$T(x) = \sum_{n=0}^{\infty} x^n \tag{1.265}$$

$$T = 1 + x + x^2 + \cdots = 1 + x(1 + x + x^2 + \cdots) = 1 + xT \tag{1.266}$$

$$T = \frac{1}{1-x} \tag{1.267}$$

So

$$\int_0^x S(x')\,\mathrm{d}x' = T - 1 = \frac{1}{1-x} - 1 = \frac{1-(1-x)}{1-x} = \frac{x}{1-x} \tag{1.268}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x S(x')\,\mathrm{d}x' = S(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{1-x} = \frac{(1-x)-x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} \tag{1.269}$$

And hence

$$S(x) = \frac{1}{(1-x)^2} \tag{1.270}$$

1.28 Hermite Generating Function

A generating function $F(x,t)$ of the Hermite polynomial $H_n(x)$ is

$$F(x,t) = e^{x^2-(t-x)^2} = \sum_{k=0}^{\infty} H_k(x)\frac{t^k}{k!} \tag{1.271}$$

(a) Express $H_n(x)$ as a contour integral.

(b) Prove that $H_n(x)$ satisfies Hermite's differential equation

$$\frac{\mathrm{d}^2 H}{\mathrm{d}x} - 2x\frac{\mathrm{d}H}{\mathrm{d}x} + 2nH = 0 \tag{1.272}$$

(c) Deduce the relation

$$\frac{\mathrm{d}H_n}{\mathrm{d}x}(x) = 2nH_{n-1}(x) \tag{1.273}$$

Solution:

(a) We can take an integral of the form

$$\oint \mathrm{d}t\ \frac{e^{x^2-(t-x)^2}}{t^{j+1}} = \oint \mathrm{d}t\ \sum_{k=0}^{\infty}\frac{H_k(x)}{k!}t^{k-(j+1)} = \oint \mathrm{d}t\ \sum_{k=0}^{\infty}\frac{H_k(x)}{k!}t^{k-j-1} \tag{1.274}$$

From Cauchy's theorem, we can simply do the residue of the integral on the right (say around the unit circle) and see that the only contribution will be from $j = k$ as the other powers will vanish. Thus

$$\frac{H_k(x)}{k!}\oint \mathrm{d}t\ t^{-1} = \frac{H_k(x)}{k!}\int_0^{2\pi}\mathrm{d}\theta\ \frac{ie^{i\theta}}{e^{i\theta}} = 2\pi i\frac{H_k(x)}{k!} \tag{1.275}$$

and so

$$H_k(x) = \frac{k!}{2\pi i}\oint \mathrm{d}t\ \frac{e^{x^2-(t-x)^2}}{t^{k+1}} \tag{1.276}$$

(b) Note $(x^2-(t-x)^2 = -x^2-(t^2-2xt+x^2) = t^2-2xt)$

$$\frac{\partial F}{\partial x} = -2te^{-t^2-2xt} \tag{1.277}$$

$$\frac{\partial F}{\partial t} = (-2t-2x)e^{-t^2-2xt} = \left(\frac{\mathrm{d}}{\mathrm{d}x}-2x\right)e^{-t^2-2xt} = \left(\frac{\mathrm{d}}{\mathrm{d}x}-2x\right)F \tag{1.278}$$

$$\frac{\partial^2 F}{\partial x^2} = 4t^2e^{-t^2-2xt} \tag{1.279}$$

$$\frac{\partial^2 F}{\partial x^2} = -2t\frac{\partial F}{\partial t} + 4xtF = -2t\frac{\partial F}{\partial t} - 2\frac{\partial F}{\partial x} \tag{1.280}$$

Thus

$$\frac{\partial F}{\partial t} - \frac{\partial F}{\partial x} + 2xF = 0 \tag{1.281}$$

This must be true order by order in t so

$$(k+1)H_{k+1} - H_k' + 2xH_k = 0 \tag{1.282}$$

And from $\frac{\partial^2 F}{\partial x^2} + 2t\frac{\partial F}{\partial t} + 2\frac{\partial F}{\partial x} = 0$ we find

$$H_k'' + 2kH_k + 2xH_k' = 0 \tag{1.283}$$

which is the differential equation we desired.

(c) Use the contour integral expression. We see $(x^2-(t-x)^2 = -x^2-(t^2-2xt+x^2) = t^2-2xt)$

$$\frac{\mathrm{d}H_k}{\mathrm{d}x} = \frac{k!}{2\pi i}\oint \mathrm{d}t\ \frac{e^{-t^2}}{t^{k+1}}\mathrm{d}xe^{-2tx} = \frac{k!}{2\pi i}\oint \mathrm{d}t\ \frac{-2te^{-t^2-2tx}}{t^{k+1}} = -2k\frac{(k-1)!}{2\pi i}\oint \mathrm{d}t\ \frac{e^{-t^2-2tx}}{t^k} = -2kH_{k-1}(x) \tag{1.284}$$

1.29 Legendre Generating Function

A generating function for the Legendre polynomials $P_l(x)$ is

$$\frac{1}{(1-2xr+r^2)^{1/2}} = \sum_{l=0}^{\infty} r^l P_l(x) \tag{1.285}$$

with $x = \cos\theta$ and $|r| \leq 1$. Prove that $xP_l'(x) = P_{l-1}'(x) + lP_l(x)$ where $P_l'(x) = \frac{\mathrm{d}P_l(x)}{\mathrm{d}x}$

Solution:

Consider $F(x,r) = \left(1 - 2xr + r^2\right)^{-1/2}$. Then

$$\frac{\partial F}{\partial x} = \frac{-1}{2}(-2r)(1 - 2xr - r^2)^{-3/2} = \frac{r}{(1 - 2xr - r^2)}F \tag{1.286}$$

$$\frac{\partial F}{\partial r} = \frac{-1}{2}(-2x + 2r)(1 - 2xr - r^2)^{-3/2} = \frac{x - r}{(1 - 2xr - r^2)}F \tag{1.287}$$

$$\tag{1.288}$$

Hence we have

$$\frac{(1 - 2xr - r^2)}{r}\frac{\partial F}{\partial x} = \frac{(1 - 2xr - r^2)}{x - r}\frac{\partial F}{\partial r} \tag{1.289}$$

$$(x - r)\frac{\partial F}{\partial x} = r\frac{\partial F}{\partial r} \tag{1.290}$$

$$\sum_{l=0}^{\infty}(x - r)r^l P_l'(x) = \sum_{l=0}^{\infty} r^{l+1}P_l(x) \tag{1.291}$$

$$\sum_{l=0}^{\infty}\left[(x - r)r^l P_l'(x) - r^{l+1}P_l(x)\right] = 0 \tag{1.292}$$

arranging them order by order in r^l we see

$$\sum_{l=0}^{\infty}\left[r^l\left\{xP_l'(x) - P_{l-1}'(x) - P_{l-1}(x)\right\}\right] = 0 \tag{1.293}$$

and so we get

$$xP_l'(x) = P_{l-1}'(x) + P_{l-1}(x) \tag{1.294}$$

as desired.

1.30 Integral Formulation of Bessel Function

Given the Laurent series for $e^{(\mu/2)(z-1/z)}$ as $\sum_{n=-\infty}^{\infty} A_n z^n$ where $A_n = J_n(\mu)$, obtain an expression for the Bessel function $J_n(\mu)$ as an integral from $-\pi$ to π.

Solution:

We use that

$$\oint \mathrm{d}z\ z^{-j-1}e^{(\mu/2)(z-1/z)} = \oint \mathrm{d}z\ \sum_{n=-\infty}^{\infty} A_n z^{n-j-1} \tag{1.295}$$

Clearly, only when $n = j$ will there be a contribution to the expression on the right. Thus,

$$\oint \mathrm{d}z \; z^{-n-1} e^{(\mu/2)(z-1/z)} = 2\pi i A_n \tag{1.296}$$

Let us be on the unit circle so $z = e^{i\theta}$ and then

$$i \int_{-\pi}^{\pi} \mathrm{d}\theta \; e^{i\theta} e^{-(n-1)\theta} e^{\frac{\mu}{2} e^{i\theta} - e^{-i\theta}} = 2\pi i A_n \tag{1.297}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \; e^{in\theta} e^{i\mu \sin\theta} = A_n \tag{1.298}$$

and so the expression is

$$J_n(\mu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \; e^{i(\mu \sin\theta - n\theta)} \tag{1.299}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \;\; \cos\left[\mu \sin\theta - n\theta\right] + \frac{i}{2\pi} \int_{-\pi}^{\pi} \sin\left[\mu \sin\theta - n\theta\right] \tag{1.300}$$

Note that because $\mu \sin\theta - n\theta \equiv g(\theta)$ is an odd function then $\sin(g(\theta))$ is an odd function and so that integral is identically zero. Also because cos is even then $\cos(g(\theta))$ is even.

The proof is simple. For $f(x)$ even in x $[f(-x) = f(x)]$ and $g(x)$ odd in x $[g(-x) = -g(x)]$, then $h(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = h(x)$. So $h(x)$ is even. Let $d(-x) = -d(x)$ then $c(-x) = d(g(-x)) = d(-g(x)) = -d(g(x)) = -c(x)$ and so c is odd. So

$$J_n(\mu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \;\; \cos\left[\mu \sin\theta - n\theta\right] = \frac{1}{\pi} \int_{0}^{\pi} \mathrm{d}\theta \;\; \cos\left[\mu \sin\theta - n\theta\right] \tag{1.301}$$

1.31 Laplace Equation on a Plane

The function $\phi(x, y)$ is given on the plane $z = 0$. Find for $z > 0$, a solution $\psi(x, y, z)$ of Laplace's equation that reduces to $\phi(x, y)$ on the plane $z = 0$.

Solution:

Laplace's equation is given by

$$\nabla^2 \psi = 0 \tag{1.302}$$

We are given that at $z = 0$ the solution is $\psi = \phi$. We are basically using that we want a physical solution and that the symmetry of the problem will require a certain combination of ϕ as the general solution for $z > 0$. This is often called the method of images.

Suppose we put a charge above the $z = 0$ plane (say at $z = a$) that creates a potential at $\phi(x, y)$. Then if we put a negative charge at $z = -a$, we note that we have not changed the potential at $z = 0$, as the potential of these two charges is given by

$$\phi_{charges} = \frac{-q + q}{a} = 0 \tag{1.303}$$

Use $\nabla^2 G(\mathbf{x}-\mathbf{x}') = 0$ where $G(\mathbf{x}-\mathbf{x}') = -\epsilon_0\delta(\mathbf{x}-\mathbf{x}')$. Then we need to find G where $G(z=0)=0$. Using

$$\epsilon_0\psi(x) = -\int_V \mathrm{d}^3x' \; \rho(x') + \int_S \mathrm{d}^2x' \; \hat{\mathbf{n}} \cdot [\psi(x')\,\nabla' G - G\,\nabla'\psi(x')] \tag{1.304}$$

Note that the volume integral must equal zero as $\rho(x') = \epsilon_0\,\nabla^2\psi = 0$ since this is a Laplace equation.

We have G is zero on our surface, and $\hat{\mathbf{n}}$ will be $\hat{\mathbf{z}}$ on the plane, where the only contributions from the surface integral will occur. Thus

$$\int_S \mathrm{d}^2x' \; \psi(x')\frac{\partial G}{\partial z'} \tag{1.305}$$

From the method of images, it's clear that $r(+z) = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ and $r(-z') = \sqrt{(x-x')^2+(y-y')^2+(z+z')^2}$

$$G = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \tag{1.306}$$

Thus,

$$\frac{\partial G}{\partial z'} = \frac{-1}{2}\frac{-2(z-z')(-1)}{r(+z')^{3/2}} - \frac{-1}{2}\frac{-2(z+z')}{r(-z')^{3/2}} \tag{1.307}$$

$$= \frac{-(z-z')}{r(+z')^{3/2}} - \frac{(z+z')}{r(-z')^{3/2}} \tag{1.308}$$

Now we are at $z'=0$ for the integral so this reduces to

$$\frac{\partial G}{\partial z'} = \frac{-2z}{r^2} = \frac{-2z}{((x-x')^2+(y-y')^2+(z)^2)^{3/2}} \tag{1.309}$$

So the answer is

$$\psi = \frac{-2z}{\epsilon_0}\int_S \mathrm{d}x' \; \mathrm{d}y' \; \frac{\phi(x',y')}{((x-x')^2+(y-y')^2+(z)^2)^{3/2}} \tag{1.310}$$

1.32 Integral 11

Show that

$$K_0(x) = \int_0^\infty e^{-x\cosh\phi}\,\mathrm{d}\phi \tag{1.311}$$

satisfies Bessel's equation of zeroth order and imaginary argument, that is $K_0(x) \equiv J_0(ix)$. Show that $K_0(x)$ has the asymptotic form $De^{-x}/\sqrt{x}$ for very large x; give the value of the constant D.

Solution:

Modified Bessel's differential equation is (for $n = 0$)

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - x^2 y = 0 \tag{1.312}$$

We use from the above definition that

$$K_0'(x) = -\int_0^\infty \mathrm{d}\phi\ e^{-x\cosh\phi}\cosh\phi \tag{1.313}$$

$$K_0''(x) = \int_0^\infty \mathrm{d}\phi\ e^{-x\cosh\phi}\cosh^2\phi \tag{1.314}$$

and so

$$x^2 \frac{\mathrm{d}^2 K_0}{\mathrm{d}x^2} + x\frac{\mathrm{d}K_0}{\mathrm{d}x} - x^2 K_0 = \int_0^\infty (x^2\cosh^2\phi - x\cosh\phi - x^2)e^{-x\cosh\phi} \tag{1.315}$$

We use $\cosh^2(\phi) - 1 = \sinh^2\phi$

$$x^2 \frac{\mathrm{d}^2 K_0}{\mathrm{d}x^2} + x\frac{\mathrm{d}K_0}{\mathrm{d}x} - x^2 K_0 = \int_0^\infty (x^2\sinh^2\phi - x\cosh\phi)e^{-x\cosh\phi} \tag{1.316}$$

Now realize $u = e^{-x\cosh\phi}$, $\mathrm{d}v = -x\cosh\phi$ and

$$-\int_0^\infty \mathrm{d}\phi\ x\cosh(\phi)e^{-x\cosh\phi} = \cancel{-e^{-x\cosh\phi}x\sinh(\phi)\Big|_0^\infty} - \int_0^\infty \mathrm{d}\phi\ (-x\sinh\phi)(-x\sinh\phi)e^{-x\cosh\phi}\,\mathrm{d}\phi \tag{1.317}$$

$$= -\int_0^\infty \mathrm{d}\phi\ x^2\sinh^2(\phi)e^{-x\cosh\phi} \tag{1.318}$$

And so

$$x^2 \frac{\mathrm{d}^2 K_0}{\mathrm{d}x^2} + x\frac{\mathrm{d}K_0}{\mathrm{d}x} - x^2 K_0 = \int_0^\infty (x^2\sinh^2\phi - x^2\sinh^2\phi)e^{-x\cosh\phi} = 0 \tag{1.319}$$

Thus K_0 is a solution.

Now to determine the solution for large x. We can note that $e^{-x\cosh\phi}$ is a controlling factor, and that most of the integral comes from 0 to some ϵ that is small (so use Taylor approximations for $\cosh\phi$ around 0) and we see

$$K_0(x) = \int_0^\epsilon e^{-x\left(1+\frac{\phi^2}{2}+\cdots\right)}\,\mathrm{d}\phi \tag{1.320}$$

We can then extend the integration over to ∞, which won't contribute much to the result anyway, since the exponential quickly dies away. We use $u\sqrt{x/2} = \phi$ and so

$$K_0(x) \sim \int_0^\epsilon e^{-x}e^{\frac{-x\phi^2}{2}}\,\mathrm{d}\phi = \frac{e^{-x}}{\sqrt{x/2}}\int_0^\epsilon \mathrm{d}u\ e^{-u^2} \sim \frac{e^{-x}\sqrt{2}}{\sqrt{x}}\int_0^\infty \mathrm{d}u\ e^{-u^2} = \frac{e^{-x}\sqrt{\pi}}{\sqrt{2x}} \tag{1.321}$$

where $\int_0^\infty \mathrm{d}x\ e^{-x^2} = \sqrt{\pi}/2$ was used. Thus $D = \sqrt{\pi/2}$.

1.33 Toroidal Surface

Calculate $\int \mathbf{r} \cdot \mathrm{d}\mathbf{A}$ over the surface of a torus.

Solution:

There are two straightforward ways to do this. The simpler method recognizes the above integral as a divergence theorem application. Thus

$$\int_S \mathbf{r} \cdot \mathrm{d}\mathbf{A} = \int_V \mathrm{d}V\ \boldsymbol{\nabla} \cdot (\mathbf{r}) = \int_V \mathrm{d}V\ 3 = 3V_{\text{torus}} = 6\pi R(\pi r^2) = 6\pi^2 R r^2 \tag{1.322}$$

where R is the major radius and r is the minor radius of the torus.

Alternatively, use in primitive toroidal coordinates (r, θ, ζ) where r is the minor radius, θ is poloidal and ζ is toroidal. We then have

$$x = (R + r\cos\theta)\cos\zeta \tag{1.323}$$
$$y = -(R + r\cos\theta)\sin\zeta \tag{1.324}$$
$$z = r\sin\theta \tag{1.325}$$

Then clearly

$$\mathbf{r} = \hat{\mathbf{x}}(R + r\cos\theta)\cos\zeta + \hat{\mathbf{y}}\left(-(R + r\cos\theta)\sin\zeta\right) + \hat{\mathbf{z}}(r\sin\theta) \tag{1.326}$$

and so a surface vector is given by

$$\frac{\partial \mathbf{r}}{\partial \theta} \equiv \mathbf{F} = -r\sin\theta\cos\zeta\hat{\mathbf{x}} + r\sin\theta\sin\zeta\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}} \tag{1.327}$$
$$\frac{\partial \mathbf{r}}{\partial \zeta} \equiv \mathbf{G} = -(R + r\cos\theta)\sin\zeta\hat{\mathbf{x}} + -(R + r\cos\theta)\cos\zeta\hat{\mathbf{y}} \tag{1.328}$$
$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \zeta} = \mathbf{F} \times \mathbf{G} \tag{1.329}$$
$$= (F_y \cancel{G_z} - F_z G_y)\hat{\mathbf{x}} + (F_z G_x - F_x \cancel{G_z})\hat{\mathbf{y}} + (F_x G_y - F_y G_x)\hat{\mathbf{z}} \tag{1.330}$$
$$= -(-1) r\cos\theta(R + r\cos\theta)\cos\zeta\hat{\mathbf{x}} + r\cos\theta(-[R + r\cos\theta]\sin\zeta)\hat{\mathbf{y}}$$
$$+ \left[r\sin\theta\cos^2\zeta(R + r\cos\theta) + r\sin\theta\sin^2\zeta(R + r\cos\theta)\right]\hat{\mathbf{z}} \tag{1.331}$$
$$= r\cos\theta\cos\zeta(R + r\cos\theta)\hat{\mathbf{x}} - r\cos\theta\sin\zeta(R + r\cos\theta)\hat{\mathbf{y}} + r\sin\theta(R + r\cos\theta)\hat{\mathbf{z}}$$
$$\mathbf{n} \cdot \mathbf{n} = r^2\cos^2\theta\cos^2\zeta(R + r\cos\theta)^2 - r^2\cos^2\theta\sin^2\zeta(R + r\cos\theta)^2 + r^2\sin^2\theta(R + r\cos\theta)^2 \tag{1.332}$$
$$= r^2\cos^2\theta(R + r\cos\theta)^2 + r^2\sin^2\theta(R + r\cos\theta)^2 \tag{1.333}$$
$$= r^2(R + r\cos\theta)^2 \tag{1.334}$$

And so

$$\hat{\mathbf{n}} = \cos\theta\cos\zeta\hat{\mathbf{x}} - \cos\theta\sin\zeta\hat{\mathbf{y}} + \sin\theta\hat{\mathbf{z}} \tag{1.335}$$

We also need the coordinate transformation's "Jacobian" $|\mathbf{F} \times \mathbf{G}| = r(R + r\cos\theta)$ so

$$\int_S \mathbf{r} \cdot \mathrm{d}\mathbf{A} = \int_0^{2\pi} \mathrm{d}\zeta \int_0^{2\pi} \mathrm{d}\theta \; r(R + r\cos\theta)\mathbf{r} \cdot \hat{\mathbf{n}} \tag{1.336}$$

$$= \int_0^{2\pi} \mathrm{d}\zeta \int_0^{2\pi} \mathrm{d}\theta \; r(R + r\cos\theta) \left(\cos\theta(R + r\cos\theta)\cos^2\zeta + \cos\theta(R + r\cos\theta)\sin^2\zeta + r\sin^2\theta\right) \tag{1.337}$$

$$= \int_0^{2\pi} \mathrm{d}\zeta \int_0^{2\pi} \mathrm{d}\theta \; r(R + r\cos\theta)(R\cos\theta + r) \tag{1.338}$$

$$= \int_0^{2\pi} \mathrm{d}\zeta \int_0^{2\pi} \mathrm{d}\theta \; r(R^2\cos\theta + rR\cos^2\theta + rR + r^2\cos\theta) \tag{1.339}$$

Clearly the $\cos\theta$ terms will vanish, so

$$= rR^2 \int_0^{2\pi} \mathrm{d}\zeta \int_0^{2\pi} \mathrm{d}\theta \; (1 + \cos^2\theta) = rR^2 4\pi^2 + 2\pi r R^2 \int_0^{2\pi} \mathrm{d}\theta \; \cos^2\theta \tag{1.340}$$

We use $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ and thus

$$\int_0^{2\pi} \mathrm{d}\theta \; \frac{1 + \cos(2\theta)}{2} = \pi - \cancel{\frac{\sin(2\theta)}{4}} \tag{1.341}$$

So we get

$$\int_S \mathbf{r} \cdot \mathrm{d}\mathbf{A} = rR^2(4\pi^2 + 2\pi^2) = 6\pi^2 r R^2 \tag{1.342}$$

in agreement with our previous answer.

1.34 Volume of 4D Sphere

Calculate the volume V of a four dimensional unit sphere.

$$x_1 = r\sin\phi_2 \sin\phi_1 \cos\phi \tag{1.343}$$

$$x_2 = r\sin\phi_2 \sin\phi_1 \sin\phi \tag{1.344}$$

$$x_3 = r\sin\phi_2 \cos\phi_1 \tag{1.345}$$

$$x_4 = r\cos\phi_2 \tag{1.346}$$

First let's form the inverse Jacobian.

$$\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(r, \phi_1, \phi_2, \phi_3)} = \begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \phi} & \frac{\partial x_1}{\partial \phi_1} & \frac{\partial x_1}{\partial \phi_2} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \phi} & \frac{\partial x_2}{\partial \phi_1} & \frac{\partial x_2}{\partial \phi_2} \\ \frac{\partial x_3}{\partial r} & \frac{\partial x_3}{\partial \phi} & \frac{\partial x_3}{\partial \phi_1} & \frac{\partial x_3}{\partial \phi_2} \\ \frac{\partial x_4}{\partial r} & \frac{\partial x_4}{\partial \phi} & \frac{\partial x_4}{\partial \phi_1} & \frac{\partial x_4}{\partial \phi_2} \end{bmatrix} \tag{1.347}$$

$$= \begin{bmatrix} \sin\phi_2\sin\phi_1\cos\phi & -r\sin\phi_2\sin\phi_1\sin\phi & r\sin\phi_2\cos\phi_1\cos\phi & r\cos\phi_2\sin\phi_1\cos\phi \\ \sin\phi_2\sin\phi_1\sin\phi & r\sin\phi_2\sin\phi_1\cos\phi & r\sin\phi_2\cos\phi_1\sin\phi & r\cos\phi_2\sin\phi_1\sin\phi \\ \sin\phi_2\cos\phi_1 & 0 & -r\sin\phi_2\sin\phi_1 & r\cos\phi_2\cos\phi_1 \\ \cos\phi_2 & 0 & 0 & -r\sin\phi_2 \end{bmatrix} \tag{1.348}$$

So then

$$\left|\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(r, \phi_1, \phi_2, \phi_3)}\right| = r^3 \sin\phi_1 \sin^2\phi_2 \tag{1.349}$$

And so (we integrate from 0 to π except for ϕ because otherwise we are going over the same area twice)

$$\int_V \mathrm{d}^4x\ 1 = \int_0^1 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\phi_1 \int_0^{\pi} \mathrm{d}\phi_2\ r^3 \sin\phi_1 \sin^2\phi_2 \tag{1.350}$$

$$= \int_0^{\pi} \mathrm{d}\phi_2\ \sin^2\phi_2 \left[\int_0^1 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\phi_1\ \sin\phi_1\right] \tag{1.351}$$

$$= \int_0^{\pi} \mathrm{d}\phi_2\ \frac{1-\cos(2\phi_2)}{2}\frac{4\pi}{4} = \frac{\pi^2}{2} \tag{1.352}$$

1.35 Concentration of Air in a Pipe

Gaseous helium is flowing without turbulence at a velocity v down a pipe and into the atmosphere. Within a very short distance from the end of the pipe, the helium is rapidly diluted to essentially zero concentration.

Set up and solve the differential equation for the concentration of air in the pipe as a function of distance from the end of the pipe. Assume equilibrium conditions, neglect wall friction and end effects, assume no temperature difference, and assume that the coefficients of diffusion of O_2 and N_2 into He are the same and equal to D.

Solution:

We must have conservation of particles in the pipe.

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{V}) = S \tag{1.353}$$

where $n = n_h + n_a$ with n_h the helium and n_a the air number density. Put $x = 0$ at the end of the pipe. The S is a source.

We know in steady state that we must have sources to keep the helium and air continually supplied. Consider n_a.

If there is diffusion of air into the pipe, we can note that they contribute by

$$S = D\frac{\partial^2 n}{\partial x^2} \tag{1.354}$$

Thus in steady state we have

$$v\frac{\partial n_a}{\partial x} = D\frac{\partial^2 n_a}{\partial x^2} \tag{1.355}$$

Hence,

$$\frac{v}{D}n_a = \frac{\partial n_a}{\partial x} \tag{1.356}$$

$$n_a = C_0 e^{vx/D} + C \tag{1.357}$$

At $x = -\infty$ we want $n_a = 0$ so $C = 0$ and we see

$$n_a = ne^{vx/D} \tag{1.358}$$

C_0 is set by choosing what the density at $x = 0$, which is purely air so n. Then the concentration of helium would be given by

$$\frac{n_h}{n} = \frac{n - n_a}{n_a + n_h} = n\frac{1 - e^{vx/D}}{n} = 1 - e^{vx/D} \tag{1.359}$$

and for air

$$\frac{n_a}{n} = e^{vx/D} \tag{1.360}$$

for $x < 0$ in both cases.

1.36 Neutron Density in Reactor

The equation describing the neutron density in a chain reacting pile is $\nabla^2 n + K^2 n = 0$.

(a) With the boundary condition that it vanish outside the pile, find the radius for a spherical pile of a given value of K.

(b) Now suppose that a thin layer of material of thickness t is added to the surface, and that the neutron density in the layer is described by $\nabla^2 n - \mu^2 n = 0$. Assume the boundary conditions at the interface are that n and ∇n are continuous. Demanding that n vanish outside the pile and material layer, find for fixed values of K, μ and t, an expression for the radius of the internal region. Assuming $K \ll \mu$, derive an approximate relation for the difference between the radii without and with the layer.

Solution:

(a)

We have Dirichlet boundary conditions that we want enforced on a sphere. We remember that on a sphere, that this equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial n}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 n}{\partial\varphi^2} + K^2 n = 0 \tag{1.361}$$

We know via spherical symmetry that there is no θ or φ dependence. Thus

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial n}{\partial r}\right) + r^2K^2 n = 0 \tag{1.362}$$

$$r^2 n'' + 2rn' + r^2K^2 n = 0 \tag{1.363}$$

$$\tag{1.364}$$

We can recognize this as almost the equation for a Spherical Bessel function. Let $x = \kappa r$ we find

$$r^2 n'' + 2rn' + r^2 K^2 n = 0 \tag{1.365}$$

$$x^2 \frac{\mathrm{d}^2 n}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}n}{\mathrm{d}x} + x^2 n = 0 \tag{1.366}$$

which is a spherical Bessel function which has solution j_k for

$$x^2 \frac{\mathrm{d}y^2}{\mathrm{d}x^2} + 2r\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - k(k+1))y = 0 \tag{1.367}$$

so our solution is

$$j_0(x) = \frac{\sin(x)}{x} = \frac{\sin(Kr)}{Kr} \tag{1.368}$$

If we want $n = 0$ outside some radius we require

$$\frac{\sin(Kr)}{Kr} = 0 \tag{1.369}$$

So $Kr = \pi$ is the smallest value for our solution so at $r = \pi/K$. If we chose some other value, then we would have negative densities inside the radius, which would make little sense.

(b)

Note that if we take $\mu = im$ then the equation is a spherical Bessel function j_0 and so the solution is of the form (let R_i be the internal radius)

$$n = \frac{\sin(mr)}{mr} = \frac{\sin(-i\mu r)}{-i\mu r} = \frac{e^{i(-i\mu r)} - e^{-i(-i\mu r)}}{2i(-i\mu r)} = \frac{e^{\mu r} - e^{-\mu r}}{2\mu r} = \frac{\sinh(\mu r)}{\mu r} \tag{1.370}$$

we want at $r = R_i + t$ that this vanishes. The only way for this to occur is if we let $\mu r \to [\mu(r - R_i - t)]$ so

$$n = \frac{A\sinh(\mu(r - R_i - t)}{r} \tag{1.371}$$

for some A.

Let the inner solution have a constant B.

Now we require that for some internal Radius R_i that

$$\frac{B\sin(KR_i)}{R_i} = \frac{A\sinh(-\mu t)}{R_i} = \frac{-A\sinh(\mu t)}{R_i} A = -B\frac{\sin(KR_i)}{\sinh(\mu t)} \tag{1.372}$$

$$\tag{1.373}$$

The derivative condition requires

$$B\frac{KR_i\cos(KR_i) - \sin(KR_i)}{R_i^2} = A\frac{\mu R_i\cosh(\mu t) + \sinh(\mu t)}{R_i^2} \tag{1.374}$$

$$B\frac{KR_i\cos(KR_i) - \sin(KR_i)}{R_i^2} = -B\frac{\mu R_i\coth(\mu t)\sin(KR_i) + \sin(KR_i)}{R_i^2} \tag{1.375}$$

$$KR_i\cos(KR_i) = -\mu R_i\coth(\mu t)\sin(KR_i) \tag{1.376}$$

$$KR_i\cot(KR_i) = -\mu R_i\coth(\mu t) \tag{1.377}$$

$$K\tanh(\mu t) = -\mu\tan(KR_i) \tag{1.378}$$

So our solution is given by

$$KR_i = \tan^{-1}\left(\frac{-K}{\mu}\tanh(\mu t)\right) \tag{1.379}$$

if $K \ll \mu$ then we can use that for $x \ll 1$

$$\tan^{-1}(x) \approx x \tag{1.380}$$

$$KR_i \approx \frac{-K}{\mu}\tanh(\mu t) \tag{1.381}$$

We recognize this solution is unphysical. If we more carefully look at tan we see that it repeat every π so we can add multiples of π to our answer. Clearly, then,

$$R_i \approx \frac{\pi}{K} - \frac{1}{\mu}\tanh(\mu t) \tag{1.382}$$

as that way as $t \to 0$ we see we recover the solution of $R_i = \pi/K$ as from (a).

1.37 Neutron Flux

A point source of neutrons on the axis of a long square column of graphite 150 cm on a side emits 10^6 neutrons per second. Calculate the flux of neutrons at a point on the axis 1 m from the source if the diffusion coefficient of the neutrons is $D = \lambda v/3$, v is their velocity, and $\lambda = 2.8$ cm is the mean free path for scattering. Neglect the effects of slowing down and capture.

Solution:

This question is not very well-worded. A "long square column" would be clearer as a "long, square column". I assume that it means there is a small hole in the graphite cube, such that the neutrons can exit along the axis of the hole unimpeded.

Place the axis (say the z along the point source), then we have diffusion in the form (placing the point source at the origin and $Q = 10^6$)

$$D\,\nabla^2 n = Q\delta(r) \tag{1.383}$$

For x and y let's put boundary conditions $x = y = \pm a$ that $n = 0$. For z we want $z = \pm\infty$ that $n = 0$.

$$n = \sum_{l,k} e^{ilx+iky}A_{lk}(z) \tag{1.384}$$

The boundary condition at $x = \pm a$ implies $e^{\pm ila} = 0$ so $la = (2m+1)\pi/2$, and similarly $ka = (2n+1)\pi/2$ [because we care about the real part in the end]. We then find

$$\nabla^2 n = \sum_{l,k} e^{ilx+iky}((-l^2 - k^2)A_{lk} + A''_{lk}) = C\delta(\mathbf{r}) \tag{1.385}$$

$$\nabla^2 n = \sum_{l,k} e^{i\frac{(2m+1)\pi}{2a}x + i\frac{(2n+1)\pi}{2a}y}((-l^2 - k^2)A_{lk} + A''_{lk}) = C\delta(\mathbf{r}) \tag{1.386}$$

where $C = Q/D$. Integrate while using the orthogonality of $e^{il'x}$ and $e^{ik'y}$ (changing from x, y, to u, v via $u = \pi x/(a)$ and $v = \pi y/(a)$ yields a factor of $a/(\pi)$ for each and orthogonality for both picks up a two factors of (π)) That is $\beta = (2m+1)/2$ and $\gamma = (2n+1)/2$

$$\int_{-a}^{a} \mathrm{d}x \ \cos\left(\frac{(2m+1)\pi x}{2a}\right) \cos\left(\frac{(2n+1)\pi x}{2a}\right) \tag{1.387}$$

$$= \frac{a}{\pi}\int_{-\pi}^{\pi} \mathrm{d}u \ \cos\left(\frac{(2m+1)u}{2}\right) \cos\left(\frac{(2n+1)u}{2}\right) \tag{1.388}$$

$$= \frac{a}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}u \ \left[\cos\left([\beta+\gamma]u\right) + \cos\left([\beta-\gamma]u\right)\right] \tag{1.389}$$

$$= \frac{a}{2\pi}\left[-\frac{\sin\left([\beta+\gamma]u\right)}{\beta+\gamma} - \frac{\sin\left(\beta-\gamma\right)}{\beta-\gamma}\right]_{-\pi}^{\pi} = 0 \text{ for } \beta \neq \gamma \text{ and } \beta \neq -\gamma \tag{1.390}$$

if $\beta = \gamma$ or $\beta = -\gamma$ we find

$$\int_{-a}^{a} \mathrm{d}x \ \cos\left(\frac{(2m+1)\pi x}{2a}\right) \cos\left(\frac{(2n+1)\pi x}{2a}\right) \tag{1.391}$$

$$= \frac{a}{2\pi} 2\pi = a \tag{1.392}$$

proving the orthogonality factor above is indeed a^2.

$$(-l^2 - k^2)A_{lk} + A''_{lk}(z) = \frac{C}{a^2}\delta(z) \tag{1.393}$$

Integrating z over a small region around zero will yield (let's assume A is continuous, since we want n to not be discontinous)

$$[\![A'_{lk}(z)]\!] = C/a^2 \tag{1.394}$$

Thus, the jump in the derivative of $A_{lk} = C/a^2$ and also have $A''_{lk} = fA_{lk}$ where $f = (l^2 + k^2)$. Thus for A^+ in $z > 0$ and A^- in $z < 0$ our boundary conditions require

$$A^{\pm}_{lk} = C_0 e^{\mp\sqrt{f}z} \tag{1.395}$$

$$A^{\pm\prime}_{lk} = \mp C_0\sqrt{f}e^{\mp\sqrt{f}z} \tag{1.396}$$

Our jump condition indicates

$$-\left[-C_0\sqrt{f}e^{-\sqrt{f}0} - C_0\sqrt{f}e^{\sqrt{f}0}\right] = C/a^2 \tag{1.397}$$

$$2C_0\sqrt{f} = C/a^2 \tag{1.398}$$

$$C_0 = \frac{C}{2a^2\sqrt{f}} \tag{1.399}$$

Thus, altogether, we find

$$n = \sum_{l,k} e^{ilx+iky}\frac{Q}{2Da^2\sqrt{l^2+k^2}}e^{-\sqrt{l^2+k^2}|z|} \tag{1.400}$$

with $l = \frac{(2m+1)\pi}{2a}$ and $k = \frac{(2n+1)\pi}{2a}$. Thus the flux along z will be given by

$$j_z = D\frac{\partial n}{\partial z} = \sum_{l,k} \frac{Q}{2a^2} e^{-\sqrt{l^2+k^2}|z|} \frac{|z|}{z} \tag{1.401}$$

$$D\frac{\partial n}{\partial z} = \sum_{n,m} \frac{Q}{2a^2} e^{-\frac{\pi}{2a}\sqrt{(2m+1)^2+(2n+1)^2}|z|} \frac{|z|}{z} \tag{1.402}$$

Evaluating the sum, we see that the larger m and n get, the smaller the contribution.

Note n, m go over 0 to ∞ at this point. So we get

$$\sum_{j=0}^{10}\sum_{k=0}^{10} e^{\frac{-1}{0.75}\frac{\pi}{2a}\sqrt{(2m+1)^2+(2n+1)^2}} \approx 0.188 \tag{1.403}$$

$$j_z = \frac{Q}{2a^2}(0.188) = \frac{10^6}{(0.75)^2}(0.188) \approx 3.3 \times 10^5 \tag{1.404}$$

Chapter 2

Mechanics

2.1 Stokes's Law

Derive the form of Stokes's law by dimensional analysis. Assume that the force is independent of the density ρ of the fluid. What happens when the assumption is dropped?

(Note: Stokes's law is the force due to a fluid flowing past a sphere)

Solution:

What do we have? The radius of the sphere R, the kinematic viscosity $\nu = \mu/\rho$ where μ is the dynamic viscosity, and the velocity of the fluid around the sphere v.

Using $[Q]$ to denote units with L, M, S standing for length, mass, and time, we see

$$[F] = \frac{ML}{S^2} \tag{2.1}$$
$$[R] = L \tag{2.2}$$
$$[v] = L/S \tag{2.3}$$
$$[\nu] = \frac{L^2}{S} \tag{2.4}$$
$$[\mu] = \frac{M}{LS} \tag{2.5}$$
$$\tag{2.6}$$

So clearly a combination of

$$[R]^\alpha [v]^\beta [\mu]^\gamma = MLS^{-2} \tag{2.7}$$
$$L^\alpha L^\beta S^{-\beta} M^\gamma L^{-\gamma} S^{-\gamma} = MLS^{-2} \tag{2.8}$$
$$L^{\alpha+\beta-\gamma} M^\gamma S^{-\beta-\gamma} = MLS^{-2} \tag{2.9}$$
$$\alpha + \beta - \gamma = 1 \tag{2.10}$$
$$-\beta - \gamma = -2 \tag{2.11}$$
$$\gamma = 1 \tag{2.12}$$

Thus $\gamma = 1$, $\beta = 1$ and $\alpha = 1$. So we have

$$F \sim Rv\mu = Rv\nu\rho \tag{2.13}$$

If the force is independent of the fluid density that means that the ρ above does not contribute except as a constant factor and so for scaling we have

$$F \sim Rv\nu \tag{2.14}$$

$$\mathrm{Re} \equiv \frac{Rv}{\nu} \tag{2.15}$$

$$F \sim \mathrm{Re}\nu^2 \tag{2.16}$$

Here $\mathrm{Re} = Rv/\nu$ is dimensionless and so the scaling parameter for F must be able to be written as a function of the number, often called the Reynolds number Re.

Small Reynolds number indicates viscosity is dominant and non-turbulent flow is occurring. Large Reynolds number indicates turbulence, which will complicate our analysis and make it incorrect if it occurs. That is because there are turbulent eddies and so v won't be a good measure of what's happening as there are no nice streamline flows.

2.2 Dimensional Analysis Explosion

A gas bubble from a deep explosion under water oscillates with a period $T \sim p^a d^b e^c$ where p is static pressure, d the water density, and e the total energy of the explosion. Find, a, b, and c.

Solution:

Use dimensional analysis

$$[T] = S \tag{2.17}$$

$$[p] = ML^{-1}S^{-2} \tag{2.18}$$

$$[d] = ML^{-3} \tag{2.19}$$

$$[e] = ML^2S^{-2} \tag{2.20}$$

$$[p]^a[d]^b[e]^c = M^aL^{-a}S^{-2a}M^bL^{-3b}M^cL^{2c}S^{-2c} = M^{a+b+c}L^{-a-3b+2c}S^{-2a-2c} \tag{2.21}$$

and so we have

$$a + b + c = 0 \tag{2.22}$$

$$-a - 3b + 2c = 0 \tag{2.23}$$

$$-2a - 2c = 1 \tag{2.24}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -3 & 2 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.25}$$

We see that two times the first equation plus the last equation yields $2b = 1$ or $b = 1/2$. So

$$-a + 2c = \frac{3}{2} \tag{2.26}$$

$$-2a - 2c = 1 \tag{2.27}$$

$$-3a = \frac{5}{2} \tag{2.28}$$

$$a = \frac{-5}{6} \tag{2.29}$$

Thus

$$\frac{-5}{6} + \frac{1}{2} + c = 0 \tag{2.30}$$

$$c = \frac{2}{6} = \frac{1}{3} \tag{2.31}$$

Altogether $a = \frac{-5}{6}$, $b = \frac{1}{2}$ and $c = \frac{1}{3}$.

Double checking we have

$$\frac{5}{6} - \frac{3}{2} + \frac{2}{3} = \frac{5 - 9 + 4}{6} = 0 \tag{2.32}$$

$$-2\left(\frac{-5}{6} + \frac{1}{3}\right) = -2\frac{-3}{6} = 1 \tag{2.33}$$

so

$$T \sim p^{-5/6} d^{1/2} e^{1/3} \tag{2.34}$$

This tells us, for example, the larger the static pressure the shorter the period.

2.3 Satellite Circular Orbit

A satellite is put into a circular orbit at a distance R_0 above the center of the earth. A viscous force resulting from the thin upper atmosphere has a magnitude $F_v = Av^\alpha$, where v is the velocity of the satellite. It is noted that this results in a rate of change in the radial distance r given by $\mathrm{d}r/\mathrm{d}t = -C$ where C is a positive constant, sufficiently small so that the loss of energy per orbit is small compared to the total kinetic energy. Obtain expressions for A and α.

Solution:

We use that for a circular orbit, we must have the centripetal force balanced by gravity, so

$$\frac{mv^2}{r} = \frac{M_E m G}{r^2} \tag{2.35}$$

$$v = \sqrt{\frac{M_E G}{r}} \tag{2.36}$$

We can then use that the tangential force on the satellite is given by (note that since this is a drag force, we expect F_v to have a negative sign overall)

$$F = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{1}{2}m\sqrt{\frac{M_E G}{r^3}}(-C) = \frac{Cm}{2}\sqrt{\frac{M_E G}{r^3}} \tag{2.37}$$

$$= \frac{Cm}{2M_E G}\sqrt{\frac{M_E G}{r}}^3 = \frac{Cm}{2M_E G}v^3 \tag{2.38}$$

Note that the F above is $-F_v$ that we desire. So

$$A = -\frac{Cm}{2M_E G} \tag{2.39}$$

$$\alpha = 3 \tag{2.40}$$

Alternatively we can use

$$E = -\frac{M_E m G}{r} + \frac{1}{2}mv^2 = -\frac{M_E m G}{r} + \frac{M_E G m}{2r} = \frac{-M_E m G}{2r} \tag{2.41}$$

and so

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{F}\cdot\mathbf{v} = Av^{\alpha+1} = \frac{\mathrm{d}E}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{M_E m G}{2r}\right)(-C) = -\frac{CM_E m G}{r^2} \tag{2.42}$$

$$Av^{\alpha+1} = A\left(\frac{M_E G}{r}\right)^{(\alpha+1)/2} = -\frac{CM_E m G}{2r^2} \tag{2.43}$$

For the powers of r to match, we must have $\alpha = 3$ and we find

$$A = \frac{-C}{2M_E G} \tag{2.44}$$

2.4 Mass on a String around a Cylinder

A point mass m under no external forces is attached to a weightless cord fixed to a cylinder of radius R. Initially the cord is completely wound up so that the mass touches the cylinder. A radially-directed impulse is now given to the mass, which starts unwinding.

(a) Find the equation of motion in terms of some suitable generalized coordinate.

(b) find the general solution satisfying the initial condition.

(c) find the angular momentum of the mass about the cylinder axis using the result of (b)

Solution:

(a)

Let φ be the angle from the top of the cylinder to where the string begins to leave the cylinder. Then the length of the string off the cylinder is $L = R\varphi$ where we allow φ to go beyond 2π as an angle.

Let's write out what the position of the particle is (using $\dot{L} = R\dot{\varphi}$)

$$x = R\sin\varphi - L\cos\varphi \tag{2.45}$$

$$y = R\cos\varphi + L\sin\varphi \tag{2.46}$$

$$\begin{aligned}\dot{x} &= R\cos\varphi\dot{\varphi} - \dot{L}\cos\varphi + L\sin\varphi\dot{\varphi} \\ &= \dot{L}\cos\varphi - \dot{L}\cos\varphi + L\sin\varphi\dot{\varphi} \\ &= L\sin\varphi\dot{\varphi}\end{aligned} \tag{2.47}$$

$$\begin{aligned}\dot{y} &= -R\sin\varphi\dot{\varphi} + \dot{L}\sin\varphi + L\cos\varphi\dot{\varphi} \\ &= -\dot{L}\sin\varphi + \dot{L}\sin\varphi + L\cos\varphi\dot{\varphi} \\ &= L\cos\varphi\dot{\varphi}\end{aligned} \tag{2.48}$$

$$v^2 = L^2\dot{\varphi}^2 = L^2\frac{\dot{L}^2}{R^2} \tag{2.49}$$

So without a potential energy we find

$$\mathcal{L} = T = \frac{mL^2\dot{L}^2}{2R^2} \tag{2.50}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{L}}\right) = \frac{\partial\mathcal{L}}{\partial L} \tag{2.51}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mL^2}{R^2}\dot{L}\right) = \frac{m\dot{L}^2L}{R^2} \tag{2.52}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L^2\dot{L}\right) = \dot{L}^2L \tag{2.53}$$

$$2L\dot{L}\dot{L} + L^2\ddot{L} = \dot{L}^2L \tag{2.54}$$

$$L^2\ddot{L} = -\dot{L}^2L \tag{2.55}$$

$$\ddot{L} = -\frac{\dot{L}^2}{L} \tag{2.56}$$

We see that this can be rewritten

$$L\ddot{L} + \dot{L}^2 = 0 \tag{2.57}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L\dot{L}\right) = 0 \tag{2.58}$$

(b)

We see that we need something satisfying $L\dot{L} = C$ for some constant C such that at time $t = 0$ we have $L = 0$ ($L_0 = 0$). We see

$$\int_{L_0}^{L(t)} \mathrm{d}L\ L = \int \mathrm{d}t\ C \tag{2.59}$$

$$\frac{L(t)^2 - L_0^2}{2} = Ct \tag{2.60}$$

$$L^2(t) = \frac{2Ct + L_0^2}{2} = Ct \tag{2.61}$$

$$L(t) = \sqrt{Ct} \tag{2.62}$$

We must have with initial velocity v_0 that

$$L'(t) = \frac{C}{2L} = \frac{Rv_0}{L} \tag{2.63}$$

so

$$L^2(t) = 2Rv_0 t \tag{2.64}$$
$$L(t) = \sqrt{2Rv_0 t} \tag{2.65}$$

So $L\dot{L} = Rv_0$.

(c)

To find the angular momentum, we need the distance from the center of mass and then we multiply by the velocity at that time. This is clearly

$$mvr = m\frac{L\dot{L}}{R}\sqrt{L^2 + R^2} = mL\dot{L}\sqrt{\frac{L^2}{R^2} + 1} \tag{2.66}$$
$$= m\sqrt{2Rv_0 t}\frac{\sqrt{Rv_0}}{\sqrt{2t}}\sqrt{\frac{2v_0 t}{R} + 1} \tag{2.67}$$
$$= mv_0 R\sqrt{\frac{2v_0 t}{R} + 1} \tag{2.68}$$

The book says

$$mvr = mvL = m\sqrt{2Rv_0^3 t} \tag{2.69}$$

which is clearly incorrect. The distance from the center of the cylindrical axis is always $R^2 + L^2$, and so I have no idea how they can say $r = L$. My answer makes a lot more sense as it says intially that $mvr = mv_0 R$ as you'd expect at $t = 0$.

The book's answer only makes sense if you go from where the cylinder and string meet, which isn't a stationary axis.

2.5 Lawn Sprayer

Consider a lawn sprayer consisting of a spherical cap ($\alpha_0 = 45°$) provided with a large number of equal holes through which water is ejected with velocity v_0. The lawn is not uniformly sprayed if these holes are evenly spaced. How must $\rho(\alpha)$, the number of holes per unit area, be chosen to achieve uniform spraying of a circular area? Assume the radius of the sprinkling cap is very much less than the radius of the area to be sprayed, and the surface of the cap is at the level of the lawn.

Solution:

With α measured from the upward direction, we see that we can look at the distribution based on how far a particle goes when ejected from a hole. This will be determined by having

$$x_d = t_d v_0 \sin\alpha \tag{2.70}$$

To find the amount of time in the air we use that there is an acceleration downwards and so

$$y = -\frac{1}{2}gt^2 + v_0 \cos\alpha t \tag{2.71}$$

$$0 = -gt_d/2 + v_0 \cos\alpha \tag{2.72}$$

$$t_d = \frac{2v_0 \cos\alpha}{g} \tag{2.73}$$

So

$$x = v_0 \sin\alpha \frac{2v_0 \cos\alpha}{g} = v_0^2 \frac{2\sin\alpha\cos\alpha}{g} = \frac{v_0^2 \sin(2\alpha)}{g} \tag{2.74}$$

Because of spherical symmetry it's clear this x is also good for any r. Let $r_d = x_d$. Then the area is proportional to $r_d\,\mathrm{d}r$. We can convert this into α and find

$$2\pi r_d\,\mathrm{d}r = 2\pi \frac{v_0^2 \sin(2\alpha)}{g}\,\mathrm{d}r \tag{2.75}$$

$$\mathrm{d}r = \frac{v_0^2 2\cos(2\alpha)}{g}\,\mathrm{d}\alpha = \frac{2v_0^2 \cos(2\alpha)}{g}\,\mathrm{d}\alpha \tag{2.76}$$

$$r_d\,\mathrm{d}r = \frac{v_0^2 \sin(2\alpha)}{g}\frac{2v_0^2 \cos(2\alpha)}{g}\,\mathrm{d}\alpha = \frac{2v_0^4 \sin(2\alpha)\cos(2\alpha)}{g^2}\,\mathrm{d}\alpha = \frac{v_0^4}{g^2}\sin(4\alpha)\,\mathrm{d}\alpha \tag{2.77}$$

Now, we know that in the form of α that given $\mathrm{d}\alpha$, the amount of water on a given angle spread is proportional to $\rho(\alpha)\sin\alpha\,\mathrm{d}\alpha$ because this is the water passing through a solid angle $\rho(\alpha)\,\mathrm{d}\Omega \sim \rho(\alpha)\sin\alpha\,\mathrm{d}\alpha$.

Thus, equating these two differentials yields

$$\rho(\alpha)\sin(\alpha) \sim \frac{v_0^4}{g^2}\sin(4\alpha) \tag{2.78}$$

$$\rho(\alpha) \sim \frac{\sin(4\alpha)}{\sin(\alpha)} \tag{2.79}$$

2.6 Constraining Surface

Find the differential equation for the contour of a constraining surface on which a point mass will oscillate with a period independent of the amplitude.

Solution:

We consider some surface with gravity. Then align the y axis with the force of gravity. Then let s be a coordinate along the constraining surface, that is the path length. This implies for $y = y(x)$

that

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \tag{2.80}$$

$$\dot{s} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\dot{x} \tag{2.81}$$

$$|v|^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \dot{x}^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\dot{x}\right)^2 = \dot{x}^2\left(1 + \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]^2\right) \tag{2.82}$$

$$= \dot{s}^2\frac{\left(1 + \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]^2\right)}{\left(1 + \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]^2\right)} = \dot{s}^2 \tag{2.83}$$

And so the equation of motion is given by letting $f(s) = \left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2$

$$\mathcal{L} = \frac{1}{2}m|v|^2 - mgy(s) = \frac{m}{2}\dot{s}^2 - mgy(s) \tag{2.84}$$

And so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{s}}\right) = \frac{\partial\mathcal{L}}{\partial s} \tag{2.85}$$

$$m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{s}) = -mg\frac{\mathrm{d}y}{\mathrm{d}s} \tag{2.86}$$

$$m\ddot{s} = -mg\frac{\mathrm{d}y}{\mathrm{d}s} \tag{2.87}$$

$$\ddot{s} = -g\frac{\mathrm{d}y}{\mathrm{d}s} \tag{2.88}$$

$$\tag{2.89}$$

If we want it to oscillate then we must have $s(t) \sim e^{-i\omega t}$ and so with $x = s$ and $y = y(s)$

$$\ddot{s} = -\omega^2 s \tag{2.90}$$

Note that the book arrives at this by saying $\ddot{s} = -g\sin\theta = -g\frac{\mathrm{d}y}{\mathrm{d}s}$ and so

$$\ddot{s} = -g\frac{\mathrm{d}y}{\mathrm{d}s} \tag{2.91}$$

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\omega^2}{g}s \tag{2.92}$$

$$y = \frac{\omega^2}{2g}s^2 \tag{2.93}$$

2.7 Equilateral Mass Triangle

Three masses (m_1, m_2, m_3) forming the corners of an equilateral triangle, attract each other according to Newton's Law. Determine the rotational motion which will leave the relative position of these masses uncharged.

Solution:

Define

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j = (x_i - x_j)\hat{\mathbf{x}} + (y_i - y_j)\hat{\mathbf{y}} \tag{2.94}$$

$$|r_{ij}| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \equiv a \tag{2.95}$$

Let $\mathbf{r}_i$ point to the ith particle.

The force on m_i is then given by

$$\mathbf{F}_1 = \frac{m_1 G}{a^3}\left[m_2\mathbf{r}_{21} + m_3\mathbf{r}_{31}\right] \tag{2.96}$$

$$\mathbf{F}_2 = \frac{m_2 G}{a^3}\left[m_1\mathbf{r}_{12} + m_3\mathbf{r}_{32}\right] \tag{2.97}$$

$$\mathbf{F}_3 = \frac{m_3 G}{a^3}\left[m_1\mathbf{r}_{13} + m_2\mathbf{r}_{23}\right] \tag{2.98}$$

Let $\mathbf{R}$ point to the center of mass. Then $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)/(m_1 + m_2 + m_3)$ with $M = m_1 + m_2 + m_3$. Then the above can be written as (i, j, k) is an even permutation of $(1, 2, 3)$

$$\mathbf{F}_i = \frac{m_i G}{a^3}\left[M\mathbf{R} - M\mathbf{r}_i\right] \tag{2.99}$$

$$= \frac{m_i M G}{a^3}\left(\mathbf{R} - \mathbf{r}_i\right) \tag{2.100}$$

In the center of mass frame $\mathbf{R} = 0$ and so

$$\mathbf{F}_i = -\frac{m_i M G}{a^3}\mathbf{r}_i \tag{2.101}$$

In the rotating frame we must have no forces at all. In a rotating frame we will have

$$\mathbf{F}_{\text{rotating}} = \mathbf{F}_{\text{inertial}} - m\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{r} = \mathbf{0} \tag{2.102}$$

as there is no velocity in the rotating frame and also we have $\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{dt}} = 0$.

Choose $\hat{\mathbf{z}}$ such that it is normal to the plane formed by the equilateral triangle. Thus we require for any particle

$$-\frac{m_i M G}{a^3}\mathbf{r}_i = m_i\Omega^2\hat{\mathbf{z}} \times \hat{\mathbf{z}} \times \mathbf{r}_i = -m_i\Omega^2\mathbf{r}_i \tag{2.103}$$

and so we see

$$\Omega^2 = \frac{MG}{a^3} \tag{2.104}$$

with $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ where $\hat{\mathbf{z}}$ is pointing normal to the plane formed by the equilateral triangle.

We can prove perpendicularity by using

$$\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{r} = \boldsymbol{\Omega}(\boldsymbol{\Omega} \cdot \mathbf{r}) - \Omega^2\mathbf{r}_i \tag{2.105}$$

and so we see from matching that $(\boldsymbol{\Omega} \cdot \mathbf{r}) = 0$, showing that $\boldsymbol{\Omega}$ points normal to the equilateral triangle plane.

2.8 Circular Orbit Central Potential

A mass m moves in a circular orbit of radius r_0 under the influence of a central force whose potential is $-km/r^n$. Show that the circular orbit is stable under small oscillations (that is, the mass will oscillate about the circular orbit) if $n < 2$.

Solution:

We use that this is a central force potential. So we can use there is an effective potential with that term given by $V = \frac{L^2}{2mr^2}$, thus

$$\mathbf{F}_{\text{eff}} \equiv \mathbf{F} = -(\nabla f + \nabla V) = -\left(-\frac{-nkm}{r^{n+1}} + \frac{-L^2}{mr^3}\right)\hat{\mathbf{r}} \tag{2.106}$$

To determine if this is a stable orbit, we make a small perturbation δr to r_0 and see

$$\hat{\mathbf{r}} \cdot \mathbf{F}(r + \delta r) = \frac{-nkm}{(r_0 + \delta r)^{n+1}} + \frac{L^2}{m(r_0 + \delta r)^3} = \frac{-nkm}{r_0^{n+1}} \frac{1}{(1 + \frac{\delta r}{r_0})^{n+1}} + \frac{L^2}{mr_0^3(1 + \frac{\delta r}{r_0})^3} \tag{2.107}$$

$$\approx \frac{-nkm}{r_0^{n+1}}\left(1 - \frac{(n+1)\delta r}{r_0}\right) + \frac{L^2}{mr_0^3}\left(1 - \frac{3\delta r}{r_0}\right) \tag{2.108}$$

We use that the $\hat{\mathbf{r}} \cdot \mathbf{F}(r)$ term must cancel at equilibrium so

$$\frac{n(n+1)km}{r_0^{n+2}}\delta r - \frac{L^2}{mr_0^4}\delta r \tag{2.109}$$

Thus for stability we require

$$\frac{n(n+1)km}{r_0^{n+2}} - \frac{3L^2}{mr_0^4} < 0 \tag{2.110}$$

We use that the equilibrium term must cancel so

$$\frac{-nkm}{r_0^{n+1}} + \frac{L^2}{mr_0^3} = 0 \tag{2.111}$$

and so

$$\frac{(n+1)L^2}{mr_0^3 r_0} - \frac{L^2}{mr_0^4} < 0 \tag{2.112}$$

$$(n+1-3)\frac{L^2}{mr_0^4} < 0 (n-2)\frac{L^2}{mr_0^4} < 0 \tag{2.113}$$

Thus $n < 2$ is stable.

2.9 Collision After Circular Orbits

Two particles move about each other in circular orbits under the influence of gravitational forces, with a period τ. The motion is suddenly stopped at a given instant of time, and the particles are then released and allowed to fall into each other. Prove that they collide after a time $\tau/(4\sqrt{2})$.

Solution:

Define

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j = (x_i - x_j)\hat{\mathbf{x}} + (y_i - y_j)\hat{\mathbf{y}} \tag{2.114}$$

$$|r_{ij}| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \equiv a \tag{2.115}$$

Let $\mathbf{r}_i$ point to the ith particle.

The force on m_i is then given by (because the distance between them is constant)

$$\mathbf{F}_1 = \frac{m_1 m_2 G}{a^3}\mathbf{r}_{21} \tag{2.116}$$

$$\mathbf{F}_2 = \frac{m_1 m_2 G}{a^3}\mathbf{r}_{12} \tag{2.117}$$

Let $\mathbf{R}$ point to the center of mass. Then $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$ with $M = m_1 + m_2$. Then the above can be written as (i, j) is an even permutation of $(1, 2)$ Then

$$\mathbf{F}_i = \frac{m_i G}{a^3}\left(m_j\mathbf{r}_j - m_j\mathbf{r}_i\right) = \frac{m_i G}{a^3}\left(M\mathbf{R} - M\mathbf{r}_i\right) \tag{2.118}$$

$$\mathbf{F}_i = \frac{m_i M G}{a^3}\left(\mathbf{R} - \mathbf{r}_i\right) \tag{2.119}$$

Thus if we choose the center of mass frame, we see $\mathbf{R} = \mathbf{0}$ and

$$\mathbf{F}_i = -\frac{m_i M G}{a^3}\mathbf{r}_i \tag{2.120}$$

In order for there to be circular motion by each, we must have

$$-\frac{m_i M G}{a^3}\mathbf{r}_i = \frac{m_i v_i^2}{a^2}\mathbf{r}_j \tag{2.121}$$

it is simple to see that in the center of mass frame that $\mathbf{r}_i = -\mathbf{r}_j$ and so

$$\frac{m_i M G}{a^3} = \frac{m_i v_i^2}{a^2} \tag{2.122}$$

$$MG = a v_i^2 \tag{2.123}$$

$$a = \frac{MG}{v_i^2} \tag{2.124}$$

The period is given by

$$\tau = \frac{2\pi a}{v_i} = \frac{2\pi MG}{v_i^3} \tag{2.125}$$

$$\tau^2 = \frac{4\pi^2 a^2}{v_i^2} = \frac{4\pi^2 a^3}{MG} \tag{2.126}$$

$$a^3 = \frac{MG\tau^2}{4\pi^2} \tag{2.127}$$

If they are stopped then we can find the position by using that it is now 1D motion and we need only find when they hit the center of mass.

We can solve the problem as where **r** is the relative distance between the particles and $\mu = m_1 m_2/M$ is the reduced mass. In this system, it is one dimensional, and so using conservation of energy, we have

$$\frac{\mu}{2}\dot{r}^2 - \frac{Gm_1m_2}{r} = \frac{-Gm_1m_2}{a} \tag{2.128}$$

$$\dot{r} = -\sqrt{\frac{2Gm_1m_2}{\mu}}\sqrt{\frac{1}{r} - \frac{1}{a}} \tag{2.129}$$

$$\int_0^a \frac{\mathrm{d}r}{\sqrt{\frac{1}{r} - \frac{1}{a}}} = -\int_t^0 \sqrt{2GM}\,\mathrm{d}t \tag{2.130}$$

$$t\sqrt{2GM} = \int_0^a \frac{\mathrm{d}r}{\sqrt{\frac{1}{r} - \frac{1}{a}}} = \int_0^a \frac{\sqrt{ra}\,\mathrm{d}r}{\sqrt{a-r}} \tag{2.131}$$

Try $x = r/a$ and we find

$$= \int_0^1 \frac{a\sqrt{x}}{\sqrt{a}\sqrt{1-x}} a\,\mathrm{d}x = a^{3/2}\int_0^1 \frac{\sqrt{x}\,\mathrm{d}x}{\sqrt{1-x}} \tag{2.132}$$

If we try $x = \sin^2\theta$ then $\mathrm{d}x = 2\sin\theta\cos\theta\,\mathrm{d}\theta = 2\sin\theta\sqrt{1-x}\,\mathrm{d}\theta$ and

$$= a^{3/2}\int_0^{\frac{\pi}{2}} 2\sin\theta\sin\theta\,\mathrm{d}\theta = 2a^{3/2}\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\,\frac{1-\cos(2\theta)}{2} = a^{3/2}\frac{\pi}{2} \tag{2.133}$$

and so

$$t = \sqrt{\frac{a^3\pi^2}{4(2GM)}} = \sqrt{\frac{MG\tau^2}{4\pi^2}\frac{\pi^2}{8GM}} = \frac{\tau}{\sqrt{32}} = \frac{\tau}{\sqrt{2^4}\sqrt{2}} = \frac{\tau}{4\sqrt{2}} \tag{2.134}$$

as desired.

2.10 Moon Jump

If r_e and ρ_e are the earth's radius and density, respectively, the corresponding quantities for the moon are $r_m = 0.275 r_e$ and $\rho_m = 0.604\rho_e$. A man standing on Earth bends his knees, lowering his center of mass 50 cm. Exerting his maximum strength he jumps straight up, raising his center of mass 60 cm above its height at his normal erect posture. How much higher can he jump in this manner, on the moon?

Solution:

The small height differences mean we can use a constant force/acceleration for these calculations. Then use that the gravitational force is the same as if both are considered at their center of mass.

Then

$$g_e = \frac{M_E G}{r_e^2} = \frac{4\pi r_e^3 G\rho_e}{3r_e^2} = \frac{4\pi r_e\rho_e G}{3} \quad (2.135)$$

$$g_m = \frac{4\pi r_m\rho_m G}{3} \quad (2.136)$$

$$(2.137)$$

Put the zero of the gravitational potential at the man's center of mass when he is bending down. Then the energy he uses on earth is

$$\delta E = mg_e h \quad (2.138)$$

with m the mass of the man and $h = 60\,\text{cm} + 50\,\text{cm} = 1.1\,\text{m}$. On the moon, place the zero at where the man's knees are bent and we have

$$h_m = \frac{\Delta E}{mg_m} = \frac{mg_e h}{mg_m} = \frac{g_e}{g_m}h = \frac{\frac{4\pi r_e\rho_e G}{3}}{\frac{4\pi r_m\rho_m G}{3}}h = \frac{r_e\rho_e}{r_m\rho_m}h \quad (2.139)$$

$$= \frac{1}{(0.275)(0.604)}h \approx 6.02h \quad (2.140)$$

Thus the man can go up to $6.02(1.1\,\text{m}) \approx 6.6\,\text{m}$ above his center of mass when bending his knees.

From his center of mass normally standing, he can then jump $6.6\,\text{m} - 0.5\,\text{m} = 6.1\,\text{m}$ up on the moon.

2.11 Force on Balance Beam

A uniform thin rigid rod of weight W is supported horizontally by two vertical props at its ends. At $t = 0$ one of these supports is kicked out. Find the force on the other support immediately thereafter.

Solution:

Let the rod be $2d$ long. When it was in static equilibrium, we know that there was a torque on the left side support of Wd. This means that the other support must be providing a torque of $-Wd$. Since it is a distance $2d$ from the left support then the force must be $-W/2$.

This tells us that before we kick out the support each support supports half the load of W.

When we kick out the support we have (measure from the remaining support for torque)

$$I\ddot{\theta} = Wd \quad (2.141)$$

where m is the mass of the entire rigid rod. The moment of inertia for a rigid rod of length L is

$$I = \frac{m}{L}\int_0^L \mathrm{d}x\; x^2 = \frac{M}{L}\frac{L^3}{3} = \frac{mL^2}{3} \quad (2.142)$$

Thus we have

$$\ddot{\theta} = \frac{Wd}{\frac{m(2d)^2}{3}} = \frac{3W}{4md} \tag{2.143}$$

We must also have for a short period of time that at the center of the rod that

$$F - W = m\ddot{y} \tag{2.144}$$

with y pointing upward and where F is the force due to the support. It's fairly clear that for small angles that $y \approx d\theta$ since $y = d\sin\theta$ and so $\ddot{y} \approx d\ddot{\theta}$ and we have

$$\frac{F-W}{md} = \frac{3W}{4md} \tag{2.145}$$

$$F = \frac{W}{4}(3-4) = \frac{-W}{4} \tag{2.146}$$

so the magnitude of the force on the support is $W/4$, half of what it was just a moment before.

2.12 Three Cylinders Minimum Angle

Three identical cylinders with parallel axes are in contact with each other on a rough plane with two cylinders lying on the plane and the third resting on top of them. What is the minimum angle which the direction of the force acting between the cylinders and the plane makes with the vertical? (What they mean is, what is the force between one of the bottom cylinders and the plane. . . Very poorly worded. . .)

Solution:

By symmetry all the weight must be supported vertically by the two bottom cylinders evenly. If each cylinder weighs W, then $3W/2$ is the total force on any of the bottom cylinders.

Take the bottom right cylinder then, and we see that

$$F_x = F_T\cos\frac{\pi}{3} - F_{fx} + F_L = 0 \tag{2.147}$$

$$F_y = -F_T\sin\frac{\pi}{3} - W + \underbrace{\frac{3W}{2}}_{F_{fy}} = 0 \tag{2.148}$$

$$(F_y) : F_T\frac{\sqrt{3}}{2} = \frac{W}{2} \tag{2.149}$$

where F_T is the force from the top cylinder, F_L is the force from the bottom left cylinder and F_f is the horizontal force from the floor. So

$$F_x = \frac{1}{\sqrt{3}}W\frac{1}{2} - F_{fx} + F_L = 0 \tag{2.150}$$

$$F_f = \frac{1}{2\sqrt{3}}W + F_L \tag{2.151}$$

We see that the smallest angle from the floor will be given for F_f being as small as possible. The smallest F_f could be is for $F_L = 0$ and then (using θ measured from the vertical

$$\tan\theta = \frac{F_{fx}}{F_{fy}} = \frac{\frac{1}{2\sqrt{3}}W}{\frac{3W}{2}} = \frac{1}{3\sqrt{3}} \tag{2.152}$$

$$\theta \approx 0.1901 \approx 10.9^\circ \tag{2.153}$$

Strangely the book argues that there is an extra F_f force exerted where F_T hits the bottom right cylinder but pointing tangent and to the left of that point.

I think it's pretty obvious that the floor is not exerting an extra force at that point, and so I don't know why they think it is there. It is extremely puzzling.

The book gets $\tan\theta \approx \frac{2+\sqrt{3}}{3}$ which would yield

$$\theta \approx 0.8937 \approx 51.2^\circ \tag{2.154}$$

and even if I assume that's an angle measured from the horizontal, so that we should have $\tan\theta \approx \frac{3}{2+\sqrt{3}}$ we'd get $\theta \approx 0.6771 \approx 38.8^\circ$ which all seem rather large for the minimum angle.

2.13 Pull on a Yo-Yo

A yo-yo rests on a level surface. A gentle horizontal pull is exerted on the cord so that the yo-yo rolls without slipping (the cord is wrapped around the axle and so the force is exerted at a radius $r < R$ where R is the radius of the yo-yo, and the cord is pulled such that it pulls from the lower part of the yo-yo). Which way does it roll and why?

Solution:

We are exerting a force to the right on the yo-yo. This means the center of mass of the yo-yo and string system must move to the right. The string is basically massless in comparison to the yo-yo and so this means the yo-yo must begin rolling to the right. The point P may move to the left, so long as the center of mass of the yo-yo moves to the right.

2.14 Dog Walks in Horizontal Circular Disk

A horizontal circular disk of mass M is free to rotate about a vertical axis through a point on its rim. If a dog of mass m walks once around the rim, show that the disk turns through an angle given by the expression

$$\int_0^\pi \frac{4m\cos^2\gamma\,\mathrm{d}\gamma}{3M/2 + 4m\cos^2\gamma} \tag{2.155}$$

Solution:

The moment of inertia of the uniform circular disk at its center of mass would be (use $\rho(r,\theta) = \frac{M}{\pi R^2}$ with R the radius of the disk)

$$\int_0^R \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \; r\rho(r,\theta) r^2 = \frac{2\pi M}{\pi R^2} \int_0^R \mathrm{d}r \; r^3 = \frac{2M}{R^2}\frac{R^2}{4} = \frac{MR^2}{2} \tag{2.156}$$

and so by the parallel axis theorem, we have that moving the rotation point R to a point on the rim of the disk that

$$I = I_{cm} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3MR^2}{2} \tag{2.157}$$

Because there was initially no angular momentum, then at all further times there is no angular momentum for the system. Let α be the angle the disk rotates about it's fixed point and β be the angle of the dog from the rotation point. The fixed disk will have it's point farthest from the rotation axis rotate around a circle of radius $2R$ from the rotation axis. I assume this is what is meant by the angle α, but the book seems to want some other angle that makes no sense as they don't define it. Then we must have from conservation of angular momentum that

$$\underbrace{-I_{\text{disk}}\dot{\alpha}}_{L_{\text{disk}}} + \underbrace{I_{\text{dog}}\omega}_{L_{\text{dog}}} = 0 \tag{2.158}$$

using that the dog can be reduced to a point mass then $I_{\text{cm,dog}} = mr^2$ and $I_{\text{disk}} = \frac{3MR^2}{2}$. Here ω is the angular velocity of the dog. Note $-\dot{\alpha}$ because I am defining counterclockwise as positive and if $\omega > 0$ then $\dot{\alpha}$ must be point in the opposite direction.

We need r the distance of the dog from the part of the rotational axis of the disk. The distance between a point fixed on a circle and another point on the circle and angle β is $2R\sin(\beta/2)$. Thus the we have

$$I_{\text{dog}} = 4mR^2 \sin^2(\beta/2) \tag{2.159}$$

If we use α as I conceive it (in Figures 2.1 and 2.2) then we can calculate the position of the dog to find ω and so

$$\mathbf{r} = \sin\beta\hat{\mathbf{x}}' + (1-\cos\beta)\hat{\mathbf{y}}' \tag{2.160}$$

$$\hat{\mathbf{x}}' = \cos\alpha\hat{\mathbf{x}} - \sin\alpha\hat{\mathbf{y}} \tag{2.161}$$

$$\hat{\mathbf{y}}' = \sin\alpha\hat{\mathbf{x}} + \cos\alpha\hat{\mathbf{y}} \tag{2.162}$$

$$\mathbf{r} = [\sin(\beta-\alpha) + \sin\alpha]\,\hat{\mathbf{x}} + [\cos\alpha - \cos(\beta-\alpha)]\,\hat{\mathbf{y}} \tag{2.163}$$

$$\dot{\mathbf{r}} = \Big[\cos(\beta-\alpha)(\dot{\beta}-\dot{\alpha}) + \cos\alpha(\dot{\alpha})\Big]\,\hat{\mathbf{x}} + \Big[-\sin\alpha(\dot{\alpha}) + \sin(\beta-\alpha)(\dot{\beta}-\dot{\alpha})\Big]\,\hat{\mathbf{y}} \tag{2.164}$$

$$\boldsymbol{\omega} = \frac{\mathbf{r}\times\dot{\mathbf{r}}}{|\mathbf{r}|^2} \tag{2.165}$$

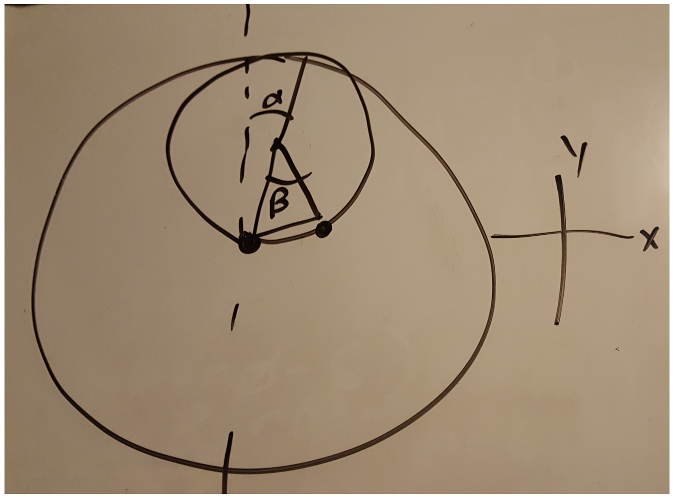

Figure 2.1: Main coordinates x and y for the loop.

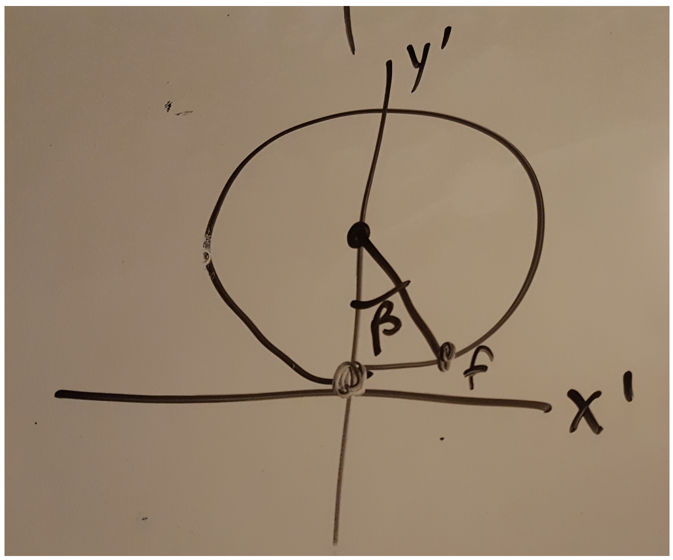

Figure 2.2: Sub coordinates x' and y' on the loop itself ignoring the pivot point.

$$\begin{aligned}
\hat{\mathbf{z}} \cdot \mathbf{r} \times \dot{\mathbf{r}} &= [\sin(\beta-\alpha)+\sin\alpha]\left[-\sin\alpha(\dot\alpha)+\sin(\beta-\alpha)(\dot\beta-\dot\alpha)\right] \\
&- [\cos\alpha-\cos(\beta-\alpha)]\left[\cos(\beta-\alpha)(\dot\beta-\dot\alpha)+\cos\alpha(\dot\alpha)\right] \\
&= -\dot\alpha\sin\alpha\sin(\beta-\alpha)+\sin^2(\beta-\alpha)(\dot\beta-\dot\alpha)-\dot\alpha\sin^2\alpha+\sin\alpha\sin(\beta-\alpha)(\dot\beta-\dot\alpha) \\
&- \left[\cos\alpha\cos(\beta-\alpha)(\dot\beta-\dot\alpha)+\dot\alpha\cos^2\alpha-\cos^2(\beta-\alpha)(\dot\beta-\dot\alpha)-\dot\alpha\cos\alpha\cos(\beta-\alpha)\right] \\
&= -\dot\alpha\left[\sin\alpha\sin(\beta-\alpha)+\sin^2\alpha+\cos^2\alpha-\cos\alpha\cos(\beta-\alpha)\right] \\
&+ (\dot\beta-\dot\alpha)\left[\sin^2(\beta-\alpha)+\sin\alpha\sin(\beta-\alpha)-\cos\alpha\cos(\beta-\alpha)+\cos^2(\beta-\alpha)\right] \\
&= \dot\alpha\left[-1-\sin\alpha\sin(\beta-\alpha)+\cos\alpha\cos(\beta-\alpha)\right] \\
&+ (\dot\beta-\dot\alpha)\left[1+\sin\alpha\sin(\beta-\alpha)-\cos\alpha\cos(\beta-\alpha)\right] \\
&= \dot\alpha\left[-1+\cos(\beta-\alpha+\alpha)\right]+(\dot\beta-\dot\alpha)\left[1-\cos(\beta-\alpha+\alpha)\right] \\
&= \dot\alpha\left[\cos\beta-1\right]+(\dot\beta-\dot\alpha)\left[1-\cos\beta\right] \\
&= 2\dot\alpha\left(\cos\beta-1\right)+\dot\beta\left(1-\cos\beta\right)
\end{aligned} \tag{2.166}$$

$$
\begin{aligned}
|\mathbf{r}|^2 &= \sin^2(\beta-\alpha)+\sin^2\alpha+2\sin\alpha\sin(\beta-\alpha)+\cos^2\alpha+\cos^2(\beta-\alpha)-2\cos\alpha\cos(\beta-\alpha)\\
&= 2+2\left[\sin\alpha\sin(\beta-\alpha)-\cos\alpha\cos(\beta-\alpha)\right]\\
&= 2(1-\cos\beta)
\end{aligned}
\tag{2.167}
$$

and so

$$
\omega = -\dot\alpha + \frac{\dot\beta}{2} \tag{2.168}
$$

We then have (note my sign convention for α and β is such that as β increases α decreases)

$$
-\frac{3MR^2}{2}\dot\alpha + 4mR^2\sin^2(\beta/2)\left[-\dot\alpha+\frac{\dot\beta}{2}\right] = 0 \tag{2.169}
$$

$$
\dot\alpha\left[\frac{3MR^2}{2}+4mR^2\sin^2(\beta/2)\right] = \dot\beta\left[\frac{4mR^2}{2}\sin^2(\beta/2)\right] \tag{2.170}
$$

$$
\dot\alpha = \dot\beta\frac{\frac{4mR^2}{2}\sin^2(\beta/2)}{\frac{3MR^2}{2}+4mR^2\sin^2(\beta/2)} = \dot\beta\frac{2m\sin^2(\beta/2)}{\frac{3M}{2}+4m\sin^2(\beta/2)} \tag{2.171}
$$

$$
\int_0^{t_f}\mathrm{d}t\ \dot\alpha = \int_0^{t_f}\mathrm{d}t\ \dot\beta\frac{2m\sin^2(\beta/2)}{\frac{3M}{2}+4m\sin^2(\beta/2)} \tag{2.172}
$$

$$
\int_0^{\alpha}\mathrm{d}\alpha\ = \int_0^{2\pi}\mathrm{d}\beta\ \frac{2m\sin^2(\beta/2)}{\frac{3M}{2}+4m\sin^2(\beta/2)} \tag{2.173}
$$

$$
\alpha = \int_0^{2\pi}\mathrm{d}\beta\ \frac{2m\sin^2(\beta/2)}{\frac{3M}{2}+4m\sin^2(\beta/2)} \tag{2.174}
$$

Then we introduce $\gamma=\beta/2$, $2\,\mathrm{d}\gamma = \mathrm{d}\beta$ and

$$
\alpha = \int_0^{\pi}\mathrm{d}\gamma\ \frac{4m\sin^2\gamma}{\frac{3M}{2}+4m\sin^2\gamma} \tag{2.175}
$$

just as the book desires. (Because their γ is an angle that is $\pi/2$ off from my γ, so we need to use the periodic properties of sin and cos.)

2.15 Dust Layer

A layer of dust is formed h feet thick (h small compared to the Earth's radius) by the fall of meteors reaching the Earth from all directions. Show, by considering angular momentum, that the change in the length of the day is approximately $5hd/(RD)$ of a day, where R is the radius of the Earth, and D and d the densities of Earth and dust, respectively. Use a notation in which the initial quantities carry subscript zero, final quantities a subscript 1. The moment of inertia of a sphere about an axis through its center is $(2/5)MR^2$, that of a thin walled, hollow sphere of mass M and radius R is $(2/3)MR^2$.

Solution:

Let's find the moment of inertia of a sphere with an axis through its center. The distance as a function of z from the center to the edge of the sphere will be given by $r^2 + z^2 = R^2$ with R the radius of the sphere.

Thus

$$I_{\text{sphere}} = \int_0^{2\pi} \mathrm{d}\theta \int_{-R}^{R} \mathrm{d}z \int_0^{\sqrt{R^2-z^2}} \mathrm{d}r\ r\rho r^2 = 2\pi \frac{M}{\frac{4}{3}\pi R^3} \int_{-R}^{R} \mathrm{d}z\ \frac{(R^2-z^2)^2}{4} \tag{2.176}$$

$$= \frac{3M\pi}{8\pi R^3} \int_{-R}^{R} \mathrm{d}z\ (R^4 - 2z^2R^2 + z^4) = \frac{3M}{8R^3}\left[2R^5 - \frac{4R^3R^2}{3} + \frac{2R^5}{5}\right] \tag{2.177}$$

$$= \frac{3MR^2}{8}\left[\frac{30-20+6}{15}\right] = \frac{3MR^2}{8}\frac{16}{15} = \frac{6MR^2}{15} = \frac{2MR^2}{5} \tag{2.178}$$

For a thin-walled hollow sphere, however we'd require (using $r^2 = R^2 - z^2$) remembering that a shell will have $2\pi r R\,\mathrm{d}\theta$ as its "volume" we use $r = R\sin\theta$ and so

$$I_{\text{h. sphere}} = \int_0^{\pi} \mathrm{d}\theta\ \underbrace{\frac{M}{4\pi R^2}}_{\rho} \underbrace{R^2\sin^2\theta}_{r^2} \underbrace{2\pi R^2 \sin\theta}_{\mathrm{d}A/\,\mathrm{d}\theta} = \frac{MR^2}{2}\int_0^{\pi} \sin^3\theta = \frac{MR^2}{2}\left[2 - \frac{2}{3}\right] = \frac{2MR^2}{3} \tag{2.179}$$

Let earth initially have angular momentum

$$L_0 = I_0\omega_0 = \frac{2M_E R^2}{5}\omega_0 \tag{2.180}$$

The final angular momentum of the new system is

$$I_1 = I_0 + \frac{2M_D R^2}{3} \tag{2.181}$$

if we assume uniform density then

$$I_0 = \frac{2R^2}{5}\frac{4\pi R^3 D}{3} = \frac{8\pi R^5 D}{15} \tag{2.182}$$

$$I_1 = \frac{8\pi R^5 D}{15} + \frac{2R^2}{3} 4\pi R^2 hd = 8R^4\pi\left(\frac{RD}{15} + \frac{hd}{3}\right) \tag{2.183}$$

so with the period $\tau = 2\pi/\omega$ we find

$$L_0 = \frac{8\pi R^5 D}{15}\omega_0 = L_1 = 8\pi R^4\left(\frac{RD}{15} + \frac{hd}{3}\right)\omega_1 \tag{2.184}$$

$$\frac{2\pi\tau_1}{2\pi\tau_0} = \frac{8\pi R^4\left(\frac{RD}{15} + \frac{hd}{3}\right)}{\frac{8\pi R^5 D}{15}} \tag{2.185}$$

$$\frac{\tau_1}{\tau_0} = \frac{15}{RD}\left(\frac{RD}{15} + \frac{hd}{3}\right) = 1 + \frac{5hd}{RD} \tag{2.186}$$

Thus $\tau_1 - \tau_0$ is

$$\tau_1 - \tau_0 = \frac{5hd}{RD}\tau_0 \tag{2.187}$$

2.16 Gyrocompass

A simple gyrocompass consists of a gyroscope spinning about its axis with angular velocity ω. The moment of inertia about this axis is C, that about a transverse axis is A. The gyroscope suspension floats on a pool of mercury so that the only torque acting on the gyroscope is one constraining its axis to remain in a horizontal plane. If the gyro is placed at the earth's equator, the angular velocity of the earth being Ω, show that the axis of the gyro will oscillate about the north-south direction; and for small amplitudes of oscillation, find this period. Remember that $\omega \gg \Omega$ is an excellent approximation.

Solution:

Let's make a coordinate system for the earth. Let $\hat{\mathbf{y}}$ be the coordinate pointing along the earth's spin axis (and the original gyroscope direction). Let $\hat{\mathbf{z}}$ then be "up" on the Earth and then $\hat{\mathbf{x}}$ be perpendicular to these two directions for a right-handed coordinate system.

We then must calculate the torque on the system in this rotating system.

We have in an inertial frame.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\tau} \tag{2.188}$$

with τ the torque. In any inertial frame, we can switch a quantity to its rotational form via

$$\underbrace{\mathbf{Q}}_{\text{inertial}} = \left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{\text{rot.}} + \boldsymbol{\omega}' \times \mathbf{Q} \tag{2.189}$$

Since we are rotating in a frame where $\omega' = \Omega$ we have

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{\text{rot.}} = \boldsymbol{\tau} - \boldsymbol{\Omega} \times \mathbf{L} \tag{2.190}$$

Consider ϕ to be the angel in the xy plane that gives the precession. Then using that ϕ increasing is leads to negative L_z we write

$$L_x = C\omega \sin\varphi \tag{2.191}$$
$$L_y = C\omega \cos\varphi + \cancel{\Omega} \tag{2.192}$$
$$L_z = -A\dot{\varphi} \tag{2.193}$$

and for small angle ϕ $\sin\varphi \approx \phi$ and $\cos\varphi \approx 1$ so that we have for the z component that

$$-A\ddot{\varphi} = N_z + \Omega C\omega\varphi \tag{2.194}$$

and $N_z = 0$ since it is free to rotate in the x and y plane. so $\varphi \sim e^{\varphi\Omega C\omega/A}$ and so the frequency is

$$\nu^2 = \frac{\omega\Omega C}{A} \tag{2.195}$$

2.17 Sphere Surface Vibrating

The surface of a sphere is vibrating slowly in such a way that the principal moments of inertia are harmonic functions of time:

$$I_{zz} = \frac{2mr^2}{5}(1+\epsilon\cos\omega t) \tag{2.196}$$

$$I_{xx} = I_{yy} = \frac{2mr^2}{5}\left(1-\epsilon\frac{\cos\omega t}{2}\right) \tag{2.197}$$

where $\epsilon \ll 1$. The sphere is simultaneously rotating with angular velocity $\mathbf{\Omega}(t)$. Show that the z-component of $\mathbf{\Omega}$ remains approximately constant. Show also that $\mathbf{\Omega}(t)$ precesses around z with a precession frequency $\omega_p = \frac{3\epsilon\Omega_z}{2}\cos\omega t$ provided $\Omega_z \gg \omega$.

Solution:

We must have

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{\text{rot.}} = \boldsymbol{\tau} - \mathbf{\Omega}\times\mathbf{L} \tag{2.198}$$

from before. No torques are given, so

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{\text{rot.}} = -\mathbf{\Omega}\times\mathbf{L} \tag{2.199}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[I_{xx}\Omega_x] = -\Omega_y L_z + \Omega_z L_y = \Omega_z I_{yy}\Omega_y - \Omega_y I_{zz}\Omega_z \tag{2.200}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[I_{yy}\Omega_y] = -\Omega_z L_x + \Omega_x L_z = \Omega_x I_{zz}\Omega_z - \Omega_z I_{xx}\Omega_x \tag{2.201}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[I_{zz}\Omega_y] = -\Omega_x L_y + \Omega_y L_x = \Omega_y I_{xx}\Omega_x - \Omega_x I_{yy}\Omega_y \tag{2.202}$$

and so for each component we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[I_{xx}\Omega_x] = \Omega_y\Omega_z(I_{yy}-I_{zz}) \tag{2.203}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[I_{yy}\Omega_y] = \Omega_x\Omega_z(I_{zz}-I_{xx}) \tag{2.204}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[I_{zz}\Omega_z] = \Omega_x\Omega_y\left(I_{xx-I_{yy}}\right) = 0 \tag{2.205}$$

So $L_z = I_{zz}\Omega_z$ is constant in time. Let the initial Ω_z be Ω_{z0} then

$$\Omega_z = \frac{I_{zz}(0)\Omega_{z0}}{I_{zz}} = \frac{\frac{2mr^2}{5}\Omega_{z0}}{\frac{2mr^2}{5}(1+\epsilon\cos\omega t)} = \frac{\Omega_{z0}}{1+\epsilon\cos(\omega t)} \tag{2.206}$$

Thus for $\epsilon \ll 1$ we have

$$\Omega_z \approx \Omega_{z0}(1-\epsilon\cos(\omega t)) \tag{2.207}$$

so $\Omega_z \approx \Omega_{z0}$.

Then we can take a derivative of the x component and find

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[I_{xx}\Omega_x\right] = \Omega_z \frac{\mathrm{d}}{\mathrm{d}t}\left[\Omega_y\left(I_{yy} - I_{zz}\right)\right] \tag{2.208}$$

If we ignore terms $\mathcal{O}(\omega)$ this equation becomes

$$I_{xx}\frac{\mathrm{d}^2\Omega_x}{\mathrm{d}t^2} = \Omega_z\left(I_{yy} - I_{zz}\right)\frac{\mathrm{d}\Omega_y}{\mathrm{d}t} = \Omega_z\left(I_{yy} - I_{zz}\right)\Omega_x\Omega_z\frac{I_{zz} - I_{xx}}{I_{yy}} \tag{2.209}$$

$$\frac{\mathrm{d}^2\Omega_x}{\mathrm{d}t^2} = -\Omega_z^2\frac{\left(I_{yy} - I_{zz}\right)^2}{I_{xx}^2}\Omega_x \tag{2.210}$$

and using (throw away $\mathcal{O}(\epsilon^2)$ terms)

$$\frac{I_{yy} - I_{zz}}{I_{yy}} \approx \frac{\frac{-3\epsilon\cos\omega t}{2}}{\left(1 - \epsilon\cos\omega t + \mathcal{O}(\epsilon^2)\right)} = \frac{-3\epsilon\cos\omega t}{2} + \mathcal{O}(\epsilon^2) \tag{2.211}$$

Thus we have, assuming that on the long times scales we are talking about that $\cos(\omega t)$ barely varies so that it is essentially a constant and we have

$$\Omega_x \sim e^{-i\sqrt{\omega_p}t} \tag{2.212}$$

$$\omega_p \sim \frac{3\epsilon\Omega_z}{2}\cos(\omega t) \tag{2.213}$$

Clearly due to the coupling of Ω_x and Ω_y this will be true for Ω_y as well, and so that is the precession frequency.

2.18 Rigid Sphere Normal Modes

Three rigid spheres are connected in a line by light, flexible rods with relative masses $m_1 : m_2 : m_3 = 1 : 2 : 1$. Describe all the normal modes of the system and state whatever you can about the relative frequencies.

Solution:

The first normal mode, due to symmetry, is the middle mass remaining still and the two side masses vibrating in and out. Say this frequency is ω_1.

The second normal mode will be the big mass moving back and forth towards the little masses (and them moving slightly as well. Say this frequency is ω_2.

A "third" normal mode is all of them moving together, so no frequency.

If we allow planar motion, then we will have the big mass moving up/down and the small masses moving down/up as another mode.

We can figure this out by forming the Lagrangian. Let y_i be normal to the rods, and x_i displacement of m_i from its equilibrium position. Let k be the spring coefficient along x and k_y be the coefficient in the y direction. Note that we need the y term to represent the bending of the rod away from the straight line equilibrium in x so we cannot allow arbitrary changes in y_i. What we want is a force if the rods do not form a straight line. Thus if the change in y isn't the same for (y_2, y_1) and (y_3, y_2).

$$\begin{aligned} 2\mathcal{L} = m_1\dot{x_1}^2 + m_2\dot{x_2}^2 + m_3\dot{x_3}^2 + m_1\dot{y_1}^2 + m_2\dot{y_2}^2 + m_3\dot{y_3}^2 \\ - k\left[(x_1 - x_2)^2 + (x_2 - x_3)^2\right] - k_y\left[(y_1 - y_2) - (y_2 - y_3)\right]^2 \end{aligned} \tag{2.214}$$

$$\begin{aligned} \mathcal{L}' \equiv \frac{2\mathcal{L}}{m_1} = \dot{x_1}^2 + 2\dot{x_2}^2 + \dot{x_3}^2 + \dot{y_1}^2 + 2\dot{y_2}^2 + \dot{y_3}^2 \\ - \frac{k}{m_1}\left[(x_1 - x_2)^2 + (x_2 - x_3)^2\right] - \frac{k_y}{m_1}\left[(y_1 - y_2) - (y_2 - y_3)\right]^2 \end{aligned} \tag{2.215}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}'}{\partial\dot{x}_{1,3}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(2\dot{x_{1,3}}) = \frac{\partial\mathcal{L}'}{\partial x_{1,3}} = -2\frac{k}{m_1}(x_{1,3} - x_2) \tag{2.216}$$

$$\ddot{x}_{1,3} = -k(x_{1,3} - x_2) \tag{2.217}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}'}{\partial\dot{x}_2} = 4\ddot{x}_2 = \frac{\partial\mathcal{L}'}{\partial x_2} = -2\frac{k}{m_1}\left[(x_2 - x_3) - (x_1 - x_2)\right] = -2\frac{k}{m_1}(2x_2 - x_1 - x_3) \tag{2.218}$$

Summarizing

$$\ddot{x}_1 = \frac{-k}{m_1}(x_1 - x_2) \tag{2.219}$$

$$\ddot{x}_2 = \frac{-k}{2m_1}(2x_2 - x_3 - x_1) \tag{2.220}$$

$$\ddot{x}_3 = \frac{-k}{m_1}(x_3 - x_2) \tag{2.221}$$

To find the frequencies here, we say $x_i \sim e^{-i\omega t}$ and find

$$\begin{bmatrix} -\omega^2 + \frac{k}{m_1} & -\frac{k}{m_1} & 0 \\ -\frac{k}{2m_1} & -\omega^2 + \frac{k}{m_1} & -\frac{k}{2m_1} \\ 0 & -\frac{k}{m_1} & -\omega^2 + \frac{k}{m_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.222}$$

Taking the determinant we'd find

$$-\omega^2\left(\frac{k}{m_1} - \omega^2\right)\left(\frac{2k}{m_1} - \omega^2\right) = 0 \tag{2.223}$$

yielding frequencies (from above)

$$\omega_1 = \pm\sqrt{\frac{k}{m_1}} \tag{2.224}$$

$$\omega_2 = \pm\sqrt{\frac{2k}{m_1}} \tag{2.225}$$

$$\omega_3 = 0 \tag{2.226}$$

with $\omega_3 = 0$ as we guessed.

For the y direction we do the same thing and find a frequency

$$\omega_4 = \pm\sqrt{\frac{4k_y}{m}} \tag{2.227}$$

Namely, $(k_y' = k_y/m_1)$

$$\frac{\partial \mathcal{L}'}{\partial \dot{y}_1} = 2\ddot{y}_1 = \frac{\partial \mathcal{L}'}{\partial y_1} = -2k_y'(y_1 - 2y_2 + y_3) \tag{2.228}$$

$$\frac{\partial \mathcal{L}'}{\partial \dot{y}_2} = 4\ddot{y}_3 = \frac{\partial \mathcal{L}'}{\partial y_2} = 4k_y'(y_1 - 2y_2 + y_3) \tag{2.229}$$

so for $y \sim e^{-i\omega t}$

$$\begin{bmatrix} -\omega^2 + k_y' & -2k_y' & k_y' \\ -k_y' & -\omega^2 + 2k_y' & -k_y' \\ k_y' & -2k_y' & -\omega^2 + k_y' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.230}$$

So the determinant gives

$$\omega^4 \left(4k_y' - \omega^2\right) = 0 \tag{2.231}$$

giving what I stated before.

2.19 Bar on Springs

A rigid uniform bar of mass M and length L is supported in equilibrium in a horizontal position by two massless springs attached one at each end. The springs have the same force constant k. The motion of the center of the gravity is constrained to move parallel to the vertical X axis. Find the normal modes and frequencies of vibration of the system, if the motion is constrained to the XZ plane.

Solution:

We have for small perturbations that the center of mass will accelerate under

$$M\frac{\ddot{x}_1 + \ddot{x}_2}{2} = Mg - k(x_1 + x_2) \tag{2.232}$$

while the torque can be calculated if in directions from using

$$I = \int_{-L/2}^{L/2} \mathrm{d}r \; \frac{M}{L} r^2 = \frac{M}{L}\frac{2L^3}{3(2)^3} = \frac{ML^2}{12} \tag{2.233}$$

Thus we have the torque as

$$I(\ddot{x}_1 - \ddot{x}_2)/L = k(x_1 - x_2)\frac{L}{2} \tag{2.234}$$

Writing these in matrix form and assuming $X_i \sim e^{-i\omega t}$, we find

$$\begin{bmatrix} -\omega^2\frac{M}{2}+k & -\omega^2\frac{M}{2}+k \\ \frac{ML}{12}\omega^2-\frac{kL}{2} & -\omega^2\frac{ML}{12}+\frac{kL}{2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} Mg \\ 0 \end{bmatrix} \tag{2.235}$$

the Mg term can be defined away with a new coordinate system. Thus we find

$$\left(\frac{2k}{M}-\omega^2\right)\left(\frac{6k}{M}-\omega^2\right)=0 \tag{2.236}$$

giving frequencies

$$\omega_1^2 = \frac{2k}{M} \tag{2.237}$$

$$\omega_2^2 = \frac{6k}{M} \tag{2.238}$$

corresponding to eigenvector

$$v_1 = X_1 + X_2 \tag{2.239}$$

$$v_2 = X_2 - X_1 \tag{2.240}$$

2.20 Particle on Uniform String

A particle of mass M hangs from one end of a uniform string of mass m and length L; the other end of the string is fixed. The particle is given a small lateral displacement δ and released from rest. Set up the differential equations and boundary conditions to determine the motion of string and particle. Set up a transcendental equation that determines the natural frequencies, and solve the equation for the case $m \ll M$.

Solution:

Let $y(x,t)$ represent the horizontal motion of the string at point x along the string at time t.

Then are boundary conditions are clearly

$$y(0,t) = 0 \tag{2.241}$$

$$y(L,0) = \delta \tag{2.242}$$

$$\frac{\partial y}{\partial t}(x,0) = 0 \tag{2.243}$$

$$\frac{\partial^2 y}{\partial x^2}(L,t) = 0 \tag{2.244}$$

where the last boundary condition comes from the bend in the string not be discontinuous. In general, for any line/string we know tension will be what pushes on any small element, so

$$\mu\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x)\frac{\partial y}{\partial x}\right] \tag{2.245}$$

with $T(x) = g(m(1 - x/L) + M)$ is the tension (since a small displacement initially, this is the equilibrium value).

We see the third boundary condition yields $y = f(x)\cos\omega t$. Thus our equation becomes

$$-\omega^2 \mu f = \frac{\partial}{\partial x}\left[g\left(m\left\{1 - \frac{x}{L}\right\} + M\right)\frac{\mathrm{d}f}{\mathrm{d}x}\right] \tag{2.246}$$

If we assume $f(x) = h(\sqrt{T(x)})$ then we get

$$h'' + \frac{h'}{\sqrt{T}} + \mu\left(\frac{2L\omega}{mg}\right)^2 h = 0 \tag{2.247}$$

which is is a Bessel equation of order zero and so has solutions

$$f(x) = AJ_0\left(\frac{2\omega}{g}\sqrt{\frac{T(x)}{\mu}}\right) + BN_0\left(\frac{2\omega}{g}\sqrt{\frac{T(x)}{\mu}}\right) \tag{2.248}$$

Then we can use boundary conditions to find y and set up the transcendental equation.

In the large argument limit of $m \ll M$ we get a solution that $y = \delta x\frac{\cos\omega t}{L}$.

2.21 Membrane with Surface Tension

Set up a variational principle for the frequency ω of a membrane with surface tension T of mass σ per unit area, and with fixed edges that is , find an integral over the area of the membrane, of which the extreme value is the frequency of the membrane.

Solution:

This is basically going to be Rayleigh's quotient. We have

$$\sigma\frac{\partial^2 y}{\partial t^2} = T\,\nabla^2 y \tag{2.249}$$

If we assume $y \sim e^{-i\omega t}$ then

$$-\sigma\omega^2 y = T\,\nabla^2 y \tag{2.250}$$

Then $y = u(x, y)e^{-i\omega t}$ and we can then view this as an eigenvalue problem for y with eigenvalue $\sigma\omega^2/T$. Then we know that the minimum of the Rayleigh quotient is given by

$$\lambda = -\frac{\sigma\omega^2}{T} = \frac{\int \mathrm{d}A\; u\,\nabla^2 u}{\int \mathrm{d}A\; u^2} \tag{2.251}$$

now we can find good approximations by putting in trial functions that satisfy the boundary conditions and iterating.

2.22 Watch Moved to High Altitude

If a watch is moved to a high altitude, does it run fast or slow?

Solution:

Depends on the watch. For a mechanical watch, they usually work by tension in a string, and so temperature is more important than anything else. However if they use a flywheel, then the viscosity of air pushes against the flywheel, so that when there's less air this is less of a force and the flywheel releases more energy causing the clock to go slightly faster. Apparently Fermi calculated this.

Of course, today with quartz watches, this effect would make no difference at all, as the electrical components will be almost completely unaffected by air viscosity.

2.23 Release Mass on String

A mass m is attached to a weightless string of length L, cross section S, and tensile strength T. The mass is suddenly released from a point near the fixed end of the string. How small should the Young's modulus, Y, of the string be, in order that it not break?

Solution:

The mass will exert a force mg on the string eventually. The Young's modulus is the effective spring constant for a material ($Y\frac{S}{L}$ makes it a spring constant k). We have

$$Y = \frac{FL}{\Delta L\, S} \Rightarrow F = \frac{SY\Delta L}{L} \tag{2.252}$$

where ΔL is the change in length of the object under force F.

We can find ΔL from conservation of energy.

$$mg(L + \Delta L) = \frac{YS}{2L}\Delta L^2 \tag{2.253}$$

$$\Delta L^2 - mg\frac{2L}{YS}\delta L - mgL\frac{2L}{YS} = 0 \tag{2.254}$$

$$\Delta L = \frac{mgL}{YS} \pm \sqrt{\frac{m^2g^2L^2}{Y^2S^2} + \frac{2mgL^2}{YS}} \tag{2.255}$$

Clearly only the + sign makes sense, and so

$$\Delta L = \frac{mgL}{YS}\left(1 + \sqrt{1 + \frac{2YS}{mg}}\right) \tag{2.256}$$

We must have

$$F < ST \quad (2.257)$$

$$Y\frac{S\Delta L}{L} < ST \quad (2.258)$$

$$Y\frac{mg}{YS}\left(1+\sqrt{1+\frac{2YS}{mg}}\right) < T \quad (2.259)$$

$$\frac{mg}{S}\left(1+\sqrt{1+\frac{2YS}{mg}}\right) < T \quad (2.260)$$

$$\sqrt{1+\frac{2YS}{mg}} < \frac{ST}{mg} - 1 \quad (2.261)$$

$$1+\frac{2YS}{mg} < \left(\frac{ST}{mg}-1\right)^2 \quad (2.262)$$

$$\frac{2YS}{mg} < \left(\frac{ST}{mg}-1\right)^2 - 1 \quad (2.263)$$

$$Y < \frac{mg}{2S}\left(\frac{ST}{mg}-1\right)^2 - \frac{mg}{2S} \quad (2.264)$$

$$Y < \frac{mg}{2S}\left(\frac{S^2T^2}{m^2g^2} - 2\frac{ST}{mg} + \cancel{1}\right) - \cancel{\frac{mg}{2S}} \quad (2.265)$$

$$Y < \frac{ST^2}{2mg} - T \quad (2.266)$$

2.24 Train Into a Spring

A train of mass M, moving with velocity v is to be stopped with a coil-spring buffer of uncompressed length l_0 and spring constant k_0, which remains constant until the spring is fully compressed. At this point $l \ll l_0$ the spring constant k suddenly becomes very much greater than k_0. Assuming a free choice of k_0, what is the minimum value of l_0 if the absolute value of the maximum deceleration is not to exceed $a_{\max}$?

Solution:

Clearly we have to get there before $x \to l$ and the last point this is possible is then

$$a_{\max} = \frac{k_0}{M}(l_0 - l) \quad (2.267)$$

We know want the train stopped at this point for $l \ll l_0$. From energy, at standstill we will have

$$\frac{M}{2}v^2 = \frac{k_0}{2}(l_0 - l)^2 \quad (2.268)$$

we can plug in k_0 here from the acceleration equation and so

$$\frac{M}{2}v^2 = \frac{1}{2}\frac{Ma_{\max}}{l_0 - l}(l_0 - l)^2 = \frac{M}{2}a_{\max}(l_0 - l) \tag{2.269}$$

$$l_0 = \frac{v^2}{a_{\max}} + l \approx \frac{v^2}{a_{\max}} \tag{2.270}$$

using $l \ll l_0$.

2.25 Cylinder in Fluid

(a) A cylinder of radius R, length h, and density ρ floats upright in a fluid of density ρ_0. If it is given a small downward displacement of amplitude x, find the circular frequency ω of the resulting (undamped) harmonic motion.

(b) Show that for small oscillations, the motion of the fluid near the oscillating cylinder extends for a distance $\delta \sim \sqrt{\eta/(\rho_0\omega)}$ from the edge of the cylinder. The maximum gradient of velocity near the cylinder is thus $\mathrm{d}V/\mathrm{d}r \approx \omega x/\delta$. Neglecting the friction at the bottom of the cylinder, show that the maximum viscous retarding force on the cylinder is $F \approx 2\pi R h \rho(\eta\omega^3/\rho_0)^{1/2}x$.

Solution:

(a) The small displacement will cause a displacement of fluid. It will displace a volume of $x\pi R^2$. Then there will be a buoyant force on the cylinder from the surrounding fluid to push the can back towards equilibrium. This buoyant force should be the mass of the displaced fluid times g. Thus

$$h\pi R^2\rho\ddot{x} = -xg\pi R^2\rho_0 \tag{2.271}$$

$$\ddot{x} = -x\frac{g\pi R^2\rho_0}{h\pi R^2\rho} \tag{2.272}$$

$$\omega^2 = \frac{g\rho_0}{h\rho} \tag{2.273}$$

(b) Imagine a slab of fluid next to the cylinder of area A (extending $\mathrm{d}r$ away). Then the viscous force on this slab is given by (η is the dynamic viscosity)

$$A\eta\frac{\partial^2 v}{\partial r^2}\,\mathrm{d}r \tag{2.274}$$

with η the viscosity of the water. By Newton's second law this force must equal the acceleration of this fluid slab

$$A\rho_0\,\mathrm{d}r\frac{\partial v}{\partial t} = A\eta\frac{\partial^2 v}{\partial r^2}\,\mathrm{d}r \tag{2.275}$$

Given a harmonic response in time $v \sim e^{-i\omega t}$ we find

$$-i\omega\rho_0 v = \eta\frac{\partial^2 v}{\partial r^2} \tag{2.276}$$

So that

$$v = v_0 e^{\sqrt{-i}\sqrt{\omega\rho_0}r/\sqrt{\eta}} = v_0 e^{\frac{(1-i)}{\sqrt{2}}\sqrt{\omega\rho_0}r/\sqrt{\eta}} = v_0 e^{(1-i)r/\delta} \tag{2.277}$$

if we make $\delta = \sqrt{2\eta/(\rho_0\omega)}$ as one e-folding period. Then the viscous force on the cylinder is given by (the $(\rho/\rho_0)^2$ is to correct for the difference in force due to mass density differences between the fluid column surrounding the cylinder and the cylinder itself)

$$|F|^2 = \frac{\rho^2}{\rho_0^2}A^2\left|\eta\frac{\partial v}{\partial r}\right|^2 = \frac{\rho^2}{\rho_0^2}A^2|\eta\frac{(1-i)}{\delta}v_0|^2 = 2\frac{\rho^2}{\rho_0^2}A^2\eta^2\frac{\omega\rho_0}{2\eta}v_0^2 \tag{2.278}$$

We clearly need a v_0 which will be given by our previous problem with $\dot{x} = -i\omega x$ and so $v_0^2 = \omega^2 x^2$

$$|F|^2 = (2\pi Rh)^2\frac{\rho^2}{\rho_0}\eta\omega^3 x^2 \tag{2.279}$$

$$|F| = 2\pi Rh\rho\sqrt{\frac{\eta\omega^3}{\rho_0}}|x| \tag{2.280}$$

2.26 Surface Tension Between Loops

A liquid film of surface tension τ is stretched between two circular loops of radius a.Find the equation $r(z)$. For what ratio d/a is the configuration indicated in the figure stable?

Solution:

Set the z axis in between the centers of the two loops, so that each loop center is a distance d from the $z = 0$ point.

Consider some infinitesimal area then we will have an angle formed from the vertical and the curve inwards from the surface tension curve. (At the top let the angle be θ_1 and at the bottom θ_2). We have force balance and so in the vertical direction we must have

$$2\pi r_1\tau\cos\theta_1 = 2\pi r_2\tau\cos\theta_2 \tag{2.281}$$

This implies $r_i\cos\theta_i = C$ for some constant C. Drawing the r-z triangle for this infinitesimal area we see that

$$\cos\theta_i = \frac{\mathrm{d}z}{\sqrt{\mathrm{d}r^2 + \mathrm{d}z^2}} = \frac{1}{\sqrt{1 + \left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)^2}} \tag{2.282}$$

And so we find that the curve is given by

$$\frac{r}{\sqrt{1 + \left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)^2}} = C \tag{2.283}$$

$$\frac{r^2}{C^2} = 1 + \left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)^2 \tag{2.284}$$

$$\frac{\mathrm{d}r}{\mathrm{d}z} = \sqrt{\frac{r^2}{C^2} - 1} \tag{2.285}$$

$$\int \mathrm{d}r\,\frac{1}{\sqrt{\frac{r^2}{C^2} - 1}} = z - z_0 \tag{2.286}$$

Substitute $\frac{r}{C} = \cosh x$ then $\mathrm{d}r = C \sinh x \, \mathrm{d}x$ so

$$\int \mathrm{d}x \, \frac{C \sinh x}{(\sqrt{\sinh^2 x})} = \int \mathrm{d}x \; C = Cx - Cx_0 = C \operatorname{arccosh}\left(\frac{r}{C}\right) - C \operatorname{arccosh}\left(\frac{r_0}{C}\right) \tag{2.287}$$

Let's eliminate z_0 and the extra cosh term by setting $z_0 = 0$ and $C = r_0$ (that is, at $z = 0$ we have the radius as r_0). Then

$$\operatorname{arccosh}\left(\frac{r}{r_0}\right) = \frac{z}{r_0} \tag{2.288}$$

$$r = r_0 \cosh\left(\frac{z}{r_0}\right) \tag{2.289}$$

We require

$$a = r_0 \cosh\left(\frac{d}{r_0}\right) \tag{2.290}$$

$$\frac{a}{r_0} = \cosh\left(\frac{d}{r_0}\right) \tag{2.291}$$

which determines r_0. We see that these only sometimes intersect depending on the value of a and d. Define $\alpha = \frac{d}{r_0}$ and we see the above equation is equivalent to

$$\frac{a}{d}\alpha = \cosh \alpha \tag{2.292}$$

This only has solutions when a/d is large enough. We can find it by using that $a/d = (\cosh \alpha)/\alpha$ which has a minimum. We find the minimum via

$$\frac{\mathrm{d} \cosh \alpha/\alpha}{\mathrm{d}\alpha} = \frac{\alpha \sinh \alpha - \cosh \alpha}{\alpha^2} = 0 \tag{2.293}$$

$$\alpha \tanh \alpha = 1 \tag{2.294}$$

Using a Newton Iteration with a guess of about 1.2 we find $\alpha \approx 1.1997$ and so then the smallest a/d possible is $\cosh(1.1997)/1.1997 \approx 1.51$. The maximum d/a is therefore $d/a = 0.663$.

chapter2/NewtonApprox.py

```
#!/usr/bin/env python2

import numpy as np
import math
import matplotlib.pyplot as plt

def newtonit(val, derivative, tol=1e-9):
    x    = val
    xprev = x
    fval = x*np.tanh(x)-1.
    j=0
    while ((np.abs(fval) > tol) | (np.abs(x-xprev)>tol)):
        xprev = x
        x = derivative(x)
        fval = x*np.tanh(x)-1.
        j+=1
#         print j,x

```

```
    return x,fval,j

def xidiff(x):
    f  = x*np.tanh(x)-1.
    fp = np.tanh(x)+x/np.cosh(x)**2
    return x - f/fp

x = 1.2
print newtonit(x,xidiff)
```

2.27 Vertical Strut

A straight vertical strut, having length l and a square cross section with side a, is firmly fixed to the ground. Show that the maximum weight it can carry on the free end without bending is given by $W = \pi^2 a^4 Y/(48l^2)$, where Y is the Young's modulus for the material of the strut.

Solution:

We consider a small bending of the strut horizontally. Then we have to calculate the restoring force. We have to calculate the restoring force at a height x and horizontal distance (bending) y.

Consider the bending moment from the center of the beam. We then must have (note that it the length of the bent part will be $(R+z)\theta$ for bending θ so that the strain $\frac{\Delta L}{L} = \frac{(R+z)\theta}{R\theta}$ with z being measured from the square plane center and

$$F = \int_{-a/2}^{a/2} \mathrm{d}y \int_{-a/2}^{a/2} \mathrm{d}z\, Y \frac{(R+z)}{R} z = \frac{Y a}{R}\left[\frac{2(a/2)^3}{3}\right] = \frac{Y a}{R}\frac{a^3}{12} = \frac{Y a^4}{12R} \tag{2.295}$$

We can use that R is the radius of curvature and given by

$$\frac{1}{R} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\left[1 + \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)^2\right]^{-3/2} \approx \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \tag{2.296}$$

for small bending. Thus

$$F = \frac{Y a^4}{12}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \tag{2.297}$$

We must have the weight torque equal this restoring force for equilibrium. Given weight W, this implies the torque is $W(y(l) - y)$ (just $\tau = rW$ for this)

$$\frac{Y a^4}{12}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = W y(l) - W y \tag{2.298}$$

The general solution is

$$y = A\cos(\omega x) + B\sin(\omega x) + y(l) \tag{2.299}$$

with boundary conditions $y(0) = 0$, $y'(0) = 0$ and $y(l) = y(l)$ we find $A = -y(l)$, $B = 0$ and

$$-y(l)\cos(\omega l) + y(l) = y(l) \tag{2.300}$$

with $\omega^2 = 12W/(Ya^4)$. Thus

$$y = y(l)\left[1 - \cos(\omega x)\right] \tag{2.301}$$

with $y(l)\cos(\omega l) = 0$ is our solution. Clearly either $y(l) = 0$ or $\omega l = \frac{\pi}{2}$ for this to be true. The $y(l) = 0$ is no change, whereas

$$\omega l = \sqrt{\frac{12W}{Ya^4}}l = \frac{\pi}{2} \tag{2.302}$$

$$W = \frac{\pi^2}{4l^2}\frac{Ya^4}{12} = \frac{Y\pi^2 a^4}{48l^2} \tag{2.303}$$

is the other possible solution with infinitesimal deformation.

2.28 Rectangular Beam

A rectangular beam with cross section $a \times a$ and length L has one end anchored in a vertical brick wall. Calculate the deflection of its free end due to its own weight. The density is ρ and the Young's modulus is Y. Assume small bending.

Solution:

Let z be along the direction horizontal along the beam. Then $h(z)$ is the difference from the beam under its weight and the beam completely horizontal. Then we must have through a generalized Hooke's law (Young's law) that for bending a segment of area $a\,\mathrm{d}s$ and length $\mathrm{d}z$ through an angle $\mathrm{d}\theta$ (with s measured from the center of the beam), that the force required is

$$Yas\,\mathrm{d}\theta\frac{\mathrm{d}s}{\mathrm{d}z} \tag{2.304}$$

If we take the moment about a point of those forces, then

$$M = \int_{-a/2}^{a/2} \frac{as^2\,\mathrm{d}\theta\,\mathrm{d}s}{\mathrm{d}z} = \frac{Ya^3}{12}\frac{\mathrm{d}\theta}{\mathrm{d}z} \tag{2.305}$$

We can then use for small bending that $\mathrm{d}\theta = \mathrm{d}z\frac{\partial^2 h}{\partial z^2}$. For rotational equilibrium, we then must have

$$M = \int_z^L \rho g a^2 (L-u)\,\mathrm{d}u = \rho g a^2\left(L(L-z) - \frac{L^2 - z^2}{2}\right) = \rho g a^2\left(\frac{L^2}{2} - Lz + \frac{z^2}{2}\right) \tag{2.306}$$

so that

$$\frac{Ya^3}{12}\frac{\mathrm{d}\theta}{\mathrm{d}z} = \rho g a^2\left(\frac{L^2}{2} - Lz + \frac{z^2}{2}\right) \tag{2.307}$$

$$\tag{2.308}$$

and we must have $h(0) = h'(0) = 0$ and then

$$h(z) = \frac{\rho g}{2Ya}\left[6L^2z^2 - 4Lz^3 + z^4\right] \tag{2.309}$$

$$h(L) = \frac{\rho g L^4}{2Ya}\left[6 - 4 + 1\right] = \frac{3\rho g L^4}{2Ya} \tag{2.310}$$

2.29 Thin Uniform Chimney

A thin uniform chimney is pivoted at its low end. Show that a section through the chimney at any point undergoes a flexion stress, and calculate the most probable point of rupture as the chimney falls.

Solution:

As the chimney falls, it will clearly want to bend due to gravity from the weight of material above it.

The chimney as a whole will rotate from its pivot point in the ground with $I = ML^2/3$ and force on its center of mass $(MgL\sin\theta)/2$ with θ the angle from the vertical. So this yields

$$\ddot{\theta} = \frac{3g\sin\theta}{2L} \tag{2.311}$$

If we conceptually break the chimney into a part of length below of x and above of $L-x$, then we must have each part satisfy equations due to the moment from the origin and the center of mass, respectively. For the lower part x, we have M_L

$$\overbrace{M\frac{x}{L}}^{M_L}\frac{x^2}{3}\ddot{\theta} = M\frac{x}{L}\frac{gx\sin\theta}{2} + xF - \gamma \tag{2.312}$$

where γ is the restoring force from the $L-x$ piece keeping the piece together despite the Fx force from the rotation of the lower piece.

Similarly for the top piece we find, around its center of mass,

$$M\frac{L-x}{L}\frac{(L-x)^2}{12}\ddot{\theta} = \frac{(L-x)}{2}F + \gamma \tag{2.313}$$

So

$$\frac{Mx^3}{3L}\frac{3g\sin\theta}{2L} = \frac{Mx^2 g\sin\theta}{2L} + xF - \gamma \tag{2.314}$$

$$\frac{M(L-x)^3}{12L}\frac{3g\sin\theta}{2L} = \frac{L-x}{2}F + \gamma \tag{2.315}$$

or

$$\frac{Mx^3 g\sin\theta}{2L^2} - \frac{Mx^2 g\sin\theta}{2L} - xF + \gamma = 0 \tag{2.316}$$

$$\frac{M(L-x)^3 g\sin\theta}{8L^2} + \frac{x-L}{2}F - \gamma = 0 \tag{2.317}$$

are our two equations. We find F first

$$\frac{Mx^3g\sin\theta}{2L^2} - \frac{Mx^2g\sin\theta}{2L} - xF + \frac{M(L-x)^3g\sin\theta}{8L^2} + \frac{x-L}{2}F = 0 \tag{2.318}$$

$$-F\left(\frac{L+x}{2}\right) + \frac{Mg\sin\theta}{8L^2}\left(4x^3 - 4Lx^2 + (L-x)^3\right) = 0 \tag{2.319}$$

$$-F\left(\frac{L+x}{2}\right) + \frac{Mg\sin\theta}{8L^2}\left(4(x-L)x^2 + (L-x)^3\right) = 0 \tag{2.320}$$

$$-F\left(\frac{L+x}{2}\right) + \frac{Mg\sin\theta}{8L^2}\left(4x^3 - 4Lx^2 + L^3 - L^2x + Lx^2 - x^3\right) = 0 \tag{2.321}$$

$$F = \frac{1}{L+x}\frac{Mg\sin\theta}{4L^2}\left(3x^3 - 3xL^2 - L^2x + L^3\right) \tag{2.322}$$

$$F = \frac{1}{L+x}\frac{Mg\sin\theta}{4L^2}\left((L-x)^3 - 4x^2(L-x)\right) \tag{2.323}$$

so

$$\gamma = \frac{x-L}{2}F + \frac{M(L-x)^3g\sin\theta}{8L^2} \tag{2.324}$$

$$= \frac{Mg\sin\theta(L-x)^3}{8L^2} + \frac{x-L}{2}\frac{Mg\sin\theta}{4(L+x)L^2}\left[(L-x)^3 - 4x^2(L-x)\right] \tag{2.325}$$

$$= \frac{Mg\sin\theta}{8L^2}\left[(L-x)^3 + \frac{x-L}{L+x}\left\{(L-x)^3 - 4x^2(L-x)\right\}\right] \tag{2.326}$$

$$= \frac{Mg\sin\theta}{8L^2}2(L-x)^2x = \frac{Mg\sin\theta}{4L^2}x(L-x)^2 \tag{2.327}$$

It should break when this is maximum for x. Thus we require

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x(L-x)^2\right] = (L-x)^2 - 2x(L-x) = 0 \tag{2.328}$$

$$L^2 - 2Lx + x^2 - 2xL + 2x^2 = 0 \tag{2.329}$$

$$3x^2 - 4Lx + L^2 = 0 \tag{2.330}$$

$$x^2 - \frac{4}{3}Lx + \frac{L^2}{3} = 0 \tag{2.331}$$

$$x = \frac{2L}{3} \pm \sqrt{\frac{4L^2}{9} - \frac{L^2}{3}} = \frac{2L}{3} \pm \frac{L}{3} \tag{2.332}$$

Clearly, $x = \frac{2L}{3} - \frac{L}{3} = \frac{L}{3}$ is the solution that would give the correct answer.

2.30 Free Surface of Liquid

The free surface of a liquid is one of constant pressure. If an incompressible fluid is placed in a cylindrical vessel and the whole rotated with constant angular velocity ω, show that the free surface becomes a paraboloid of revolution.

Solution:

We simply need the equation of pressure for an incompressible fluid.

This is given by

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - \rho g \hat{\mathbf{z}} \tag{2.333}$$

In this case we have (R, φ, z) as the coordinates and there is no time dependence so

$$\mathbf{u} = R\omega \hat{\boldsymbol{\varphi}} = R^2 \omega \nabla \varphi \tag{2.334}$$

So that

$$\mathbf{u} \cdot \nabla \mathbf{u} = R^2\omega \nabla\varphi \cdot \nabla\left(R^2\omega \nabla\varphi\right) = R^2\omega \nabla\varphi \cdot \left(\omega \left[2R \nabla R \nabla\varphi + R^2 \nabla\nabla\varphi\right]\right) \tag{2.335}$$

$$= R^4\omega^2 \nabla\varphi \cdot \nabla\nabla\varphi = \frac{R^4\omega^2}{2} \nabla(\nabla\phi)^2 = \frac{R^4\omega^2}{2} \nabla \frac{1}{R^2} = \frac{R^4\omega^2}{2} \frac{-2}{R^3} \nabla R \tag{2.336}$$

$$= R\omega^2 \nabla R \tag{2.337}$$

And so

$$R\omega^2 \nabla R = -\nabla p - \rho g \hat{\mathbf{z}} \tag{2.338}$$

$$R\omega^2 = -\frac{\partial p}{\partial R} \tag{2.339}$$

$$p - p_0 = -\omega^2 \frac{R^2 - R_0^2}{2} + g(z) \tag{2.340}$$

And so the surfaces of constant tension are those where R is fixed, and we have $p \propto R^2$ so it is a parabola in R which is a paraboloid of revolution in Cartesian coordinates.

We must also of course have $\frac{\partial p}{\partial z} = \rho g$ or $p = f(R) + \rho g z$. This yields

$$p = \omega^2 \frac{R^2 - R_0^2}{2} - \rho g z + p_0 \tag{2.341}$$

which does not change our conclusion in any way, it just adjusts the pressure based on height z.

2.31 Hangar Door Force

An aircraft hangar of semi-cylindrical shape (with length L and radius R) is exposed to wind directly perpendicular to its axis at infinity with a velocity v_∞. What force is exerted on this hangar if the door, located at A is open (A is at the bottom of the hangar, and when open the velocity points into the door)? The velocity potential is given by

$$\phi = -v_\infty \left(r + \frac{R^2}{r}\right) \cos\theta \tag{2.342}$$

$$L = 70\,\text{m};\ R = 10\,\text{m};\ v = 72\,\text{km/hr};\ \text{air density} = 1.2\,\text{kg/m}^3 \tag{2.343}$$

Solution:

The velocity potential yields

$$v_r = \frac{\partial \phi}{\partial r} = -v_\infty \left(1 - \frac{R^2}{r^2}\right) \cos\theta \tag{2.344}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = v_\infty \left(1 + \frac{R^2}{r^2}\right) \sin\theta \tag{2.345}$$

Thus the velocity at the open door $(r, \theta) = (R, \pi)$ is

$$v_r(A) = v_\theta(A) = 0 \tag{2.346}$$

$$v^2 = v_\infty^2 \left(1 - \frac{R^2}{r^2}\right)^2 \cos^2\theta + v_\infty^2 \left(1 + \frac{R^2}{r^2}\right)^2 \sin^2\theta \tag{2.347}$$

$$= v_\infty^2 \left[\cos^2\theta(1 - \frac{2R^2}{r^2} + \frac{R^4}{r^4}) + \sin^2\theta(1 + \frac{2R^2}{r^2} + \frac{R^4}{r^4})\right] \tag{2.348}$$

$$= v_\infty^2 \left[1 + \frac{R^4}{r^4} + 2(\sin^2\theta - \cos^2\theta)\frac{R^2}{r^2}\right] \tag{2.349}$$

Pressure must be equal inside and outside the hangar. Call it P_0 on the inside. So using Bernoulli's principle, the pressure along a streamline on the outside must have

$$P_0 = P + \frac{\rho}{2} v^2 \tag{2.350}$$

Thus the pressure difference $P - P_0$ is given by $-\frac{\rho}{2}v^2$. The force due to this pressure is (we evaluate only along $r = R$ eliminating the v_r component

$$|F| = \int \mathrm{d}A \, \frac{\rho}{2} v^2 = \frac{L\rho}{2} \int_0^\pi \mathrm{d}\theta \; R \left(4v_\infty^2 \sin^2\theta\right) = 2LR\rho v_\infty^2 \int_0^\pi \mathrm{d}\theta \; \frac{1 - \cos\theta}{2} = \pi L R \rho v_\infty^2 \tag{2.351}$$

The pressure is higher in the hangar, so the force is outward when in the hangar. Its magnitude is

$$|F| = \pi (70\,\mathrm{m})(10\,\mathrm{m})(20\,\mathrm{m/s})^2 (1.2\,\mathrm{kg/m^3}) = 1.056 \times 10^6\,\mathrm{kg\,m/s^2} = 1.056\,\mathrm{MN} \tag{2.352}$$

2.32 Gravity Waves in Air

An air mass of $T = 280\,\mathrm{K}$ is separated by a horizontal plane from an air mass at $T = 300\,\mathrm{K}$, lying above it. Assume the presence of gravity waves of wavelength λ and small amplitude, causing a sinusoidal wave on the interface. Find the velocity of the wave as a function of the wavelength, assuming the interface is far from other horizontal interfaces. Treat the oscillations of the air masses as incompressible.

Solution:

From general concerns, it's clear that the velocity will be given by $v = \lambda\nu$ where ν is the frequency of the wave.

For the gravity waves, let's assume that it's due to the pressure and gravitational difference between the two air masses.

Let the top have pressure P_1 and the bottom interface P_2. The momentum equation yields

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla P \tag{2.353}$$

If we take the divergence and use incompressibility we see that $\nabla^2 P = 0$.

Assume the amplitude of the wave (in the z direction, is given by $\alpha e^{i(kx-\omega)}$ so that $P \sim e^{i(kx-\omega t)}$ on the two sides of the interface. (x is along the interface and z is up from the ground with the boundary at $z = 0$). These then yield

$$P_1 = -\rho_1 g z + \beta_1 e^{-k(z)+i(kx-\omega t)} \tag{2.354}$$

$$P_2 = -\rho_2 g z + \beta_2 e^{kz+i(kx-\omega t)} \tag{2.355}$$

We must now match across the boundary. We need $P_1 = P_2$ at $z = z_0$. We can then use the momentum condition near the boundary (ignoring the non-linearity which should be small for small perturbations)

$$\rho\frac{\partial \mathbf{v}}{\partial t} = \nabla P \tag{2.356}$$

In the z direction these yield

$$\rho_1\omega^2\alpha = -k\beta_1 \tag{2.357}$$

$$\rho_2\omega^2\alpha = k\beta_2 \tag{2.358}$$

$$-k\beta_1 + k\beta_2 = \omega^2\alpha\left(\rho_1+\rho_2\right) \tag{2.359}$$

$$\beta_1 - \beta_2 = -\frac{\omega^2\alpha}{k}\left(\rho_1+\rho_2\right) \tag{2.360}$$

The matching means

$$-\rho_1 g\alpha + \beta_1 = -\rho_2 g\alpha + \beta_2 \tag{2.361}$$

$$\beta_1 - \beta_2 = g\alpha(\rho_1 - \rho_2) \tag{2.362}$$

This means

$$-\frac{\omega^2\alpha}{k}(\rho_1+\rho_2) = g\alpha(\rho_1-\rho_2) \tag{2.363}$$

$$\omega^2 = \frac{gk(\rho_2-\rho_1)}{\rho_1+\rho_2} \tag{2.364}$$

Thus, the phase and group velocity of the wave is given by

$$v_p^2 = \frac{\omega^2}{k^2} = \frac{g(\rho_2-\rho_1)}{k(\rho_1+\rho_2)} \tag{2.365}$$

$$v_p = \sqrt{\frac{g(\rho_2-\rho_1)}{k(\rho_1+\rho_2)}} \tag{2.366}$$

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \sqrt{\frac{g(\rho_2-\rho_1)}{k(\rho_1+\rho_2)}} \tag{2.367}$$

so that $v_p = v_g$ in this case.

In terms of λ we have

$$v_g = \sqrt{\frac{\lambda g(\rho_2 - \rho_1)}{2\pi(\rho_1 + \rho_2)}} \tag{2.368}$$

If we have an ideal gas law, then $\rho \sim \frac{N}{V}$ so that for $P = NVT$ we can replace V with P/NT

$$v_g = \sqrt{\frac{\lambda g(\frac{1}{T_2} - \frac{1}{T_1})}{2\pi(\frac{1}{T_1} + \frac{1}{T_2})}} = \sqrt{\frac{\lambda g(\frac{T_1 - T_2}{T_1 T_2})}{2\pi(\frac{T_1+T_2}{T_1 T_2})}} = \sqrt{\frac{\lambda g(T_1 - T_2)}{2\pi(T_1 + T_2)}} \tag{2.369}$$

We note that if $\rho_2 < \rho_1$ then this is unstable, (the heavy air falls and keeps falling), or equivalently, if $T_2 > T_1$.

2.33 Pressure On Two Perpendicular Walls with Incompressible Fluid

Two perpendicular semi-infinite walls, OA and OB in the diagram, intersect at the origin O, and block the two-dimensional hydrodynamic flow of an incompressible fluid of density ρ from a point source of strength K situated at the coordinates (a, b). Calculate the pressure on the walls.

Solution:

This should be time-dependent, or so one would think. We have (α is the strength of the source)

$$\frac{\partial \rho}{\partial t} + \cancel{\mathbf{v} \cdot \nabla\rho} + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = \alpha\delta(x - a, y - b) \tag{2.370}$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{v} = -\nabla p \tag{2.371}$$

If we assume there is a time asymptotic state, then

$$\rho \boldsymbol{\nabla} \cdot \mathbf{v} = \alpha\delta(x - a, y - b) \tag{2.372}$$

$$\mathbf{v} \cdot \nabla\mathbf{v} = -\nabla p \tag{2.373}$$

Usually, one would have $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ for incompressibility, but if there's a point source, then we nearly have it so we use that.

If we assume $v = \nabla\phi$ then ($K = \alpha/\rho$)

$$\nabla^2\phi = K\delta(x - a)\delta(x - b) \tag{2.374}$$

which is known to have a solution in 2D of

$$\phi = \frac{K}{2\pi}\rho \tag{2.375}$$

with $\rho = \sqrt{(x-a)^2 + (y-b)^2}$. We also need $v_x = 0$ at OB ($x = 0$) and $v_y = 0$ at OA ($y = 0$), so then we can construct the total solution by symmetry, ($\phi = 0$ on those boundaries). Denote $\phi_{\pm\pm} = \phi(\pm x, \pm y)$, so then the solution is given by

$$\phi_{\text{tot}} = \phi_{++} + \phi_{+-} + \phi_{--} + \phi_{-+} \tag{2.376}$$

which can be imagined as putting in source strengths at $(a, -b)$, $(-a, -b)$, and $(-a, b)$ to balance out on the two axes.

Thus the solution is

$$\begin{aligned}\phi = \frac{K}{4\pi}\Big\{ &\log\left[(x-a)^2 + (y-b)^2\right] + \log\left[(x-a)^2 + (y+b)^2\right] \\ &+ \log\left[(x+a)^2 + (y+b)^2\right] + \log\left[(x+a)^2 + (y-b)^2\right] \Big\}\end{aligned} \tag{2.377}$$

The pressure is given by Bernoulli's principle, $\frac{\rho}{2}v^2 + P = P_0$. P_0 is found by finding where $v = 0$

$$\begin{aligned}\frac{2\pi}{K}\nabla\phi = \hat{\mathbf{x}}&\left[\frac{(x-a)}{(x-a)^2+(y-b)^2} + \frac{(x-a)}{(x-a)^2+(y+b)^2} + \frac{x+a}{(x+a)^2+(y+b)^2} + \frac{x+a}{(x+a)^2+(y-b)^2}\right] \\ +\hat{\mathbf{y}}&\left[\frac{(y-b)}{(x-a)^2+(y-b)^2} + \frac{(y+b)}{(x-a)^2+(y+b)^2} + \frac{y+b}{(x+a)^2+(y+b)^2} + \frac{y-b}{(x+a)^2+(y-b)^2}\right]\end{aligned} \tag{2.378}$$

2.34 Tides from Sun and Moon

Let M and m be the masses of the sun and moon, and R and r be their respective distances from the earth. What is the ratio of the tides induced by these two bodies at the equator?

Solution:

The tides should be proportional to the force acting on them through gravity.

The force of gravity on some water mass m_w will be given by (on the nearest side of the Earth)

$$|\mathbf{F}_S| = F_S = \frac{GMm_w}{R^2} \tag{2.379}$$

$$|\mathbf{F}_M| = F_M = \frac{Gmm_w}{r^2} \tag{2.380}$$

We can imagine the difference in effect due to the water being at the center of the Earth instead, and how much force there would be on such a mass of water there.

$$|\mathbf{f}_S| = f_S = \frac{GMm_w}{(R+R_E)^2} \tag{2.381}$$

$$|\mathbf{f}_M| = f_M = \frac{Gmm_w}{(r+R_E)^2} \tag{2.382}$$

Since the tide height should be proportional to the force, the tide height difference is proportional to the difference between these two. Then, because $R_E \ll r, R$, we can write

$$f_S \approx \frac{GMm_w}{R^2 + 2RR_E} \approx \frac{GMm_w}{R^2}\frac{1}{1+\frac{2R_E}{R}} \approx \frac{GMm_w}{R^2}\left(1 - \frac{2R_E}{R}\right) \tag{2.383}$$

$$f_M \approx \frac{Gmm_w}{r^2}\left(1 - \frac{2R_E}{r}\right) \tag{2.384}$$

and thus the height difference will be

$$F_S - f_S \approx \frac{GMm_w}{R^2} - \frac{GMm_w}{R^2}\left(1 - \frac{2R_E}{R}\right) \approx \frac{2GMm_W R_E}{R^3} \tag{2.385}$$

$$F_M - f_M \approx \frac{Gmm_w}{r^2} - \frac{Gmm_w}{r^2}\left(1 - \frac{2R_E}{r}\right) \approx \frac{2Gmm_W R_E}{r^3} \tag{2.386}$$

so that the sun/moon ratio will be given by

$$\frac{F_S - f_S}{F_M - f_M} = \frac{Mr^3}{mR^3} \tag{2.387}$$

Because $r/R \ll 1$ and $M \gg m$, we see that this ratio isn't clearly anything. Based on this estimate,

$$\frac{F_S - f_S}{F_M - f_M} = \frac{(1.99 \times 10^{30}\,\mathrm{kg})(3.58 \times 10^{8}\,\mathrm{m})^3}{(7.35 \times 10^{22}\,\mathrm{kg})(1.5 \times 10^{11}\,\mathrm{m})^3} \approx 0.37 \tag{2.388}$$

So we see the moon is dominant, causing $1/1.37 \approx 73\%$ of the tide height.

2.35 Water Planet Self-Oscillation

Find the fundamental period of oscillation of an isolated mass of incompressible water, having the radius of the Earth 6300 km and vibrating under its own gravitational attraction. Assume the velocity flow is irrotational.

Solution:

Let's assume that we perturb the sphere with a small perturbation of the form $r \sim \delta e^{i(\mathbf{k}\cdot\mathbf{R} - \omega t)}$.

We then have that the acceleration of some bit of water near the surface along this perturbation is given by

$$\ddot{x} = \frac{GM_W}{(R_E + r)^2} \sim \frac{GM_W}{R_E^2}\left(1 - \frac{r}{R_E}\right) \tag{2.389}$$

and so

$$\ddot{r} = -\frac{GM_W}{R_E^3} r \tag{2.390}$$

$$-\omega^2 \delta e^{i(\mathbf{k}\cdot\mathbf{R} - \omega t)} = -\frac{GM_W}{R_E^3}\delta e^{i(\mathbf{k}\cdot\mathbf{R} - \omega t)} \tag{2.391}$$

$$\omega^2 = \frac{GM_W}{R_E^3} \tag{2.392}$$

Then using $M_W = \rho_W \frac{4}{3}\pi R_E^3$ and so

$$\omega^2 = \frac{4}{3}\pi G\rho_W \approx \frac{4}{3}(3.14159)(6.67\times10^{-11}\,\mathrm{m^3/kg\,s^2})(1000\,\mathrm{kg/m^3}) \approx 2.79\times10^{-7}\,\mathrm{rad^2/s^2} \tag{2.393}$$
$$\omega \approx 5.29\times10^{-4}\,\mathrm{rad/s} \tag{2.394}$$
$$\nu = \frac{\omega}{2\pi} \approx 8.41\times10^{-5}\,\mathrm{Hz} \tag{2.395}$$

This yields a period of about 11 887 s or about a seventh of a day.

If you were to carefully analyze this problem, which is apparently what the book desires, you have to write out the entire potential energy due to the deformation in spherical harmonics and then keep second order contributions from the perturbation (to keep first order contributions in the equations of motion).

Thus, you'd write

$$U = \frac{-1}{2}\int \mathrm{d}V\mathrm{d}V' \; \frac{G\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}')|} \tag{2.396}$$

with $\rho = \rho_0 + \delta\rho$ the density and ρ_0 the homogeneous sphere of radius R density.

You write out all the terms, and use for incompressibility that

$$\int_R^{R+h} \mathrm{d}\Omega\mathrm{d}r \; r^2 = 0 \tag{2.397}$$

where h represents the perturbation in space, (and so everything beyond the sphere must average out to zero since we have incompressible flow).

This can be translated into

$$\int \mathrm{d}\Omega \; \frac{(R+h)^3 - R^3}{3} = \int \mathrm{d}\Omega \; \left(R^2 h + Rh^2\right) = 0 \tag{2.398}$$
$$R\int \mathrm{d}\Omega \; h = -\int \mathrm{d}\Omega \; h^2 \tag{2.399}$$

One then notes that

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{R}\sum \frac{4\pi}{2l+1} Y_{lm}(\hat{\mathbf{n}})Y^*_{lm}(\hat{\mathbf{n}}') \tag{2.400}$$

Because of incompressibility the $l=0$ mode only exists when other l modes does, and so is not independent. We note $l=1$ which is just a spatial displacement creates no change in gravitational energy. Then use $\mathbf{v} = \nabla\phi$ with incompressibility to find the kinetic energy, form the Hamiltonian, look at the coefficients and find

$$\omega_l^2 = \frac{8\pi\rho_0 G l(l-1)}{3(2l+1)} \tag{2.401}$$
$$\omega_2^2 = \frac{16\pi\rho_0 G}{15} \tag{2.402}$$

Note that our early, rougher estimate, is not wildly off (here $l=2$ is the lowest mode of use, $[4/3 - 16/15 = 4/15 \approx 0.26]$)

2.36 Time on Clock Between Reference Systems

The coordinate systems S_1 and S_2 move along the x-axis of a reference coordinate frame S, with velocities v_1 and v_2 respectively, referred to S. The time measured in S for the hand of a clock in S_1 to go around once, is t. What is the time interval t_2 measured in S_2 for the hand to go around?

Solution:

In S_1 the clock is stationary so that t is the proper time. We then need to find the velocity of S_2 with respect to S_1. For this, we require the rules for velocity addition in relativity.

We use (the difference in value between the S_1 and S frame is velocity v_1 and find what v_2 is in the S_1 frame)

$$v_2' = \frac{v_2 - v_1}{1 - \frac{v_1 v_2}{c^2}} \tag{2.403}$$

This gives us a $\beta_2' = v_2'/c$ for the time dilation factor. So

$$t_2 = \gamma t = \frac{1}{\sqrt{1 - \left(\frac{v_2'}{c}\right)^2}} t = \left[1 - \frac{(\beta_2 - \beta_1)^2}{(1 - \beta_1\beta_2)^2}\right]^{-1/2} t \tag{2.404}$$

$$= \left[\frac{1 - 2\beta_1\beta_2 + \beta_1^2\beta_2^2 - \beta_1^2 - \beta_2^2 + 2\beta_1\beta_2}{(1 - \beta_1\beta_2)^2}\right]^{-1/2} = (1 - \beta_1\beta_2)\left[1 + \beta_1^2\beta_2^2 - \beta_1^2 - \beta_2^2\right]^{-1/2} t \tag{2.405}$$

$$= (1 - \beta_1\beta_2)\left[(1 - \beta_1^2)(1 - \beta_2)^2\right]^{-1/2} t \tag{2.406}$$

$$= \gamma_1\gamma_2(1 - \beta_1\beta_2)t \tag{2.407}$$

with $\gamma_i = (1 - \beta_i^2)^{-1/2}$ and $\beta_i = v_i/c$.

2.37 Constant Acceleration Rocket Into Space

Solution:

A rocket is shot out from the earth into interstellar space. Except for a short time in the beginning, the acceleration of the rocket, as measured by the passengers, is constant. The rocket has been aimed at a star a fixed distance from the earth, and moves on a straight line. According to clocks inside the rocket, how long will it take to get to the star? Denote the constant distance and acceleration by D and a' respectively.

Solution:

We start with the velocity addition rule for some velocity v, and have $u = \mathrm{d}x/\mathrm{d}t$ in the Earth frame and the $u' = \frac{\mathrm{d}x'}{\mathrm{d}t'}$ in the frame going with velocity v at that particular time (corresponding to the rocket frame), and we have

$$u' = \frac{u-v}{1-\frac{uv}{c^2}} \tag{2.408}$$

$$\frac{\mathrm{d}u'}{\mathrm{d}t} = \frac{\frac{\mathrm{d}u}{\mathrm{d}t}(1-\frac{uv}{c^2}) - -\frac{\mathrm{d}u}{\mathrm{d}t}\frac{v}{c^2}(u-v)}{(1-\frac{uv}{c^2})^2} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{1-\frac{uv}{c^2}+\frac{vu}{c^2}-\frac{v^2}{c^2}}{(1-\frac{uv}{c^2})^2} \tag{2.409}$$

$$\frac{\mathrm{d}u'}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{1-\frac{v^2}{c^2}}{(1-\frac{uv}{c^2})^2} \tag{2.410}$$

Further, we use $u = v$ so that the above with $\beta = v/c$ becomes

$$\frac{\mathrm{d}u'}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{1-\beta^2}{(1-\beta^2)^2} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{1}{1-\beta^2} = \frac{\mathrm{d}u}{\mathrm{d}t}\gamma^2 \tag{2.411}$$

Then use

$$c\gamma t - \beta\gamma x = ct' \tag{2.412}$$

$$-c\beta\gamma t + \gamma x = x' \tag{2.413}$$

$$c\gamma(1-\beta^2)t = c\gamma\gamma^{-2}t = ct' - x' \tag{2.414}$$

$$c\frac{\mathrm{d}t}{\mathrm{d}t'} = c\gamma - \gamma\cancel{\frac{\mathrm{d}x'}{\mathrm{d}t'}} \tag{2.415}$$

and so

$$\frac{\mathrm{d}u'}{\mathrm{d}t} = \frac{\mathrm{d}u'}{\mathrm{d}t'}\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{\mathrm{d}u'}{\mathrm{d}t'}\gamma^{-1} = \frac{\mathrm{d}u}{\mathrm{d}t}\gamma^2 \tag{2.416}$$

$$\frac{\mathrm{d}u'}{\mathrm{d}t'} = \frac{\mathrm{d}u}{\mathrm{d}t}\gamma^3 \tag{2.417}$$

Thus, write $\frac{\mathrm{d}u'}{\mathrm{d}t'} = a'$ and $\frac{\mathrm{d}u}{\mathrm{d}t} = a$ as the accelerations in the rocket and Earth frame, respectively. Thus,

$$\int \mathrm{d}t' = \int \frac{1}{\gamma}\mathrm{d}t = \int \frac{1}{\gamma a}\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t = \int \frac{1}{\gamma a'\gamma^{-3}}\mathrm{d}u = \frac{1}{a'}\int \mathrm{d}u\,\frac{1}{1-u^2/c^2} \tag{2.418}$$

Then we have

$$T' = \int \mathrm{d}t' = \frac{c}{2a'}\ln\left(\frac{1+v_f/c}{1-v_f/c}\right) = \frac{c}{a'}\,\mathrm{arctanh}\left(\frac{v_f}{c}\right) \tag{2.419}$$

with v_f the final speed of the rocket as seen from Earth.

We can find v_f via

$$D = \int u\,\mathrm{d}t = \int \frac{u}{a}\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{a'}\int v\gamma^3\,\mathrm{d}v = \frac{c^2}{a'}\left[\frac{1}{\sqrt{1-v_f^2/c^2}} - 1\right] \tag{2.420}$$

Thus

$$v_f^2 = c^2 - \frac{c^2}{(1 + Da'/c^2)^2} \tag{2.421}$$

and so

$$T' = \frac{c}{2a'} \ln \left(\frac{1 + \sqrt{1 - \frac{1}{(1+Da'/c^2)^2}}}{1 - \sqrt{1 - \frac{1}{(1+Da'/c^2)^2}}} \right) = \frac{c}{a'} \operatorname{arctanh} \left(\sqrt{1 - \frac{1}{(1 + Da'/c^2)^2}} \right) \tag{2.422}$$

2.38 Relativistic Oscillator

A particle of rest mass m moves on the x-axis of a Galilean frame of reference and is attracted to the origin O by a force (time rate of change of momentum) $m\omega^2 x$. It performs oscillations of amplitude a. Express the period of this relativistic oscillator in terms of a definite integral, and obtain an approximate value for this integral.

Solution:

We must have the force given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(m\gamma \frac{\mathrm{d}x}{\mathrm{d}t} \right) = -m\omega^2 x \tag{2.423}$$

The period will pretty clearly be 4 times as long as the time it takes to go from $x = 0$ to $x = a$ by the symmetry of this situation.

We use

$$\frac{\mathrm{d}}{\mathrm{d}t} (\gamma v) = -\omega^2 x \tag{2.424}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} (\gamma v) = \frac{\mathrm{d}\gamma}{\mathrm{d}t} v + \gamma \frac{\mathrm{d}v}{\mathrm{d}t} = v \frac{-1(-2v/c^2)\gamma^3}{2} \frac{\mathrm{d}v}{\mathrm{d}t} + \gamma \frac{\mathrm{d}v}{\mathrm{d}t} \tag{2.425}$$

$$= (\beta^2\gamma^2 + 1)\gamma \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\beta^2 + 1 - \beta^2}{1 - \beta^2} \gamma \frac{\mathrm{d}v}{\mathrm{d}t} = \gamma^3 \frac{\mathrm{d}v}{\mathrm{d}t} \tag{2.426}$$

We can then use

$$v\gamma^3 \frac{\mathrm{d}v}{\mathrm{d}t} = c^2 \frac{\gamma^3}{2} \frac{\mathrm{d}v^2/c^2}{\mathrm{d}t} = -\frac{c^2\gamma^3}{2} \frac{\mathrm{d}(1 - v^2/c^2)}{\mathrm{d}t} = c^2 \frac{-\gamma^3}{2} \frac{\mathrm{d}\gamma^{-2}}{\mathrm{d}t} = c^2 \frac{\gamma^3}{\gamma^3} \frac{\mathrm{d}\gamma}{\mathrm{d}t} = c^2 \frac{\mathrm{d}\gamma}{\mathrm{d}t} \tag{2.427}$$

So we then have (define $k = \omega/c$) (note $\gamma = 1$ at $t = T/4$ because then $v = 0$)

$$\int \mathrm{d}t \; v \frac{\mathrm{d}(\gamma v)}{\mathrm{d}t} = -\int \mathrm{d}t \; v \frac{\omega^2}{c^2} x \tag{2.428}$$

$$\int_{T/4}^{t} \mathrm{d}t \; \frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\int_a^x \mathrm{d}x \; k^2 x = -k^2 \frac{(x^2 - a^2)}{2} = \frac{k^2}{2}(a^2 - x^2) \tag{2.429}$$

$$\gamma - 1 = k^2 \frac{a^2 - x^2}{2} \tag{2.430}$$

$$\frac{1}{\left(1+\frac{k^2(a^2-x^2)}{2}\right)^2} = 1 - \frac{1}{c^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \tag{2.431}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c\sqrt{1-\frac{1}{\left(1+\frac{k^2(a^2-x^2)}{2}\right)^2}} \tag{2.432}$$

and so

$$\int_0^{T/4}\mathrm{d}t = \frac{T}{4} = \int_0^a \frac{\mathrm{d}x}{c\sqrt{1-\frac{1}{\left(1+\frac{k^2(a^2-x^2)}{2}\right)^2}}} = \frac{1}{c}\int_0^a \frac{\mathrm{d}x}{\sqrt{1-\frac{1}{\left(1+\frac{k^2a^2(1-\frac{x^2}{a^2})}{2}\right)^2}}} \tag{2.433}$$

Let $\xi = x/a$ and $\lambda^2 = k^2a^2 = \frac{\omega^2a^2}{c^2}$ and we then have

$$\frac{T}{4} = \frac{a}{c}\int_0^1 \mathrm{d}\xi\, \frac{1}{\sqrt{1-\frac{1}{\left(1+\frac{\lambda^2}{2}(1-\xi^2)\right)^2}}} = \frac{a}{c}\int_0^1 \mathrm{d}\xi\, \frac{1+\frac{\lambda^2}{2}(1-\xi^2)}{\sqrt{\lambda^2(1-\xi^2)+\frac{\lambda^4}{4}(1-\xi^2)^2}} \tag{2.434}$$

$$= \frac{a}{c}\int_0^1 \mathrm{d}\xi\, \frac{1+\frac{\lambda^2}{2}(1-\xi^2)}{\sqrt{\lambda^2(1-\xi^2)\left[1+\frac{\lambda^2}{4}(1-\xi^2)\right]}} = \frac{1}{\omega}\int_0^1 \mathrm{d}\xi\, \frac{1+\frac{\lambda^2}{2}(1-\xi^2)}{\sqrt{(1-\xi^2)\left[1+\frac{\lambda^2}{4}(1-\xi^2)\right]}} \tag{2.435}$$

If we assume $\lambda \ll 1$ then we can write the integrand as a power series,

$$\frac{1}{\omega}\int_0^1 \mathrm{d}\xi\, \frac{1+\frac{\lambda^2}{2}(1-\xi^2)}{\sqrt{(1-\xi^2)\left[1+\frac{\lambda^2}{4}(1-\xi^2)\right]}} = \frac{1}{\omega}\int_0^1 \mathrm{d}\xi\, \left[\frac{1}{\sqrt{1-\xi^2}} + \lambda^2\left\{\frac{\sqrt{1-\xi^2}}{2} - \frac{1}{8}\sqrt{1-\xi^2}\right\} + \mathcal{O}(\lambda^4)\right] \tag{2.436}$$

$$= \frac{1}{\omega}\left[\arcsin(\xi) + \lambda^2\frac{3}{8}\frac{1}{2}\left\{\xi\sqrt{1-\xi^2} + \arcsin(\xi)\right\}\right]_0^1 \tag{2.437}$$

$$= \frac{1}{\omega}\left[\frac{\pi}{2} + \frac{3\omega^2a^2}{16c^2}\left\{\frac{\pi}{2}\right\}\right] = \frac{\pi}{2\omega} + \frac{3\omega\pi a^2}{32c^2} + \mathcal{O}\left(\frac{\omega^3a^4}{c^4}\right) \tag{2.438}$$

so

$$T = \frac{2\pi}{\omega} + \frac{3\omega\pi a^2}{8c^2} + \mathcal{O}\left(\frac{\omega^3a^4}{c^4}\right) = \frac{2\pi}{\omega}\left[1 + \frac{3\omega^2}{16c^2} + \mathcal{O}\left(\frac{\omega^4a^4}{c^4}\right)\right] \tag{2.439}$$

Note that we could have used conservation of energy at the beginning and found v via

$$m\gamma c^2 + \frac{m\omega^2x^2}{2} = mc^2 + \frac{m\omega^2a^2}{2} \tag{2.440}$$

yielding the same $v = \mathrm{d}x/\mathrm{d}t$ we found above.

2.39 Total Energy of Antiproton-Deuterium Reaction

Antiprotons are captured at rest in deuterium, giving rise to the reaction $\bar{\mathrm{p}} + \mathrm{D} \longrightarrow \mathrm{n} + \pi^0$ (In this problem we ignore other possibilities). Determine the π^0 total energy. The rest masses are $M(\bar{\mathrm{p}}) = 938.2\,\mathrm{MeV}$, $M(\mathrm{D}) = 1875.5\,\mathrm{MeV}$, $M(n) = 939.5\,\mathrm{MeV}$, and $M(\pi^0) = 135.0\,\mathrm{MeV}$.

Solution:

Let $p_{\bar{p}}$, p_D, p_n, and p_π be the four-vectors for $\bar{p}$, D, n, and π^0. By conservation of momentum, we must have ($c = 1$ units)

$$p_{\bar{p}} + p_D = p_n + p_\pi \tag{2.441}$$
$$p_{\bar{p}} + p_D - p_\pi = p_n \tag{2.442}$$
$$m_{\bar{p}} + m_D = E_n + E_\pi \tag{2.443}$$

We have (assume that n and π^0 have some momentum in the x direction)

$$p_{\bar{p}} = (m_{\bar{p}}, 0, 0, 0)^\intercal \tag{2.444}$$
$$p_D = (m_D, 0, 0, 0)^\intercal \tag{2.445}$$
$$p_n = (E_n, p_x, 0, 0)^\intercal \tag{2.446}$$
$$p_\pi = (E_\pi, -p_x, 0, 0)^\intercal \tag{2.447}$$

We then find (use the square of a particle's four vector is the mass of a particle, with $m_2 = m_{\bar{p}}+m_D$)

$$(m_{\bar{p}} + m_D + E_\pi)^2 - p_x^2 = m_n^2 \tag{2.448}$$
$$(m_2 - E_\pi)^2 = m_n^2 + p_x^2 \tag{2.449}$$
$$(m_2 - E_\pi)^2 - p_x^2 - m_n^2 = 0 \tag{2.450}$$

We can then use that $p_\pi^2 = E_\pi^2 - p_x^2 = m_\pi^2$ and write

$$(m_2 - E_\pi)^2 - E_\pi^2 + m_\pi^2 - m_n^2 = 0 \tag{2.451}$$
$$-2E_\pi m_2 + m_2^2 + m_\pi^2 - m_n^2 = 0 \tag{2.452}$$
$$E_\pi = \frac{m_\pi^2 + m_2^2 - m_n^2}{2m_2} \tag{2.453}$$

Alternatively, (and less elegantly), use conservation of energy with the original four-vector's squared We also must have $m_a^2 \equiv m_2^2 - m_n^2 - m_\pi^2$

$$m_2^2 = m_n^2 + m_\pi^2 + 2E_nE_\pi + 2p_x^2 \tag{2.454}$$
$$\frac{m_a^2}{2} = (m_2 - E_\pi)E_\pi + p_x^2 \tag{2.455}$$
$$-p_x^2 = (m_2 - E_\pi)E_\pi - \frac{m_a^2}{2} \tag{2.456}$$

Thus, substituting above, (use $m_p = m_{\bar{p}}$)

$$(m_2 - E_\pi)^2 + m_2E_\pi - E_\pi^2 - \frac{m_2^2 - m_n^2 - m_\pi^2}{2} - m_n^2 = 0 \tag{2.457}$$
$$\cancel{E_\pi^2} - 2E_\pi m_2 + m_2^2 + m_2E_\pi - \cancel{E_\pi^2} - \frac{m_2^2 - m_n^2 - m_\pi^2}{2} - m_n^2 = 0 \tag{2.458}$$
$$-E_\pi m_2 + \frac{m_2^2 - m_n^2 + m_\pi^2}{2} = 0 \tag{2.459}$$
$$\tag{2.460}$$

yielding once again

$$E_\pi = \frac{m_2^2 + m_\pi^2 - m_n^2}{2m_2} \tag{2.461}$$

Or, putting units back in, we find

$$E_\pi = \frac{(938.2 + 1875.5)^2 + (135.0)^2 - (939.5)^2}{2(938.2 + 1875.5)} \times 1\,\text{Mev} \tag{2.462}$$

$$= 1253\,\text{MeV} = 1.253\,\text{GeV} \tag{2.463}$$

2.40 Positron and Electron Pair Creation

A positron (Energy E_+, momentum $\mathbf{p}_+$) and an electron (Energy E_-, momentum $\mathbf{p}_-$) are produced in a pair-creation process. (a) What is the velocity of the frame in which the pair has zero momentum (barycentric frame)? (b) Deduce the energy either particle has in this frame, and (c) give an expression for the magnitude of the relative velocity between the particles, i.e., the velocity of one particle as seen by an observer attached to the other.

Solution:

(a)

We use the Lorentz transformation for momentum is given by (for change of velocity v, with v's direction being parallel)

$$p'_\parallel = \gamma(p_\parallel - Ev) \tag{2.464}$$

$$p'_\perp = p_\perp \tag{2.465}$$

So $\mathbf{p}' = 0$ when $p_\parallel = Ev$ and $p_\perp = 0$. Thus when

$$\mathbf{v} = \mathbf{p}/E \tag{2.466}$$

where E is the total energy and $\mathbf{p}$ the total momentum.

Thus this is the velocity of the frame in which the pair has zero momentum.

$$\mathbf{v} = \frac{\mathbf{p}_+ + \mathbf{p}_-}{E_+ + E_-} \tag{2.467}$$

(b)

Each must have equal momentum and energy. The square of the four vectors must be equal, as well, so

$$4E'^2 = (E_+ + E_-)^2 - (\mathbf{p}_+ + \mathbf{p}_-)^2 = (E_+ + E_-)^2 - (E_+ + E_-)^2\mathbf{v}^2 \tag{2.468}$$

$$= (E_+ + E_-)^2(1 - v^2) = \frac{(E_+ + E_-)^2}{\gamma^2} \tag{2.469}$$

where $\gamma = 1/\sqrt{1-v^2}$, or

$$E' = \frac{E_+ + E_-}{2\gamma} \tag{2.470}$$
$$E' = \frac{\sqrt{(E_+ + E_-)^2 - (\mathbf{p}_+ + \mathbf{p}_-)^2}}{2} \tag{2.471}$$

(c)

From the barycentric frame, we can figure out what a velocity in the opposite direction would look like to a mover in the other frame. That is, look at what velocity $-v$ looks like in the primed frame which is moving at velocity v relative to the unprimed frame.

We have u' is the velocity of the other particle in the frame of one particle.

$$u' = v_{\text{rel}} = \frac{v+v}{1+\frac{v^2}{c^2}} = \frac{2v}{1+\frac{v^2}{c^2}} = \frac{2|\mathbf{p}_+ + \mathbf{p}_-|}{(E_+ + E_-) + \frac{|\mathbf{p}_+ + \mathbf{p}_-|^2}{c^2(E_+ + E_-)}} \tag{2.472}$$

The book claims that $I = (\mathbf{p}_+ - \mathbf{p}_-)^2 - (E_+ - E_-)^2$ is in an invariant and then uses that for the electron rest frame $\mathbf{p}_- = \mathbf{0}$, $E_- = m$ and $E_+ = m/\sqrt{1-v_{\text{rel}}^2}$ and solves for v_{rel}.

$$v_{\text{rel}} = \sqrt{1 - \frac{1}{(1+\frac{I}{2m^2})^2}} \tag{2.473}$$

I am unaware of anywhere that proves that I is invariant. In fact, it would clearly be zero in the center of momentum frame, so is not invariant. If instead, they meant $I = I = (\mathbf{p}_+ + \mathbf{p}_-)^2 - (E_+ + E_-)^2$, then with their assumptions [let $u = v_{\text{rel}}$ for convenience and $\gamma^2 = 1/(1-u^2)$]

$$I = m^2\gamma^2u^2 - m^2(\gamma+1)^2 = m^2\left[\gamma^2u^2 - \gamma^2 - 2\gamma - 1\right] = m^2\left[\gamma^2(u^2-1) - 2\gamma - 1\right] \tag{2.474}$$
$$= m^2\left[-\gamma^2\gamma^{-2} - 2\gamma - 1\right] = -2m^2(\gamma+1) \tag{2.475}$$

and so then one would actually find

$$\frac{-I}{2m^2} - 1 = \gamma = \frac{1}{\sqrt{1-u^2}} \tag{2.476}$$
$$1 - u^2 = \frac{1}{(1+\frac{I}{2m^2})^2} \tag{2.477}$$
$$u = \sqrt{1 - \frac{1}{(1+\frac{I}{2m^2})^2}} \tag{2.478}$$

as they desired.

I know that my solution above must be correct, as well, since in the non-relativistic limit we must find $v_{\text{rel}} = 2v$, and for the ultrarelativistic limit we should find $u' = c$ as we do above.

From $I = (v^2-1)(E_+ - E_-)^2 = -\gamma^{-2}(E_+ + E_-)^2 = -\gamma^{-2}4\gamma^2E'^2 = -4E'^2$ we see that

$$u = \frac{\sqrt{\frac{-|I|}{m^2} + \frac{I^2}{4m^4}}}{1 - \frac{|I|}{2m^2}} = \sqrt{\frac{|I|}{m^2}}\frac{\sqrt{\frac{|I|}{4m^2} - 1}}{1 - \frac{|I|}{2m}} = \frac{2E'}{m}\frac{\sqrt{\frac{E'^2}{m^2} - 1}}{1 - \frac{E'}{m}} \tag{2.479}$$

2.41 Relativistic Electron in Capacitor

A fast (extremely relativistic) electron enters a capacitor at an angle α as shown in the sketch. V is the voltage across the capacitor and d is the distance between plates. Give an equation for the path of the electron in the capacitor.

Solution:

The x-direction velocity will be about $c\cos\alpha$ which is unaffected by the Capacitor potential. The y-direction velocity will then obviously be $c\sin\alpha$. The potential across the capacitor is V,and so the potential at some distance y above the capacitor is given by yV/d. Let's look at this in the frame where the capacitor is at rest. If we assume γ is some constant (so basically unchanged) then

$$\gamma m_e \frac{\mathrm{d}v_y}{\mathrm{d}t} = -q_e V/d \tag{2.480}$$

$$v_y = \frac{-Vq_e}{dm_e\gamma}t + v_{0y} \tag{2.481}$$

$$y = -\frac{q_e V}{dm_e\gamma}t^2 + v_{0y}t \tag{2.482}$$

while x is

$$x = v_{0x}t \tag{2.483}$$

We can see that the larger γ, the less the correction to y makes.

If we wish to be more precise we can use

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_x) = \frac{\mathrm{d}}{\mathrm{d}t}(p\cos\theta) = 0 \tag{2.484}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_y) = \frac{\mathrm{d}}{\mathrm{d}t}(p\sin\theta) = 0 \tag{2.485}$$

and can use $p\cos\theta = p_0\cos\alpha$ which becomes a constant of motion

$$p\cos\theta = p_0\cos\alpha = C_1 p\sin\theta = p\tan\theta\cos\theta = C_1\tan\theta = C_1\frac{\mathrm{d}y}{\mathrm{d}x} \tag{2.486}$$

We can use that $\frac{\mathrm{d}}{\mathrm{d}t} \approx c\frac{\mathrm{d}}{\mathrm{d}s}$ where $\mathrm{d}s = \sqrt{\mathrm{d}x + \mathrm{d}y}$ is the arclength of the path. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(p\sin\theta) = C_1 c\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{-q_e V}{d} \tag{2.487}$$

We then have to use

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\mathrm{d}f}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} + \frac{\mathrm{d}f}{\mathrm{d}y}\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}} \tag{2.488}$$

because $f = \mathrm{d}y/\mathrm{d}x$ then $\mathrm{d}f/\mathrm{d}y = 0$ so

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{-q_e V}{dcC_1}\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \tag{2.489}$$

This is a nonlinear differential equation. It happens to have a solution, of

$$y = A - \frac{dcC_1}{q_e V}\cosh\left(\frac{q_e V}{dcC_1}\left[x - a\right]\right) \tag{2.490}$$

one can find the constants via $y(0) = 0$ and $y'(0) = \tan\alpha$.

2.42 Give Probability Cone for Decaying Meson

The neutral π^0 meson, of rest mass M, decays into two photons. The angular distribution of these γ-rays is isotropic in the rest system of the π^0. If in the laboratory the π^0 travels with velocity v in the z-direction, what is the probability $P(\theta)\,\mathrm{d}\Omega$ that a photon is emitted in the solid angle $\mathrm{d}\Omega$ about θ, when the meson decays in flight? Here θ is the angle as measured in the laboratory with respect to the z-axis, and v may be comparable to the speed of light.

Solution:

We use the Lorentz transformations for the momentum-energy 4-vector and see that for a photon emitted that (the primed frame is the rest frame of the pion and the unprimed frame is the laboratory frame) using $E = p$ for the photon ($v/c = \beta$)

$$p'_z = \gamma p_z - \beta\gamma p' \tag{2.491}$$
$$p'_x = p_x \tag{2.492}$$
$$p'_y = p_y \tag{2.493}$$
$$p' = \gamma p - \beta\gamma p_z \tag{2.494}$$

We can then reformulate these into angles using $\sqrt{p_x^2 + p_y^2} = p\sin\theta$ and $p_z = p\cos\theta$ so that

$$p'\cos\theta' = \gamma p(\cos\theta - \beta) \tag{2.495}$$
$$p'\sin\theta' = p\sin\theta \tag{2.496}$$
$$p' = \gamma p(1 - \beta\cos\theta) \tag{2.497}$$
$$\sin\theta' = \frac{p\sin\theta}{p'} = \frac{\sin\theta}{\gamma(1 - \beta\cos\theta)} \tag{2.498}$$

Therefore, we find

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)} \tag{2.499}$$
$$\cos\theta' = \frac{\gamma(\cos\theta - \beta)}{\sin\theta}\sin\theta' = \frac{\gamma(\cos\theta - \beta)}{\sin\theta}\frac{\sin\theta}{\gamma(1 - \beta\cos\theta)} = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \tag{2.500}$$

We can then use that $P(\theta') = \frac{1}{4\pi}$ because of it being uniform and that the probability is conserved so that $P(\theta')\,\mathrm{d}\Omega' = P(\theta)\,\mathrm{d}\Omega$

$$P(\theta)\,\mathrm{d}\Omega = -\frac{1}{4\pi}\,\mathrm{d}\cos\theta'\,\mathrm{d}\varphi' = -\frac{1}{4\pi}\frac{\mathrm{d}\cos\theta'}{\mathrm{d}\cos\theta}\,\mathrm{d}\cos\theta\frac{\mathrm{d}\varphi'}{\mathrm{d}\varphi}\,\mathrm{d}\varphi \tag{2.501}$$

We clearly have since φ' is independent of z' that $\mathrm{d}\varphi'/\mathrm{d}\varphi = 1$ and from above we have

$$\frac{\mathrm{d}\cos\theta'}{\mathrm{d}\cos\theta} = \frac{\mathrm{d}u}{\mathrm{d}w} = \frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{w-\beta}{1-\beta w}\right) = \frac{(1-\beta w)-(w-\beta)(-\beta)}{(1-\beta w)^2} \tag{2.502}$$

$$= \frac{1-\beta^2}{(1-\beta w)^2} = \frac{1-\beta^2}{(1-\beta\cos\theta)^2} \tag{2.503}$$

Thus

$$P(\theta')\,\mathrm{d}\Omega' = -\frac{1}{4\pi}\frac{1-\beta^2}{(1-\beta\cos\theta)^2}\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi = \frac{1-\beta^2}{4\pi(1-\beta\cos\theta)^2}\,\mathrm{d}\Omega \tag{2.504}$$

so that

$$P(\theta) = \frac{1-\beta^2}{4\pi(1-\beta\cos\theta)^2} = \frac{1}{4\pi\gamma^2(1-\beta\cos\theta)^2} \tag{2.505}$$

As a check, for $\theta = \pi/2$ and $\gamma = 1$ we should simply get $1/4\pi$. Also if $\gamma \gg 1$ then $P(\pi/2) \ll 1$ because the velocity is near c in the z direction and so it become difficult for the photons to conserve momentum and have a large excursion from the z axis (because they can't exceed speed c and most of the speed is along the z axis).

$$P(\pi/2) = \frac{1}{4\pi\gamma^2} \tag{2.506}$$

which corresponds with my intuition.

2.43 Cosmic Ray Interactions

(a) If neutrons from a cosmic-ray interaction one light-year from the earth were to reach here with a probability of $1/e$ or greater, what must their minimum energy be? (b) If they then decay, what is the maximum angle to the flight path at which their decay electrons could be produced? (c) What is the maximum angle for the decay neutrinos? (d) At the angle calculated in (c), what is the maximum energy of the neutrino?

Solution:

(a)

Neutrons have a free decay half-life of $10.3\,\mathrm{min} = 618\,\mathrm{s}$. Thus, the time for free neutrons to have a $1/e$ probability is

$$\tau = \frac{t_{1/2}}{\ln 2} \approx 891.6\,\mathrm{s} \tag{2.507}$$

We see then that the time dilation must be given by

$$\gamma = t/t' \tag{2.508}$$

where t is the lab-time and t' is the time in the frame of the free neutron. For $1/e$ we require $t' = \tau$ from above. The t must be $t = d/v = d/(\beta c)$ where d is one light-year, so that

$$\gamma\beta = \frac{d}{ct'} \equiv \alpha \tag{2.509}$$

$$\gamma\sqrt{1 - \frac{1}{\gamma^2}} = \alpha \tag{2.510}$$

$$\gamma^2 - 1 = \alpha^2 \tag{2.511}$$

$$\gamma^2 = 1 + \alpha^2 \tag{2.512}$$

$$\gamma = \sqrt{1 + \alpha^2} \tag{2.513}$$

Then we find the energy via $E = m\gamma$ So

$$\gamma = \sqrt{1 + \frac{d}{ct'}} = \sqrt{1 + \left(\frac{1\,\mathrm{yr}}{891.6\,\mathrm{s}}\right)^2} \approx \sqrt{1 + \left(\frac{\pi \times 10^7}{891.6}\right)^2} \approx \sqrt{1 + 1.242 \times 10^9} \approx 35\,240 \tag{2.514}$$

Thus, using that the neutron mass is $939.5\,\mathrm{MeV/c^2}$ we get

$$E = 35\,240(939.5\,\mathrm{MeV}) \approx 3.310 \times 10^7\,\mathrm{MeV} = 33\,310\,\mathrm{GeV} = 33.310\,\mathrm{TeV} \tag{2.515}$$

(b)

The neutron will decay into a proton, an electron, and a neutrino.

Thus using four vectors in the earth frame, we see we have

$$p_n = p_e + p_\nu \tag{2.516}$$

$$p_n = (E_n, p, 0, 0) \tag{2.517}$$

$$p_e = (E_e, p_{ex}, p_{ey}, 0) \tag{2.518}$$

$$p_\nu = (E_\nu, p_{\nu x}, -p_{ey}, 0) \tag{2.519}$$

We then must have

$$p_n - p_e = p_\nu \tag{2.520}$$

$$p_n^2 + p_e^2 - 2p_n \cdot p_e = p_\nu^2 \tag{2.521}$$

$$m_n^2 + m_e^2 - 2(E_n E_e - \mathbf{p}_n \cdot \mathbf{p}_e) = m_\nu^2 \tag{2.522}$$

$$\mathbf{p}_n \cdot \mathbf{p}_e = E_n E_e + \frac{m_\nu^2 - m_n^2 - m_e^2}{2} \tag{2.523}$$

$$|p_n||p_e|\cos\theta = E_n E_e + \frac{m_\nu^2 - m_n^2 - m_e^2}{2} \tag{2.524}$$

$$\cos\theta = \frac{E_n E_e}{|p_n||p_e|} + \frac{m_\nu^2 - m_n^2 - m_e^2}{2|p_n||p_e|} \tag{2.525}$$

2.44 Special Relativistic Perihelion Precession

A precession of the perihelion of planetary trajectories has been derived from the general theory of relativity. However, even the special theory of relativity predicts such an effect because of the dependence of inertial mass on velocity. Derive a formula for the *special-relativistic* precession for a planet of given angular momentum L, rest mass m, and energy E, moving in the gravitational potential of the sun. [Hint. Use polar coordinates $u = 1/r$ and θ, and find a differential equation involving u and θ, but not involving time explicitly].

Solution:

2.45 Balloon Accelerates in Space

A helium-filled balloon floats inside a closed container filled with air at STP in interstellar space. The container accelerates in a given direction, with acceleration equal to that due to gravity at the surface of the earth. Which way does the balloon move relative to the acceleration.

Solution: